Write-up on Runge's Theorem

Chunye Yang

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Since we will need to use Runge's theorem to embed an algebraic surface to the projective space and Runge's theorem is not a standard topic in a one-variable complex analysis class, therefore we include it in a write-up so that you can read this when you need Runge's theorem. Basically we are following [NN01]. Let's state all the versions of Runge's theorem before we begin to prove them:

Theorem 0.1. (Runge's Theorem: First Form) Let Ω be open in $\mathbb{C}$ and let K be a compact subset of Ω, then the following statements are equivalent:

1. $\rho(\mathcal{H}(\Omega))$ is dense in $\mathcal{O}(K)$.
2. No connected components of $\Omega \backslash K$ is relatively compact in Ω.
3. For any $a \in \Omega$ and $a \notin K$, there exists a $f \in \mathcal{H}(\Omega)$ with:

$$|f(a)| > |f|_K$$

Here $\mathcal{H}(\Omega)$ are just holomorphic functions on Ω and we will explain what is ρ and what is $| \quad |_K$ latter.

Theorem 0.2. (Runge's Theorem: Second Form) Let $\Omega' \subset \Omega$ be two open sets in $\mathbb{C}$, then (Ω, Ω') is a **Runge's pair** iff no connected component of $\Omega \backslash \Omega'$ is relatively compact.

We will define what is a Runge's pair later but simply speaking, what this is saying is that holomorphic functions in Ω', the smaller open set can be approximated using restrictions of holomorphic functions on a larger open set.

Theorem 0.3. (Runge's Theorem: Classical Form) Let Ω be open in $\mathbb{C}$ and let $\mathbb{C} \backslash \Omega = \bigcup_{\alpha \in A} C_\alpha$ be the decomposition into connected components. The $A' \subset A$ be set where $A' = \{\alpha \in A : C_\alpha \text{ is compact.}\}$, and for each $\alpha \in A'$, choose a point $a_\alpha \in C_\alpha$.

Then for any $f \in \mathcal{H}(\Omega)$, it can be approximated uniformly on any compact subset of Ω by rational functions all of whose poles are contained in the set $\{a_\alpha\}_{\alpha \in A'}$.

A corollary of this theorem is:

Corollary 0.4. Let Ω be open in $\mathbb{C}$. Then the set of restrictions to Ω of polynomials in z is dense in $\mathcal{H}(\Omega)$ iff $\mathbb{C} \backslash \Omega$ has no compact connected component.

Before we begin our study on Runge's theorem, let's examine some preliminaries:

1 The $\frac{\partial u}{\partial \bar{z}} = \phi$ Equation

In our proof of Runge's theorem, one important tool is to understand the so called **inhomogeneous version of Cauchy's Integral formula**. To be able to set this up, we will have to examine the following partial differential equation on an open subset Ω of $\mathbb{C}$:

$$\frac{\partial u}{\partial \bar{z}} = f$$

where f is a smooth function on Ω, or we can write is as $f \in C^\infty(\Omega)$. Here the differential operator $\frac{\partial}{\partial \bar{z}}$ just means:

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$$

This equation is sometime called the $\bar{\partial}$ equation and as you will see later, a lot of complex geometry actually hides behind this equation. Turns that we won't be able to solve it in the full generality without Runge's theorem, however the study of this partial differential equation helps us prove Runge's theorem, which in turn solves the equation.

We will need the following fact on smooth functions which basically comes from the existence of a partition of unity:

Theorem 1.1. Let Ω be open in $\mathbb{R}^n$ and let $X \subset \Omega$ be a closed subset which is contained in some U open in Ω. Then there is a $\phi \in C^\infty(\Omega)$ with $0 \leqslant \phi(x) \leqslant 1$ for all $x \in \Omega$ such that $\phi|_X = 1$ and $\phi|_{\Omega \setminus U} = 0$.

In fact ϕ can be chosen to be 1 on a neighborhood of X and if X is compact, ϕ can also be chosen to be compactly supported.

What this function is telling us is that we can create a bump function which takes exactly the value 1 on some closed subset.

We will also need the following theorem on some integrations:

Lemma 1.2. Let R be a closed rectangle in $\mathbb{C}$ and let $f \in \mathbb{C}^\infty(R)$ with values in $\mathbb{C}$, which means that there is an open neighborhood of R where f is smooth as a function from that open neighborhood to $\mathbb{R}^2$. Then we have the following integral formula:

$$\iint\limits_R \frac{\partial f}{\partial \bar{z}} dz \wedge d\bar{z} = -\int_{\partial R} f dz$$

Here since $dz = dx + idy$ and $d\bar{z} = dx - idy$, we know that $dz \wedge d\bar{z} = -2idx \wedge dy$. Therefore you can just think of this as the Lebesgue integration with the standard planar Lebesgue measure.

