Algebraic Geometry on Compact Riemann Surfaces

Chunye Yang

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This is the notes of the Algebraic Geometry class taught by Prof. Harry Tamvakis during the fall semester of 2026 at the University of Maryland which mainly focuses on Compact Riemann Surface. This will be an algebraic geometry class that is basically with analytic flavor where even if Serre duality is proved using analytic methods. We begin with compact Riemann surfaces and use the theory we develop about it to transition to algebraic geometry in higher dimensions.

1 Lecture 1

2 Lecture 2

We will head to the main content of this course from this section. Recall that a 1 **dimensional complex manifold** is a 2 dimensional topological manifold X together with charts (U, φ) such that for any two such charts (U_1, φ_1) and (U_2, φ_2) are **compatible** in the sense that:

$$\varphi_2 \circ \varphi_1^{-1} : \varphi(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is holomorphic. Since $\varphi_2 \circ \varphi_1^{-1}$ is bijection, we know that it's inverse is also holomorphic from the local behavior of a holomorphic function.

A **complex structure** on X is a family of pairs of charts $\{(U_i, \varphi_i)\}_{i \in I}$ which are pairwise compatible such that $\cup_i U_i = X$. In general, we will always assume that such a structure is **maximal** which means you can not throw in more charts such that they still remain compatible (such a maximal structure always exists by Zorn's lemma). The pairs are called **holomorphic charts**.

Remark 2.1. This is a good place to review some complex variable theory. Notice that $\varphi_2 \circ \varphi_1^{-1}$ is a function from $\mathbb{R}^2$ to $\mathbb{R}^2$. For any such a real function F, recall that a criterion for it to be holomorphic at a point p is that it's differentiable in the usual real analysis sense in a neighborhood of p and that the derivative DF at any point in the neighborhood has to be **complex linear**. A priori, for a differentiable function F, we know that $DF|_p : \mathbb{R}^2 \to \mathbb{R}^2$, after we identify $\mathbb{R}^2$ with $\mathbb{C}$ via $(x, y) \mapsto x + iy$, is always real linear. If we want to check that it's complex linear, we only need to that it commutes with multiplication by i. Recall that if you think of a complex $a + bi$ as the column vector $(a, b)^T$, then multiplication by $(c + di)$ gives you the complex number (in the column vector form):

$$\begin{pmatrix} ac - bd \\ ad + bc \end{pmatrix} = \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

Therefore multiply by a complex number, thought in terms of how it works in $\mathbb{R}^2$, is actually just multiplying by a certain matrix. In particular, the matrix corresponding to i is $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, therefore we only need to check:

$$DF|_p J(a, b)^T = JDF|_p(a, b)^T, \quad \forall (a, b)^T \in \mathbb{R}$$

This simply means that we only need to check $DF|_p J = FDF|_p$ for all p. If you write down what J does to $DF|_p$, you will find that this is just the **Cauchy Riemann Equations**. Therefore the derivative matrix of F just looks like a complex number and it's just saying that the action of the derivative looks like multiply by a complex number, which should be $f'(z)$.

Remark 2.2. Here is another result on one variable complex analysis that is worth mentioning to clear possible confusion when we pass to several variable complex analysis.

Recall that f is holomorphic at a point p actually means that it's complex differentiable in a neighborhood of p. This is equivalent to f is real differentiable in this neighborhood and each derivative is $\mathbb{C}$ linear. This is also equivalent to f is C^1 in this neighborhood and the partials satisfy the Cauchy-Riemann equation.

However, if we only assume that the partials of f exists in a neighborhood of p and satisfy the Cauchy-Riemann equation, we can not conclude that f is holomorphic at p. A classic example is the function:

$$f : \mathbb{C} \to \mathbb{C}, \quad z \mapsto e^{-1/z^4}$$

You can see that it's holomorphic away from 0 thus the Cauchy-Riemann equation automatically holds. You can also check that the partials exist at the origin and C-R equation also holds there. However, $f(z)$ is not even continuous at 0, thus not holomorphic at 0.

The reason is because the real analysis picture is much more complicated than the complex analysis picture. If f is complex differentiable in a neighborhood, we can automatically conclude that all it's partials are even C^∞. However, if we only assume f is real differentiable in a neighborhood, we can not even conclude the partials are continuous. A classic example is:

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^2 \sin \frac{1}{x}$$

However, what amazingly works is that if we assume that:

$$f : \mathbb{C} \to \mathbb{C}$$

is continuous in a neighborhood of p and all the partials exists (without even assuming continuous) and satisfies the C-R equation, then f is holomorphic at p. This is the famous **Looman-Menchoff Theorem**. Again, this is something that is not true in real analysis, even if we assume that $f : \mathbb{R}^m \to \mathbb{R}^n$ is continuous in a neighborhood of p and all the partials exists, we can not assume that f is differential at p. A classic example is:

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto \frac{xy}{\sqrt{x^2 + y^2}}, \quad (0, 0) \mapsto 0$$

You can check that f is indeed continuous at 0 and all partials exist, however, it's not differentiable at 0.

The upshot of Looman-Menchoff Theorem, though we won't prove it here, is that once we assume that a complex function is continuous, C-R equation and existence of all partials automatically imply the function is complex differentiable and as a consequence, the partials are thus indeed continuous. We will refer back to this once we need it.

For some notations, if we work with a holomorphic chart U, we will generally identify U with $\varphi(U)$. Moreover, we will use z to replace the chart φ since we know that everything after we apply φ will be in $\mathbb{C}$, therefore instead of using φ, we will use z, which looks more like a complex variable we use.

Definition 2.3. A Riemann Surface is a 1 dimensional connected complex manifold.

Given a Riemann Surface X, we can define **holomorphic functions** on X. We say that a function $f : X \to \mathbb{C}$ is holomorphic if f is continuous and for any chart (U, φ) in X:

$$f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}$$

is holomorphic. This also defines holomorphic functions on open subsets $\Omega \subset X$ since $\Omega \cap U$ is also going to be a chart for Ω. We will let $\mathcal{O}(\Omega)$ denote the $\mathbb{C}$-algebra of all holomorphic functions on Ω. Notice that all these definition actually works for all complex manifolds of all dimensions, we don't have to care about if it's connected either.

Then this also enables us to define **holomorphic functions between two Riemann surfaces** X and Y. If X and Y are Riemann surfaces we say that $f : X \to Y$ is holomorphic if it's continuous and for every chart (V, ψ) on Y, the function $\psi \circ f : f^{-1}V \to \psi(V)$ is holomorphic.

Definition 2.4. A holomorphic $f : X \to Y$ is an **analytic isomorphism** or a **biholomorphism** if there is $g : Y \to X$ which is also holomorphic and $f \circ g = \mathrm{Id}_Y$ and $g \circ f = \mathrm{Id}_X$.

Again, every definition works for arbitrary complex manifolds and we are not using the assumption on connectedness as well.

Remark 2.5. Recall that any manifold is required to be second countable. However, when we give out the definition of a Riemann surface, we could omit the second countable part in the definition of a manifold. In fact, if we use the definition of a Riemann surface without the second countable part, there is indeed a theorem of Rado which asserts, the space is automatically second countable. Since we won't use it we will not prove it.

Now let's give some examples of Riemann surfaces other than open subset of $\mathbb{C}$ or open subsets of an existing Riemann surface.

Example 2.6. The first example we will give is the **Riemann sphere**, which is denoted by $\mathbb{P}^1_{\mathbb{C}}$. This notation indicates that the Riemann sphere is going to be the same as the 1 dimensional projective space over $\mathbb{C}$. But instead of using the definition of the projective sphere, we will give another definition and show that it's the same as the projective line.

To begin with, $\mathbb{P}^1$, as a set is:

$$\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$$

This is going to be a topological space if we declare that open sets of $\mathbb{P}^1$ that contains ∞ are just $\mathbb{C}\backslash K \cup \{\infty\}$ where K is any compact subset of $\mathbb{C}$. You can check that this indeed defines a topology and you can check that using this topology $\mathbb{P}^1$ is compact. This is usually also called the one point compactification of $\mathbb{C}$.

Then we define two charts on $\mathbb{P}^1$. The first chart is:

$$\mathrm{Id}_{\mathbb{C}} = \varphi_1 : \mathbb{C} \to \mathbb{C}$$

The second chart is:

$$\varphi_2 : \mathbb{P}^1\backslash\{0\} \to \mathbb{C}$$

where we will map $z \mapsto \frac{1}{z}$ if $z \in \mathbb{C}^*$ and we will map ∞ to 0. Clearly the first one is a homeomorphism. The second one is also a homeomorphism since we know that $\frac{1}{z} : \mathbb{C}^* \to \mathbb{C}^*$ is a biholomorphic function and thus a homeomorphism. This tells you that this φ_2 is actually bijective. You can prove that it's a homeomorphism by showing that it has a continuous inverse function. I will let you do it. If you consider the transition functions in the intersection, it's:

$$\varphi_2 \circ \varphi_1^{-1}\mathbb{C}^* \to \mathbb{C}^* : z \mapsto \frac{1}{z}$$

This is indeed a biholomorphic function, therefore we are done.

Remark 2.7. Let's make a short remark why under the usual definition of the projective space $\mathbb{P}^n$, in particular $\mathbb{P}^1$, is a complex manifold and why it's analytically isomorphic or biholomorphic to the above definition we make. Though we are only going to use $\mathbb{P}^1$ in the first half of the course, we will see $\mathbb{P}^n$ a lot as we go down the line.

Recall that the usual definition of the n-dimensional complex projective space is that $\mathbb{P}^n_{\mathbb{C}}$ as a set is just the set of all equivalent classes of points $(z_0 : x_1 : \cdots : z_n) \in \mathbb{C}^{n+1}$ such that not all of z_i are simultaneous zero any two points are equivalent iff they differ by a scalar multiplication. We will denote the equivalent class of the point $(z_0 : x_1 : \cdots : z_n)$ by $[z_0 : x_1 : \cdots : z_n]$. Clearly, we have a surjective map $\mathbb{C}^{n+1} \to \mathbb{P}^n$ and we will declare that the topology on $\mathbb{P}^n$ is the quotient topology. Let's try to examine some topological properties of $\mathbb{P}^n$ which helps us give charts of $\mathbb{P}^n$. The first thing we should notice is that we clearly have a filtration of $\mathbb{P}^n$ given by:

$$\mathbb{P}^0 \subset \mathbb{P}^1 \subset \cdots \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n$$

Since we know that that points in $\mathbb{P}^n$ are just $[z_0 : x_1 : \cdots : z_n]$, if we insist that $z_0 = 0$, then we know that the rest of the points are clearly $\mathbb{P}^{n-1}$ as a set. Similarly, for any $k < n$, if we insist that $z_{k+1} = z_{k+1} = \cdots = z_n = 0$, then we get $\mathbb{P}^k$ inside $\mathbb{P}^n$. These subsets of $\mathbb{P}^n$ are not just set theoretically equal to $\mathbb{P}^k$, they actually are homeomorphic. To see this, we know first that these subsets, given the subspace topology is actually a closed subset of $\mathbb{P}^n$ (I will let you see why, which is essentially using the definition of the quotient topology). Then consider the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & \mathbb{C}^k \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \mathbb{C}^n \arrow[r] \arrow[d] & \mathbb{C}^{n+1} \arrow[d] \\
\cdots \arrow[r] & \mathbb{P}^{k-1} \arrow[r] & \cdots \arrow[r] & \mathbb{P}^{n-1} \arrow[r] & \mathbb{P}^n
\end{tikzcd}$$

Here the map $\mathbb{C}^{n+1} \to \mathbb{P}^n$ is the quotient map you started with and all the other $\mathbb{C}^k$'s are the inverse image of $\mathbb{P}^{k-1}$. Then I will let you see that the subspace topology on $\mathbb{P}^{k-1}$ is the same as the quotient topology induced by the map $\mathbb{C}^k \to \mathbb{P}^{k-1}$.

One consequence of the above discussion is that $\mathbb{P}^n \backslash \mathbb{P}^{n-1}$ is open. We know that this subsets consists of those point $[z_0 : \cdots : z_n]$ where $z_n \neq 0$ an we will denote this set by $D_+(z_n)$. And we can essential let $z_i = 0$ for any coordinate and obtain another copy of $\mathbb{P}^{n-1}$ and other open subsets $D_+(z_i)$. Notice that the unions of $D_+(z_i)$ covers $\mathbb{P}^n$ and we will give charts for each $D_+(z_i)$. Notice that we have:

$$\varphi_i : D_+(z_i) \to \mathbb{C}^n, \quad [z_0 : \cdots : z_n] \mapsto (\frac{z_0}{z_i}, \cdots, \hat{1}, \cdots, \frac{z_n}{z_i})$$

This is a homeomorphism. To see it, I will let you show that this is a bijective map. It's clearly continuous since if we consider the composition:

$$\mathbb{C}^{n+1} \backslash \mathbb{C}^n \to D_+(z_i) \to \mathbb{C}^n, \quad (z_0, \cdots, z_n) \mapsto (\frac{z_0}{z_i}, \cdots, \hat{1}, \cdots, \frac{z_n}{z_i})$$

This is clearly continuous, then since $\mathbb{C}^{n+1} \backslash \mathbb{C}^n \to D_+(z_i)$ is the quotient map, this implies $D_+(z_i) \to \mathbb{C}^n$ is continuous from the universal property. It clearly has an inverse where:

$$\mathbb{C}^n \to D_+(z_i), \quad (z_0, \cdots, \hat{z_i}, \cdots, z_n) \mapsto [z_0 : \cdots : 1 : \cdots, z_n]$$

This is again continuous since we can consider the composition:

$$\mathbb{C}^n \to \mathbb{C}^{n+1} \backslash \mathbb{C}^n \to D_+(z_i)$$

Notice that each function in the composition is continuous, which means that the composition is continuous. Notice that this actually makes $\mathbb{P}^n$ a n-dim complex manifold and I will let you check that the charts are compatible.

Then we prove that $\mathbb{P}^1$ defined using the Riemann sphere or using the projective coordinates above are biholomorphic. To distinguish these two definitions, we use subscripts s and p. Consider the map:

$$\mathbb{P}^1_s \to \mathbb{P}^1_p$$

where $z \mapsto [z : 1]$ if z is in $\mathbb{C}$ and $\infty \mapsto [1 : 0]$. Recall that we only need to show that this is holomorphic and bijective, then the 1-dimensional inverse holomorphic function theorem tells you that this is indeed a biholomorphism. Consider the chart $D_+(z_1)$ in $\mathbb{P}^1_p$, notice that the inverse image is $\mathbb{C}$, and we know that the coordinate description of this map is simply:

$$\mathbb{C} \to \mathbb{C}, \quad z \mapsto z$$

This is of course holomorphic. Consider the chart $D_+(z_0)$ and we know that it's preimage will be $\mathbb{P}^1_s \backslash \{0\}$ and the coordinate description of this map is again:

$$\mathbb{C} \to \mathbb{C}, \quad z \mapsto z$$

Therefore in our future we will not distinguish the two definitions since they are essentially the same if only care about biholomorphisms. For simplicity, we will always use the Riemann sphere description.

Remark 2.8. This is just a side remark. Our discussion of the topological properties of $\mathbb{P}^n$ actually can be used to give $\mathbb{P}^n$ a **CW-complex** structure which enables you to show more topological properties of $\mathbb{P}^n$ using the tool of algebraic topology. We mention it here just for the sake of general mathematic culture, so you should skip this remark if you are not interested.

Recall that a to specify a CW-complex structure, we first need to give a filtration of $\mathbb{P}^n$ and of course, we will just use the filtration we talked about before:

$$\mathbb{P}^0 \subset \mathbb{P}^1 \subset \cdots \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n$$

Now we also need to specify the cells of every dimension and give the gluing/attaching maps. I claim that there will be only one cell at dimension $0, 2, 4, \cdots, 2n-2, 2n$ and no cells for other dimensions. Clearly, we know that $\mathbb{P}^0$ is just a point, therefore it's just the zero cell with the trivial attaching map. Then we consider $\mathbb{P}^0 \subset \mathbb{P}^1$, we define the attaching map by:

$$\varphi_1 : \partial\overline{\mathbb{D}^2} \to \mathbb{P}^0$$

where we map all of the boundary of $\overline{\mathbb{D}^2}$ to the only point in $\mathbb{P}^0$. Then we define the characteristic map:

$$\Phi_1 : \overline{\mathbb{D}^2} \to \mathbb{P}^1$$

Recall that $\overline{\mathbb{D}^2}$ can be consider as the closed unit disk in the complex plane, then the map is obviously just $z \mapsto [z, \sqrt{1-|z|^2}]$. Let's examine this map $\frac{z}{\sqrt{1-|z|^2}}$ a bit when $0 \leqslant |z| < 1$. Notice that this map preserves the angle of z and if $|z| = r$, it outputs a value with absolute value $\frac{r}{\sqrt{1-r^2}}$, clearly it's increasing when r increases and when $r = 0$ it's 0 and when r approach 1, it approaches $+\infty$. Therefore this implies $\frac{z}{\sqrt{1-|z|^2}}$ can take every complex value when $0 \leqslant |z| < 1$. Clearly this map is also injective since $\frac{r}{\sqrt{1-r^2}}$ increases when r increases.

Clearly this maps $z \mapsto [z, \sqrt{1-|z|^2}]$ the boundary to $\mathbb{P}^0 = [1 : 0]$ and this is a continuous since we can first think about the inverse image of $D_+(z_0)$ and show it's continuous there and then consider the inverse image of $D_+(z_1)$ and show that it's continuous there (using our knowledge that $D_+(z_i)$ is homeomorphic to $\mathbb{C}$ via certain maps). Then consider the map induced by the characteristic map:

$$\overline{\mathbb{D}^2} \coprod \mathbb{P}^0 / \sim \to \mathbb{P}^1$$

Clearly it's continuous and surjective (because $\frac{z}{\sqrt{1-|z|^2}}$ takes every complex value) and it's also injective (because $\frac{r}{\sqrt{1-r^2}}$ is also injective) and we know that the boundary of $\overline{\mathbb{D}^2}$ is identified with $\mathbb{P}^2$. Then we know that this is a bijective continuous from a compact space to a Hausdorff space, thus automatically a homeomorphism. This explains the CW-structure of $\mathbb{P}^0$ and $\mathbb{P}^1$.

For higher n, we essentially do something similar, here the attaching map is going to be:

$$\varphi_n : \partial\overline{\mathbb{D}^{2n}} \to \mathbb{P}^{n-1}$$

which comes from the characteristic map:

$$\Phi_n : \overline{\mathbb{D}^{2n}} \to \mathbb{P}^n, \quad \omega = (\omega_1, \cdots, \omega_n) \mapsto [\omega_1, \cdots, \omega_n, \sqrt{1-|\omega|^2}]$$

Again, you prove it's continuous by using the charts $D_+(z_i)$ on $\mathbb{P}^n$. And we know that on the boundary, we get a surjective map to $\mathbb{P}^{n-1}$. In the interior, I claim Φ_n attains every value in $D_+(z_n) = \mathbb{P}^n \backslash \mathbb{P}^{n-1}$ only once. This is essentially studying the tuples:

$$\left(\frac{\omega_1}{\sqrt{1-|\omega|^2}}, \cdots, \frac{\omega_n}{\sqrt{1-|\omega|^2}}\right)$$

Essentially we want to show that it takes every value in $\mathbb{C}^n$ only once. Indeed, giving any $z \in \mathbb{C}^n$, consider:

$$\omega = \frac{z}{\sqrt{1+\sqrt{z}^2}}$$

Then we know that $|\omega|^2 = \frac{|z|^2}{1+\sqrt{z}^2}$ thus $\frac{\omega}{\sqrt{1-|\omega|^2}} = \frac{z}{\sqrt{1+|z|^2}} \cdot \sqrt{1+|z|^2} = z$, therefore we know that it's surjective and injective. Then you can show that the map:

$$\overline{\mathbb{D}^{2n}} \coprod \mathbb{P}^{n-1} / \sim \rightarrow \mathbb{P}^n$$

is a bijective continuous map from compact to Hausdorff space, therefore is a homoemorphism.

Example 2.9. Here is another example called the **complex tori** that will be extremely useful for finding all meromorphic functions on a Riemann surface later.

Let ω_1, ω_2 be two $\mathbb{R}$-linearly independent complex numbers and let Λ be the lattice it generates. Then consider the quotient map:

$$\pi : \mathbb{C} \rightarrow \mathbb{C}/\Lambda$$

I claim that if we give $\mathbb{C}/\Lambda$ the quotient topology, then π becomes a covering map and $\mathbb{C}/\lambda$ is going to be called the complex tori. Π is the fundamental parallelogram, which is:

$$\Pi = \{a\omega_1 + b\omega_2 : a, b \in [0, 1]\}$$

To see why this is a covering map, recall that we need to find for each $[z] \in \mathbb{C}/\Lambda$ an evenly covered neighborhood. Note that $\mathbb{C}/\Lambda$ is a topological group with the usual addition inherited from $\mathbb{C}$. If we consider a point $[y] \in \mathbb{C}/\Lambda$ where y is not on the lattice, we know that since y only has a unique representation in terms of $y = \lambda + x$ where $\lambda \in \Lambda$ x is in the interior of Π. Then clearly, we can pick a neighborhood U of y such that it's still contained in the interior of Π and we know that the inverse image of this neighborhood is $\Lambda + U = \bigcup_{\lambda \in \Lambda} \lambda + U$ which is open therefore $[U]$ is open in $\mathbb{C}/\Lambda$ and for each $\lambda \in \Lambda$, we know that $\pi : U + \lambda \rightarrow [U]$ is bijective because can choose U so small that any two distinct element in U can not have a difference in Λ. It's continuous because it's just a restriction. Since it's the restriction of a quotient map to an open subset, it's open, therefore a homeomorphism.

Now if y is on the lattice Λ, the situation is similar, we can again pick a U, it's just this time, it's not contained in the interior of Π. If we consider $[U]$, we know that it's if x is mapped to U, then there is a $z \in U$ such that:

$$x = z + \lambda$$

This implies that $x \in U + \Lambda$. We again consider the map $\pi : U + \lambda \rightarrow [U]$ for some specific $\lambda \in \Lambda$. It's continuous and open, so we only need to show it's bijective. It's clearly surjective, the obstruction here is that if we suppose that $[z_1] = [z_2]$, then $z_1 - z_2 \in \Lambda$, then if $x_1 + \lambda \mapsto [z_1], x_2 + \lambda \mapsto [z_2]$, then we know that $(x_1 + \lambda) - (x_2 + \lambda) \in \Lambda$. This implies that $x_1 - x_2 \in \Lambda$. But if we choose U small enough and assume $x_1 - x_2 \neq 0$, I will let you show that $x_1 - x_2$ can not be in λ.

In general, if $\pi : X \rightarrow Y$ is a covering map with X a complex manifold, then Y automatically assumes a complex structure from that of X, given some conditions on the covering map. So we can not just say that $\mathbb{C}/\Lambda$ assumes a complex structure just from the fact that $\mathbb{C}$ is Riemann surface. Instead, we know that we can choose a cover of $\mathbb{C}$ by disks V_i such that $\pi : V_i \rightarrow U_i = \pi(V_i)$ are homeomorphisms. Then I declare the charts are:

$$\varphi_i : U_i \rightarrow V_i, \quad \varphi_i = \pi^{-1}$$

Let's check that these charts are indeed compatible. Consider the diagram:

$$\begin{tikzcd}
\varphi_i(U_i \cap U_j) \arrow[rr, dashed] & & \varphi_j(U_i \cap U_j) \\
& U_i \cap U_j \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

We want to prove that $\varphi_i(U_i \cap U_j) \rightarrow \varphi_j(U_i \cap U_j)$ is holomorphic. Notice that if $z \in \varphi_i(U_i \cap U_j)$ is mapped to $[z]$ and it's again mapped to $\varphi_j([z]) = z'$, we know that $z - z' = \lambda \in \Lambda$. Then I claim that for all $z \in \varphi_i(U_i \cap U_j)$, the λ's are the same, which makes the map $z \mapsto z + \lambda$, which is indeed biholomorphic. This fact indeed again comes from that we can take each V_i, V_j very small.

Now we will try to generalize some of our familiar one-variable complex analysis results to Riemann surfaces.

Theorem 2.10. (Riemann Extension Theorem) Let X be a Riemann surface and let $a \in X$, let $f : X\backslash\{a\} \to \mathbb{C}$ be a holomorphic function such that f is bounded on $U\backslash\{a\}$ for some neighborhood of $\{a\}$, then there is a a holomorphic function $F : X \to \mathbb{C}$ such that $F|_{X\backslash\{a\}} = f$.

Notice that this is completely a local result and we can thus use a chart to put everything in our familiar one variable complex analysis situation. We omit the details here.

Theorem 2.11. (Principle of Analytic Extension) Let X, Y be Riemann surfaces and let $f : X \to Y$ be a holomorphic map. If there is a nonempty open $U_0 \subset X$ such that $f||_{U_0}$ is constant y_o, then $f(x) = y_0$ for all $x \in X$.

Remark 2.12. This formulation is equivalent in dimension 1 case to the accumulation points formulation but it also generalizes to higher dimensions.

Indeed, we know that if we have some points on which f is constant and these points have an accumulation point. Then we pick a neighborhood of this accumulation point and consider the Taylor expansion around that point, which tells you that the function must be constant possibly in a smaller neighborhood. Then you get the formulation in the above theorem.

The proof of this uses in an essentially way X is connected. Again, we use charts to transfer everything back to our familiar case in $\mathbb{C}$. The details are omitted.

Now we introduce meromorphic functions on a Riemann surface. We will first give the definition which is easy to use but doesn't generalize to higher dimensions. The correct definition we will use in higher dimension will be mentioned later.

Definition 2.13. (Meromorphic Functions Ver.1) Let X be a Riemann surface and let $X' \subset X$ be an open subset. We say that a holomorphic function $f : X' \to \mathbb{C}$ is a **meromorphic function** on X if:

1. $X\backslash X'$ is a discrete set of points in X.
2. For any $p \in X\backslash X'$, the limit of $f(x)$ as $x \to p$ is ∞.

In particular, all holomorphic functions are meromorphic. Clearly since X is Hausdorff, given two meromorphic functions, we can consider their sum, product and this clearly gives us a ring structure of all the meromorphic functions. We will use $\mathcal{M}(X)$ to denote the $\mathbb{C}$-algebra of all meromorphic functions.

We will soon see that $\mathcal{M}(X)$ is a field. Now, if (U, z) is a coordinate neighborhood of a pole p of f with $z(p) = 0$ and here we already are using our previous convention that the map φ for the chart are often written as z to make it looks more like what we see in complex analysis. Then if we consider the function $f(z)$ from $z(U)$ to $\mathbb{C}$, we know that it has a **Laurent expansion**:

$$f(z) = \sum_{\nu=-k}^{\infty} c_\nu z^\nu, \quad c_{-k} \neq 0$$

This integer k is called **the order of the pole** at p of f. We can show that it's independent of any choice of coordinates because if we have any other coordinate charts (U', z') of p such that $z'(p) = 0$, and consider the function $f(z')$. We know that there is a biholomorphic function g such that $z = g(z')$ and $f(z') = f(g(z'))$. We know that another way to characterize the order of the pole is to use the lower power of z we need to multiply such that f becomes a nonvanishing holomorphic function. Then we know that if we assume that the order of pole of $f(z')$ is k'. Then we know that after we multiply by $z'^{k'}$, it becomes a nonvanishing holomorphic function. Therefore we know that $(z')^{k'} f(g(z'))$ is also such a function. However, we know that:

$$f(g(z')) = \sum_{\nu=-k}^{\infty} c_\nu g(z')^\nu, \quad c_{-k} \neq 0$$

Notice that $z' = 0$ is only a degree 1 zero, in other words, $z'/g(z')$ is a nonvanishing holomorphic function. Then consider:

$$(z')^k f(z') = (z')^k f(g(z')) = c_{-k} \frac{(z')^k}{g(z')} + \cdots$$

which is indeed a nonvanishing holomorphic function. Therefore we know that $k' = k$.

Remark 2.14. As what we have mentioned, this definition of meromorphic functions doesn't work in higher dimension. For example, in higher dimensions, there are no isolated zeros and poles.

The definition that turns out to work will be that a meromorphic function is a function that is locally $\frac{f}{g}$ where f, g are holomorphic functions. We will discuss this in details when we need it.

Theorem 2.15. If X is a Riemann surface, and f is a meromorphic function on X, for each pole p of f, we let $f(p) = \infty$, then $f : X \to \mathbb{P}^1$ is holomorphic.

Conversely, given any holomorphic function $f : X \to \mathbb{P}^1$. Then either $f = \infty$ for all $x \in X$ or $f^{-1}\infty$ is a discrete set of points in X. $f : X \backslash f^{-1}\infty \to \mathbb{C}$ gives a meromorphic function on X.

Hence we can identify the meromorphic functions on X with holomorphic functions to $\mathbb{P}^1$.

Proof. Let f be a meromorphic function on X and let P be the set of poles, we consider the map to $\mathbb{P}^1$ induced by f. First it's continuous since we know that if we take any open U of $\mathbb{P}^1$ that contains ∞, we know that it will be the complement of a compact set K in $\mathbb{C}$ union the infinity point. If we don't consider the infinity point, we know that the inverse image is an open set such that P is contained in the closure of this open set. Indeed, suppose that K is contained in $\mathrm{B}(0, r)$, take any open neighborhood of a point p in P, we know that since as the limit goes to p $|f| \to \infty$, then there is some z in the neighborhood such that $|f(z)| > 2r$, which means that $f(z) \in \mathbb{C} \backslash K$, which means that z is in the inverse image of U. Using this argument, we can even find a punctured small disk neighborhood of p such that it's contained in the inverse image of U, then we know that if we add this point p, it will still be open. If U doesn't contain the infinity point, then we can use the fact that f is continuous itself as a function to $\mathbb{C}$.

Then we have to prove that it's holomorphic. Consider a chart $(\mathbb{P}^1 \backslash \{\infty\}, \psi)$ on $\mathbb{P}^1$ and a chart (U, φ) on X such that the $f(U) \subset U'$, then consider the map:

$$g : \psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(U')$$

Notice that f is in $\mathcal{O}(X \backslash P)$, then we know that g is in $\mathcal{O}(\varphi(U) \backslash \varphi(P))$, notice that $\varphi(P)$ is a discrete set of points in $\varphi(U)$ and g clearly is bounded near each point in $\varphi(U)$ since we know that the chart ψ is $\frac{1}{z}$, then by Riemann extension theorem, we know that g is actually holomorphic on all of $\varphi(U)$ with a unique holomorphic extension. Since g is itself continuous, then the extension from **Riemann Extension Theorem** should be g itself. This shows that f is holomorphic.

Conversely, suppose that $f : X \to \mathbb{P}^1$ is holomorphic, if f is not constantly ∞, we know pick a chart (V, ψ) of ∞ and we know that if we can cover the preimage $f^{-1}(V)$ by charts (U, φ) such that $f(U) \subset V$ and we know that if we consider the composition $g : \psi \circ f \circ \varphi^{-1}$, it will be a holomorphic function which is not constant. Then we know that if we assume that $\psi(\infty) = z'$, we know that g only takes ∞ at isolated points (by **analytic continuation principle**), which means that the inverse image $f^{-1}(\infty)$ in U is isolated. This holds for every U, therefore $f^{-1}(\infty)$ is isolated and closed in X. Then consider the map $f : X \backslash f^{-1}(\infty) \to \mathbb{C}$, it's just a restriction of a holomorphic function to open subsets, therefore is of course still holomorphic. $\square$

Remark 2.16. Recall that we proved that for holomorphic maps between Riemann surfaces, analytic continuation holds. Then for a meromorphic function on X, since it can be thought of as a holomorphic function to $\mathbb{P}^1$, analytic continuation holds. In the sense that if there is a nonempty open of X on which f is constant, then f is constant on the entirety of X. In particular, a meromorphic function on X which is not constant can only have isolated zeros.

This in particular proves that $\mathcal{M}(X)$ is a field. To see why, recall that there is a holomorphic function on $\mathbb{P}^1$:

$$\frac{1}{z} : \mathbb{P}^1 \to \mathbb{P}^1$$

which is defined by $z \mapsto \frac{1}{z}$ if $z \in \mathbb{C}^*$, and we map 0 to ∞ and ∞ to 0. I will let you check this is a biholomorphism in charts.

But if we consider the composition:

$$X \xrightarrow{f} \mathbb{P}^1 \xrightarrow{1/z} \mathbb{P}^1$$

This is again a holomorphic function, which we will denote by $\frac{1}{f}$. Notice that if we think of this function as a meromorphic function to $\mathbb{C}$, it will just be $\frac{1}{f}$ for where $f(z)$ is not 0 and ∞. Since we already know that if we denote the set of zeros of f by Z, it's discrete. Then we know that the meromorphic function $\frac{1}{f}$ is:

$$\frac{1}{f} : X \backslash Z \to \mathbb{C}$$

where $\frac{1}{f}(p)$ for $p \in P$ are now mapped to 0.

Now if we consider the product $\frac{1}{f} \cdot f$, we know that it's constantly 1 except not defined at a discrete set $P \cup Z$, then by analytic continuation, we know that the product is actually 1 and can be thought of as the constant function from X to $1 \in \mathbb{P}^1$.

Remark 2.17. We didn't really tell you how to multiply two meromorphic functions actually since everything should be pretty intuitive and whether define it or not rigorously or not doesn't affect our discussion later. However, if you think it's really important, the correct definition involves taking the orders of zeros and poles into account.

Then let's study the local behavior of the holomorphic maps:

Theorem 2.18. Suppose that X, Y are Riemann surfaces and $f : X \to Y$ is a holomorphic nonconstant map. If $a \in X$ and $b = f(a) \in Y$, then there is an integer $k \geqslant 1$ and charts $\varphi : U \to V$ on X and $\psi : U' \to V'$ on Y such that:

1. $a \in U, \varphi(a) = 0$ and $b \in U', \psi(b) = 0$
2. F is just a power of the coordinate chart, i.e.:

$$F : \psi \circ f \circ \varphi^{-1} : V \to V', \quad z \mapsto z^k$$

Proof. Clearly, we know that charts satisfying condition 1 exists. Then we know that it suffice to show that if $f : \Omega \to \mathbb{C}$ is a holomorphic function where $\Omega \subset \mathbb{C}$ is a neighborhood of 0 with $f(0) = 0$. There is a neighborhood U of 0 in the target and a biholomorphism $\psi : U \to U' \subset \mathbb{C}$ such that:

$$f \circ \psi(z) = z^k$$

This is just pretty standard one variable complex analysis. We know that since we assumed that f is not constant, then f only has a finite order 0 at $z = 0$, which lets us assume that:

$$f(z) = z^k g(z)$$

where $g(z)$ is a holomorphic function on Ω such that $g(z) \neq 0$. Then consider the following diagram where $\mathbb{D}$ is just a small enough open disk centered at $z = 0$:

$$\begin{tikzcd} & \mathbb{C} \arrow[dr, "e^z"] & \\ \mathbb{D} \arrow[rr, "g"'] & & \mathbb{C}^* \end{tikzcd}$$

Recall that existence of various branches of log functions makes e^z a cover of $\mathbb{C}^*$, again since $\mathbb{C}$ is simply connected, we know that there is a unique lift from $\mathbb{D}$ to $\mathbb{C}$ denoted by $\tilde{g}(z)$, and you can easily prove that it's holomorphic since e^z locally is a biholomorphism and g is holomorphic. This implies that locally on $\mathbb{D}$:

$$g(z) = e^{\tilde{g}(z)}$$

Then consider $h(z) = e^{\tilde{g}(z)/k}$, which means that $g(z)$ locally is $h(z)^k$. This implies:

$$f(z) = (zh(z))^k, \quad z \in \mathbb{D}$$

Consider $\varphi(z) = zh(z)$ on $\mathbb{D}$. Notice that $\varphi'(0) \neq 0$, then this is a biholomorphism by the **holomorphic inverse function theorem** at least locally from some $U' \subset \mathbb{D}$ to some other neighborhood of 0, say $U'' \subset \mathbb{C}$. Then consider:

$$f \circ \varphi^{-1} : U'' \to \mathbb{C}$$

we know that $f \circ \varphi^{-1}(z) = \varphi^k(\varphi^{-1}(z)) = z^k$. ☐

Remark 2.19. The above integer k can be characterized as follows: for any open neighborhood U_0 of $b = f(a)$, there is a smaller neighborhood of a, denoted by $U \subset U_0$ and a neighborhood W of $f(a)$ such that for all $y \in W$ and $y \neq b$, $f^{-1}(y) \cap U$ has exactly k elements. Indeed, recall that since z^k is a nonconstant holomorphic function, so by the usual **open mapping theorem** from complex analysis, we know that the image of z^k of U_0 is going to be open and we call it W. Then the above characterization is obvious. This integer k will be called the **local multiplicity** of f and this is a coordinate independent way to characterize this number.

2.1 Homework

Below are the homework of this section:

Problem 2.20. Let Λ, Λ' be two lattices in $\mathbb{C}$ and

$$X = \mathbb{C}/\Lambda, \qquad X' = \mathbb{C}/\Lambda'$$

the corresponding complex tori.

Prove that any holomorphic map

$$f : X \to X'$$

is induced by a linear map

$$g : \mathbb{C} \to \mathbb{C}$$

of the form

$$g(z) = \alpha z + \beta,$$

where $\alpha \in \mathbb{C}$ is such that

$$\alpha\Lambda \subset \Lambda'.$$

The map f is biholomorphic if and only if

$$\alpha\Lambda = \Lambda'.$$

Proof. $\Rightarrow$: We will use $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ and $\pi' : \mathbb{C} \to \mathbb{C}/\Lambda'$ to denote the two projections. Let's first assume that we have $g(z) : \mathbb{C} \to \mathbb{C}$ which is $\alpha z + \beta$ where $\alpha\Lambda \subset \Lambda'$. Note first there is a continuous map from $\mathbb{C}/\Lambda$ to $\mathbb{C}/\Lambda'$ by the universal property of the quotient topology. Indeed, since:

$$\alpha(z + \lambda) + \beta = \alpha z + \alpha\lambda + \beta$$

with $\alpha\lambda \in \Lambda'$, we know that $[\alpha z + \alpha\lambda + \beta] = [\alpha z + \beta] \in \mathbb{C}/\Lambda'$. Therefore we know that g is constant over the fiber of π, which induces f uniquely from the quotient topology.

We now need to prove that f holomorphic. Recall that the complex structure on a complex tori is given by the fact that π and π' are local homeomorphisms. Then we know if we are given a point in $\mathbb{C}/\Lambda'$, the chart (V, ψ) looks like:

$$\psi : V \to \psi(V), \quad [z] \mapsto z + \lambda_z$$

where λ'_z is the unique $\lambda'_z \in \Lambda'$ such that $z + \lambda_z$ is in $\psi(V)$ (it's unique because ψ is a homeomorphism). We pick charts (U, φ) for $f^{-1}(V)$ similarly, but we make a restriction that U are so small that the sum or difference of any two distinct elements in U should not be in Λ'. Then we know that the map f in charts looks like:

$$z \mapsto [z] \mapsto [\alpha z + \beta] \mapsto \alpha z + \beta + \lambda_{\alpha z+\beta}$$

Now I claim that for distinct $z, z' \in U$, $\lambda_{\alpha z+\beta} = \lambda_{\alpha z'+\beta}$, this is because if they are not equal, we will have:

$$z - z' = \lambda_{\alpha z+\beta} - \lambda_{\alpha z'+\beta} \in \Lambda'$$

which contradicts our assumption that $z - z'$ is never an element in Λ'. Since for different z, $\lambda_{\alpha z+\beta}$ actually doesn't on z, we will denote this $\lambda_{\alpha z+\beta}$ directly by λ. This shows that f is holomorphic because in charts we chose above, f looks like $z \mapsto \alpha z + \beta + \lambda$.

$\Leftarrow$: Conversely, we assume that we have f, then consider the following diagram:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\pi"] & \mathbb{C}/\Lambda \arrow[d, dashed, "f"] \\ \mathbb{C} \arrow[r, "\pi'"] & \mathbb{C}/\Lambda' \end{tikzcd}$$

Consider $0 \in \mathbb{Z}$, then we pick β in $\mathbb{C}$ to be an arbitrary lift of $f([0])$. Notice that $\mathbb{C}$ is simply connected, then we know that if we consider $f \circ \pi$, we know that:

$$0 = (f \circ \pi)_*(\pi_1(\mathbb{C}, 0)) \subset \pi'_*(\pi_1(\mathbb{C}, \beta)) = 0$$

Then we know that the lifting criterion gives us a unique continuous lift $g : \mathbb{C} \to \mathbb{C}$ such that 0 is mapped to β. I claim that g must be holomorphic. Indeed, holomorphicity is a local property, and if we consider the diagram:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\pi"] \arrow[d, dashed, "g"] & \mathbb{C}/\Lambda \arrow[d, dashed, "f"] \\ \mathbb{C} \arrow[r, "\pi'"] & \mathbb{C}/\Lambda' \end{tikzcd}$$

We for any point $p \in \mathbb{C}$ downstairs, we can pick a small disk U' such that $\pi' : U' \to \pi'(U')$ is biholomorphic by the complex structure of $\mathbb{C}/\Lambda'$. We consider $g^{-1}(U')$, we know that for each point q in $g^{-1}(U')$, we can again choose a small disk U such that $\pi : U \to \pi(U)$ is a biholomorphism, then we get the following diagram:

$$\begin{tikzcd} U \arrow[r, "\pi"] \arrow[d, "g"'] & \pi(U) \arrow[d, "f"] \\ U' \arrow[r, "\pi'"'] & \pi'(U') \end{tikzcd}$$

We know that $f \circ \pi$ is holomorphic and π' is a biholomorphism, therefore g is holomorphic.

Then we prove that g is in fact linear. First notice that for any elements $\lambda \in \Lambda$, we know that:

$$g(z + \lambda) - g(z) \in \Lambda'$$

Notice that $g(z + \lambda) - g(z)$ is continuous but Λ' is discrete, therefore $g(z + \lambda) - g(z)$ is a constant, which means:

$$g(z + \lambda)' - g(z)' = 0$$

This implies that $g'(z)$ is an elliptic function with out any poles, which means that it's a constant α, which means that:

$$g(z) = \alpha z + b, \quad b \in \mathbb{C}$$

However we assumed that $g(0) = \beta$, thus $b = \beta$, thus $g(z) = \alpha z + \beta$.

In other words, we show that the all holomorphic functions from $\mathbb{C}/\Lambda$ to $\mathbb{C}/\Lambda'$ comes from some linear function g on $\mathbb{C}$. This is some surjective result. However, this is clearly not injective since αz and $\alpha z + \lambda'$ for a nonzero $\lambda' \in \Lambda'$ clearly induces the same f.

Finally, if $\alpha\Lambda = \Lambda'$, consider the function:

$$g(z) : \mathbb{C} \to \mathbb{C}, \quad z \mapsto \alpha z$$

This induces a holomorphic $f : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$. We can consider the function:

$$g'(z) : \mathbb{C} \to \mathbb{C}, \quad z \mapsto \frac{1}{\alpha} z$$

. This induces another $f' : \mathbb{C}/\Lambda' \to \mathbb{C}/\Lambda$. Clearly $f'f([z]) = f'([\alpha z]) = [z]$ and $ff'([z]) = f([\frac{1}{\alpha}z]) = [z]$, therefore f, f' are indeed inverses, thus biholomorphisms. ☐

Problem 2.21. Show that every torus $\mathbb{C}/\Lambda$ is isomorphic to a torus of the form:

$$X(\gamma) = \mathbb{C}(\mathbb{Z} + \mathbb{Z}\gamma), \quad \mathrm{Im}(\gamma) > 0$$

Then give a criterion when two $X(\gamma), X(\gamma')$ are biholomorphic.

Proof. Suppose that Λ is a lattice spanned by ω_1, ω_2 with $\mathrm{Im}(\frac{\omega_1}{\omega_2}) > 2$, then let $\gamma = \frac{\omega_1}{\omega_2}$ and consider the following diagram:

$$\begin{tikzcd} \mathbb{C} \arrow[r] \arrow[d, "z \mapsto \frac{z}{\omega_2}"'] & \mathbb{C}/\Lambda \arrow[d, dashed] \\ \mathbb{C} \arrow[r] & \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\gamma \end{tikzcd}$$

Notice that since $\frac{1}{\omega_2}\Lambda = \mathbb{Z} \oplus \mathbb{Z}\gamma$, from the previous problem, we know that this linear map on $\mathbb{C}$ induces an biholomorphism between $\mathbb{C}/\Lambda$ and $X(\gamma)$.

Finally, we know that $X(\gamma) \simeq X(\gamma')$ iff there a complex number α such that $\alpha(\mathbb{Z} \oplus \mathbb{Z}\gamma) = \mathbb{Z} \oplus \gamma'\mathbb{Z}$, this tells you that $\alpha, \alpha\gamma$ is another integral basis of $\mathbb{Z} \oplus \gamma'\mathbb{Z}$, this tells you there is:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$$

such that $A(\alpha, \alpha\gamma)^T = (1, \gamma')$ or in other words:

$$a\alpha + b\gamma\alpha = 1, \quad \gamma' = c\alpha + d\alpha\gamma$$

Solve for γ', we see that:

$$\gamma' = \frac{d\gamma + c}{b\gamma + a}$$

Notice that the matrix that defines this linear fraction transformation is again in $\mathrm{GL}_2(\mathbb{Z})$. I claim that this matrix can be chosen to lie in $\mathrm{SL}_2(\mathbb{Z})$. A priori, we know that this matrix has determinant 1 or -1, so we are just saying that we want the determinant to be 1. Indeed, since they are keeping the orientation of $1, \gamma$, we know that it has to be determinant 1.

Conversely, if we have some matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{Z})$ such that:

$$\gamma' = \frac{a + b\gamma}{c + d\gamma}$$

Then we first know that the lattice $(1.\gamma)$ is the same as the lattice $\Lambda' = (a + b\gamma, c + d\gamma)$. Then we know that if we set $\alpha = \frac{1}{c+d\gamma}$, we know that $\alpha\Lambda' = (\gamma', 1)$, thus we are done.

Therefore if we consider the usual action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half plane, we know that $X(\gamma')$ is biholomorphic to $X(\gamma)$ if γ and γ' are in the same orbit. ☐

Problem 2.22. Let

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$$

be a lattice in $\mathbb{C}$. An *elliptic function* is a meromorphic function f on $\mathbb{C}$ such that

$$f(z+\omega) = f(z)$$

for all $\omega \in \Lambda$ and every $z \in \mathbb{C}$ where f is holomorphic.

For $z_0 \in \mathbb{C}$ let

$$P(z_0) := \{z_0 + t_1\omega_1 + t_2\omega_2 \mid 0 \leqslant t_1 \leqslant 1,\ 0 \leqslant t_2 \leqslant 1\}$$

and assume that the boundary $\partial P(z_0)$ does not contain any pole of f.

4. (a) Prove that

$$\int_{\partial P(z_0)} f(z)\, dz = 0.$$

   (b) If $\partial P(z_0)$ does not contain any zero of f, prove that the interior of $P(z_0)$ contains equally many zeroes as poles of f, all counted with their multiplicities.