Proof. Suppose that $R = [a,b] \times [c,d]$, then we know that:

$$\iint\limits_R \frac{\partial f}{\partial \bar{z}} dz \wedge d\bar{z} = (-i)\int_c^d (\int_a^b \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} dx)dy$$

We know that if we compute:

$$\int_c^d \int_a^b \frac{\partial f}{\partial x} dxdy$$

It's just:

$$\int_c^d f(b,y) - f(a,y)dy$$

Let call the boundary of R, positively oriented by $\gamma_1, \cdots, \gamma_4$, in counter clockwise. Recall that if we parametrize the left and right vertical side of the rectangle by $t \mapsto b + ti$ and $t \mapsto a + (d+c)i - ti$, then we know that:

$$\int_{\gamma_2} f(z)dz = i\int_c^d f(b,t)dt, \quad \int_{\gamma_4} f(z)dz = -i\int_c^d f(a, d+c-t)dt$$

Notice that for the second term, we do a change of variable by $y = d + c - t$, we know that $dy = -dt$ and the integral is:

$$-i\int_c^d f(a, d+c-t)dt = -i\int_d^c f(a,y) - dy = -i\int_c^d f(a,y)dy$$

Therefore we know that:

$$\int_c^d \int_a^b \frac{\partial f}{\partial x} dxdy = \int_c^d f(b,y) - f(a,y)dy = \frac{1}{i}\int_{\gamma_2} f(z)dz + \frac{1}{i}\int_{\gamma_4} f(z)dz$$

Similarly, we can compute:

$$i\int_c^d \int_a^b \frac{\partial f}{\partial y} dxdy$$

Since we are dealing with continuous functions on compact subset, we know that we can either use the Fubini from multivariable calculus or the Fubini from Lebesgue integration and switch the integral order and get:

$$i\int_c^d \int_a^b \frac{\partial f}{\partial y} dydx = i\int_a^b \int_d^c \frac{\partial f}{\partial y} dydx$$

Then I will let you do the parametrization and see that this is indeed:

$$\frac{-1}{i}(\int_{\gamma_1} f(z)dz + \int_{\gamma_3} f(z)dz)$$

□

Using the above lemma and the theorem on bump functions, we get:

Proposition 1.3. Let R, R' be a closed rectangles and let $R' \subset \mathring{R}$. If $f \in C^\infty(R\backslash\mathring{R}')$ again with values in $\mathbb{C}$ then:

$$\iint\limits_{R\backslash\mathring{R}'} \frac{\partial f}{\partial \bar{z}} dz \wedge d\bar{z} = -\int_{\partial R} f dz + \int_{\partial R'} f dz$$

Proof. Clearly if f is smooth on R, then we know that we can apply the previous lemma to:

$$\iint\limits_{R} \frac{\partial f}{\partial \bar{z}} dz \wedge d\bar{z} - \iint\limits_{R'} \frac{\partial f}{\partial \bar{z}} dz \wedge d\bar{z} = -\int_{\partial R} f dz + \int_{\partial R'} f dz$$

However, we only assumed that $f \in C^\infty(R\backslash\mathring{R}')$, therefore we need to do some work here.

By our assumption, there is some open $U \supset R\backslash\mathring{R}'$ such that f is smooth on U. Recall that using the theorem about the bump function, there is some function $\alpha \in C^\infty(U)$ such that $\alpha = 1$ on $R\backslash\mathring{R}'$ and α is supported in U, then if we consider the function $g(z)$ which is defined by $g(z) = \alpha(z) \cdot f(z)$ if $z \in U$ and is defined to be 0 outside U. Notice that this definition actually glues to a function g on all of $\mathbb{C}$ and you can check that g is actually in $C^\infty(\mathbb{C})$, then we can apply the previous lemma to g and get:

$$\iint\limits_{R\backslash\mathring{R}'} \frac{\partial g}{\partial \bar{z}} dz \wedge d\bar{z} = \iint\limits_{R} \frac{\partial g}{\partial \bar{z}} dz \wedge d\bar{z} - \iint\limits_{R'} \frac{\partial g}{\partial \bar{z}} dz \wedge d\bar{z} = -\int_{\partial R} g dz + \int_{\partial R'} g dz$$

However, we know that on $R\backslash\mathring{R}'$, $g(z) = f(z)$ since $\alpha = 1$, then we actually just have:

$$\iint\limits_{R\backslash\mathring{R}'} \frac{\partial f}{\partial \bar{z}} dz \wedge d\bar{z} = -\int_{\partial R} f dz + \int_{\partial R'} f dz$$

□

Here is another little lemma we need:

Lemma 1.4.

$$\iint\limits_{|z|<1} \frac{1}{|z|} dxdy < \infty$$

In other words, $\frac{1}{|z|} \in \mathrm{L}^1(D(0,1))$.