5. Assume that $\partial P(z_0)$ does not contain any zero or pole of f, and let

$$z_1, \ldots, z_n \qquad \text{(resp. } w_1, \ldots, w_n)$$

   be the zeroes (resp. poles) of f in the interior of $P(z_0)$, counted with their multiplicities. Prove that

$$\sum_{i=1}^{n}(z_i - w_i) \in \Lambda.$$

Proof. We omit the details first and second part of this problem. For the first one, just use doubly periodicity and the fact that on the opposite sides, the orientation is different. For the second part, consider the integral of:

$$\frac{1}{2\pi i}\int_{\partial P(z_0)}$$

For the last part, we consider the following integral:

$$\frac{1}{2\pi i}\int_{\partial P(z_0)} \frac{zf'(z)}{f(z)}dz = \sum_{i=1}^{n}(z_i - \omega_i)$$

Then it's only left to prove that this is indeed in Λ. Notice that after we parametrize the above integral it now looks like:

$$\frac{1}{2\pi i}\int_0^1 (z_0+\omega_1 t)\frac{f'(z_0+\omega_1 t)}{f(z_0+\omega_1 t)}\omega_1 dt + \frac{1}{2\pi i}\int_0^1 (z_0+\omega_1+\omega_2 t)\frac{f'(z_0+\omega_1+\omega_2 t)}{f(z_0+\omega_1+\omega_2 t)}\omega_2 dt$$

$$-\frac{1}{2\pi i}\int_0^1 (z_0+{+\omega_2}+\omega_1 t)\frac{f'(z_0+\omega_2+\omega_1 t)}{f(z_0+\omega_2+\omega_1 t)}\omega_1 dt - \frac{1}{2\pi i}\int_0^1 (z_0+\omega_2 t)\frac{f'(z_0+\omega_2 t)}{f(z_0+\omega_2 t)}\omega_2 dt$$

Using the fact that $f'(z)$ and $f(z)$ are still doubly periodic, we know that we can simply the above formulation as:

$$\frac{\omega_2}{2\pi i}\int_0^1 \omega_1 \frac{f'(z_0+\omega_1 t)}{f(z_0+\omega_1 t)}dt + \frac{\omega_1}{2\pi i}\frac{f'(z_0+\omega_1 t)}{f(z_0+\omega_2 t)}\omega_2 dt$$

Notice that $f'(z_0+\omega_1 t)/f(z_0+\omega_1 t)$ and $f'(z_0+\omega_2 t)/f(z_0+\omega_2 t)$ are all closed loops in $\mathbb{C}$ where the integral is it's winding number, therefore it's a integer. This show that this lies in $m_1\omega_1 + m_2\omega_2$. $\square$

3 Lecture 3

Let f be a nonconstant map of Riemann surfaces $f : X \to Y$. Then from last time, we saw that locally around a point $a \in X$, f looks like $f(z) = z^k$. This k is called the local multiplicity of this given point $a \in X$. We now prove a few corollaries of this result:

Corollary 3.1. Let $f : X \to Y$ be a nonconstant holomorphic map, then f is open.

Proof. Suppose that $U \subset X$ is open, we want to prove that $f(U)$ is open. We know that for each $f(a) \in f(U)$, since it locally looks like z^k, we know that there is a small open U_0 of a, which can be taken to be a small disk, such that $f(U_0)$ is open, we are done. □

Corollary 3.2. Let X, Y be Riemann surfaces with X compact. Then if $f : X \to Y$ is a nonconstant holomorphic map, then f is surjective and Y is compact.

Proof. We already see f is open. Since X is compact and Y is Hausdorff, we know that f is closed. This tells us that $f(X)$ is open and closed. Since Y is connected, $f(X) = Y$. Since X is compact, Y is compact. □

Corollary 3.3. If X is a compact Riemann surface and f is a holomorphic function to $\mathbb{C}$. Then f has to be constant. In other words, $\mathcal{O}(X) \simeq \mathbb{C}$.

Proof. Since $\mathbb{C}$ is not compact, we know that f can not be nonconstant from the previous corollary. □

Corollary 3.4. Let X be a Riemann surface and let $f : X \to \mathbb{C}$ be a nonconstant holomorphic function, then $|f|$ has no maximum.

Proof. f is open since it's nonconstant, then we know that if $f(X) \subset \overline{B(0,r)}$ for some $r > 0$, then we know that no $f(x)$ is allowed to be on the boundary, therefore $f(X) \subset B(0,r)$. This tells you that if M is the maximum value, then $f(X) \subset \overline{B(0,M)}$, but we also know that $f(X) \subset B(0,M)$ which means that $|f(z)| < M$ for all z, a contradiction. □

The above result tells you that the structure of holomorphic functions on a compact Riemann surface is very boring as they are just the complex numbers. This generalizes our intuition that functions defined on all of $\mathbb{P}^1$ should be constant. However, if we think about the meromorphic functions on X, for example on $\mathbb{P}^1$. Things become much more interesting. Our next goal is to find out what is $\mathcal{M}(X)$ for a compact Riemann surface. Let's start by studying what is $\mathcal{M}(\mathbb{P}^1)$.

Corollary 3.5. Every meromorphic function on $\mathbb{P}^1$ is rational. That is, if f is a meromorphic function on $\mathbb{P}$, then there are two polynomial g, h, such that $f = \frac{g}{h}$. In particular, $\mathcal{M}(\mathbb{P}^1) \simeq \mathbb{C}(z)$.

Proof. Apparently, any rational function is a meromorphic function. Conversely, if $f \in \mathcal{M}(\mathbb{P}^1)$, then we already showed that f induces a unique holomorphic map to $\mathbb{P}^1$:

$$\tilde{f} : \mathbb{P}^1 \to \mathbb{P}^1$$

Notice that the inverse image of ∞ is a closed subset of a compact subset, which is compact. Then we know that it's also discrete since f is meromorphic, thus $\tilde{f}^{-1}(\infty)$ is finite. This implies f has only finitely many poles.

Now we can assume ∞ is not a pole of f, otherwise, we can just consider $\frac{1}{f}$ and if we prove $\frac{1}{f}$ is a rational function, then so is f. Let $\alpha_1, \cdots, \alpha_n \in \mathbb{C}$ be the poles of f. Then consider the restriction of f to $\mathbb{C}$, and we know that at each α_ν, it has a principal part:

$$p_\nu(z) = \sum_{j=-k_\nu}^{-1} c_{\nu_k}(z - a_{\nu_j})^j$$

Notice that each p_ν is a well defined meromorphic function on $\mathbb{P}^1$ where $p_\nu(\infty) = 0$. Then consider:

$$f - p_1 - \cdots - p_n$$

We know that this will be a holomorphic function on $\mathbb{P}^1$, which is a constant c, therefore:

$$f = p_1 + \cdots + p_\nu + c$$

This is indeed a rational function. □

We can also study the automorphisms of $\mathbb{P}^1$:

Proposition 3.6. The automorphisms of $\mathbb{P}^1$:

$$\mathrm{Aut}(\mathbb{P}^1) = \{f : \mathbb{P}^1 \to \mathbb{P}^1 : \ f \text{ is biholomorphic}\}$$

is isomorphic to the projective linear group of rank 2 $\mathrm{PGL}_2(\mathbb{C})$ over $\mathbb{C}$.

Proof. **** the idea to choose the charts like $\mathbb{P}^1/K$ is correct, but need to be more careful when chooing K*First, given a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$, we get a map $f(z) = \frac{az+b}{cz+d}$ where $ab - bc \neq 0$, we get a meromorphic function on $\mathbb{P}^1$. Clearly, we know that when the function is a meromorphic function on $\mathbb{C}$:

$$\frac{az+b}{cz+d} : \mathbb{C} \to \mathbb{C}$$

Let's first assume that $c \neq 0$. Clearly we know that $z = \frac{-d}{c}$ is an order 1 pole of this function. This gives a holomorphic function to $\mathbb{P}^1$ if we define $z = \frac{-d}{c}$ is mapped to ∞. Then consider the map:

$$\mathbb{P}^1 \to \mathbb{P}^1$$

where for $z \in \mathbb{C}$, we define it according to the above meromorphic function. We will define ∞ is mapped to $\frac{a}{c}$. Notice this definition actually covers when $c = 0$ since we can think that if $c = 0$, then $\frac{-d}{c}$ is ∞, therefore we are mapping ∞ to ∞.

This is holomorphic since we know that the map maps $\mathbb{C}\backslash\{\frac{-d}{c}\}$ to $\mathbb{C}$, and we clearly know that in charts, it looks like $\frac{az+b}{cz+d}$, therefore is holomorphic. We know that if $c = 0$, then this is a linear function, then we know that $\frac{-d}{c} = \infty$ and we know that the map maps to $\mathbb{C}$ to $\mathbb{C}$ holomorphically. On $\mathbb{P}^1\backslash\{0\}$, we know that it maps everything to $\mathbb{P}^1\backslash\{b\}$. We now can pick the a chart for the domain looks like $\mathbb{P}^1\backslash K$ where K is a closed ball in $\mathbb{C}$ whose radius is large enough that for any z not in this ball, $az + b \neq 0$. Then we know that in this chart, the maps looks like:

$$z \mapsto \frac{z}{a + zb}, \quad z \mapsto \frac{a}{z} + b$$

and this are indeed holomorphic. This proves the case when $c = 0$.

If $c \neq 0$, we know that this is a meromorphic function from $\mathbb{C}$ to $\mathbb{C}$, which is a holomorphic function from $\mathbb{C}$ to $\mathbb{P}^1$. Now we consider the chart $\mathbb{P}^1\backslash\{0\}$ for the domain, and we know that it's mapped to $\mathbb{P}^1\backslash\{\frac{b}{d}\}$. We can also choose the charts for the domain as $\mathbb{P}^1\backslash K$ such that every z outside K is not a root for $a + bz$ and $c + dz$. Then we know that on charts for the codomain, the maps looks like:

$$z \mapsto \frac{c + dz}{a + bz}, \quad z \mapsto \frac{a + bz}{c + dz}$$

these are all holomorphic. Therefore this linear fraction transformation map is holomorphic.

Since $ad - bc$ is not 0, we know that the matrix has an inverse A^{-1}, then it's easy to check that the linear fraction transformation defined by A^{-1} is the inverse using the Riemann Extension theorem. Finally, we know that if we multiply the matrix A by any nonzero constant, we get the same map, therefore we obtain a map:

$$\mathrm{PGL}_2(\mathbb{C}) \to \mathrm{Aut}(\mathbb{P}^1)$$

Conversely, suppose that f is an automorphism of $\mathbb{P}^1$, if the $f(\infty) = \infty$, we know that f gives an automorphism of $\mathbb{C}$, and from complex analysis, we know that an automorphism of $\mathbb{C}$ is $az + b$ for some nonzero $a \in \mathbb{C}$. Then we know that f is indeed just $az + b$.

Now assume that ∞ is mapped to 0, then we know that after composing with the automorphism $\frac{1}{z}$, $\frac{1}{f}$ is of the form $az+b$, which means that $f(z) = \frac{1}{az+b}$ for $a \neq 0$.

Now suppose that $\alpha \in \mathbb{C}$ is mapped to ∞. In this case, we know that ∞ is mapped to some $\beta \in \mathbb{C}$, then we know that if we consider the restriction of f to $\mathbb{C}$ it gives you an automorphism function:

$$f : \mathbb{C}\backslash\{\alpha\} \to \mathbb{C}\backslash\{\beta\}$$

Notice that we can consider translations to get an automorphism g of $\mathbb{C}^*$ where $z = 0$ is a pole. Recall that from complex analysis, such a automorphism can only be $\frac{\gamma}{z}$ for some $\gamma \neq 0$. This means that:

$$f(z) = g(z+\alpha) + \beta = \frac{\gamma}{z+\alpha} + \beta = \frac{\beta z + \gamma + \beta\alpha}{z+\alpha}$$

Then to make f on $\mathbb{P}^1$ continuous, we can only set ∞ to γ and α to ∞. This is the f we started with. Consider the matrix:

$$\begin{pmatrix} \beta & \gamma + \beta\alpha \\ 1 & \alpha \end{pmatrix}$$

Since γ is not 0, we get a matrix in $\mathrm{PGL}_2(\mathbb{C})$. These two constructions are clearly inverses of each other. ☐

Now after we are finished with the study of meromorphic functions on $\mathbb{P}^1$, we want to study it for a general compact Riemann surface X. To do this, we will need some topological preparations.

Let X and Y be topological manifolds, the most important part is locally compact and Hausdorff. Recall that we say a space is locally compact if it has a neighborhood U such that $\overline{U}$ is closed. Clearly, every topological manifold is locally compact.

We say that a continuous map $f : X \to Y$ is **proper** if $f^{-1}K$ is proper for every compact $K \subset Y$.

We say that $f : X \to Y$ is a **local homeomorphism** if for any $x \in X$, there is an open neighborhood U_x of x such that $f(U_x)$ is open in Y and $f : U_x \to f(U_x)$ is a homeomorphism.

We say that f is a **covering map** if for all $y \in Y$, there is an open neighborhood V of y such that it's evenly covered by f in the sense that $f^{-1}(V) = \coprod_{j\in K} U_j$ is a disjoint union and $f : U_j \to V$ is a homeomorphism. In this case, we say X is a **covering** of Y.

Remark 3.7.
1. $\#J$ is locally constant on Y which means that $\#f^{-1}(y)$ is locally constant on Y. Then if Y is connected, then $\#f^{-1}$ is constant.
2. In general we will assume that Y is connected and we say that a covering is **finite** if $f^{-1}(y)$ is finite and we say that this is a n**-sheeted covering** if the number $\#f^{-1}(y) = n$.
3. Let $f : X \to Y$ and $\tilde{f} : X \to Y$ be two coverings, we say that they are isomorphic as coverings if there is a homeomorphism $\varphi : \tilde{X} \to X$ such that the following diagram commutes:

$$\begin{tikzcd} \tilde{X} \arrow[rr, "\varphi"] \arrow[dr, "\tilde{f}"'] & & X \arrow[dl, "f"] \\ & Y & \end{tikzcd}$$

Example 3.8. Consider the space $\mathbb{D}^*$, the punctured unit disk, I claim that any n-sheeted covering of $\mathbb{D}^*$ from a connected space is isomorphic to $z \mapsto z^n$.

This is easy to prove this once we know that $\pi_1(\mathbb{D}, 0) = \mathbb{Z}$, therefore we know that any n-sheeted covering should correspond to a index n subgroup of $\mathbb{Z}$ (since $\mathbb{Z}$ is abelian). Then recall that there is only one index n subgroup in $\mathbb{Z}$, which is $n\mathbb{Z}$. Therefore if we find one degree n cover, then we are done. Clearly z^k is such a cover.

Lemma 3.9. If $f : X \to Y$ is proper with X, Y locally compact Hausdorff, then f is closed.

Problem 3.10. Let F be closed in F, we want to show that $f(F)$ is closed. Recall that our space X is locally compact, therefore to prove $f(F)$ is closed, I claim that we only need show that for all compact sets K of Y, $K \cap f(F)$ is closed. Indeed, if $y \notin f(F)$, we know that we can find a compact neighborhood of y, say K, then since $K \cap f(F)$ is closed, we know that $K \backslash \{K \cap f(F)\}$ is open in K and it contains y. Since y is contained in the interior of K, we know that we know that the interior intersect $K \backslash \{K \cap f(F)\}$ is open in the interior, therefore open in the whole space since the interior of K is open in Y. Then we are done.

Then notice that $f(F) \cap K = f(F \cap f^{-1}(K))$. Since $f^{-1}K$ is compact and X is Hausdorff, then it's closed, then $F \cap f^{-1}(K)$ is a closed subset of a compact subset, which is again compact, then it's image is compact, since closed since Y is Hausdorff.

Proposition 3.11. Suppose that $p : X \to Y$ is a local homeomorphism with Y connected and X, Y locally compact and Hausdorff, then P is a finite covering map iff it's proper.

Proof. $\Rightarrow$: Let K be a compact set in Y and consider $p^{-1}K$. Suppose that we are given an open cover of K, say by O_α, we need a subcover. Now fix $y \in K$, we know that since this is a finite sheet cover, $f^{-1}y = \{x_1, \cdots, x_n\}$. Notice that we can choose an open cover U of y such that it's inverse image are $V_1, \cdots, V_n$ and each of them is homeomorphic to U. Notice that for each x_i, there is some O_{α_i} such that $x \in O_{\alpha_i}$. Then consider $\mathcal{O}_{\alpha_i} \cap V_i$ which is not empty and we consider it's image W_i in U, which is again open. Now let $W_y = W_1 \cap \cdots \cap W_n$, which is open and nonempty. Then we know that $p^{-1}W_y \cap V_i \subset O_{\alpha_i}$. This implies that $p^{-1}W_y$ is covered by $\mathcal{O}_{\alpha_1} \cup \cdots \cup \mathcal{O}_{\alpha_n}$. Now consider W_y for each y, since K is compact, we know that we can choose finitely many W_y to cover K. Then we know that $p^{-1}K$ is covered by finitely many $p^{-1}W_y$, each of them covered by finitely many elements in $\mathcal{O}_\alpha$, therefore we get a finite subcover of $p^{-1}K$.

$\Leftarrow$: Now assume that p is proper, and we want to show it's a covering. Since p is proper and we know that in a Hausdorff space a single point is closed and compact, we know that p^{-1} of a single point $y \in Y$ is closed and compact. Recall that p is a local homeomorphism and X is Hausdorff, we can find a finite open cover of p^{-1} each of them is homeormorphic to some open neighborhoods of y. Clearly, since they are local homeomorphisms, each such open up stairs only contain one point in $p^{-1}y$. Therefore $p^{-1}y$ is finite. Now by shrinking these neighborhoods, we can assume they are disjoint from Hausdorffness.

Let's call these neighborhoods upstairs $U_1', \cdots, U_n'$, the neighborhood down stairs $V_1, \cdots, V_n$ and we consider the closed point $X \backslash \cup_i U_i'$. Recall that since X, Y are locally compact and Hausdorff, we know that p is closed, therefore $F = f(X \backslash \cup_i U_i')$ is closed and doesn't contain y. Notice that $V' = Y \backslash F$ is now an open neighborhood of y and we know that $p^{-1}V = U_1' \cup \cdots \cup U_n'$. Now define $V = V' \cap V_1 \cap \cdots \cap V_n$ and V is going to be evenly covered by $p^{-1}V \cap U_i'$. ☐

Using these topological consequences, we now talk about **branched** and **unbranched coverings**.

Definition 3.12. Let $f : X \to Y$ be a nonconstant holomorphic map between Riemann surfaces. A point $x \in X$ is a **branch point** or **ramification point** if there is no neighborhood U_x of x such that $f|_{U_x}$ is injective.

If f has no branch points/ramification points, we will say that f is a **unbranched holomorphic map**.

Remark 3.13. First, I claim that if f is a holomorphic function betwen two compact Riemann surfaces, then it is a local homeormorphism iff f has no branch points. Indeed, we know that if f is a local homeomorphism then we know that locally near at point, it's bijective. Conversely, from last time, we know that, for each $x \in X$, we can choose local charts for x and $f(x)$ such that the holomorphic function locally looks like $z \mapsto z^k$. Then we know that if f has no branch points, then k has to be 1, which means that locally it's a homeomorphism.

If we consider the set A of all the branched points, I claim that if f is not constant, then it's closed and discrete. It's clearly closed no matter if f is constant or not since if we assume that x is outside A, then by definition, there is a open neighborhood of x such that f is injective there. Then we know that for all points x' in this neighborhood, x' is not a branch point. We also know that A has to be discrete if f is not constant. Again this comes from the local behavior of f. If x is a branch point, we know that locally f looks like $z \mapsto z^k$ for some finite integer k. We know that for $z \neq 0$, the map is indeed locally injective since the complex derivative is nonzero.

Now let's assume that f is also a proper map. For example if we are looking at maps between two compact Riemann surfaces, then every holomorphic map is proper. Then we know that $f(A) = B$ is also closed and discrete since f is open (nonconstant). B is called the set of all **critical values** of f. This tells you that if we consider $Y' = Y\backslash B$ and let $X' = X\backslash f^{-1}(B)$, then we know that if we restrict f to X' and Y', it's now a proper local homeomorphism, which gives you a finite sheeted cover. Indeed, f is still proper since we know that if K is compact in Y', then we know that it's inverse image if compact in X but is contained in the open subset X', therefore K is still compact in X'.

Let's call the restriction $f' : X' \to Y'$, which is proper holomorphic and unbranched. Moreover I claim Y' again connected. Indeed, suppose that Y' is not, then we know that we can separate it into two disjoint nonempty open subsets U and V. Now for each $p \in B$, we know that there is a coordinate disk D_p of p such that this disk doesn't intersect with any other points in B. Then consider $D\backslash\{p\}$, it's connected, since the punctured disk is connected, then we know that it's has to be contained in either U or V. Then consider:

$$B_U = \{p \in B : D_p\backslash\{p\} \subset U\}, \quad B_V = \{p \in B : D_p\backslash\{p\} \subset V\}$$

We will let you check that $U' = U \cup B_U$ and $V' = V \cup B_V$ are going to be open, therefore we know that Y now is separated into U', V' both are nonempty opens, a contradiction.

This implies that $f' : X' \to Y'$ is a finite sheet covering of a connected space, which means it has a well defined degree or sheet of covering. Say it's a n-sheeted covering, we know that every point $p \in Y'$ are assumed exactly n times.

We will now extend the above statement to include the critical points. In other words, we want to say that in some sense, the critical points are also taken exactly n times, but with a different meaning. Suppose that $f : X \to Y$ is a proper holomorphic nonconstant map between Riemann surfaces, we will use $v(f,x)$ to denote the local multiplicity of f at x. Recall that this is defined using the local behavior of f and is characterized by there is an open neighborhood U of x and V of $f(x)$ such that every $y \in V$ not equal to $f(x)$ has exactly k values in U. We will say that f takes the value $c \in Y$ **counting multiplicity** m times on X if $m = \sum_{x \in f^{-1}c} v(f,x)$.

Theorem 3.14. Suppose that X, Y are Riemann surfaces and $f : X \to Y$ is a proper nonconstant holomorphic map. Then there is a natural number $n > 0$ such that f takes every value $c \in Y$ counting multiplicity n times.

Proof. Let n be the number of sheets of covering we defined above. We already know that for $y \in Y'$, we exactly have n preimage in X', each with multiplicity 1. Therefore, we only need to consider the critical points. Let $b \in B$ be a critical value, and let $f^{-1}b = \{x_1, \cdots, x_r\}$ be the preimage. Indeed we know that since f is proper, the preimage is closed and compact. Moreover, it's also discrete, therefore is finite. Let k_j be the multiplicity $v(f, x_j)$. Notice that by the local behavior of holomorphic functions, we know that there are neighborhood U_j of x_j and V_j of b such that for every $c \in V_j\backslash\{b\}$ we know $f^{-1}c \cap U_j$ has exactly k_j elements. Notice that we can find a $V \subset V_1 \cap \cdots \cap V_r$ such that $f^{-1}V \subset U_1 \cup \cdots \cup U_r$.

Indeed, this is the same trick we used before when we want to prove a proper local homeomorphism is a covering. We know that we can first take $f(X\backslash(U_1 \cup \cdots \cup U_r))$, which is closed, then it's complement is going to be open, which contains b. Then we know that if we let $V' = Y\backslash f(X\backslash(U_1 \cup \cdots \cup U_r))$ and let:

$$V = V' \cap V_1 \cap \cdots \cap V_r$$

We now see that the inverse image of V is contained in $U_1 \cup \cdots \cup U_r$.

Now consider $V \cap Y'$, notice that since B is discrete, we know that $V \cap Y'$ is not empty and we know that for every $c \in V \cap Y'$ where $c \neq b$, we have $k_1 + \cdots + k_r$ elements. On the other hand, we know that $f^{-1}c$ has n elements for all $c \in Y'$. ☐

Corollary 3.15. Let X be a compact Riemann surface, then every nonconstant meromorphic function $f : X \to \mathbb{P}^1$ has as many 0 as poles, counting multiplicity.

Proof. Recall that the multiplicity of the pole is equal to the order of the pole. Indeed, we know that after we choose a coordinate chart of X, we can assume that the function $f(z)$ has the Laurent expansion:

$$f(z) = \sum_{\nu=-k}^{\infty} c_\nu z^\nu, c_{-k} \neq 0$$

Then we know that if we consider the chart of $\mathbb{P}^1$ which is $\mathbb{P}^n \backslash \{0\}$, we know that the function looks like:

$$z \mapsto \frac{1}{f(z)}$$

Then we can prove that $z = 0$ now is an order k zero, which means that we can find another holomorphic coordinate change such that the function looks like $z \mapsto z^k$, therefore the multiplicity of the pole is the order of the pole.

The order of zeros is just the multiplicity of these zeros. Then we know that since f is a map between compact Riemann surfaces, it's automatically proper, then by the above theorem, we know that f takes the value 0 and ∞ the same number of times counting multiplicity, which means that the order of zeros plus the order of poles has to be equal. □

Those above theories are indeed very good, but it still didn't answer the question of what is $\mathcal{M}(X)$ for a compact Riemann surface. In next section, we will use these results to study meromorphic functions on the tori $\mathbb{T} = \mathbb{C}/\Omega$ where Ω is a lattice. Turns out that the theory on $\mathbb{T}$ actually generalizes to all compact Riemann surfaces X.

4 Lecture 4

Recall that we have studies a bit of elliptic functions in the homework problems. We will now dive deeper and see more properties they have.

Definition 4.1. Recall that let Ω be a lattice in $\mathbb{C}$, then a meromorphic function from $\mathbb{C}$ to $\mathbb{P}^1$ is **elliptic** with respect to this lattice Ω if $f(z + \omega) = f(z)$ for any $\omega \in \Omega$.

Clearly, if z_0 is a pole of such a function, then so is $f(z_0 + \omega)$ for any $\omega \in \Omega$ just from that fact that poles in dimension 1 case are defined using limits. Then we can consider the following diagram:

$$\begin{tikzcd}
\mathbb{C} \arrow[rr, "f"] \arrow[dr] & & \mathbb{P}^1 \\
& T = \mathbb{C}/\Omega \arrow[ur, dashed] &
\end{tikzcd}$$

Again, since f is holomorphic and $\mathbb{C} \to T$ is a covering map and a local biholomorphism, we know that the uniquely induced map is automatically holomorphic. Therefore every elliptic function on $\mathbb{C}$ correspond to a holomorphic function from T to $\mathbb{P}^1$.

Note that T is compact since we know that every element in $\mathbb{C}$ can be written as the sum of something in the fundamental parallelogram and something in the lattice, therefore we know that T is the image of the fundamental parallelogram in $\mathbb{C}$, which is compact. Then we know that the function $f : T \to \mathbb{P}^1$ is now a holomorphic function between two compact Riemann surfaces, therefore it's automatically proper, thus we can apply the results we obtianed in the last section:

Definition 4.2. The **order** of an elliptic function $f : T \to \mathbb{P}^1$, denoted by $\mathrm{ord}(f)$, is the number of solutions of $f(z) = \infty$ on T counting multiplicity.

Notice that if $\mathrm{ord}(f) = 0$, then this means that the image of f in $\mathbb{P}^1$ is just a compact subset of $\mathbb{C}$, which means that it's bounded. Then we know that f considered as a function on $\mathbb{C}$ is bounded and entire, then Liouville's theorem tells you that f is constant.

We will later use P to denote the fundamental parallelogram of Ω. Sometimes we will assume that P to have vertices $t, t + \omega_1, t + \omega_2, t + \omega_1 + \omega_2$ where t is a complex number chosen such that there are no zeros and poles of f on the boundary of P. Notice that even if translating the parallelogram with vertices $0, \omega_1, \omega_2, \omega_1 + \omega_2$ by t did have an effect if we think in terms of $\mathbb{C}$, however, if we think of f as a function on T. Then notice that the image of this translation is still all of T. The only difference it makes is that if now we consider the zeros and poles in the translated parallelogram, none of their differences is in Ω, while if you don't have this property if you don't do the translation.

This translation makes it easier for us to think of the class of zeros and poles of f in T since after the translation no distinct zeros or poles are getting mapped to the same class in T. Recall that they only have finitely many poles and zeros in the parallelogram without the shifting, so we can indeed choose a t. Then if we compute the integral, we know that the residue inside P sums to 0.

Corollary 4.3. There is no elliptic functions with order 1.

Proof. If there is, then we know that the function only has one pole inside P, which means that the sum of the residue can not be 0. $\square$

Corollary 4.4. If an elliptic function has order $N > 0$, then it takes every value in $\mathbb{P}^1$ exactly N times counting multiplicity in T.

Proof. Note that f is a holomorphic function from T to $\mathbb{P}^1$ with both compact. Then we can think of it as a branched covering where N is the number of sheets. $\square$

Corollary 4.5. If f, g are two elliptic functions of the same zeros and poles counted with multiplicity, then I claim that $f = \lambda g$ where $\lambda \in \mathbb{C}$ is a nonzero constant.

Proof. We can consider the elliptic function $\frac{f}{g}$, which has no poles and no zeros. The fact that it has no poles means that it's constant. The fact that it has no zeros means that the constant is not 0. $\square$

We have proved the following theorem in the previous homework and we are just restating it below:

Theorem 4.6. Let f be an elliptic function and let $[\alpha_1], \cdots, [\alpha_r]$ mod Ω be the distinct zeros of f counted with multiplicity $k_1, \cdots, k_r$ and let $[\beta_1], \cdots, [\beta_r]$ be the distinct poles of f with multiplicity $l_1, \cdots, l_s$ respectively. Then we have:

$$\sum_{i=1}^{r} k_i \alpha_1 - \sum_{j=1}^{s} l_j \beta_j \in \Omega$$

Now we have discussed the general theory of elliptic functions, we can ask that if we can produce a concrete elliptic function. We have already seen that a degree 1 such function can not exist. So our next goal is to produce a degree 2 such function. To do it, turns out that producing an order 3 function first is helpful.

Lemma 4.7. The series $\sum_{n \geqslant 1} \frac{1}{n^s}$ converges iff $s > 1$.

Proof. Clearly if $s = 1$, then we already know that $\sum \frac{1}{n}$ doesn't proof, then for smaller powers, we know that $\frac{1}{n^s} \geqslant \frac{1}{n}$, therefore they don't converge. If $s > 1$, recall that we can use Cauchy condensation to test if a decreasing series converges or not, and in this case we get it converges iff the following series converges:

$$\sum_k 2^k \frac{1}{2^{ks}} = \sum_k 2^{k-sk}$$

We know that this converges iff $1 - s < 0$ or in other words $s > 1$. $\square$

The reason we mention this elementary lemma is because what we will need below is a 2 dimensional generalization of the above proof:

Lemma 4.8. If $s \in \mathbb{R}$ and Ω is a lattice, then the series:

$$\sum_{\omega \in \Omega *} \frac{1}{|\omega|^s}$$

converges iff $s > 2$. Here $\Omega^* = \Omega \backslash \{0\}$.

Proof. Consider the following set for an integer $r \geqslant 1$:

$$\mathrm{P}_r = \{a\omega_1 + b\omega_2 \mid a, b \in \mathbb{R}, \max\{|a|, |b|\} = r\}$$

Notice that we have:

$$\Omega^* = \cup_{r \geqslant 1} \Omega \cap \mathrm{P}_r$$

Then we know that we will group the summands in the order of $r \geqslant 1$ and we know that each $\Omega \cap \mathrm{P}_r$ has $8r$ points. Let M, m be the maximum and minimum of $|\omega|$ for $\omega \in \Omega \cap \mathrm{P}_1$. Then we know that for those points in $\Omega \cap \mathrm{P}_r$, the maximum and minimum moduli will be rM and rm. Therefore we know that all the moduli of the points in $\Omega \cap \mathrm{P}_r$ will be:

$$rm \leqslant |\omega| \leqslant rM$$

Then if we define:

$$\sigma_r = \sum_{\omega \in \Omega \cap \mathrm{P}_r} \frac{1}{|\omega|^s}$$

We have:

$$8r(rM)^{-s} = 8M^{-s}\frac{1}{r^{s-1}} \leqslant \sigma_r \leqslant 8r(rm)^{-s} = 8m^{-s}\frac{1}{r^{s-1}}$$

We also know that $\sum_r \sigma_r$ is what we are looking for. We know that it converges iff $\frac{1}{r^{s-1}}$ converges iff $s-1 > 1$ that is iff $s > 2$. □

Using the above lemma, we can produce an elliptic function with order $N \geqslant 3$. Consider the summation:

$$F_N(z) = \sum_{\omega \in \Omega} \frac{1}{(z-\omega)^N}$$

Proposition 4.9. For $N \geqslant 3$, we get an elliptic functions of order 3 with a pole of order 3 at each lattice points.

Proof. Let K be a compact subset of $\mathbb{C}\backslash\Omega$, then we know that for all but finitely many $\omega \in \Omega$, they don't lie in K. Since finitely many terms in the summation doesn't affect the convergence. Let's assume that for all $\omega \in \Omega$, we have $|\omega| \geqslant 2|z|$ for all $z \in K$.

We will show that this series absolutely and uniformly converges. Here absolute convergence makes switching the order of summation possible and we need uniform convergence to conclude that the limit is holomorphic on all of $\mathbb{C}\backslash\Omega$.

Notice that we know:

$$\sup_{z \in K} |\frac{1}{\omega - z}| \leqslant \frac{1}{2^N}\frac{1}{|\omega|^N}$$

Then we know that since $\sum_{\omega \in \Omega} \frac{1}{|\omega|}$ converges when $N \geqslant 2$, we know that:

$$|\sum_{\omega \in \Omega} \frac{1}{(z-\omega)^N}| \leqslant \sum_{\omega \in \Omega} \frac{1}{2^N|\omega|^N}$$

This shows that this series converges absolutely. Moreover, since we know that $\frac{1}{2^N|\omega|^N}$ is a majorant of $\frac{1}{|(z-\omega)^N|}$, then we know that this series converges uniformly, which implies that $F_N(z)$ is indeed holomorphic on $\mathbb{C}\backslash\Omega$.

Now we prove that $F_N(z)$ is doubly periodic, we know that for each $\omega' \in \Omega$, $F_N(z+\omega')$ is:

$$\sum_{\omega \in \Omega} \frac{1}{(z+\omega'-\omega)^N}$$

Recall that since this series is absolutely convergent, we can switch the order of summation in the series so that $\sum_{\omega \in \Omega} \frac{1}{(z+\omega'-\omega)^N} = \sum_{\omega \in \Omega} \frac{1}{(z-\omega)^N}$ as we know that each term showing up in the latter summation is also in the first summation.

Now clearly, we know that this $F_N(z)$ has a pole of order 2 at every lattice point because if ω' is such a point, after we subtract $\frac{1}{(z-\omega)^N}$, the rest of the summation is going to be holomorphic at ω' and of course, it's still going to be absolutely and uniformly convergent by exact the same argument as above, therefore we know that $F_N(z) - \frac{1}{(z-\omega)^N}$ is holomorphic at ω, therefore the order of the pole of $F_N(z)$ at ω' is N. □

Notice that since the only poles of the elliptic function are at lattice points with order 3, we know that after we pass to a holomorphic function T to $\mathbb{P}^1$, we know that this function indeed has order 3.

Now we begin to look for order 2 elliptic functions.

Definition 4.10. The **Weierstrass $\wp$ function** is defined to be:

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Omega*} (\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2})$$

Note:

$$\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} = \frac{2\omega z - z^2}{\omega^2(z-\omega)^2}$$

This implies that for any z in a compact subset of $\mathbb{C}\backslash\Omega$, we can assume that this has order of magnitude $\frac{1}{|\omega|^3}$ for all but finitely many $\omega \in \Omega^*$. Then the exactly same proof as above shows that that $\wp(z)$ is a meromorphic function with an order 2 pole at each lattice point.

Proposition 4.11. $\wp(z)$ is doubly periodic with respect to Ω.

Proof. Notice that this is not immediately clear from the definition of $\wp(z)$ so we will need to do something. First, since the series defining $\wp(z)$ is absolutely convergent, which allows us to rearrange the summation, therefore we know that $\wp(-z) = \wp(z)$. This shows $\wp(z)$ is an even function.

Recall that since this series also converges uniformly on every compact subset, we know that we can compute the derivative term by term, which measn that:

$$\wp'(z) = -2 \sum_{\omega \in \Omega} \frac{1}{(z-\omega)^3}$$

We already proved that this is an elliptic function therefore we know that:

$$\wp'(z+\omega) - \wp'(z) = 0, \quad \omega \in \Omega$$

Then we know that $\wp(z+\omega) - \omega(z) = C$ for all $z \in \Omega\backslash\Omega$ and some constant $C \in \mathbb{C}$.

Notice that if we let $z = -\frac{\omega}{2}$, then we know:

$$\wp(\frac{\omega}{2}) - \wp(-\frac{2}{\omega}) = \wp(\frac{\omega}{2}) - \wp(\frac{\omega}{2}) = 0 = C$$

□

We will use $\mathcal{M}(\Omega)$ or $\mathcal{M}(T)$ to be the set of all elliptic functions with respect to Ω. Clearly, the summation, difference and product and division still remain elliptic, we know that $\mathcal{M}(\Omega)$ is actually a field. Now I claim that we have:

$$\mathcal{M}(\Omega) = \mathbb{C}(\wp(z), \wp'(z))$$

That is to say that any elliptic functions can be written as a rational function with $\mathbb{C}$ coefficients in terms of $\wp(z)$ and $\wp'(z)$. To prove this, we need to study further properties of $\wp(z)$.

Theorem 4.12. If P is a fundamental parallelogram (possibly translated) such that $0, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2} = \frac{\omega_1+\omega_2}{2}$ are in the interior of P and there are nonzeros and poles on the boundary of P. Then the only zeros of of $\wp'(z)$ are $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$ each with multiplicity 1. In particular, 0 is not a critical value of $\wp'(z)$.

Proof. Notice that since the order of $\wp'(z)$ is 3, there should be 3 zeros of $\wp'(z)$ in this fundamental parallelogram. Notice that since $\wp'(z)$ is odd and doubly periodic, we know that:

$$\wp'(\frac{\omega_i}{2}) = \wp'(-\frac{\omega_i}{2}) = -\wp'(\frac{\omega_i}{2}), \quad i = 1, 2, 3$$

Notice that $\frac{\omega_i}{2}$ are not poles of $\wp'(z)$, thus we know that:

$$\wp'(\frac{\omega_i}{2}) = 0$$

Notice that we have three of them, therefore each of them has to be multiplicity 1 as the sum of all multiplicity should add up to 3. ☐

Notice that the zeros of $\wp'(z)$ are the branch points of $\wp(z)$ and we will call the images $e_i = \wp(\frac{\omega_i}{2})$ for $i = 1, 2, 3$ and these are going to be the critical values of $\wp(z)$ in $\mathbb{C}$ (we already know that ∞ is also a critical point of $\wp(z)$). We also know that e_1, e_2, e_3 must be distinct, otherwise, we know that $\wp(z) - e_i$ would have two solutions each of them with order 2, therefore this will make the function $\wp(z)$ of order at least 4, which is a contradiction.

You probably notice that these values are defined using a basis of Ω. So the question is that if we choose a different basis of the lattice Ω, do we get different e_1, e_2, e_3. The answer is, if we only care these three values as a set $\{e_1, e_2, e_3\}$, then it doesn't depend on the choice of basis.

Note that if we let S be:

$$S = [\frac{\omega_1}{2}] \cup [\frac{\omega_2}{2}] \cup [\frac{\omega_3}{2}]$$

where each bracket is the set of equivalence classes mod Ω. Notice that this is the set of all zeros of $\wp'(z)$ on $\mathbb{C}$. Then if we consider the image $\wp(S)$, which is independent of the choice of basis, we recover $\{e_1, e_2, e_3\}$. Therefore this set only depends on the lattice, not on the basis.

Corollary 4.13. For each point $c \in \mathbb{P}^1 \backslash \{e_1, e_2, e_3, \infty\}$, the equation $\wp(z) = c$ has 2 simple solutions in T.

Proof. We know that besides the critical point ∞, we only have three others, which are e_1, e_2, e_3. Then we know from the branched covering that besides the critical points, the rest should be a 2-sheeted covering, therefore we know that the inverse image has exactly 2 points. ☐

Theorem 4.14. If f is an even elliptic function then there is some rational function $R_1(z) \in \mathbb{C}(z)$ such that $f = R_1(\wp(z))$. Therefore we know that:

$$\mathcal{M}^{\text{even}}(\Omega) = \mathbb{C}(\wp(z))$$

Proof. Let f be a nonconstant elliptic function, otherwise there is nothing to prove. Now let's assume that the order of f is $N > 0$. Notice that if we consider the zeros of f, there are only finitely many in the fundamental parallelogram P and no zeros or poles of f are on the boundary. Notice that if $k \in \mathbb{C}$ is a critical value of f with $f'(z) = 0$. Since we know that there are only finitely many zeros of $f'(z)$ in P, we know that there can only be finitely many $k \in \mathbb{C}$ such that k is a critical value. Therefore, we may choose two complex numbers $c, d \in \mathbb{C}$ with $c \neq b$. Such that $f(z) = c$ and $f(z) = d$ only have simple solutions.

Moreover, we can choose c, d such that the solutions $f(z) = c$ and $f(z) = d$ has no solution that is congruent to $0, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$ mod Ω. Indeed, we can assumed that $0, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$ are in the interior of P such that we know that if some z is congruent to one of them, we know that z has to be $0, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$ plus some $\omega \in \Omega$ since if you subtract an element in P with something in the interior of P, you never get something on the lattice. Therefore when we choose c, d we still only to avoid at most 3 more elements in $\mathbb{C}$.

Since $f(z)$ is even, we know that a complete set of solutions of $f(z) = c$ looks like:

$$\{a_1, -a_1, a_2, -a_2, \cdots, a_n, -a_n\}$$

where no a_i is congruent to $0, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$. We need to explain this a bit. Suppose that a_1 is a root of $f(z) - c$, we know that since $f(z)$ is even, then $-a_1$ is also a root of f. However, we don't know if $-a_1$ is in P or not, so we would like to do a translation by lattice points so that $-a_1$ lies in P. Notice that after the translation, the point can not be a_1, otherwise we will have $2a_1$ is congruence to 0. This implies that either $a_1 = 0$ or a_1 is congruent to $0, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$, but any of these violates our assumptions. Moreover, we don't have to worry about that there are two ways to translate $-a_1$ to P since if there are such two ways, the two points will differ by something in the lattice, therefore they are the same point in T, therefore we only need to record one of them. For notation simplicity, we will still use $-a_1$ to denote the translated point, but it is actually in P and we actually further think that it's in T.

In particular, we see that the order of an even elliptic function has to be even.

Similarly, we know that a complete set of solutions of $f(z) = -d$ looks like:

$$\{b_1, -b_1, b_2, -b_2, \cdots, b_n, -b_n\}$$

and no solution is congruent to $0, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$. And again every solution above is considered to lie in T.

Then let's consider the elliptic function:

$$g(z) = \frac{f(z) - c}{f(z) - d}$$

What is important is that subtracting a finite complex number doesn't affect the poles of f, but it does affect the zeros of f, therefore we know that $g(z)$ is an elliptic function which has only simple zeros and simple poles. We know that it has simple zeros at:

$$\{a_1, -a_1, a_2, -a_2, \cdots, a_n, -a_n\}$$

has simple roots at:

$$\{b_1, -b_1, b_2, -b_2, \cdots, b_n, -b_n\}$$

Recall that we said that $\wp(z)$ only has critical values at points congruent to $0, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$. Then we know that:

$$\wp(z) - \wp(a_i), \quad \wp(z) - \wp(b_i)$$

all have simple roots at $\pm a_i$ and $z = \pm b_i$. Therefore we know that if we consider;

$$h(z) = \frac{\prod_{i=1}^{n}(\wp(z) - \wp(a_i))}{\prod_{i=1}^{n}(\wp(z) - \wp(b_i))}$$

We know that it has simple zeros at $\pm a_i$ and simple poles at $\pm b_i$, which is the same as $g(z)$. Then by what we have proved, there is a nonzero λ such that:

$$\frac{f(z) - c}{f(z) - d} = \lambda \frac{\prod_{i=1}^{n}(\wp(z) - \wp(a_i))(\wp(z) - \wp(-a_i))}{\prod_{i=1}^{n}(\wp(z) - \wp(b_i))(\wp(z) - \wp(-b_i))}$$

Then we know that we have:

$$f(z) = \frac{dh(z) - c}{h(z) - 1}$$

which is still a rational function in $\wp(z)$. $\square$

Once we are done with even function, we also know how to deal with odd functions: if $f(z)$ is an odd elliptic function, we know that $\frac{f(z)}{\wp'(z)}$ is even, then we know that there will be $R_2(\wp(z))$ such that $\frac{f(z)}{\wp'(z)} = R_2(\wp(z))$, which implies $f(z) = \wp'(z)R_2(\wp(z))$. Then for a general elliptic function $f(z)$, we know that we always have:

$$f(z) = \frac{f(z) + f(-z)}{2} + \frac{f(z) - f(-z)}{2}$$

where the first term is even and the second term is odd. We then conclude that for any elliptic function $f(z)$, we can write it as $R_1(\wp(z)) + \wp'(z)R_2(\wp(z))$.

All of the above discussion are good. However, recall that T is a Riemann surface, which is a dimension 1 complex manifold. Then if we think of $\mathcal{M}(T)$, which is it's field of meromorphic functions, which is just $\mathcal{M}(\Omega)$. We should better expect that it's also a dimension 1 object, in the sense that it can be generated by only one independent variable. Here we know that $\mathcal{M}(\Omega) = \mathbb{C}(\wp(z), \wp'(z))$. Therefore we better hope that there is some algebraic relation between $\wp(z)$ and $\wp'(z)$. it turns out that there is indeed such a relation, which carries more meaning than just matching with our intuition. In order to show the algebraic relation, we need to study the following function:

Definition 4.15. The **Weierstrass ζ function** relative to Ω is:

$$\zeta_\Omega(z) = \frac{1}{z} + \sum_{\omega\in\Omega^*}(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2})$$

Notice that if we do the algebra we find that:

$$\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} = \frac{\omega^2 + \omega(z-\omega) + z(z-\omega)}{(z-\omega)\omega^2} = \frac{z^2}{\omega^2(z-\omega)}$$

Then we know that on any compact set of $\mathbb{C}\backslash\Omega$, it going to have magnitude of $\frac{1}{|\omega|^3}$ therefore it converges absolutely and uniformly, therefore defines a meromorphic function on $\mathbb{C}\backslash\Omega$.

Then let D be the largest disk centered at 0 such that it doesn't contain any lattice points. Then, for $\omega \in \Omega^*$, we have:

$$\frac{1}{z-\omega} = \frac{-1}{\omega}\frac{1}{1-\frac{z}{\omega}}$$

Notice that on D, $\frac{z}{\omega}$ has moduli less than 1, thus the above formula has a series:

$$\frac{-1}{\omega}(1 + \frac{z}{\omega} + \frac{z}{\omega}^2 + \frac{z}{\omega}^3 + \cdots)$$

Moreover, we know that the above series converges absolutely and uniformly on any compact subset of D. Recall that in $\zeta_\omega(z)$, each term is $\frac{z^2}{\omega^2}\cdot\frac{1}{(z-\omega)}$. Then we can substitute using the above equation and since everything is absolutely convergent, we can rearrange the summation and get:

$$\zeta_\Omega(z) = \frac{1}{z} + \sum_{\omega\in\Omega^*}(-\frac{z^2}{\omega^2} - \frac{z^3}{\omega^4} - \cdots) = \frac{1}{z} - G_3 z^2 - G_4 z^3 - \cdots$$

This requires a bit more explanation. First, consider the double series:

$$\sum_{\omega\in\Omega^*}\sum_{n\geqslant 2}\frac{-z^n}{\omega^{n+1}}$$

This double series make sense for each $z \in D$ because we know that by the definition of a double series, first do the summation inside, which gives you $\frac{z^2}{\omega^2(z-\omega)}$ and we already know that the summation of $\frac{z^2}{\omega^2(z-\omega)}$. So for a single value $z \in D$, this double series makes sense. The issue here is that we want to switch the order of the summation from $\sum_{\omega\in\Omega^*}\sum_{n\geqslant 2}\frac{-z^n}{\omega^{n+1}}$ to $\sum_{n\geqslant 2}\sum_{\omega\in\Omega^*}\frac{-z^n}{\omega^{n+1}}$. For this to work, we will need the Fubini theorem but for series. The condition is that the convergence is absolute in the sense that:

$$\sum_{\omega\in\Omega^*}\sum_{n\geqslant 2}|\frac{-z^n}{\omega^{n+1}}| < \infty$$

Indeed for any compact $K \subset D$, we know that $|\frac{z}{\omega}| \leqslant r$ for some $r < 1$, therefore we know that:

$$\sum_{n\geqslant 2}\frac{-z^n}{\omega^{n+1}} \leqslant \frac{1}{|\omega|^3}\frac{R^2}{1-r}$$

Then we know that the double sum now is $\leqslant \frac{R^2}{1-r}\sum_{\omega\in\Omega^*}\frac{1}{|\omega|^3} < \infty$ where $|z| \leqslant R$ for all $z \in D$. Then we know that by Fubini, we can interchange the summation order. Therefore, we know that the formula:

$$\zeta_\Omega(z) = \frac{1}{z} - G_3 z^2 - G_4 z^3 - \cdots$$

makes sense for each $z \in D$. Now again suppose that we work on a compact subset K of D, and we want to ask is the above convergence uniform on K. I will let you show that one sufficient condition is that:

$$\sum_n\sum_{\omega\in\Omega^*}\sup_{z\in K}|\frac{-z^n}{\omega^{n+1}}| < \infty$$

which is something we already observed above, therefore we again know that the convergence is uniform.