Proof. Notice that the interval $[0,1)$ is just a measure 0 set in $D(0,1)$, we we know that the integration can be rewritten as:

$$\iint\limits_{D(0,1)\backslash[0,1)} \frac{1}{|z|}dxdy$$

Then recall that we have the **Lebesgue integration change of variable formula**, where we have a diffeomorphism:

$$(r,\theta) : (0,1) \times (0,2\pi) \to D(0,1)\backslash[0,1), \quad (r,\theta) \mapsto re^{i\theta}$$

Then we have:

$$\iint\limits_{D(0,1)\backslash[0,1)} \frac{1}{|z|}dxdy = \int_0^1 \int_0^{2\pi} d\theta dr = 2\pi$$

□

Then we can state the theorem that gives us the inhomogeneous Cauchy's integral formula we need to prove Runge's theorem:

Theorem 1.5. Let $f \in C_0^\infty(\mathbb{C})$ (a compactly supported smooth function on $\mathbb{C}$ with values in $\mathbb{C}$), then for any $\omega \in \mathbb{C}$, we have:

$$f(\omega) = \frac{1}{2\pi i} \iint\limits_{\mathbb{C}} \frac{\partial f}{\partial \overline{z}} \frac{1}{z-\omega} dz \wedge d\overline{z}$$

Writing it in a different form:

$$-\pi f(\omega) = \iint\limits_{\mathbb{C}} \frac{\partial f}{\partial \overline{z}} \frac{1}{z-\omega} dxdy$$

Notice that this integral actually makes sense since $\frac{\partial f}{\partial \overline{z}}$ is continuous and compactly supported, thus is integrable on $\mathbb{C}$. We also know $\frac{1}{z-\omega}$ is integrable on $\mathbb{C}$ by the previous lemma we proved, therefore their product is integrable, or in other words even in $L^1(\mathbb{C})$.

Proof. Again if we want to prove:

$$-\pi f(\omega) = \iint\limits_{\mathbb{C}} \frac{\partial f}{\partial \overline{z}} \frac{1}{z-\omega} dxdy$$

We now that we can consider the diffeomorphism which translates x and y such that after we do this diffeomorpohism, we get:

$$\iint\limits_{\mathbb{C}} \frac{\partial f}{\partial \overline{z}}(z+\omega)\frac{1}{z}dxdy$$

Since $f(z+\omega)$ is still compactly supported, we can choose a closed rectangle R such that $f(z+\omega)$ has support contained in $\mathring{R}$ and $0 \in R$.

Then we again choose a ϵ small and a rectangle R_ϵ with sides length of ϵ centered at 0. Then notice that:

$$\iint\limits_{\mathbb{C}} \frac{\partial f}{\partial \overline{z}}(z+\omega)\frac{1}{z}dxdy = \iint\limits_{\mathbb{C}} \frac{\partial}{\partial \overline{z}}(\frac{f(z+\omega)}{z})dxdy$$

Indeed, we know that since $\frac{1}{z}$ is holomorphic on $\mathbb{C}\backslash\{0\}$, then $\frac{\partial \frac{1}{z}}{\partial \overline{z}} = 0$ for $z \neq 0$. Then we know that since:

$$\frac{\partial}{\partial \overline{z}}(\frac{f(z+\omega)}{z}) = \frac{\partial \frac{1}{z}}{\partial \overline{z}} f(w+z) + \frac{\partial f}{\partial \overline{z}}(z+\omega) \cdot \frac{1}{z}$$

We know that these two functions agree when $z \neq 0$. However we know that $\{0\}$ is only a measure 0 subset, therefore it doesn't affect the integral. Therefore let's just compute $\iint_{\mathbb{C}} \frac{\partial}{\partial \overline{z}}(\frac{f(z+\omega)}{z})dxdy$.

Recall that the function:

$$|\frac{\partial f(z+\omega)}{\partial \overline{z}}\frac{1}{z}| \overset{\text{a.e.}}{=} |\frac{\partial}{\partial \overline{z}}(\frac{f(z+\omega)}{z})| \in \mathrm{L}^1(\mathbb{C})$$

We can also think about the function:

$$g_\epsilon(z) = \frac{\partial}{\partial \overline{z}}(\frac{f(z+\omega)}{z})1_{R\backslash \mathring{R}_\epsilon}$$

where 1 is the indicator function. Then we know that $g_\epsilon(z)$ converges to $\frac{f(z+\omega)}{z}$ almost everywhere on $\mathbb{C}$, therefore since:

$$|g_\epsilon(z)| \leqslant |\frac{\partial}{\partial \overline{z}}(\frac{f(z+\omega)}{z})|$$

Using dominated convergence theorem, we know that:

$$\iint\limits_{\mathbb{C}} \frac{\partial}{\partial \overline{z}}(\frac{f(z+\omega)}{z})dxdy = \lim_{\epsilon \to 0} \iint\limits_{\mathbb{C}} g_\epsilon(z)dxdy$$

Then for each $g_\epsilon(z)$, we know that the integral is:

$$\iint\limits_{\mathbb{C}} g_\epsilon(z)dxdy = \iint\limits_{R\backslash \mathring{R}_\epsilon} \frac{\partial}{\partial \overline{z}}(\frac{f(z+\omega)}{z})dxdy = \frac{-1}{2i} \iint\limits_{R\backslash \mathring{R}_\epsilon} \frac{\partial}{\partial \overline{z}}(\frac{f(z+\omega)}{z})$$

Then using the previous result, we have:

$$\frac{-1}{2i} \iint\limits_{R\backslash \mathring{R}_\epsilon} \frac{\partial}{\partial \overline{z}}(\frac{f(z+\omega)}{z})dz \wedge d\overline{z} = \frac{-1}{2i}(-\int_{\partial R}(\frac{f(w+z)}{z})dz + \int_{\partial R_\epsilon}(\frac{f(w+z)}{z})dz)$$