Definition 4.16. Given a lattice Ω and for each $k \geqslant 3$, we define the **Eisenstein series** of the lattice to be:

$$G_k(\Omega) = \sum_{\omega \in \Omega*} \frac{1}{\omega^k}$$

Notice that since the lattice is symmetric, for odd k, $G_k(\Omega) = 0$. Therefore we know that for $z \in D$, we have:

$$\zeta_\Omega(z) = \frac{1}{z} - \sum_{n=2}^{\infty} G_{2n} z^{2n-1}$$

This is the Laurent expansion of $\zeta_\Omega(z)$ around 0 due to the uniqueness of Laurent series. We will use this to develop further theories next lecture.

Below are the homework of this week:

4.1 Homework

Problem 4.17. Suppose that X is a compact Riemann surface and $\{p_1, \cdots, p_n\}$ are finitely many points on X. Define $X' = X \backslash \{p_1, \cdots, p_n\}$ and suppose that $f : X' \to \mathbb{C}$ be a nonconstant holomorphic function. Then the image of f comes arbitrarily close to any complex number.

Solution. We can assume that these points are poles. To see why, if p_i is an essentially singularity, then we can use the usual Casorati-Weierstrass theorem to conclude that even in a coordinate ball of p_i, it comes arbitrarily close to any complex number. If one of them is a removable singularity we just throw it away. However, these points can't all be removable singularities since if so, we will get nonconstant holomorphic function from X to $\mathbb{C}$, which has to be constant by what we discussed in lecture 3.

Therefore now assume that these are poles of f, which means that we get a nonconstant holomorphic map from X to $\mathbb{P}^1$. Therefore it has to be surjective. Then we know that if now we go back to $f' : X' \to \mathbb{C}$ with $p_1, \cdots, p_n$ not defined on X'. Since f is surjective, everything z_0 in $\mathbb{C}$ comes from some element in X which are not the poles. Then assume that it comes from some removable singularity p_i, we know that $\lim_{z \to p_i} f(z) = z_0$, therefore $f : X' \to \mathbb{C}$ comes arbitrarily closed to z_0.

Problem 4.18. Prove $\tan : \mathbb{C} \to \mathbb{P}^1$ is a local homeomorphism.

Solution. Recall that tan is defined by $\frac{\sin}{\cos}$. Moreover, we know that:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2}, \quad \cos z \frac{e^z + e^{-z}}{2i}$$

For example we know that the only zeros of $\sin z$ are those zeros of the real $\sin x$. Indeed if:

$$e^{iz} = e^{ix} e^{-y} = e^{-z} = e^{-ix} e^{y} \Rightarrow e^{2ix} = e^{y}$$

This implies that x has to be $k\pi$ for $k \in \mathbb{Z}$. Similarly, we can conclude that the zeros of $\cos z$ are only those real zeros.

This implies that the zeros of $\tan z$ are $k\pi$, then if we consider the derivative of tan, we know that:

$$\tan' z = \sec^2 z$$

Then we know that since when $\sin z = 0$, $\cos z \neq 0$, then we know that every 0 is a simple zero. We also know similarly that every pole is a simple pole. Moreover, we know that besides the zeros of $\cos z$, the derivative of $\tan z$ is not 0, which means that if we consider:

$$\tan : \mathbb{C} \backslash \{\frac{(2k+1)\pi}{2} : k \in \mathbb{Z}\} \to \mathbb{C}$$

This is already a local homeomorphism. Then if we consider the induced holomorphic map to $\mathbb{P}^1$. We only have to show that written in coordinates, $\infty \in \mathbb{P}^1$ is not a critical value to show it's a local homeomorphism. We consider for example the chart on $\mathbb{P}^1$ which is $\mathbb{P}^1 \backslash \{0\}$, then we know that the map looks like:

$$z \mapsto 1/\tan z = \frac{\cos z}{\sin z}$$

Then we know that when z is close to $\frac{(2k+1)\pi}{2}$, we know that the derivative of this function is not 0. Therefore we know that ∞ is also not a critical point.

Notice that we know that since $\mathbb{C}$ is not compact, we we can not conclude that the map $\tan z$ from $\mathbb{C}$ to $\mathbb{P}^1$ is surjective. In particular, we can not just conclude from it's a local homeomorphism that it's a covering map. However, we have:

Problem 4.19. Show that $\tan(\mathbb{C}) = \mathbb{P}^1 \backslash \{i, -i\}$ and we have $\tan z : \mathbb{C} \to \mathbb{P}^1 \backslash \{i, -i\}$ is covering map.

Solution. Indeed, we know that $\pm i$ is not in the image. Because if we try to compute:

$$\frac{\sin z}{\cos z} = c, \quad c\mathbb{C}$$

We are indeed solving for:

$$\sin z = c \cos z \Leftrightarrow e^{iz} - e^{iz} = \frac{c}{i}(e^{iz} + e^{-iz})$$

This is equivalent to:

$$e^{2iz}(1 - \frac{c}{i}) = (1 + \frac{c}{i})$$

Notice that e^{2iz} takes every complex value except 0. So if $c \neq \pm 1$, we know that there is a solution of:

$$e^{2iz} = \frac{(1 + \frac{c}{i})}{(1 - \frac{c}{i})}$$

If $c = i$, then clearly $0 \neq (1 + \frac{c}{i})$ and if $c = -i$, clearly there is no solution of $e^{2iz} = 0$. This indeed shows that $\tan(\mathbb{Z}) = \mathbb{P}^1 \backslash \{i, -i\}$.

Then we show that this is a covering map. Suppose that we have $c \in \mathbb{P}^1 \backslash \{i, -i\}$ which is a complex number, i.e. not ∞. Then if we consider it's preimage, we know it looks like:

$$z_0 + \pi\mathbb{Z}$$

where z_0 is a nonzero complex number. Recall that we already proved that $\tan z$ is a local homeomorphism, therefore we know that there will be an open neighborhood U of z_0 such that $\tan z$ homeomorphically takes that to an open neighborhood $\tan U$ of c. I claim that $\tan^{-1} \tan U$ is just:

$$U + \pi\mathbb{Z}$$

This is basically to notice that $\tan z$ is a periodic function with period π. Then we know that if U is mapped homeomorphically to $\pi(U)$, then $U + \pi k$ where $k \in \mathbb{Z}$ should also be homeomorphically mapped to $\pi(U)$ as we have the following composition:

$$\pi k + U \stackrel{-k\pi}{\to} U \stackrel{\tan z}{\to} \pi(U)$$

The composition is still $\tan z$ due to the periodicity. In order to show that $U + \pi\mathbb{Z}$ is all the inverse image. Suppose that X is mapped to some element in $\pi(U)$, that means that some $Y \in U$ makes $\tan X = \tan Y$, but this already implies that $X = k\pi + Y$ since the periodicity. Then we are done.

Then we need to worry about the point ∞. We know that the inverse image of ∞ is $\{\frac{(2k+1)\pi}{2} : k \in \mathbb{Z}\}$. But this time in the new coordinates, instead of looking at $\tan z$, we are using the chart of $\mathbb{P}^1 \backslash \{0\}$ and think of the function as $\frac{1}{\tan z}$. Then you know that for a fixed point in $\{\frac{(2k+1)\pi}{2} : k \in \mathbb{Z}\}$, there is again a neighborhood U such that it's homeomorphically mapped to $\tan(U)$. Then I again claim that $U + \pi\mathbb{Z}$ gives the even cover of $\tan U$. Indeed, we only have to show that $\tan^{-1} \tan U$ is $U + \pi\mathbb{Z}$, but this is again the periodicity of tan.

Problem 4.20. Suppose that X is a compact Riemann surface such that there exist a meromorphic function on X with exactly one pole of order 1, then X is biholomorphic to $\mathbb{P}^1$.

Solution. Suppose that this meromorphic function is f and I claim that if we think of this function as $f : X \to \mathbb{P}^1$, then it's a biholomorphism. I claim that this f is a biholomorphism. We already know that it's holomorphic, therefore we will only show that it's bijective.

Recall that this f is already surjective since X is compact and f is nonconstant. Then we only need to show that f is injective. But this is clear since we know that for the point ∞ of $\mathbb{P}^1$, it's preimage is a single point with multiplicity 1, which means that this branched covering has order 1. This means that for every point in $\mathbb{P}^1$, the preimage is a set of points with total multiplicity 1, and in particular, it only can be a singleton. We are done.

Problem 4.21. Determine the branch points of the map:

$$f : \mathbb{C} \to \mathbb{P}^1, \quad z \mapsto \frac{1}{2}(z + \frac{1}{z})$$

Solution. Notice that at the point 0, we have a pole of order 1 and for other nonzero points z, we know that the value of $f(z)$ is in $\mathbb{C}$.

So now, assume that $z \neq 0$, we will need to find points such that $f'(z) = 0$.

Recall that:

$$f'(z) = \frac{1}{2} + \frac{-1}{2z^2}$$

If we set this to be 0, we know that we get:

$$z^2 = 1$$

We know that we only have $z = \pm 1$ two solutions. Therefore we know that $z = \pm 1$ are branch points of $f(z)$. When z is closed to 0, we know that we can consider the chart $\mathbb{P}^1 \backslash \{0\}$ and the function in this chart looks like:

$$z \mapsto \frac{2z}{z^2 + 1} = z\frac{2}{z^2 + 1}$$

You can compute that the derivative is not 0 at $z = 0$, thus we only have 2 ramification points, which are ± 1.

Problem 4.22. The **degree** of a nonconstant holomorphic map f between two compact Riemann surface is defined to be the number of sheets of the associated unbranched covering, which is usually denoted by $\deg(f)$. Prove that if $f : X \to Y$ and $g : Y \to Z$ are both nonconstant holomorphic maps between compact Riemann surfaces, then the degree of $g \circ f$ is $\deg(f) \cdot \deg(g)$.

Solution. It's clear that the composition is still holomorphic so let's first first that $g \circ f$ is also nonconstant.

Indeed, we know that if we consider the two associated unbranched covering:

$$f : X' \to Y', \quad g : Y'' \to Z''$$

Notice that Y' is just Y with a few points taken out and Y'' is also a few points taken out with both f and g surjective on X', Y'', then we know that by possibly taking out some more finitely many points on Y', the composition can easily be seen to be nonconstant.

Then consider the composition $g(f(z))$ and we know that it's derivative is computed by:

$$(g \circ f(z))' = g(f(z))f'(z)$$

Then we know that for the composition, we get a branch point z of $g \circ f$ if $z \in X$ is a branch point of f or if $f(z)$ is a branch point of g. Say this set is A', with image in Z to be B'. Assume that the branch point of $f : X \to Y$ is A, then we know that this A might have a few more points than A, but only finitely many. Then we know that we have a unbranched covering:

$$g \circ f : X \backslash (g \circ f)^{-1}(B') \to Z \backslash B'$$

We know that for each $z \in Z \backslash B'$, we know that it's preimage in Y does not contain any critical point of f. Since we know that $f(A') \subset Y$ already contained all the critical points. Therefore we know that if we assume:

$$g^{-1}z = \{y_1, \cdots, y_{\deg(g)}\}$$

We know that:
$$(g \circ f)^{-1}(z) = f^{-1}y_1 \cup \cdots \cup f^{-1}y_{\deg(g)}$$
Notice that since y_i are not the critical points of f, we know that each of $f^{-1}y_i$ will have $\deg(f)$ points and for different y_i, the preimage will be disjoint. Then we know that we have $\deg(f) \cdot \deg(g)$ number of sheets for the covering, which is the degree.

Problem 4.23. Let X be a compact Riemann surface and let $\sigma : X \to X$ be a biholomorphism with $\sigma(a) \neq a$ for some $a \in X$. If X admits a meromorphic function $f : X \to \mathbb{P}^1$ which is holomorphic as a function to $\mathbb{C}$ on $X\backslash\{a\}$ with a a degree k pole of f, then σ at most have $2k$ fixed points.

Solution. Notice that we can consider the meromorphic function:

$$g(z) = f(z) - f \circ \sigma(z) : X\backslash\{a, \sigma^{-1}a\} \to \mathbb{C}$$

This gives a holomorphic function to $\mathbb{P}^1$ and we know that we get two poles at $a, \sigma^{-1}a$ each of them with order k. Indeed, we know first that for $f \circ \sigma$, $\sigma^{-1}a$ is a pole of order k. Indeed, we know that the order of the pole is just the local multiplicity, which is characterized by the number of points in the inverse image, which is preserved under any biholomorphism.

Then we know that this is equivalent of saying that there is a local coordinates on the source such that $\sigma^{-1}a$ is mapped to 0 and $f \circ \sigma(z)$ at $\sigma^{-1}a$ can be written as $z \mapsto \frac{1}{z^k}$ since f is holomorphic at $\sigma^{-1}a$, we know that locally we can write $f - f \circ \sigma$ as $\frac{1}{z^k} + \varphi(z)$ where z is holomorphic at $z = 0$. Then we know that $f - f \circ \sigma$ still has a pole of order k at $\sigma^{-1}(a)$. Similarly, we know that the order of pole at a is also k. Then we know that the degree of $g(z) : X \to \mathbb{P}^1$ is $2k$. This means that $g^{-1}(0)$ at most has $2k$ points. Now notice that if b is a fixed point of σ, then $g(b) = 0$, thus b is going to be a zero of g, i.e. $b \in g^{-1}(0)$, which implies that there are at most $2k$ such b.

5 Lecture 5

In this section, we pick up on what we left off last time and study Weierstrass Zeta function $\zeta(z)$ further.

Recall that from last time we already showed that if D is the largest open disk centered at the origin such that D doesn't contain any lattice points, then:

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \in \Omega*} \left(-\frac{z^2}{\omega^3} - \frac{z^3}{\omega^4} - \cdots\right)$$

We can also write it as:

$$\zeta(z) = \frac{-1}{z} - \sum_{n=2}^{\infty} G_{2n} z^{2n-1}$$

where for $n \geqslant 4$, $G_n = \sum_{\omega \in \Omega*} \frac{1}{\omega^n}$ and it's called the Eisenstein series. The above two series both converge uniformly to $\zeta(z)$ on any compact subset of D.

Recall that the initial definition of $\zeta(z)$ is:

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \in \Omega*} \left(\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right)$$

Since this converges uniformly on any compact subset of $\mathbb{C}\backslash\Omega$, we know that we can compute it's derivatives term by term and I will let you check:

$$-\zeta'(z) = \wp(z)$$

However, we know that in D, we can also compute the derivative of ζ using the series involving the Eisenstein series, and it's:

$$-\zeta'(z) = \frac{1}{z^2} + \sum_{n=2}^{\infty}(2n-1)G_{2n}z^{2n-2}$$

If you write this series out for the first few terms, you have:

$$\wp(z) = \frac{1}{z^2} + 3G_4z^2 + 5G_6z^4 + \cdots$$

Then we know that we can compute the derivative of $\wp'(z)$ by further differentiating term by term:

$$\wp'(z) = \frac{-2}{z^3} + 6G_4z + 20G_6z^3 + \cdots$$

Note:

$$(\wp'(z))^2 = \frac{4}{z^6} - \frac{24G_4}{z^2} - 80G_6 + z^2\varphi_1(z)$$

where $\varphi_1(z)$ is a holomorphic function in D such that $\varphi_1(0) \neq 0$.
Then we can compute the series of $(\wp(z))^3$ which is:

$$4(\wp(z))^3 = \frac{4}{z^6} + \frac{36G_4}{z^2} + 60G_6 + z^2\varphi_2(z)$$

where $\varphi_2(z)$ is also a holomorphic function on D.
Finally, we have:

$$60G_4\wp(z) = \frac{60G_4}{z^2} + z^2\varphi_3(z)$$

with φ_3 also holomorphic on D. Then we know that an obvious algebraic relation is:

$$4(\wp(z))^3 - (\wp'(z))^2 + 60G_4\wp(z) + 140G_6 = z^2(\varphi_2 - \varphi_1 + \varphi_3)$$

Notice that the stuff on the left hand side is still elliptic and the only poles are possibly at the lattice points. However, since $z^2(\varphi_2 - \varphi_1 + \varphi_3)$ is a holomorphic function at $z = 0$, we know that we have no poles at 0, which means that there will be no poles for the left hand side on the entire complex plane. Therefore $4(\wp(z))^3 - (\wp'(z))^2 + 60G_4\wp(z) + 140G_6$ is a constant. Let $z \to 0$, we find that this constant is 0. Therefore we have the algebraic relation:

$$4(\wp(z))^3 - (\wp'(z))^2 + 60G_4\wp(z) + 140G_6 = 0$$

Usually people will just write $g_2 = 60G_4$ and $g_3 = 140G_6$ so we have the following differential equation:

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

Remark 5.1. This differential also tells us something why $\wp(z)$ is called the elliptic function. Notice that if we write $\wp(z)$ simply as $w(z)$, then we know that we can always locally take the square root of $(w'(z))^2$ and we know that by choosing a local branch of the square root function, we have:

$$w'(z) = \pm\sqrt{4w^2 - g_2w - g_3}$$

This is just:

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \pm\sqrt{4w^2 - g_2w - g_3} \Rightarrow \mathrm{d}z = \frac{\mathrm{d}w}{\pm\sqrt{4w^2 - g_2w - g_3}}$$

We know this at least holds locally for an open neighborhood of z, therefore we know that locally we can integrate with respect to z and get:

$$z - z_0 = \int_{w(z_0)}^{w(z)} \frac{\mathrm{d}w}{\pm\sqrt{4w^2 - g_2w - g_3}}$$

Notice that this is the integral, at least locally, is the famous elliptic integral which people use to compute the **arc length of an ellipse**.

We just show that the points $(\wp(z), \wp'(z))$ lie on the points defined by an algebraic equation:

$$y^2 = 4x^3 - g_2x - g_3$$

Notice that if we compute the discriminant of the right hand side, we get:

$$\Delta = 16(g_2^2 - 27g_3^2)$$

Note that if we consider this polynomial $4x^3 - g_2x - g_3$ as in $\mathbb{C}[x]$, then we know that it has three solutions e_1, e_2, e_3. The reason why we call them e_1, e_2, e_3 is because they are exactly just the critical values of $\wp(z)$. Indeed, if $\wp(z) = e_i$, then by the equation it satisfies, we know that $\wp'(z) = 0$, but we know that the places $\wp'(z)$ are exactly at $\frac{\omega_i}{2}$. Therefore we know that if you ask what z such that $\wp(z) = e_i$, we already know that $z = \frac{\omega_i}{2}$. Recall that we already know that e_1, e_2, e_3 are distinct since $\wp$ is of order 2 and we already know that $\wp'(z) = 0$, therefore each of $\wp(z) - e_i$ has a double root at $\frac{\omega_i}{2}$. Let's order e_i in the way that $\wp(\frac{\omega_i}{2}) = e_i$.

Proposition 5.2. Given any lattice Ω, the above Δ associated to this lattice is not 0. Equivalently, the three solutions e_1, e_2, e_3 of this cubic polynomial are mutually distinct.

Proof. Let $h_j(z) = \wp(z) - e_j$ for $j = 1, 2, 3$. Notice that these h_j's are still elliptic and they have the same poles as $\wp(z)$. This implies that it assumes 2 zeros on the torus with multiplicity considered.

Note that we have:

$$h_j(\frac{\omega_j}{2}) = h_j'(\frac{\omega_i}{2}) = \wp'(\frac{\omega_i}{2}) = 0$$

This implies h_j has a double 0 at ω_j and no other zeros. Therefore for $i \neq j$, we know:

$$h_j(\frac{\omega_i}{2}) = e_i - e_j \neq 0$$

We are done. □

Remark 5.3. Put $p(x) = 4x^3 - g_2x - g_3$, then we know at least formally:

$$\wp'(z) = \sqrt{p(\wp(z))}$$

In the next lecture we will ue this relation to give a biholomorphism from $\mathbb{C}/\Omega$ to X where X is the Riemann surface in $\mathbb{P}^1$ defined by the equation:

$$y^2 = p(x)$$

Conversely, one can show that if $p(x)$ is any cubic polynomial with distinct roots, then there is a lattice Ω in $\mathbb{C}$ such that the associated Weierstrass $\wp(z)$ satisfies:

$$(\wp'(z))^2 = p(\wp(z))$$

But again, we will do that latter.

Now we want to study the so-called **Weierstrass σ function**. Let's recall some back grounds on the **infinite product** first.

Definition 5.4. Recall that if $\{b_n\}$ is a sequence of nonzero complex numbers, we say that the infinite product:

$$\prod_n^\infty b_b$$

converges to a complex number p if:

1. $p \neq 0$.
2. $\lim_{N\to\infty} \prod_n^N b_n = p$.

Remark 5.5. It's clear that if $\prod_n^\infty$ converges then we know that:

$$\lim_{N\to\infty} b_N = \lim_{N\to\infty} \frac{\prod_n^N b_n}{\prod_n^{N-1} b_n} = 1$$

If we define the sequence $\{c_n\}$ by $b_n = 1 + c_n$, then from the above remark, we know that if the infinite product converges, then $\lim_{n\to\infty} c_n = 0$. Recall that since for large N, b_N is close to 1, we know that we can choose a branch of the log function such that it's defined for all of complex numbers with positive real parts. For example, we can use the principal branch:

$$\text{Log}z = \log|z| + i\arg z, \quad -\pi < \arg z < \pi$$

Proposition 5.6. If we keep our assumption that $b_n \neq 0$, then $\prod_{n=1}^\infty b_n$ converges iff $\sum_n^\infty \text{Log}b_n$ converges and:

$$\prod_n^\infty b_n = \exp(\sum_n^\infty \text{Log}b_n)$$

In particular, if $\sum_n^\infty \text{Log}b_n$ converges to $c \in \mathbb{C}$, then the limit of $\prod_n^\infty b_n$ is not 0 as it's e^c.

Proof. This is basically just applying the definition. Notice that we can always ignore finitely many terms both in the product and in the summation. Therefore we will assume that all b_n are in the domain for Log. Then clearly we know that if the series converges then apply exp gives that the product converges.

Now assume the infinite product converges to $p \neq 0$, then we know that let's consider the tail sums:

$$\sum_{n=m}^N \text{Log}b_n = \text{Log}(\prod_{n=m}^N b_n)$$

We know that if we denote $\prod_{n=m}^N b_n$ by $Q_{m,N}$, from the fact that the infinite product converges to $p \neq 0$, we know that:

$$Q_{m,N} = (\prod_n^N b_n)/(\prod_n^m b_n)$$

When we say that m, N are larger than some large number n_0, we know that the fact that $p \neq 0$ makes the quotient:

$$(\prod_n^N b_n)/(\prod_n^m b_n)$$

as close to 1 as possible (which also justifies why we can apply the branch Log to $Q_{m,N}$), therefore we know that:

$$\text{Log}Q_{m,N} \to 0$$

Then, we know that using the Cauchy sequence criterion, the series converges. □

Remark 5.7. The proof above shows one reason why we want the limit p to be a nonzero number.

Definition 5.8. We say that $\prod_n^\infty b_n$ **converges absolutely** iff the associated series $\sum_{n=1}^\infty \text{Log}b_n$ converges absolutely.

Proposition 5.9. The product $\prod_{n=1}^\infty b_n$ converges absolutely iff $\sum_{n=1}^\infty |c_n|$ converges.

Proof. Recall that when $|z|$ is small, we have a very useful inequality:

$$\frac{|z|}{2} \leqslant |\text{Log}(1+z)| \leqslant 2|z|$$

Recall that for $|z| < 1$, we have the following expansion of $\text{Log}(1+z)$:

$$z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$$

Then we know that when $|z| < 1$, we have a Taylor series of $\frac{2\text{Log}(1+z)}{z} = 2 - z + \frac{2z}{3} - \cdots$. This implies that when z is small, the moduli of the quotient is about 2. Similarly, you can show that $|\text{Log}(1+z)| \leqslant 2|z|$ also holds.