Notice that by our choice of R, we know that:

$$-\int_{\partial R}(\frac{f(w+z)}{z})dz = 0$$

Therefore the above integral is just:

$$\frac{-1}{2i} \iint\limits_{R\backslash \mathring{R}_\epsilon} \frac{\partial}{\partial \overline{z}}(\frac{f(z+\omega)}{z})dz \wedge d\overline{z} = \frac{-1}{2i} \int_{\partial R_\epsilon} \frac{f(w+z)}{z}dz$$

However, notice that:

$$\int_{\partial R_\epsilon} \frac{f(w+z)}{z}dz = \int_{\partial R_\epsilon} \frac{f(w+z) - f(w) + f(w)}{z}dz = 2\pi i f(w) + \int_{\partial R_\epsilon} \frac{f(z+w) - f(w)}{z}dz$$

This gives us:

$$\iint\limits_{\mathbb{C}} \frac{\partial}{\partial \overline{z}}(\frac{f(z+\omega)}{z})dxdy = \lim_{\epsilon \to 0} \frac{-1}{2i}(\int_{\partial R_\epsilon} \frac{f(z+w) - f(w)}{z}dz + 2\pi i f(w))$$

Notice that since we assumed that f is smooth on $\mathbb{C}$, then we have a bound $M > 0$ such that:

$$|\frac{f(z+w) - f(w)}{z}| < M$$

Then we know that we have:

$$|\lim_\epsilon \int_{\partial R_\epsilon} \frac{f(z+w) - f(w)}{z}dz| \leqslant \lim_\epsilon \int_{\partial R_\epsilon} Mdz = 0$$

Therefore we eventually get:

$$\iint\limits_{\mathbb{C}} \frac{\partial}{\partial \overline{z}}(\frac{f(z+\omega)}{z})dxdy = \pi f(w)$$

which is exactly what we desired. $\square$

This let's partially solve the $\overline{\partial}$ equation:

Theorem 1.6. Let $f \in C_0^\infty(\mathbb{C})$, then if we define:

$$u(\omega) = \frac{1}{2\pi i} \iint\limits_{\mathbb{C}} \frac{f(z)}{z-\omega} dz \wedge d\overline{z}$$

Then $u \in C^\infty(\mathbb{C})$ and we have:

$$\frac{\partial u}{\partial \overline{z}}(w) = f(w), \quad \forall w \in \mathbb{C}$$

Proof. Again, we can change variable and obtain:

$$u(w) = \frac{1}{2\pi i} \iint\limits_{\mathbb{C}} \frac{f(z+w)}{z} dz \wedge d\overline{z}$$

The recall that:

$$\frac{\partial u}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y})$$

Let's first compute what is $\frac{\partial u}{\partial x}$. For $h > 0$ a real number, we consider:

$$\frac{u(w+h) - u(w)}{h} = \frac{1}{2\pi i} \iint\limits_{\mathbb{C}} \frac{f(z+w+h) - f(z+w)}{hz} dz \wedge d\overline{z}$$

Again, we if we tale the limit and let $h \to 0$, using dominated convergence theorem, we know:

$$\frac{\partial u}{\partial x}(w) = \lim_{h \to 0} \frac{u(w+h) - u(w)}{h} = \frac{1}{2\pi i} \iint\limits_{\mathbb{C}} \frac{\partial f}{\partial x}(w+z) \cdot \frac{1}{z} dz \wedge \overline{z}$$

Similarly, we know that:

$$\frac{\partial u}{\partial y}(w) = \frac{1}{2\pi i} \iint\limits_{\mathbb{C}} \frac{\partial f}{\partial y}(w+z) \cdot \frac{1}{z} dz \wedge \overline{z}$$

Notice that since $\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}$ and in fact all the partials of f are smooth and compactly supported, we know that all the partials of u exists, therefore u is in fact smooth on $\mathbb{C}$. Moreover, we know that:

$$\frac{\partial u}{\partial \overline{z}}(w) = \frac{1}{2\pi i} \iint\limits_{\mathbb{C}} \frac{\partial f}{\partial \overline{z}}(z+w) \frac{1}{z} dz \wedge d\overline{z} = f(w)$$

□

Note that all the above computation are for a function with compact support, i.e. a function $f \in C_0^\infty(\mathbb{C})$ and the assumption of compact support is essential in our proof. In order to solve the $\frac{\partial u}{\partial \overline{z}} = f$ for some $f \in C^\infty(\mathbb{C})$ or $f \in C^\infty(\Omega)$ for some open subset Ω of $\mathbb{C}$. We will need Runge's theorem. However, the above partial results indeed gives us something useful:

Theorem 1.7. (Inhomogeneous Cauchy's Integral Formula) Let Ω be open in $\mathbb{C}$ and K a compact subset of Ω, let $\alpha \in C_0^\infty(\Omega)$ be such that $\alpha = 1$ on a neighborhood of K then for any $f \in \mathcal{H}(\Omega)$ (again defined via partition of unity) we have:

$$f(w) = \frac{1}{2\pi i} \iint\limits_{\mathbb{C}} \frac{\partial \alpha}{\partial \overline{z}} f(z) \frac{1}{z-w} dz \wedge d\overline{z}$$

Proof. We again do the change of variable so that the integral is:

$$\frac{1}{2\pi i}\iint\limits_{\mathbb{C}} \frac{\partial \alpha}{\partial \overline{z}} f(z+w)\frac{1}{z} dz \wedge d\overline{z}$$

Now consider the function $\phi(z)$ which is defined to be $f(z)\alpha(z)$ when $z \in \Omega$ and 0 when $z \notin \Omega$. Notice that since α is supported on a compact subset of Ω, we know that this definition actually glues to a smooth function with compact support on $\mathbb{C}$. Moreover, we know that if $w \in K$, we have:

$$f(w) = \phi(w) = \frac{1}{2\pi i}\iint\limits_{\mathbb{C}} \frac{\partial \phi}{\partial \overline{z}}(z)\frac{1}{z-w} dz \wedge d\overline{z} = \frac{1}{2\pi i}\iint\limits_{\Omega} \frac{\partial \phi}{\partial \overline{z}}(z)\frac{1}{z-w} dz \wedge d\overline{z}$$

Now note that when $z \in \omega$:

$$\phi(z) = \alpha(z)f(z)$$

This implies:

$$\frac{\partial \phi}{\partial \overline{z}}(z) = \frac{\partial \alpha}{\partial \overline{z}}(z)f(z) + \frac{\partial f}{\partial \overline{z}}(z)\alpha(z)$$

Notice that by our assumption, $f \in \mathcal{H}(\Omega)$, therefore we know that if $z \in \Omega$:

$$\frac{\partial f}{\partial \overline{z}}(z)\alpha(z) = 0$$

Then we know that the above integral now equals to:

$$f(w) = \phi(w) = \frac{1}{2\pi i}\iint\limits_{\Omega} \frac{\partial \phi}{\partial \overline{z}}(z)\frac{1}{z-w} dz \wedge d\overline{z} = \frac{1}{2\pi i}\iint\limits_{\Omega} \frac{\partial \alpha}{\partial \overline{z}}(z)f(z)\frac{1}{z-w} dz \wedge d\overline{z}$$

This is the desired form in the statement. □

Remark 1.8. This is called inhomogeneous Cauchy's integral formula not just because it looks like some modification of the usual Cauchy's integral formula. Notice that since $\alpha(z)$ is always 1 on a neighborhood of K, we know that for each $z \in K$, $\frac{\partial \alpha}{\partial \overline{z}} = 0$, therefore we know that this integral can actually be done on the region:

$$\text{Supp}(\alpha)\backslash K$$

You can imagine that this is a strip which is sort of a thickened line rather than just a line which is used in the usual Cauchy's integral formula.

Using this result, we can go and study the first form of Runge's theorem:

2 Runge's Theorem: First Form

Let Ω be an open subset of $\mathbb{C}$ and let K be a compact subset in Ω. For any continuous function ϕ on K, we set:

$$|\phi|_K = \sup_{z\in K} |\phi(z)|$$

Let $\mathcal{H}(\Omega)$ be the set of all holomorphic functions on Ω. We can define a topology on $\mathcal{H}(\Omega)$ by declaring that a basis of $f \in \mathcal{H}(\Omega)$ is given by:

$$B_{\epsilon,K} = \{g \in \mathcal{H}(\Omega) : |f-g|_K < \epsilon\}$$

here notice that for each compact subset K and each $\epsilon > 0$, we obtain a $B_{\epsilon,K}$ and the basis is obtained when K ranges over all compact subsets and ϵ runs over all positive real numbers. This is indeed a basis, since we know that the intersection:

$$B_{\epsilon_1,K_1} \cap B_{\epsilon_2,K_2} = B_{\epsilon_3,K_1\cup K_2}$$

where ϵ_3 is the minimum of ϵ_1, ϵ_2. This tells you that the for any finite intersection, there is a smaller element in the system that is contained in the intersection. Then we define the topology on $\mathcal{H}(\Omega)$ to be those generated by these basis at each point.

This topology is in fact metrizable: Recall that for any open subset of $\mathbb{C}$, here we use Ω, there exists a sequence of compact subsets K_n such that K_n is contained in the interior of K_{n+1} and $\cup_n K_n = \Omega$. Then for $f, g \in \mathcal{H}(\Omega)$, we define:

$$d(f,g) = \sum_{n=1}^{\infty} 2^{-n} \frac{|f-g|_{K_n}}{1+|f-g|_{K_n}}$$

Clearly that by our definition for each n, $\frac{|f-g|_K}{1+|f-g|_K}$, therefore the above series indeed converges. It's easy to check this is indeed a metric and this metric in fact define the same topology. To see why, we only need to show that at a point, they give equivalent basis. Given a compact set K and ϵ, we know that it's contained in some K_N in the sequence, then we know that if we consider $B_{\epsilon,K}$, we can consider $\{g : d(f,g) < \epsilon/n\}$. We know that if g is in $\{g : d(f,g) < \epsilon' \cdot 2^{-N}\}$, then we know that:

$$\sum_n 2^{-n} \frac{|f-g|_{K_n}}{1+|f-g|_{K_n}} < \epsilon' 2^{-N}$$

In particular, we know that $\frac{|f-g|_{K_n}}{1+|f-g|_{K_n}} < \epsilon'$, this implies:

$$|f-g|_{K_N} < \epsilon' + \epsilon'|f-g|_{K_n} \Rightarrow |f-g|_{K_N} < \frac{\epsilon'}{1-\epsilon'}$$