Then we know that if we assume $\prod_n b_n$ converges absolutely, by definition the series $\sum_n^\infty \text{Log}(b_n) = \sum_n^\infty \text{Log}(1+c_n)$ converges absolutely, which means that:

$$\frac{1}{2}\sum_n^\infty |c_n| \leqslant \sum_n^\infty |\text{Log}(1+c_n)|$$

converges absolutely. Converse, if we know $2\sum_n^\infty |c_n|$ converges absolutely, then:

$$\sum_n^\infty |\text{Log}(1+c_n)| \leqslant 2\sum_n^\infty |c_n|$$

converges absolutely. □

Sometimes, we will need to allow some terms in the product to be 0, and if there are only finitely many such elements, i.e. if there is a N such that for all $n \geqslant N$, $b_n \neq 0$. We will define $\prod_n^\infty$ converges if $\prod_{n\geqslant N}^\infty b_n$ converges to a **nonzero** $p \in \mathbb{C}$. Then the value of $\prod_{n=1}^\infty b_n$ is defined to be:

$$b_1 \cdots b_{N-1} \prod_{n\geqslant N} b_n = b_1 \cdots b_{N-1} p$$

Now if $\{f_n\}$ is a sequence of holomorphic functions on some open $U \subset \mathbb{C}$. Again, we write it as $f_n = 1 + F_n$. Suppose that for any $z \in U$, only finitely many $f_n(z)$ is 0. Then we can define the infinite product:

$$\prod_n f_n(z) = \prod_n (1 + F_n(z))$$

We will say that this infinite product of functions converges uniformly on U if $P_n(z) = \prod_n^N f_n(z)$ converges uniformly on every compact subset of U. What we will typically assume is that for any compact subset $K \subset U$ and for all $z \in K$, there are only finitely many f_n such that $f_n(z) = 0$ for any $z \in K$.

Definition 5.10. Using the above assumption, for $K \subset U$, we say that the infinite product **converges uniformly on** K iff the associated logarithm series:

$$\sum_n \text{Log}(b_n)$$

converges uniformly.

Then recall that in this case:

$$\prod_n f_n(z) = \exp(\sum_n^\infty \text{Log}(f_n(z)))$$

Since $\sum_n^\infty \text{Log}(f_n(z))$ converges uniformly, we get that the series is also holomorphic, therefore after we post-compose with exp, it still should be holomorphic, therefore this definition of uniform convergence still gives that the function defined by the uniform infinite product is holomorphic. We will say that the infinite product of function converges **absolutely and uniformly** if the associated series in logarithms converges absolutely and uniformly. That is to say:

$$\sum_n |\text{Log} f_n(z)|$$

converges uniformly. Clearly, if $\sum_n |\text{Log} f_n(z)|$ converges uniformly, then so does $\sum_n \text{Log} f_n(z)$, therefore the function defined by the infinite product is still holomorphic.

Proposition 5.11. Using the above assumption, $\prod_n f_n(z)$ converges uniformly absolutely on U iff the series $\sum_n F_n(z)$ converges absolutely and uniformly on U.

Proof. Given a compact subset K of U, we know that since there are only finitely many bad f_n, we can just ignore them since finitely many terms doesn't impact uniform convergence. This means that we can assume that on K, $f_n(z)$ are all nonzero. Then let's first assume that the infinite product absolutely and uniformly converges, which is also nonzero on K. This means that, by the definition we gave above, the following series converges uniformly:

$$\sum_n |\mathrm{Log}(1+F_n(z))|$$

Using the uniform Cauchy sequence criterion, we know that this is equivalent to say that the tail:

$$\sum_{n=m}^{N} |\mathrm{Log}(1+F_n(z))|$$

uniformly converges to 0. Then in particular we know that $|\mathrm{Log}(1+F_n(z))|$ converges to 0 uniformly, which implies that $F_n(z)$ converges to 0 uniformly on K. Now consider the following tail term we are interested in:

$$\sum_{n=m}^{N} |F_n(z)|$$

We know that the previous inequality:

$$\frac{|z|}{2} \leqslant |\mathrm{Log}(1+z)| \leqslant 2|z|$$

We know that:

$$\frac{1}{2}\sum_{n=m}^{N} |F_n(z)| \leqslant \sum_{n=m}^{N} |\mathrm{Log}(1+F_n(z))| \leqslant 2\sum_{n=m}^{N} |F_n(z)|$$

I will leave the rest of the proof to you. □

Remark 5.12. I will let you show that if an infinite product is absolutely and uniformly convergent, then we we actually don't care about the order of the product. We can freely arrange the product and get the same holomorphic limit function in the end.

Moreover, using the fact that:

$$\prod_n f_n(z) = \exp(\sum_n \mathrm{Log} f_n(z))$$

and the series $\sum_n \mathrm{Log} f_n(z)$ converges uniformly, we know that we can do the logarithm derivative of $\prod_n f_n(z)$ via:

$$\log \prod_n f_n(z) = \sum_n \mathrm{Log} f_n(z)$$

where we choose a branch of log locally for $\prod_n f_n(z)$, then we know that if we compute the derivative, we know that:

$$\frac{(\prod_n f_n(z))'}{\prod_n f_n(z)} = \sum_n \frac{f_n'(z)}{f_n(z)}$$

Moreover this still converges uniformly, thus the logarithmic derivative is still going to be holomorphic.

Now we are ready to define the Weierstrass σ functions. Let Ω be a lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ in $\mathbb{C}$. We define:

$$\sigma(z) = z\prod_{\omega\in\omega*}(1-\frac{z}{\omega})\cdot e^{\frac{z}{\omega}+\frac{z^2}{2\omega^2}}$$

Remark 5.13. Note that you can tell from out construction of $\sigma(z)$ that we want a function such that it's holomorphic on all of $\mathbb{C}$ and all of it's zeros are simple and are at the lattice points. As you will see in the following discussion, the exponential factor we put later is for the infinite product to converge absolutely and uniformly.

Now let $K \subset \mathbb{C}$ be a compact subset and if we just ignore finitely many terms in the product, we can assume that:

$$|\omega| > 2|z|, \quad \forall \omega \in \Omega^*, \quad \forall z \in K$$

Now let's denote $g_\omega(z) = (1 - \frac{z}{\omega})e^{\frac{z}{\omega} + \frac{z^2}{2\omega}}$

Clearly, we know first that when $|\omega| \to \infty$, this $g_\omega(z)$ converges to 1 on K uniformly. Therefore for large ω, we can apply the principal branch to all $g_\omega(z)$ for all $z \in K$ and we have:

$$|\mathrm{Log} g_\omega(z)| = |\mathrm{Log}(1 - \frac{z}{\omega}) + \frac{z}{\omega} + \frac{z^2}{2\omega^2}|$$

Again, since we assume $2|z| < |\omega|$, we know that we can use the Taylor series of $\mathrm{Log}(1 - \frac{z}{\omega})$ where:

$$\mathrm{Log}(1 - \frac{z}{\omega}) = -(\frac{z}{\omega} + \frac{z^2}{2\omega^2} + \frac{z^3}{3\omega^3} + \frac{z^4}{4\omega^4} + \cdots)$$

Therefore we know:

$$|\mathrm{Log}(g_\omega(z))| = |\frac{z^3}{3\omega^3} + \frac{z^4}{4\omega^4} + \cdots| \leqslant \frac{1}{3}|\frac{z^3}{\omega^3}| \cdot |1 + \frac{1}{2} + \frac{1}{4} + \cdots| \leqslant |\frac{z^3}{\omega^3}|$$

Since $\sum_{\omega \in \Omega^*} \frac{1}{|\omega^3|}$ converges and $|z| \leqslant R$ for some $R > 0$, we know that the series with $|\mathrm{Log}(g_\omega(z))|$ converges and uniformly on K. This implies that the function defined by $x \prod_{\omega \in \Omega^*} g_\omega(z)$ is holomorphic on $\mathbb{C}$ and we don't care about the order of multiplication. This also shows that the only zeros are the lattice point, since we know that if we take out the term $g_{\omega'}(z)$, we can still run the same argument above and the limit is not going to be 0 since the series of logarithms converges.

Since we know that this infinite product absolutely converges, we don't care the order in which we do multiplication, therefore if we replace z with $-z$, we have:

$$-\sigma(z) = \sigma(-z)$$

Therefore we know that σ is an odd function with simple zeros at all lattice points.

Recall that we can compute the logarithmic derivative of $\sigma(z)$ and we have:

$$\frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum_{\omega \in \Omega^*} \frac{g_\omega(z)'}{g_\omega(z)} = \frac{1}{z} + \sum_{\omega \in \Omega^*} \frac{z^2}{(z - \omega)\omega^2}$$

Notice that the summation on the right hand side is just $\zeta(z)$.

Remark 5.14. Notice that we already know that the series defining $\zeta(z)$ converges absolutely and uniformly, therefore we know that the summation doesn't care about order. But a priori, when we are doing the summation of $\frac{g_\omega*(z)'}{g_\omega(z)}$, we have to choose an order of summation, even though in the end, it turns out the order is not relevant.

Note that in general, there is no reason for $\zeta(z)$ and $\sigma(z)$ to be elliptic. For one thing, we know that one of them only has degree 1 poles at the lattices and another one only has degree 1 zeros at the lattices and there are no order 1 elliptic functions. However, we know that:

$$\zeta'(z) = \wp(z)$$

which is an elliptic function. Therefore, we know that for each $\omega \in \Omega$, we have:

$$(\zeta(z + \omega) - \omega(z))' = 0$$

Therefore we know that for each ω and all $z \in \mathbb{C} \backslash \Omega$, we have $\zeta(z + \omega) - \omega(z) = \eta(\omega)$ for some constant $\eta(\omega)$ that only depends on ω (note that $\mathbb{C}$ minus the lattice points is still connected).

Notice that if we compute:

$$\zeta(z + \omega_1 + \omega_2) = \zeta(z) + \eta(\omega_1 + \omega_2)$$

On the other hand, we also have:

$$\zeta(z + \omega_1 + \omega_2) = \zeta(z + \omega_1) + \eta(\omega_2) = \zeta(z) + \eta(\omega_1) + \eta(\omega_2)$$

Therefore this $\eta : \Omega \to \mathbb{C}$ is $\mathbb{Z}$-linear. This implies that if we denote $\eta_1 = \eta(\omega_1)$ and $\eta_2 = \eta(\omega_2)$, we know that:

$$\eta(\omega) = \eta(m\omega_1 + n\omega_2) = m\eta(\omega_1) + n\eta(\omega_2) = m\eta_1 + n\eta_2$$

Theorem 5.15. (Legendre's Relation) Using the above notation, we have:

$$\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$$

Proof. Let's permute the fundamental parallelogram P a bit and suppose that 0 is contained in the interior. Then since $\zeta(z)$ is meromorphic with a degree 1 pole at 0 and residue 1, we know that:

$$\int_{\partial \mathrm{P}} \zeta(z)dz = 2\pi i$$

We can try to write this integral in terms of:

$$\int_{\gamma_1} \zeta(z)dz + \int_{\gamma_2} \zeta(z)dz + \int_{\gamma_3} \zeta(z)dz + \int_{\gamma_4} \zeta(z)dz$$

where γ_1 is the curve that parametrizes the bottom side of P and we go counter-clockwise to get the other γ_i.

Notice that:

$$\int_{\gamma_3} \zeta(z)dz = -\int_{\gamma_1} \zeta(z + \omega_2)dz = -\int_{\gamma_1} \zeta(z) + \eta_2 dz$$

This implies that:

$$\int_{\gamma_1} \zeta(z)dz + \int_{\gamma_3} \zeta(z)dz = -\omega_1\eta_2$$

Similarly, we know that:

$$\int_{\gamma_2} \zeta(z)dz + \int_{\gamma_4} \zeta(z)dz = \omega_2\eta_1$$

Therefore we get:

$$\omega_2\eta_1 - \omega_1\eta_2 = 2\pi i$$

□

Next we will study how $\sigma(z)$ behaves under translation. We will need the relation:

$$\frac{\sigma'(z)}{\sigma(z)} = \zeta(z)$$

We know that:

$$\frac{\sigma'(z + \omega)}{\sigma(z + \omega)} = \zeta(z) + \eta(\omega) = \frac{\sigma'(z)}{\sigma(z)} + \eta(\omega), \quad \forall \omega \in \Omega^*$$

Then by choosing appropriate branches of log functions, we can at least locally conclude that:

$$\frac{d \log(\sigma(z + \omega))}{dz} = \frac{d \log(\sigma(z))}{dz} + \eta(\omega)$$

Then at least locally we can do an integration to conclude that:

$$\log(\sigma(z + \omega)) = \log \sigma(z) + \eta(\omega)z + C \quad C \in \mathbb{C}$$

We exponentiate it to obtain:

$$\frac{\sigma(z+\omega)}{\sigma(z)} = e^{\eta(\omega)z+C}$$

Therefore we want to figure out what is e^C.

First, we want to conclude that C is a constant that only depends on ω and doesn't depend on z. However, the above argument is not ideal since we know that for one reason, our log functions are not defined globally. To resolve this problem we know that we can define:

$$F_\omega(z) = \frac{\sigma(z+\omega)}{e^{\eta z}\sigma(z)}$$

Notice that this is defined and holomorphic for all z that are not the lattice points. Moreover, we know that for such z, $F_\omega(z) \neq 0$. Then we know that if we consider:

$$\frac{F'_\omega(z)}{F_\omega(z)} = \frac{\sigma'(z+\omega)}{\sigma(z+\omega)} - \frac{\sigma'(z)}{\sigma(z)} - \eta = 0$$

Therefore we know that $F'_\omega(z) = 0$. Since $\mathbb{C}$ minus the lattice points are still connected, we know that this implies $F_\omega(z)$ is C for all z not lattice points. Moreover, this C clearly doesn't depend on z, thus denoted by C_ω, therefore:

$$\sigma(z+\omega) = C_\omega e^{\eta z}\sigma(z)$$

Then our job is to figure out what is this C_ω.

Let's first assume that $\frac{\omega}{2}$ is not in ω. This implies $\sigma(\frac{\omega}{2}) \neq 0$. Consider $z = \frac{-\omega}{2}$ and we can use the fact that σ is odd to conclude that:

$$\sigma(\frac{\omega}{2}) = C_\omega e^{-\eta\frac{\omega}{2}} - \sigma(\frac{\omega}{2})$$

Since $\sigma(\frac{\omega}{2}) \neq 0$, we conclude that:

$$C_\omega e^{-\frac{\eta\omega}{2}} = -1$$

This implies $C_\omega = -e^{\frac{\eta\omega}{2}}$

Then we know:

$$\sigma(z+\omega) = e^{\eta(z+\frac{\omega}{2})}\sigma(z)$$

Now for any $\omega \in \Omega$, we define:

$$\epsilon(\omega) = \frac{\sigma(z+\omega)}{\sigma(z)}e^{-\eta(z+\frac{\omega}{2})} = C_\omega e^{\frac{\omega}{2}}$$

We already know that if $\frac{\omega}{2} \notin \Omega$, then $\epsilon(\omega) = -1$. Moreover, this $\epsilon(\omega)$ only depends on ω and not on z.

Notice that:

$$\frac{\sigma(z+2\omega)}{\sigma(z)} = \frac{\sigma(z+2\omega)}{\sigma(z+\omega)} \cdot \frac{\sigma(z+\omega)}{\sigma(z)}$$

Then we know that:

$$\epsilon(2\omega) = \epsilon(\omega)^2$$

So if $\omega/2$ lies in Ω, then we know that $\epsilon(\omega) = \omega(\omega/2)^2$. Then if $\omega/4$ still lies in Ω, we know $\epsilon(\omega) = \epsilon(\frac{\omega}{4})^4$. Then until we get $\frac{\omega}{2^n}$ doesn't lie in Ω, and we know that $\epsilon(\omega) = 1$. Therefore we know that:

$$\epsilon(\omega) = \pm 1$$

If $\omega/2 \notin \Omega$, then it's -1 and if $\omega/2 \in \Omega$, it's 1.

Then we conclude that:

$$\frac{\sigma(z+\omega)}{\sigma(z)} = e^{\eta(z+\frac{\omega}{2})}\epsilon(\omega)$$

We also have a formula for $\epsilon(\omega)$ if $\omega = m\omega_1 + n\omega_2$, then:

$$\epsilon(\omega) = (-1)^{mn+m+n}$$

Now using all of the above discussion, we can answer the following question: is there an elliptic function $f \in \mathcal{M}(\Omega)$ with $[a_1], \cdots, [a_r]$ as zeros and $[b_1], \cdots, [b_s]$ as poles with multiplicity $k_1, \cdots, k_r$ and $l_1, \cdots, l_s$ respectively.

Notice that these points must satisfies the following necessary conditions:

1. $\sum_i k_i = \sum_j l_j$ using the the summation of the order of all the zeros and all the order of poles should equal to the order of the elliptic function.
2. $[a_1], \cdots, [a_r]$ and $[b_1], \cdots, [b_s]$ should be disjoint.
3. We already saw in the first homework that:
$$\sum_i a_i k_i \sim \sum_j b_j l_j \mod \Omega$$

It turns out that these conditions are all sufficient:

Theorem 5.16. If the above necessary conditions are satisfied, there is an elliptic function, unique up to scalar multiple, with the prescribed zeros and poles with no others.

We will prove this in the next lecture.

6 Lecture 6

Let's pick up the proof of the theorem from last lecture.

Proof. Recall that one of the necessary conditions is:

$$\sum_i a_i k_i \sim \sum_j b_j l_j \mod \Omega$$

Since we assumed that $[a_i] \neq [a_j]$, we can at least assume that their representative a_i are all distinct are all from one fundamental parallelogram (possibly shifted a bit). let's use $u_1, \cdots, u_n$ to denote the zeros with repeats, for example, $u_1 = u_2 = \cdots = u_k$ if some zero is with multiplicity k. Similarly, we will let $v_1, \cdots, v_n$ be the poles with repeats. Then we have:

$$\sum_i u_i = \sum_j v_i + \omega, \quad \omega \in \Omega$$

Then if we replace u_1 with $u_1 + \omega$, we can assume that:

$$\sum_i u_i = \sum_i v_i$$

Define:

$$f(z) = \frac{\sigma(z-u_1)\cdots\sigma(z-u_n)}{\sigma(z-v_1)\cdots\sigma(z-v_n)}$$

Notice that this is a meromorphic function that has simple zeros at each point in $\Omega + u_i$ and has a simple pole at each point in $\Omega + v_i$. Moreover, $f(z)$ doesn't have any other zeros or poles. Therefore we know that replacing u_1 with $u_1 + \omega$ doesn't harm this definition of f at least in terms of where the simple zeros and poles are. We only need to show that $f(z)$ is elliptic then we are done.

Recall that:

$$\sigma(z+\omega_i) = -\sigma(z)e^{\eta_1(z+\frac{\omega_i}{2})}, \quad \omega_i = \omega_1, \omega_2$$

Then we know that if we shift f by ω_i, we get:

$$f(z+\omega_i) = \frac{\sigma(z-u_1)\cdots\sigma(z-u_n)}{\sigma(z-v_1)\cdots\sigma(z-v_n)}e^{\eta_i \sum_j (u_j - v_j)} = \frac{\sigma(z-u_1)\cdots\sigma(z-u_n)}{\sigma(z-v_1)\cdots\sigma(z-v_n)}$$

The last equality holds since $\sum_j(u_j - v_j) = 0$. This shows $f(z+\omega_i) = f(z)$, thus $f(z) \in \mathcal{M}(\Omega)$. □

Recall that if two elliptic functions have the same zeros and poles, then they differ by a constant multiple. This tells us that, at least if we are satisfied with constant multiple differences, the above proof of the theorem gives us a recipe for how to construct elliptic functions explicitly:

Corollary 6.1. For any $a \in \mathbb{C}$ not in Ω, we have:

$$\wp(z) - \wp(a) = -\frac{\sigma(z-a)\sigma(z+a)}{\sigma^2(z)\sigma(a)^2}$$

Proof. Notice that the two zeros of $\wp(z)$ are given by $-a$ and a. If a is not the mid lattice points, then we know that $[-a], [a]$ are distinct and if a is a mid lattice point, $[a] = [-a]$ and it have a double zero. But recall that from the above proof, we don't care if it's a double zero or simples zeros, we only need to pick the representatives such that the summation of the representatives of zeros is equal to the sum of the representatives of poles. We know that it has double poles at 0, therefore the representatives of poles we choose will be $0, 0$. Then we know that $-a, a$ will be a good choice of the zeros. Therefore:

$$C\frac{\sigma(z-a)\sigma(z+a)}{\sigma^2(z)} = \wp(z) - \wp(a)$$

To figure out the constant C, we multiply both sides by z^2 and we know that we have:

$$C\frac{\sigma(z-a)\sigma(z+a)}{\sigma^2(z)/z^2} = z^2(\wp(z) - \wp(a))$$

Recall that by construction of $\sigma(z)$ and $\wp(z)$, we know that $\lim_{z\to 0}\sigma(z)/z = 1$ and $\lim_{z\to 0} z^2\wp(z) = 1$, therefore we know that:

$$-C\sigma(a)^2 = \lim_{z\to 0} C\frac{\sigma(z-a)\sigma(z+a)}{\sigma^2(z)} = \lim_{z\to 0}\wp(z) - \wp(a) = 1$$

This implies:

$$C = -\frac{1}{\sigma^2(a)}$$

□

To proceed further we will have to introduce the minimum amount of **several variables complex analysis** since our next topic will be to study the Riemann surface defined by an algebraic equation.

Recall that in one variable complex analysis, if we assume that $f : \Omega \to \mathbb{C}$ is continuous on an open subset Ω of $\mathbb{C}$ and it's 1st-order partials all exist (may not even be continuous), then if we set:

$$\frac{\partial f}{\partial z}(a) = \frac{1}{2}(\frac{\partial f}{\partial x}(a) - i\frac{\partial f}{\partial y}(a)), \quad \frac{\partial f}{\partial \bar{z}}(a) = \frac{1}{2}(\frac{\partial f}{\partial x}(a) + i\frac{\partial f}{\partial y}(a))$$

Then f is holomorphic on Ω iff the Cauchy-Riemann equation, i.e. $\frac{\partial f}{\partial \bar{z}}(a) = 0$ holds on Ω. See the remark in the first section for more discussion on why we don't need the partials to be continuous. A similar definition will be given to several variable complex functions, though it will be equivalent to this more convenient definition:

Definition 6.2. Let Ω be an open subset in $\mathbb{C}^n$, we say that:

$$f : \Omega \to \mathbb{C}$$

is **holomorphic on** Ω if for every $a = (a_1, \cdots, a_n) \in \Omega$, there is a neighborhood $U \subset \Omega$ of a and constants c_α for $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n_{\geqslant 0}, \alpha_j \in \mathbb{Z}_{\geqslant 0}$ such that:

$$f(z) = \sum_\alpha c_\alpha (z_1 - a_1)^{\alpha_1} \cdots (z_n - a_n)^{\alpha_n}, \quad \forall z \in U$$

We say that f is **holomorphic at a point** p if there is a neighborhood U_p of p such that f is holomorphic on U_p.

Since we work with multi-index series, therefore we will need to specify an order of summation if we don't assume absolutely convergent. The following remark deals with this issue:

Remark 6.3. Given any multi-index series:

$$\sum_\alpha c_\alpha (z_1 - a_1)^{\alpha_1} \cdots (z_n - a_n)^{\alpha_n}$$

We can show that if it converges (it doesn't need to be absolutely convergent and the convergence can be in any order) at $(w_1, \cdots, w_n)$ where $w_i \neq a_i$ for all i, then it converges absolutely and uniformly on any compact subset of the **polydisk**:

$$\{z \in \mathbb{C}^n : |z_j - a_j| < |w_j - a_j|, 1 \leqslant j \leqslant n\}$$

Indeed, when z is in this polydisk, we know that:

$$\sum_\alpha |c_\alpha||z_1 - a_1|^{\alpha_1} \cdots |z_n - a_n|^{\alpha_n} = \sum_\alpha |c_\alpha||w_1 - a_1|^{\alpha_1} \cdots |w_n - a_n|^{\alpha_n} \frac{|z_1 - a_1|}{|w_1 - a_1|} \cdots \frac{|z_n - a_n|}{|w_n - a_n|}$$

Recall that since we assumed that:

$$\sum_\alpha c_\alpha (w_1 - a_1)^{\alpha_1} \cdots (w_n - a_n)^{\alpha_n} < \infty$$

we know that the absolute value of every term is bounded by a common $M > 0$, which implies:

$$\sum_\alpha |c_\alpha||z_1 - a_1|^{\alpha_1} \cdots |z_n - a_n|^{\alpha_n} \leqslant M \sum_\alpha r_1^{\alpha_1} \cdots r_n^{\alpha_n}$$

where $r_i = \frac{|z_i - a_i|}{|w_i - a_i|}$.