Then we know that if we take ϵ' to be small enough, we have $\frac{\epsilon'}{1-\epsilon'} < \epsilon$. Conversely, if we suppose that:

$$\sum_n 2^{-n} \frac{|f-g|_{K_n}}{1+|f-g|_{K_n}} < \epsilon$$

Then we know that there is at least one N such that $|f-g|_{K_N} < \epsilon$, then we know that we can just take $B_{K_N,\epsilon}$.

This metric also makes $\mathcal{H}(\Omega)$ complete. Indeed, if we assume that $\{f_n\}$ is a Cauchy sequence, which means that for any 2^{-2M} there is N such that all $m, n > N$ we have:

$$d(f_n - f_m) = \sum_{i=1}^{\infty} 2^{-i} \frac{|f_n - f_m|_{K_i}}{1+|f_n - f_m|_{K_i}} < 2^{-M}$$

Then this implies that:

$$\sum_i 2^{M-i} \frac{|f_n - f_m|_{K_i}}{1+|f_n - f_m|_{K_i}} < 2^{-M}, \quad \forall m, n$$

This implies that we must have $\frac{|f_n-f_m|_{K_M}}{1+|f_n-f_m|_{K_M}} < 2^{-M}$ which roughly means that $|f_n - f_m|_{K_M} < 2^{-M}$. Now if K is a compact subset of Ω, we know that when we take M to be large enough $K \subset K_M \subset K_{M+1} \subset \cdots$ and we have:

$$|f_m - f_n|_K < 2^{-M}$$

Therefore we know that for any $\epsilon > 0$, there is a N such that for all $m, n > N$, we have:

$$|f_m - f_n|_K < \epsilon$$

Therefore we know that f_n converges uniformly on each compact subset of Ω, which means that the limit function is also holomorphic. Therefore we know that this is indeed a complete metric space.

The above topology on $\mathcal{H}(\Omega)$ is often called the topology of uniform convergence on compact sets. It's called this name because if we consider a sequence of holomorphic functions $\{f_n\}$, then it converges to an element in $\mathcal{H}(\Omega)$ iff this sequence converges uniformly on any compact subset of Ω. I will let you to fill in the detials, and it shouldn't be too hard given our above discussion.

Now let K be a compact subset of $\mathbb{C}$, we use $\mathcal{O}(K)$ to denote the space of continuous functions ϕ on K such that there is an open subset U of $\mathbb{C}$ which contains K and a function $f \in \mathcal{H}(U)$ with $f|_K = \phi$. That is to say that f is the restriction of some holomorphic function on a larger open subset.

Finally, we will use $C(K)$ to denote the space of all continuous functions on K with the norm $||\phi|| = |\phi|_K$. Then this makes $C(K)$ a Banach-space over $\mathbb{C}$ (since uniform limits of continuous functions on K is still continuous). We will consider $\mathcal{O}(K)$ as a subspace of $C(K)$ with the same norm. However, it's usually not closed.

Finally, if Ω is open in $\mathbb{C}$ and K is compact in Ω, we denote by $\rho = \rho_{\Omega,K}$ the restriction map:

$$\rho : \mathcal{H}(\Omega) \to \mathcal{O}(K), \quad \rho(f) = f|_K$$

Then we have the first form of Runge's Theorem:

Theorem 2.1. (Runge's Theorem: First Form) Let Ω be open in $\mathbb{C}$ and let K be a compact subset of Ω, then the following statements are equivalent:

1. $\rho(\mathcal{H}(\Omega))$ is dense in $\mathcal{O}(K)$.
2. No connected components of $\Omega \backslash K$ is relatively compact in Ω.
3. For any $a \in \Omega$ and $a \notin K$, there exists a $f \in \mathcal{H}(\Omega)$ with:

$$|f(a)| > |f|_K$$

Before we do the proof, it's good to have a feeling about what exactly is relatively compact. Recall that given any topological space X, a subset A is relatively compact in X if the closure of A in X is compact. And we usually write it as $A \Subset X$. In our case, if Ω is an open subset of $\mathbb{C}$ and A is a subset, we know that the closure of A in Ω is just the intersection of the closure of A in $\mathbb{C}$ with Ω. This implies that if $\overline{A}$ is not contained in Ω, then A is not relatively compact. We must have $\overline{A}$ is compact and contained in Ω to make sure A is relatively compact. The following two pictures might help:

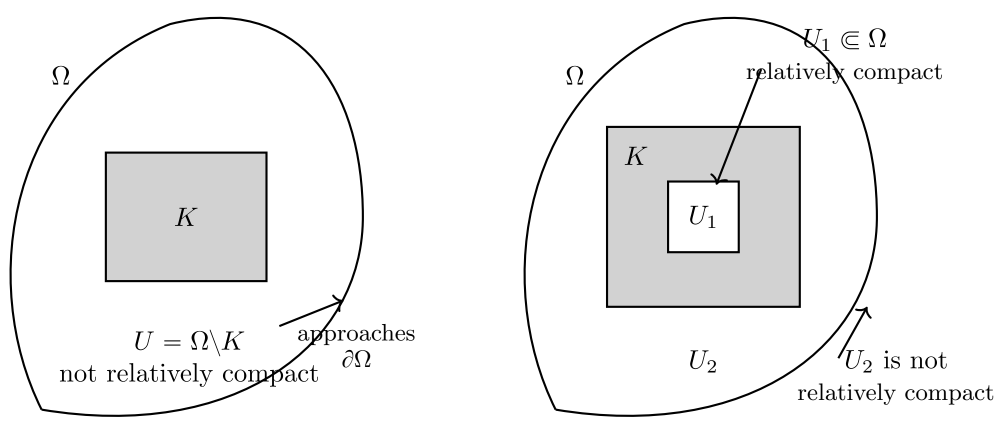

Figure 1: Connected components of $\Omega\backslash K$. On the left, K has no hole. On the right, the hole in K produces a relatively compact connected component U_1.

Proof. Let first deal with $1 \Rightarrow 2$: Recall that a set in Ω is relatively compact if it's closure is compact. Suppose that $\Omega\backslash K$ has a component U which is relatively compact. let's prove that $\partial U \subset K$. To see why, let $a \in \partial U$ and for the sake of contradiction, we assume $a \notin K$. Since it's not in K, there is a disk D around of a such that $D \subset \Omega \backslash K$ and since a is in the boundary of U, we know that there is a point in U such that it's also in the disk. Consequently, $D \cup U$ is connected, but U is already the connected component, therefore $U \subset U$, which means that $D \cap K = \varnothing$, therefore $a \notin \partial U$.

Now let $z_0 \in U$ and let $f(z) = \frac{1}{(z-z_0)}$ which is holomorphic on $\Omega \backslash \{z_0\} \supset K$, which implies that $f||_K \in \mathcal{O}(K)$, then suppose that 1 is true for the sake of contradiction, then there will be a sequence of function $\{f_n\}$ in $\mathcal{H}(\Omega)$ such that after you restrict them to K they converges uniformly to $f|_K$. However, by the maximum principle, we know:

$$\sup_{\overline{U}} |f_n - f_m| = \sup_{\partial U} |f_n - f_m| \leqslant \sup_K |f_n - f_m|$$

However, we already know that on K, f_n uniformly converges to $f|_K$, therefore $\sup_K |f_n - f_m|$ goes to 0 if m, n goes to infinity, therefore we know that f_n also converges uniformly on $\overline{U}$, to a holomorphic function $g \in \mathcal{H}(U)$. Note that we are using the assumption that U is relatively compact to apply the maximum principle which is only valid if the open set is relatively compact.

Now think about $(z - z_0)f_n$, which converges to 1 uniformly on $\partial U \subset K$, then again by the maximum principle we know that $(z - z_0)f_n$ converges to 1 uniformly on U. This implies that:

$$(z - z_0)g(z) = 1, \forall z \in U$$

But this absurd since we know for $z = z_0$, the function is 0. This contradiction tells you $1 \Rightarrow 2$.

Then we deal with $2 \Rightarrow 1$. Let $\rho(\mathcal{H}(\Omega))$ and we know $E \subset \mathcal{O}(K) \subset C(K)$. Then by the Hahn-Banach theorem applied to $C(K)$, we know that E is dense in $\mathcal{O}(K)$ iff the following holds:

Let λ be a continuous linear functional on $C(K)$ then if $\lambda|_E = 0$, we have $\lambda|_{\mathcal{O}(K)} = 0$. For those that are not too familiar with functional analysis, we state the version of Hahn-Banach we are using here.

Theorem 2.2. (Hahn-Banach Theorem) Suppose that X is a normed vector space (not necessarily complete) over $\mathbb{C}$ with Y a subspace. If $f : Y \to \mathbb{C}$ is a continuous linear functional then f extends to a continuous functional on X.

One consequence of this theorem is what we are using above in the proof:

Corollary 2.3. Let M be a subspace of a normed vector space X over $\mathbb{C}$ and let x_0 be a point in X not in $\overline{M}$. Then there is a continuous linear functional $\Lambda \in X^*$ such that:

$$\Lambda|_M = 0, \quad \Lambda(x_0) = 1$$

The way you prove this corollary using Hahn-Banach is to define a linear functional $f : \overline{M} + \mathbb{C}x_0$ by $f(m + ax_0) = a$, then you check that this is a continuous functional on $\overline{M} + \mathbb{C}x_0$ then use Hahn-Banach to extend to to X. Then we go back to the proof:

This corollary above says that if every continuous linear function λ defined on $C(K)$ which is zero on E is also 0 on $\mathcal{O}(K)$, then if $\overline{E}$ is not $\mathcal{O}(K)$, then we can create a counter-example using the above corollary, which is a contradiction.