Note that;

$$\sum_\alpha r_1^{\alpha_1} \cdots r_n^{\alpha_n} = \prod_{i=1}^n \sum_n r_i^n = \prod_{i=1}^n \frac{1}{1 - r_i} < \infty$$

Therefore we indeed have absolute convergence. To show it also converges uniformly on every compact subset of the polydisk, we only need to give an uniform error term of the sum of all absolute values which means that for example, we can try to give estimates to:

$$\sum_{\alpha, \alpha_i \geqslant N} |c_\alpha||z_1 - a_1|^{\alpha_1} \cdots |z_n - a_n|^{\alpha_n} \leqslant \sum_{\alpha, \alpha_i \geqslant N} |c_\alpha||w_1 - a_1|^{\alpha_1} \cdots |w_n - a_n|^{\alpha_n} \frac{|z_1 - a_1|}{|w_1 - a_1|} \cdots \frac{|z_n - a_n|}{|w_n - a_n|}$$

Again, the second term can be estimated by:

$$M \sum_{\alpha, \alpha_i \geqslant N} r_1^{\alpha_1} \cdots r_n^{\alpha_n}$$

which is uniform.

Remark 6.4. If we assume that f is holomorphic in Ω, then for any $a = (a_1, \cdots, a_n) \in \Omega$, we know that there is an open neighborhood U of a such that there are constants c_α and:

$$f(z) = \sum_\alpha c_\alpha (z_1 - a_1)^{\alpha_1} \cdots (z_n - a_n)^{\alpha_n}, \quad \forall z \in U$$

Then we know that in this neighborhood U, we can pick any z in U which makes the series converges in some order of summation. Then by what we discussed above, we have a smaller neighborhood where the series actually converges absolutely. Then we know that actually for any $z \in \Omega$, this series converges absolutely since Ω is open. Therefore, we know that the order of the multi-index summation is not that important as only as we assume that there is some convergence order at each $p \in \Omega$.

Remark 6.5. Using the notation and assumptions from the previous remark, we know that if we fix all z_i's in the series and only let of of them vary, we still obtain some absolutely and uniformly convergent series in one variable. Then from the theory of one variable complex analysis, we can do differentiation term by term and the result still converges absolutely and uniformly, possibly on a smaller neighborhood to the derivative of f with only one variable varying.

However, if we only let one of the variable vary and take the derivative, this is nothing by the partial derivative of f with respect to this variable. Then I will let you see that we can repeat this process and let other variable vary to get the following result:

$$c_\alpha = \frac{1}{\alpha!} \mathrm{D}^\alpha f(a)$$

where $\alpha! = \alpha_1! \cdots \alpha_n!$ and:

$$\mathrm{D}^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} z_1 \cdots \partial^{\alpha_n} z_n}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n$$

The following theorem establishes the connection between this definition and the C-R equation:

Theorem 6.6. Let $\Omega \subset \mathbb{C}^n$ be an open subset and let:

$$f : \Omega \to \mathbb{C}$$

be a continuous function such that the partial derivatives exists and:

$$\frac{\partial f}{\partial \overline{z_j}} = \frac{1}{2}(\frac{\partial f}{\partial x_j} + i \frac{\partial f}{\partial y_j}) = 0$$

then f is holomorphic on Ω.

Proof. Let $r > 0$ be a small positive real number such that the closed polydisk $\overline{P} = \prod_{j=1}^n \overline{\mathbb{D}(a_j, r)}$ is contained in Ω.

Now we know that if we fix $z_2, \cdots, z_n$ where $z_i \in \mathbb{D}(a_i, r)$ and only let z_1 vary, we get a one variable holomorphic map since we assumed the C-R equation holds and f is continuous, which means that we can use Cauchy's formula and for each $w_1 \in \mathbb{D}(a_1, r)$:

$$f(w_1, z_2, \cdots, z_n) = \frac{1}{2\pi i} \int_{C_1} \frac{f(z_1, \cdots, z_n)}{z_1 - w_1} dz_1, \quad C_1 = \partial \mathbb{D}(a_1, r)$$

Then we can do this to $f(w_1, \cdots, z_n)$ where we only let $z_2, \cdots, z_n$ vary and fix other variables, which means that we have:

$$f(w_1, \cdots, w_n) = \frac{1}{(2\pi i)^n} \int_{C_n} \cdots \int_{C_1} \frac{f(z_1, \cdots, z_n)}{(z_1 - w_1) \cdots (z_n - w_n)} dz_1 \cdots dz_n$$

since $w_i \in \mathbb{D}(a_i, r)$, we know that $|w_i - a_i| < r$ and we have:

$$\frac{1}{z_j - w_j} = \frac{1}{z_j - a_j + a_j - w_j} = \frac{1}{1 - \frac{w_j - a_j}{z_j - a_j}}$$

Since $|\frac{w_j - a_j}{z_j - a_j}| < 1$, we know that we can write it in a power series:

$$\frac{1}{z_j - w_j} = \sum_{m=0}^{\infty} \frac{(w_j - a_j)^m}{(z_j - a_j)^m}$$

Notice that for a fixed w_j this converges uniformly on the compact subset C_j. Then we know that we can do the similar power series expansion for all other $\frac{1}{z_i - w_i}$ and since we know that integral and uniform convergence series can interchange, we get $f(w_1, \cdots, w_n) =$:

$$\frac{1}{(2\pi i)^n} \sum_{\alpha_1 \in \mathbb{Z}_{\geqslant 0}} \cdots \sum_{\alpha_n \in \mathbb{Z}_{\geqslant 0}} (w_1 - a_1)^{\alpha_1} \cdots (w_n - a_n)^{\alpha_n} \int_{C_1} \cdots \int_{C_n} \frac{f(z_1, \cdots, z_n)}{(z_1 - a_1)^{\alpha_1} \cdots (z_n - a_n)^{\alpha}} dz_1 \cdots dz_n$$

If we denote:

$$c_\alpha = \frac{1}{(2\pi i)^n} \int_{C_1} \cdots \int_{C_n} \frac{f(z_1, \cdots, z_n)}{(z_1 - a_1)^{\alpha_1} \cdots (z_n - a_n)^{\alpha}} dz_1 \cdots dz_n$$

We have the following series converges to $f(w_1, \cdots, w_n)$ for $w_i \in \mathbb{D}(a_i, r)$:

$$f(w_1, \cdots, w_n) = \sum_{\alpha_1} \cdots \sum_{\alpha_n} c_\alpha (w_1 - a_1)^{\alpha_1} \cdots (w_n - a_n)^{\alpha_n}$$

According to our remark of being holomorphic, we only need to give one order of summation where the series converges to conclude a function is holomorphic. The above convergence gives one order, thus f is holomorphic. $\square$

For several variable holomorphic functions, some standard one variable results hold while other won't hold any more. For example the following properties are still true:

1. **Weierstrass' Theorem** on uniform limits of holomorphic functions and **Montel's Theorem** on locally bounded holomorphic functions.
2. Open mapping property of nonconstant holomorphic functions.
3. Principle of analytic continuation.
4. Cauchy's formula for polydisks.
5. Cauchy's inequalities.

However, for several variables, we have the following very interesting result:

Theorem 6.7. (Hartog's Theorem) Let B_1 be an open ball in $\mathbb{C}^n$ for $n \geqslant 2$ and let B_2 be another open ball such that $\overline{B_2} \subset B_1$, then if we define $\Omega = B_1 \backslash \overline{B_2}$, then any holomorphic function on Ω has a unique analytic continuation to B_1.

This result tells you that a holomorphic function with several variables doesn't have any isolated singularities. Therefore our notion of meromorphic functions and a lot other notions need to be modified. We will do this later.

Other familiar results such as Riemann Mapping Theorem also fails. We will mention more in case we need to use them. For a detailed study, please go to a book on several variable complex analysis as we can not provide a full study here.

However, with our limited understanding of several variable holomorphic functions, we are able to study the object of our interest, the **Riemann Surface of an algebraic function**.

Now we will mainly consider polynomials in $\mathbb{C}[x, y]$ with 2 variables and say $p(x, y)$ is such a polynomial that is irreducible. We will have a few ways to determine if a polynomial like this is irreducible when we need it. For now, let's just assume that this is irreducible. Let's assume that the y-degree of $p(x, y)$ is n, which means that we can write:

$$p(x, y) = a_0(x)y^n + \cdots + a_{n-1}(x)y + a_n(x)$$

We know that intuitively, if we set $p(x, y) = 0$ and look at the solution sets, this should give us at least locally an algebraic expression of $y = y(x)$. To make this precise, we consider the following result:

Theorem 6.8. (Implicit Function Theorem of Holomorphic Functions) Let $f(x, y)$ be a holomorphic function defined on the polydisk $\{(x, y) \in \mathbb{C}^2 : |x| < r_1, |y| < r_2\}$. Assume that $f(0, 0) = 0$ but $\frac{\partial f}{\partial y}(0, 0) \neq 0$. Then there is some $\epsilon > 0, \delta > 0$ such that for any $x \in \mathbb{D}(0, \epsilon)$, there is a unique solution $y = y(x)$ such that $f(x, y) = 0$ with $|y(x)| < \delta$ and the function $x \mapsto y(x)$ is holomorphic.

Proof. Since we know that for holomorphic functions, the partials are automatically continuous, then since $\frac{\partial f}{\partial y}(0, 0) \neq 0$, we can assume that there is some $\delta > 0$ such that $f(0, y) \neq 0$ for $0 < |y| \leqslant \delta$. This is possible since we know that if we assume that no matter how small δ we choose $f(0, y) = 0$ for some y with $|y| \leqslant \delta$, then by the principle of analytic continuation, we know that $f(0, y) = 0$ for all y, which is false since we assumed that $\frac{\partial f}{\partial y}(0, 0) \neq 0$.

Then we know that since f doesn't vanish on the compact subset $\{0\} \times \{y : |y| = \delta\}$ since f is continuous, there will be a $\epsilon > 0$ such that $f(x, y) \neq 0$ for $|x| < \epsilon$ and $|y| = \delta$. Then for any fixed x with $|x| < \epsilon$, we can consider the integral:

$$\frac{1}{2\pi i} \int_{|y|=\delta} \frac{f_y(x, y)}{f(x, y)} dy$$

Since $f(x, y)$ doesn't have a pole on $|y| = \delta$ and this function is holomorphic, then we know that this is just how many zeros the function $f(x, y)$ have for $|y| < \delta$ and this fixed x which implies that in particular this is going to be an integer $n(x)$ for each x.

Notice that:

$$F(x) : \mathbb{C} \to \mathbb{C}, \quad x \mapsto \frac{1}{2\pi i} \int_{|y|=\delta} \frac{f_y(x, y)}{f(x, y)} dy$$

is continuous since we know that for a fixed y with $|y| = \delta$ and each x with $|x| < \epsilon$, we can assume that $|x| = \epsilon' < \epsilon$ and the function:

$$\frac{f_y(x, y)}{f(x, y)}$$

will be uniformly continuous on this compact set $|x| \leqslant \epsilon'$ and $|y| = \epsilon$. However, we know that for each x, the function $F(x)$ only takes integer values, therefore $F(x)$ is constant. Notice that $F(x)$ is the number of zeros of the function $f(0, y)$ within $|y| < \delta$, which is 1 by our choice of δ. Therefore we know that for each x such that $|x| < \epsilon$, $f(x, y)$ only has one zero for $|y| < \delta$. Moreover, it can be computed by:

$$y(x) = \frac{1}{2\pi i} \int_{|y|=\delta} y \frac{f_y(x, y)}{f(x, y)} dy$$

Clearly, this is a holomorphic function in x. ☐

Using this theorem, we can obtain the following useful lemma:

Lemma 6.9. Let $p(x, y) = a_0(x)y^n + \cdots + a_n(x)$ be an irreducible polynomial in $\mathbb{C}[x, y]$ such that $a_0(a) \neq 0$ and there is no $b \in \mathbb{C}$ such that $p(a, b) = \frac{\partial p}{\partial y}(a, b) = 0$. Then there is a $\epsilon > 0$ and n holomorphic functions:

$$y_1(x), \cdots, y_n(x)$$

in the disk $\{x \in \mathbb{C} : |x - a| < \epsilon\}$ with the following properties:

1. $y_i(x) \neq y_j(x')$ for $i \neq j$ and for all $x, x' \in \mathbb{D}(a, \epsilon)$.
2. If $\eta \in \mathbb{C}$ is a number such that $F(x, \eta) = 0$ for some $|x - a| < \epsilon$, then $\eta = y_i(x)$ for a unique i.

Proof. If $p(a, b) = 0$, then we know from the assumption that $\frac{\partial p}{\partial y}(a, b) \neq 0$, then since the polynomial $p(a, y) = 0$ has exactly n roots $b_1, \cdots, b_n$. I claim that these $b_1, \cdots, b_n$ are all distinct. Indeed, we know that if some $b_i = b_j$, then b_i is a repeated root of $p(a, y)$, which means that $\frac{\partial p}{\partial y}(a, b) = 0$, which is a contradiction.

We know that we can find a small enough ϵ such that:

$$y_1(x), \cdots, y_n(x)$$

are simultaneously defined for all $y_i(x)$ where $|x - a| < \epsilon$ with $y_i(a) = b_i$ and $p(x, y_i(x)) = 0$ and we know that the $y_i(x)$ are in $|y - b_i| < \delta$ and for different i, thus $|y - b_i| < \delta$ are disjoint. This proves 1.

2 is also easy to see since $F(x, \eta) = 0$ has at most n solutions and we have already given that the n distinct solutions are $y_i(x)$ for $1 \leqslant i \leqslant n$. ☐

We will continue this discussion in the next lecture.

7 Lecture 7

First, we make a topological remark on the holomorphic implicit function theorem we derived last lecture. Notice that the function $x \mapsto (x, y(x))$ actually gives a local homeomorphism between $\mathbb{D}(0, \epsilon)$ near $x = 0$ and the sets $\{(x, y) \in \mathbb{C} : f(x, y) = 0, |x| < \epsilon, |y| < \delta\}$ near $(x, y) = (0, 0)$. Indeed, we consider $x \mapsto (x, y(x))$, which is bijective, it's clearly is injective. To show it's surjective, we know that if (x, y) is a point such that $|x| < \epsilon$ and $|y| < \epsilon$ such that $f(x, y) = 0$, then since y is the unique solution then it should be $y(x)$, therefore (x, y) is the image of x. Clearly this is continuous since $x \mapsto x$ and $x \mapsto y(x)$ are both continuous, then we know that it has a clear continuous inverse which is the projection to x-axix, therefore it's indeed a homeomorphism.

We will later use this discussion and apply it to the result of the lemma we gave in the end of last lecture.

Proposition 7.1. Suppose that $P(x, y) = a_0(x)y^n + a_1(x)y^{n-1} + \cdots + a_n(x) \in \mathbb{C}[x, y]$ is an irreducible polynomial. Then there are only finitely many $a \in \mathbb{C}$ such that there is a common solution $y \in \mathbb{C}$ for $P(a, y) = \frac{\partial P}{\partial y}(a, y) = 0$.

Proof. Let's set $Q_0(x, y) = Q(x, y) \in \mathbb{C}[x, y]$ to be $\frac{\partial P}{\partial y}(x, y)$, i.e. the y partial of $P(x, y)$. Then if we treat everything in $\mathbb{C}(x)[y]$, we know that we can do Euclidean division algorithm in $\mathbb{C}(x)[y]$ and we know that there is a $Q_1(x, y)$ such that:

$$P(x, y) = h_0(x, y)Q(x, y) + Q_1(x, y)$$

where $h_0(x, y), Q_1(x, y) \in \mathbb{C}(x)[y]$ which means that they are actually some polynomial in y with coefficients which are rational functions of x, therefore we know that by clearing out the denominators, there is some $b_0(x) \in \mathbb{C}[x]$ such that we have:

$$b_0(x)P(x, y) = b_0(x)h_0(x, y)Q(x, y) + b_0(x)Q_1(x, y)$$

where $b_0(x)h_0(x, y)$ and $b_0(x)Q_1(x, y)$ are now in $\mathbb{C}[x, y]$. For notation simplicity, let's just assume that there is some $b_0(x) \in \mathbb{C}[x]$ and $h_0(x, y), Q_1(x, y) \in \mathbb{C}[x, y]$ such that we have:

$$b_0(x)P(x, y) = h_0(x, y)Q(x, y) + Q_1(x, y)$$

Recall that since we are doing Euclidean division, the y-degree $\deg_y Q_1(x, y)$ of $Q_1(x, y)$ should be at least 1 less than $\deg_y Q(x, y)$. Then since we are only multiplying them with some $b_0(x)$ to clear denominators, this doesn't change the y degree.

Notice that if $\deg_y Q(x, y) = 0$, which means that it's only a polynomial in $\mathbb{C}[x]$, then we know that $Q(x, y)$ is a unit in $\mathbb{C}(x)[y]$, therefore $Q_1(x, y)$ is 0, therefore in this case we may assume it's y degree is $-\infty$. However, if $\deg_y Q(x, y) = 0$, we know that:

$$P(x, y) = a_0(x)y + a_1(x)$$

This implies that $Q(x, y) = a_0(x)$, therefore is a is a complex number such that $P(a, y) = Q(a, y) = 0$, then we know $Q(a, y) = a_0(a) = 0$, but there are only finitely many such a, thus in this case, the assertion in the proposition is trivially true. Then let's assume that $\deg_y Q(x, y) \geqslant 1$.

Then we know that we can keep doing this process until we have:

$$b_1(x)Q_0(x, y) = h_1(x, y)Q_1(x, y) + Q_2(x, y)$$

$$\vdots$$

$$b_r(x)Q_{r-1}(x, y) = h_r(x, y)Q_r(x, y) + Q_{r+1}(x, y)$$

such that $\deg_y Q_{r+1}(x, y) = 0$. Therefore it's just $Q_{r+1}(x) \in \mathbb{C}[x]$.

Now the claim is that $Q_{r+1}(x)$ is not 0. Indeed, if $Q_{r+1}(x) = 0$, we know that $Q_r(x, y)$ divides $b_r(x)Q_{r-1}(x, y)$, therefore an irreducible factor $f(x, y)$ of $Q_r(x, y)$ with positive y degree (there is indeed such a factor since we know that the y degree of Q_r is at least 1) divides $Q_{r-1}(x, y)$.

Then if we consider:

$$b_{r-1}(x)Q_{r-2}(x,y) = h_{r-1}(x,y)Q_{r-1}(x,y) + Q_r(x,y)$$

Then since f divides $Q_r(x,y)$ and $Q_{r-1}(x,y)$, then it divides $b_{r-1}(x)Q_{r-2}(x,y)$, which means that f divides $Q_{r-1}(x,y)$. Then we can keep going until we have:

$$b_0(x)P(x,y) = h_0(x,y)Q_0(x,y) + Q_1(x,y)$$

where f divides $Q_1(x,y)$ and $Q_0(x,y)$, which implies that f divides $P(x,y)$ but this will implies f is either a constant or $P(x,y)$ itself. However, since f is an irreducible factor of Q_r which can at most have y degree $\deg_y P(x,y) - 1$ is never a constant or $P(x,y)$, this is a contradiction.

Then we know that if there is a an $a \in \mathbb{C}$ such that $P(a,y) = 0$ and $Q(a,y) = 0$ both hold for some y, then we know that this implies $Q_1(a,y) = 0$, then $Q_2(a,y) = 0$, which does all the way to $Q_{r+1}(a,y) = Q_{r+1}(a) = 0$. Since $Q_{r+1}(x)$ is not the 0 polynomial, it at most has finitely many roots, therefore we at most have finitely many a. □

Then we will define a finite subset $S \subset \mathbb{P}^1$ by:

$$S = S_0 \cup S_1 \cup \{\infty\}$$

where $S_0 = \{x \in \mathbb{C} : a_0(x) = 0\}$ and $S_1 = \{x \in \mathbb{C} : \exists y \in \mathbb{C} \text{ s.t. } P(x,y) = \frac{\partial P}{\partial y} = 0\}$. In other, we points in S_0 are those bad points such that the degree of $P(x,y)$ drops and points in S_1 are bad points in the sense that the tangent line of curve defined by $P(x,y) = 0$ at (x,y) is "vertical" on the $\mathbb{C}^2$ plane. Or in other words, these points are where the statement in the implicit function theorem of expressing y in terms of functions in x fails.

Proposition 7.2. Suppose that $P(x,y) = a_0(x)y^n + \cdots + a_{n-1}(x)y + a_n(x)$ is an irreducible polynomial in $\mathbb{C}[x,y]$ and we define:

$$V = V(P(x,y)) = \{(x,y) \in \mathbb{C}^2 :\}$$

Then if we let:

$$\pi : V \to \mathbb{C}, \quad (x,y) \mapsto x$$

be the first projection, then the map:

$$\pi|_{\pi^{-1}(\mathbb{C}\backslash S_0)} : \pi^{-1}(\mathbb{C}\backslash S_0) \to \mathbb{C}/S_0$$

is proper.

Proof. Notice that $\pi^{-1}(\mathbb{C}\backslash S_0)$ is rarely the whole space of V but for some special $P(x,y)$, we could have the fact that $pi^{-1}(\mathbb{C}\backslash S_0) = V$. Indeed, we know that if x is a point such that $a_0(x) = 0$, then if we set $P(x,y) = 0$, we are only requiring $a_1(x)y^{n-1} + \cdots + a_n(x) = 0$. Now since we have already fixed x, $a_i(x)$ are now just complex numbers. If $a_1(x)y^{n-1} + \cdots + a_n(x)$ still has positive degree y terms, then we indeed know that there are y such that $a_0(x) = 0$ and $P(x,y) = 0$. However, x clears all positive degree y terms but gives $a_n(x) \neq 0$, then we know that there is not such y such that $P(x,y) = 0$. But this is not important, as we are taking out S_0 in case the bad situation happens.

To probe this proposition, we need to show that for each $K \subset K\backslash S_0$ which is compact, then the preimage is compact. Notice that this π is clearly continuous since it's just the restriction of the projection. Therefore the preimage of K is closed in π, then we only need to show it's bounded to show it's compact in $\pi^{-1}(\mathbb{C})$. To prove this we need the following little lemma:

Lemma 7.3. Let $c_1, \cdots, c_n$ be complex numbers and suppose that we have a polynomial equation:

$$w^n + c_1 w^{n-1} + \cdots + c_{n-1}w + c_n = 0$$

If not all of $c_i = 0$, then $|w| < \max_i |c_i|^{1/i}$.

Proof. Let $c = \max_i |c_i|^{1/i}$ which is a positive real number. Then set $z = \frac{w}{c}$ and if we divide both sides of the above equation by c^n, we get:

$$z^n + \frac{c_1}{c} z^{n-1} + \cdots + \frac{c_{n-1}}{c^{n-1}} z + \frac{c_n}{c^n} = 0$$

Then we will need to show that $|z| < 2$, indeed: since we know that $|c_i| \leqslant c^i$, then this implies that we have:

$$|z^n| \leqslant |z^{n-1}| + \cdots + 1$$

If we have $|z| \geqslant 2$, then we can divide the above inequality by $|z^n|$ on both sides and obtain:

$$1 \leqslant \frac{1}{2} + \cdots + \frac{1}{2^n} < 1$$

which is a contradiction. $\square$

Using this lemma, suppose that now $K \subset \mathbb{C} \backslash S_0$ is compact. First notice that it's also compact in $\mathbb{C}$ since we know that the inclusion map is continuous and the image of a compact subset is also compact. Since each $x \in K$ makes $a_0(x) \neq 0$, we can assume that there is $\delta > 0$ such that:

$$|a_0(x)| \geqslant \delta, \quad |a_i(x)| \leqslant \frac{1}{\delta}$$

The later inequality holds because K is bounded. Then we know that if we consider the equation:

$$a_0(x) y^n + \cdots + a_n(x) = 0 \Rightarrow y^n + \frac{a_1(x)}{a_0(x)} y^{n-1} + \cdots + \frac{a_n(x)}{a_0(x)}$$

Using the above approximation since we know that the coefficients all have norm at most $\frac{1}{\delta^2}$, the above lemma tells you that:

$$|y| \leqslant 2 \frac{1}{\delta^{2/i}}$$

This implies that the set $\pi^{-1}(K)$ has bounded x and bounded y, which means that it's bounded, thus compact. $\square$

Corollary 7.4. Using the notation from above:

$$\pi|_{\pi^{-1}(\mathbb{C} \backslash S)} : \pi^{-1}(\mathbb{C} \backslash S) \to \mathbb{P}^1 \backslash S$$

is a n-sheeted covering map.

Proof. We have seen that this map is proper, and we know that since we have also taken out those points $x \in \mathbb{C}$ where there is a y such that $P(x, y) = \frac{\partial P}{\partial y}(x, y) = 0$, therefore we know that using the remark we gave in the beginning of this section: One special case of that remark is when $f(x, y)$ is given by an irreducible polynomial $P(x, y)$ given above whose y degree is n. This puts us in the situation of the lemma we discussed at the end of last lecture and we know that if we consider a point $a \in \mathbb{C}$ where $a_0(a) \neq 0$ and there is no b such that $P(a, b) = \frac{\partial P}{\partial y}(a, b) = 0$. Then we know that if we consider the projection of the x coordinate from $V(P)$ to $\mathbb{C}$, which is basically π we are considering in this corollary, then it's a local homeomorphism. Indeed, we have seen that there is a $\epsilon > 0$ such that the inverse image of $\mathbb{D}(a, \epsilon)$ is a disjoint union $(y_i(x) \neq y_j(x')$ for all $i \neq j$ and all $x, x' \in \mathbb{D}(a, \epsilon))$. In fact, these disjoint unions are characterized by the following: first $P(a, b) = 0$ has n-distinct roots $b_1, \cdots, b_n$ where near each b_i can choose a common ϵ such that the first projection restricted to $V \cap (\mathbb{D}(a, \delta) \times \mathbb{D}(b_i, \epsilon))$ gives a homeomorphism to $\mathbb{D}(a, \delta)$.

Then go back to our corollary, we know that if we take out S_1 and $\{\infty\}$ as well, then there will be so $a \in \mathbb{C}$ such that there exists a b such that $P(a, b) = \frac{\partial P}{\partial y}(a, b) = 0$, then we know that for each $(a, b) \in \pi^{-1}(\mathbb{C} \backslash S)$, we can pick a small open polydisk around it such that the projection restricts to a homeomorphism to an open disk in $\mathbb{D}$.

Recall that we discussed before that proper plus local homeomorphism plus locally compact Hausdorff imply that the map is in fact a finite sheeted covering map. Again, since for each $a \in \mathbb{P}^1 \backslash S$, there are n elements in the preimage, we know that this is indeed an n-sheeted covering map. $\square$

Remark 7.5. Notice that $\mathbb{P}^1 \backslash S$ is an open Riemann surface (Here open just means noncompact). Then given this n-sheeted covering map π, we can give $\pi^{-1}(\mathbb{C}\backslash S)$ a natural complex structure such that it's also an open Riemann surface (If it's compact, then it's surjective image is compact, which is not true).

Indeed, this can be done more generally: suppose that $p : X \to Z$ is a local homeomorphism where Z is a Riemann surface and X is a real 2-dimensional topological manifold. Then there is a unique complex structure on X such that p is holomorphic.

For an open subset $U \subset X$ such that p restricts to a homeomorphism to it's image V $p|_U : U \to V$ and $V \subset V_j$ where (V_j, ψ_j) is a chart for Z, we define a chart on U by:

$$\varphi_j : U \overset{\psi_j \circ p}{\to} \psi_j(V)$$

Notice that such U's can cover X because X is a covering map. Then we will need to show that this definition of charts is compatible. Indeed, if U_p and U_q are two such defined charts, then we know that:

$$\varphi_j \circ \varphi_i^{-1} = \psi_q \circ p \circ p^{-1} \circ \psi_p^{-1} = \psi_q \circ \psi_p^{-1}$$

This is holomorphic from the complex structure on Z.

In fact, we don't even need to assume X is a topological manifold in the beginning, the covering map naturally gives X a topological manifold structure. Moreover, you can check that suppose that you have a topological manifold structure on X, then the charts must be compatible with those defined above, therefore they belong to the same maximal unique structure. Similarly, the complex structure is unique if we want to make p holomorphic.

Now, using the above remark, we know that if we call $Y = \pi^{-1}(\mathbb{C}/\backslash S) \subset \mathbb{C}^2$, then we already have a compact structure on Y and a holomorphic map:

$$\pi : Y \to \mathbb{P}^1 \backslash S$$

However, we know that both Y and $\mathbb{P}^1/S$ are not compact. Therefore we will want to complete Y to obtain a new space X such that X is a compact Riemann surface and π will extend to a holomorphic map to $\mathbb{P}^1$. There is another issue since we don't even know if Y is completed while a Riemann surface has to be connected. We will all need to show that Y, hence X is connected to finish off the proof.

Before we do it, let's briefly say something about the geometric picture. First, the reason we take out S_0 from $\mathbb{P}^1$ is because if we allow $a_0(x) = 0$, then the degree of the polynomial drops by at least 1, then we know that at that point, the number of solutions of $y \in \mathbb{C}$ such that $P(x, y) = 0$ is at most $n - 1$. However, there is another way to look at it, which is as we approach the bad point x, some solutions of $P(x, y) = 0$ escapes to infinity. Indeed, this is why we have to stay away from these points to give a bound of y to make π proper. Given an example, think about:

$$P_1(x, y) = xy - 1$$

We know that as we approach the bad point $x = 0$, the solution $y = \frac{1}{x}$ indeed escape to infinite, thus damaging the properness of the map. We will address this problem later in the next section in a different manner by introducing the projective space.

We are also taking out S_1, which are those points where we could have $P(x, y) = \frac{\partial P}{\partial y}(x, y) = 0$. For example, if we consider the complex parabola:

$$P_2(x, y) = x - y^2 = 0$$

We find that when $x = 0$, we have $Q(0, 0) = \frac{\partial P}{\partial y}(0, 0) = 2 \cdot 0 = 0$. Here we see that $\frac{\partial P}{\partial x}(0, 0) = 1$, which implies that the curve defined by P at $(0, 0)$ is still **regular**, means it doesn't have all zeros vanish. However, we have a double root of y here. If you think about the graph of the real numbers parabola, you can find that near $(0, 0)$, we don't have a well defining covering map structure if we project to x-axis.

There is also another situation which is when the curve is actually **singular**, which means that:

$$P(x, y) = \frac{\partial P}{\partial y}(x, y) = \frac{\partial P}{\partial x}(x, y) = 0$$

For example, if we consider:

$$P_3(x,y) = x^2 - y^2$$

Then clearly all the partials vanish at $(0,0)$. If you think about the real numbers curve defined by P_3, you will notice that at $(0,0)$, two lines intersects.

We will also have other singular cases, for example we can have:

$$P_3(x,y) = y^2 - x^3$$

We will have a **cusp singularity** at $(0,0)$.

Therefore, our job latter is to try to fix the problems caused by these bad points. The way we do it is introduced first by Riemann, which is to add some points to Y and get some space X that is not embedded in $\mathbb{C}^2$. Let's introduce this method first and we will come to a more natural way in the next lecture.

Let's first deal with the question of how to complete Y. Now let $a \in S$ be a point then if a is a point in $\mathbb{C}$, we will take a small neighborhood Δ_ϵ where $\Delta_\epsilon \cap S = \{a\}$. If $a = \infty$, then we will take $\Delta = \{|z| > \frac{1}{\epsilon}\} \cup \{\infty\}$ such that again $\Delta \cap S = \{\infty\}$. Let's just call these neighborhoods Δ for simplicity and we use Δ^* to indicate that this is a punctured disk where we take out a.

Then we consider the preimage of Δ_a^*:

$$\pi^{-1}(\Delta_a^*) = U_{1a} \coprod \cdots \coprod U_{ka}$$

where each U_i is a connected component of the preimage. Note that here this k is at most n and may not be equal to n. Indeed, we know that take any $p \in \Delta_a^*$, the inverse image contains n points, each of them lies in one of U_{ka}, therefore there are at most n of them. However, this k can be strictly smaller and one example is given by:

$$z^n : \mathbb{D}(0,1) \to \mathbb{D}(0,1), \quad z \mapsto z^n$$

Now if we restrict π to one of U_i, we find that we get again a covering map. Indeed, we know that π restricted to U_i is still proper and it's again a local homeomorphism therefore it's a covering map. Therefore we know that:

$$\pi_{ja} = \pi|_{U_{ja}} \to \Delta_a^*$$

is a covering map for all $1 \leqslant j \leqslant k$, and we know from covering space theory, there is a homeomorphism $\varphi_{ja} : U_{ia} \to \Delta^*$ and an integer $k_j \geqslant 1$ such that the following diagram commutes:

$$\begin{tikzcd}
U_{ja} \arrow[rr, "\varphi_{ja}"] \arrow[dr, "\pi_{ja}"'] & & \Delta^* \arrow[dl, "z \mapsto z^{k_{ja}} + a"] \\
& \Delta_a^* &
\end{tikzcd}$$

Let D be a small punctured disk around the point a, then we know that it's inverse image is connected in U_{ja} since we know that it's inverse image in Δ^* under $z \mapsto z^{k_{ja}} + a$ is connected.

If $a = \infty$, we know that basically the same argument works since the set $\{|z| > \frac{1}{\epsilon}\}$ is homeomorphic to Δ^* for a punctured disk.

Then we will artificially add one point p_{ja} to each U_{ja} to obtain $U'_{ja} = U_{ja} \cup \{p_{ja}\}$ and we clear that X is obtained from Y by adding all p_{ja} for all $a \in S$ and all j. The topology on X is defined by:

1. $U \subset Y$ and it's open in the topology of Y.
2. For each point p_{ij}, we define a neighborhood basis to be $\{p_{ja}\} \cup \varphi_{ja}^{-1}(0 < |z| < \frac{1}{n} : n \geqslant 1)$.

Then we define the map:

$$p : X \to \mathbb{P}$$

where $p(y) = \pi(y)$ if $y \in Y$ and a point p_{ja} is mapped to a. Clearly this is a continuous map and you can check it directly. We can also give a complex structure on X by the cover:

$$\mathcal{U} = \{U'_{ja}\} \cup \{U \subset Y : U \text{ is a chart in } Y \text{ via } \pi\}$$

The chart on U'_{ja} is defined to be:

$$\varphi_{ja} : U_{ja'} \to \Delta$$

where we map p_{ja} to 0 and other points are mapped to the φ_{ja} we defined before. Then you check that this is indeed a homeomorphism. Then we need to check that this indeed gives compatible charts. Those U in Y are already compatible since Y is a Riemann surface. Therefore we only need to check the map on each $U'_{ja} \cap U$. Clearly the map on the intersection is described by: if $z \mapsto \varphi_{ja}^{-1}(z) \mapsto \pi(\varphi_{ja}^{-1}(z) = z^{k_{ja}} + a$, which is indeed holomorphic. Notice that this is fine since we are not dealing with anything that is near 0, therefore $z \mapsto z^{k_{ja}}$ is a perfectly good biholomorphism.

One thing that we do need to be aware of, if a is ∞, then we know that the transition map is now:

$$z \mapsto z^{k_{j\infty}} + a \mapsto \frac{1}{\epsilon^2 z^{k_{j\infty}}}$$

But this is again holomorphic since z is again not near 0.

Then if we consider the map:

$$p : X \to \mathbb{P}^1$$

I claim that it's holomorphic. Indeed, if we consider points in Y, we already see that it's holomorphic. For the newly added points p_{ja}, we know that we can take the neighborhood to be U'_{ja}, then this map in coordinates looks like:

$$z \mapsto z^{k_{ja}} + a, \quad z \mapsto \frac{1}{\epsilon^2 z^{k_{j\infty}}}$$

which are all holomorphic.

Moreover this p is proper. To see why, we already know that $p : Y \to \mathbb{P}^1 \backslash S$ is proper. Then suppose that we have a compact subset K in $\mathbb{P}^1$, then we know that $K \cap \mathbb{P}^1 \backslash S$ is compact, then we know that the inverse image of K in X is the inverse image of $K \cap \mathbb{P}^1 \backslash S$ in Y, plus some finite sets of closed points, which is the union of finitely many compact subsets, which is still compact. Therefore p itself is proper, therefore since $\mathbb{P}^1$ is compact, X is compact.

Then we know that this will be a n-sheeted branched cover since taken out S in $\mathbb{P}^1$ we get a n-sheeting usual covering.

It's especially interesting to think about what happens to this map $p : X \to \mathbb{P}^1$ when we look at the points around ∞ in $\mathbb{P}^1$. Essentially, what we did is just is that we added some new points to the part of the curve V which stretches to infinity according to if they are connected. You could imagine that the curve can go to infinity in different directions, and we are adding one point to to each of the directions. This will appear again in the next lecture when we embed this curve in some higher dimension projective spaces.

Essentially the function p we constructed above comes from the projection of $V \subset \mathbb{C}^2$ to the x coordinate. Then let's also study what happens to the projection to the y coordinate.

Proposition 7.6. Let $pr_2 : V \to \mathbb{C}$ be the projection where $(x, y) \mapsto y$. Then the function:

$$f = pr_2|_Y \to \mathbb{C}$$

is holomorphic and can be extended to a holomorphic function:

$$f : X \to \mathbb{P}^1$$

Proof. Clearly the projection on Y is holomorphic since in the coordinates of Y we defined before, this projection is just:

$$x \mapsto y(x)$$

Then using the implicit function theorem, this is indeed holomorphic. We need to see how we can extend it to $X \to \mathbb{P}^1$.

Let $a \in S$ and let $x \in X$ such that $p(x) = a$. Let's choose local coordinates z around x and w around a such that p becomes $z \mapsto z^m$, which is always possible since p is holomorphic an this is just the local behavior of holomorphic functions. Then we I claim that by definitions of $V(P)$ and f, we have equations:

$$f(z)^n + \frac{a_1(w)}{a_0(w)} f(z)^{n-1} + \cdots + \frac{a_n(w)}{a_0(w)} = 0, \quad z \neq 0, w = p(z)$$

Here we are thinking of a_i as meromorphic functions from $\mathbb{P}^1$ to $\mathbb{C}^1$ using the coordinates w on $\mathbb{P}^1$ we just choose. In other words, we are assuming that the coordinate z goes to $(x, y) \in Y$ and by p it's mapped to $x \in \mathbb{P}^1$, then by the coordinate it's mapped to w. Then when writing a_n in the w coordinates, we know that $a_1(w)$ is equal to $a_1(x)$, however, we know that the two a_1 carries different meaning. I hope this abuse of notation won't cause too much confusion.

Now we know that each $a_i(w)$ is meromorphic at $w = 0$. Indeed, if $a \in S$ is ∞, then clearly we know that as $w \to 0$, which means that the point (x, y) in X are escaping to infinity, then clearly $a_0(x)$ goes to infinity as a polynomial in x, then as $w \to 0$, it also goes to infinity, thus it has a pole at 0. But if $a \neq 0$, then clearly $a_i(w)$ should be holomorphic at $w = 0$. But no matter what, we know that we can assume $\frac{a_i(w)}{a_0(w)}$ is meromorphic at $w = 0$. This means there exist constants $C_i > 0$ such that:

$$|\frac{a_i(w)}{a_0(w)}| \leqslant \frac{C_i}{|w|^N}, \quad \forall 1 \leqslant i \leqslant n$$

Then we know that via the lemma, we can estimate:

$$|f(z)| \leqslant 2 \max_{1 \leqslant i \leqslant n} \frac{C_i^{1/i}}{|w|^{N/i}} \leqslant \frac{C}{|z|^K}$$

for some $C > 0$ and $K \in \mathbb{Z}_{\geqslant 1}$. This implies that $f(z)$ is meromorphic at $z = 0$ at at worst it has a pole of order K, therefore it actually extends to a function to $\mathbb{P}^1$. □

To conclude our previous work, we have constructed some compact space X with a complex structure on it and two holomorphic functions f, p on X such that for all $y \in Y$ where we have:

$$P(p(y), f(y)) = 0$$

Then recall that X and Y only differs by a finite set of isolated points, therefore if we define a function:

$$P : X \to \mathbb{C}, \quad x \mapsto P(p(x), f(x))$$

we can just extend the value 0 to the points p_{ja} and this gives us a constant function P on all of X.

Finally, we only need to show that X is connected to conclude that X is a compact Riemann surface.

Proposition 7.7. The space Y is connected thus so is X. Therefore X is a compact Riemann surface with meromorphic functions $f : X \to \mathbb{P}^1$ and $p : X \to \mathbb{P}^1$ we defined before such that:

$$P(p(x), f(x)) = 0, \quad \forall x \in X$$

Proof. Recall that X is only Y added finitely many points and each point has a neighborhood that looks like a Euclidean ball, therefore we know that if Y is connected, then X is automatically connected.

To prove Y is connected, let's assume it's not and we will show that this contradicts the fact that $P(x, y)$ we started with is irreducible.

Suppose that Y is not connected and thus with a strictly smaller connected component Y_1. We will consider the restriction of π to Y_1 and call it π_1. Similarly we will use f_1 to denote the restriction of f to Y_1.

Again, since π is proper, π_1 is proper, and since Y_1 is open in Y, π_1 is again a local homeomorphism, thus π_1 is a r-sheeted covering map for some $r \leqslant n$. In particular, the map $\pi_1 : Y_1 \to \mathbb{P}^1 \backslash S$ is still surjective.

Now take any $x \in \mathbb{P}^1 \backslash S$ and let $e_\nu(x)$ where $1 \leqslant \nu \leqslant r$ be the ν-th elementary symmetric function with respect to $y_1, \cdots, y_r$:

$$e_\nu(x) = e_\nu(y_1, \cdots, y_r)$$

where $y_1, \cdots, y_r$ are the values taken by f_1 on $\pi^{-1}(x)$.

This define a function:

$$e_\nu(x) : \mathbb{P}^1 \backslash S \to \mathbb{C}$$

Notice that if we choose coordinates z for $x \in \mathbb{P}^1 \backslash S$, we know that this function in local coordinates is in fact defined by:

$$z \mapsto e_\nu(y_1(z), \cdots, y_r(z))$$

where $y_i(z)$ are defined using the implicity function theorem which are all holomorphic. Therefore this definition actually gives us a holomorphic function:

$$e_\nu : \mathbb{P}^1 \backslash S \to \mathbb{C}$$

I claim that this can be extended to a meromorphic function:

$$e_\nu : \mathbb{P}^1 \to \mathbb{P}^1$$

To show this, note that by the definition of f_1 and π_1 we have:

$$P(x, y_j) = 0, \quad \leqslant j \leqslant r$$

Then for $a \in \mathbb{P}^1$ that is not ∞, we choose a coordinate x around a such that $0 \mapsto a$. Note that if we consider the map:

$$x \mapsto x \in \mathbb{P}^1 \backslash S \mapsto (x, y_i) \in Y_1, \quad z \neq 0$$

We can write this in coordinates by:

$$x \mapsto (x, y_i(x))$$

where $y_i(x)$ is a holomorphic function in x. Then we know that in this coordinate we have:

$$P(x, y_i(x)) = 0, \quad x \neq 0$$

Notice that for each x, y_i, we know that $x, y_i(x)$ gives us a local coordinate of V near that point where $x \neq 0$. Then we know that f_1 in this coordinate can be written as:

$$f_1 : x \mapsto y_i(x)$$

Then we know that since:

$$a_0(x) f_1(y_i(x))^n + \cdots + a_n(x) = 0$$

for all such x in the coordinate, we know that we can use the lemma to bound $f_1(y_i(x))$ and thus $f_1(y_i(x))$ naturally extends to a meromorphic function at $x = 0$. Then we know that we have the following estimate of $e_\nu(x)$ using the same coordinate:

$$|e_\nu(x)| = |\sum_{i_1 < \cdots < i_\nu} f_1(y_{i_1}(x)) \cdots f_1(y_{i_\nu}(x))|$$

Since each $f_1(y_i(x))$ are meromorphic functions in x, we know that e_ν is also a meromorphic function in x. Therefore $e_\nu(x)$ extends to $a \in S$ where $a \neq \infty$.

When $a = \infty$, similar arguments works, we again choose a coordinate z around a such that $0 \mapsto a$, for example $z \mapsto 1/z$. Then we know that using this coordinate, for each $z \neq 0$, we get some point x in $\mathbb{P}^1$ which is close to ∞, then we know that above this point, there will be $y_1, \cdots, y_n$ and we know that each of them is a holomorphic function in x since in Y_1 we excluded the bad points which makes the implicit function theorem false. Then we can essential estimate e_ν using the same argument as above and indeed e_ν extends now to $x = 0$, which is the point $a = \infty$.

However, recall that we know exactly what is a meromorphic function:

$$\mathbb{P}^1 \to \mathbb{P}^1$$

and they are just given by rational functions! If we define $Q(x,y) \in \mathbb{C}(x)[y]$ by:

$$Q(x,y) = y^r - e_1(x)y^{r-1} + \cdots + (-1)^r e_r(x) : \mathbb{P}^1 \to \mathbb{P}^1$$

Then if $x \in \mathbb{P}\backslash S$, I claim that the roots $y_1, \cdots, y_r$ of $Q(x,y)$ are also roots of $P(x,y) \in \mathbb{C}(x)[y]$.

Indeed, we know that:

$$Q(x,y) = y^r - e_1(x)y^{r-1} + \cdots + (-1)^r e_r(x) = y^r - e_1(y_1, \cdots, y_n)y^{r-1} + \cdots + e_r(y_1, \cdots, y_r)$$

Using the identity of the elementary symmetric functions, we know that:

$$y^r - e_1(y_1, \cdots, y_n)y^{r-1} + \cdots + e_r(y_1, \cdots, y_r) = \prod_{\nu=1}^{r}(y - y_\nu)$$

This implies that given any fixed $x_0 \in \mathbb{P}^1\backslash S$, we know that:

$$Q(x_0, y) \mid P(x_0, y)$$

Now if we do the Euclidean division in $\mathbb{C}(x)[y]$ and get:

$$P(x,y) = A(x,y)Q(x,y) + R(x,y)$$

with $\deg_y R(x,y) < \deg_y Q(x,y)$. Notice that for every $x_0 \in \mathbb{P}^1\backslash S$, we know that if we plug in x_0, we get:

$$P(x_0,y) = A(x_0,y)Q(x_0,y) + R(x_0,y)$$

We know that even if we need to avoids the roots of the denominators of the coefficients of $A(x,y)$ and $R(x_0,y)$, we are only avoiding finitely many of them. Therefore we know that the above equation holds for infinitely many $x_0 \in \mathbb{C}$.

However, we already see that for a specific x_0, $Q(x_0,y)$ will divide $P(x_0,y)$ in $\mathbb{C}[y]$, therefore we know that $R(x_0,y) = 0$ as a polynomial. Therefore the coefficients of $R(x,y)$ vanishes at infinitely many x_0, therefore each of them must be 0, therefore $R(x,y) = 0$, therefore we obtain the division in $\mathbb{C}(x)[y]$. However, we know that by Gauss's lemma if $P(x,y)$ is not irreducible in $\mathbb{C}(x)[x]$, it's not irreducible in $\mathbb{C}[x,y]$. $\square$

Remark 7.8. In algebraic geometry, this space X is called the **normalization** of the algebraic curve $V = V(P)$.

8 Lecture 8

This section is basically about solving the d-bar equation, which connects to the famous Runge's theorem. Since Runge's theorem is not the typical content for a standard course on one variable complex analysis, I pushed the stuff in this section to the write-up. You should read it before moving to the next lecture.

9 Lecture 9

From this section and going forward we will start working with cohomology. Let's first do a introduction of what that is via the famous Mittag-Leffler problem and it's cohomological form.

References