Now suppose that λ is a continuous linear functional on $C(K)$, we can define a function:

$$\Phi_\lambda(w) = \lambda(\phi_w)$$

where $\phi_w = \frac{1}{z-\omega}$ and $z \in K$. I claim that Φ_λ is in face in $\mathcal{H}(\mathbb{C}\backslash K)$ and $\Phi_\lambda^{(n)}(w) = n!\lambda(\phi_{w,n})$ where $\phi_{w,n} = \frac{1}{(z-w)^{n+1}}$. To see why, for any $a \notin K$, we can choose $r > 0$ such that $\overline{D(a,r)} \cap K =$, therefore we know that:

$$\phi_w(z) = \frac{1}{z-\omega} = \frac{1}{z-a}\frac{1}{1-\frac{w-a}{z-a}}$$

We know that for $|\omega - a| < r$ and $z \in K$, the series:

$$\sum_{n=0} \frac{(w-a)^n}{(z-a)^n}$$

converges uniformly for all $z \in K$, then we know that the following series converges uniformly to $\phi_w(z)$ for $|w-a| > r$:

$$\phi_w(z) = \sum_n \frac{(w-a)^n}{(z-a)^{n+1}}$$

Then we know that since λ is continuous and $\frac{1}{(z-a)^{n+1}}$ is in $C(K)$ since $a \notin K$, then we know:

$$\lambda(\phi_w(z)) = \sum_n (w-a)^n \lambda((z-a)^{-n-1})$$

Therefore we know that at least in $|w-a| < r$, we have a power series representation:

$$\Phi_\lambda(w) = \sum_n (w-a)^n \lambda(\phi_{a,n})$$

Then we know that $\Phi_\lambda \in D(a,r)$ with $\Phi^{(n)}(a) = n!\lambda(\phi_{a,n})$. Since this holds for each $a \notin K$, we know that $\Phi_\lambda \in \mathcal{H}(\mathbb{C}\backslash K)$.

Suppose that λ is a linear functional on $C(E)$ with $\lambda|_E = 0$. I claim this implies that $\Phi_\lambda = 0$ on $\mathbb{C}\backslash K$ if $\Omega\backslash K$ has no connected component that is relatively compact in Ω. To show this, let U be a connected component of $\mathbb{C}\backslash K$. Let's discuss this case by case:

1. Suppose that first U is unbounded. Choose $R > 0$ such that $K \subset D(0,R)$ and let $w \in U$ with $|w| > R$, since U is unbounded, we can choose such a w.s Now consider $\phi_w(z) = \frac{1}{z-w}$ then we know that this is:

   $$\phi_w(z) = \frac{1}{w}\frac{1}{\frac{z}{w}-1} = \frac{-1}{w}\sum_{n=0}^{\infty}\frac{z^n}{w^n} = -\sum_{n=0}^{\infty}\frac{z^n}{w^{n+1}}$$

   Notice that for all $z \in K$, this series converges uniformly to $\phi_w(z)$.

   Then by uniform convergence, for $z \in K$:

   $$\lambda(\phi_w(z)) = -\sum_n^{\infty}\frac{1}{w^{n+1}}\lambda(z^n)$$

   Notice that $z^n \in C(K)$ is in E since z^n is holomorphic on Ω. Then by our assumption, we know that $\lambda(z^n) = 0$, therefore:

   $$\Phi_\lambda(w) = \lambda(\phi_w(z)) = 0, \quad |w| > R$$

   Then we know that $\Phi_\lambda = 0$ on $U \cap \{w : |w| > R\}$. Since U is unbounded, this is a nonempty open subset. Since U is connected, we know that Φ_λ is 0 on a nonempty subset of a connected subset U, then the analytic continuation implies $\Phi_\lambda = 0$ on U.

2. The let's see what if U is bounded. We claim that U is not contained in Ω. In face, since we know that $\partial U \subset K$ which is given in the proof of $1 \Rightarrow 2$. Hence if $U \subset \Omega$ we know that $\overline{U} \subset \Omega$. Moreover, we know that if $U \subset \Omega$, then we know that since U is disjoint with K, then it has to be a connected component of $\Omega\backslash K$ as U is even a connected component of $\mathbb{C}\backslash K$. Therefore we know $\overline{U}$ is compact and is contained in Ω and since U is a connected component of $\Omega\backslash K$, we know that it's in fact relatively compact in Ω. This contradicts to the hypothesis. Thus U is not contained in Ω.

   Now let $a \in U$ which is not in Ω, we have for $n \geqslant 0$:

   $$\Phi_\lambda^{(n)}(a) = n!\lambda(\phi_{a,n}) = 0$$

   This is because that for $a \notin \Omega$, we know that:

   $$\phi_{a,n} = \frac{1}{(z-a)^{n+1}}$$

   is in $\mathcal{H}(\Omega)$. Therefore we know that $\phi_{a,n} \in E$, therefore $\lambda(\phi_{a,n}) = 0$.

   However, if Φ is holomorphic on $U \subset \mathbb{C}\backslash K$ then it's holomorphic on U, which implies that if all of it's derivative is 0 at $a \in U$ with U connected, then $\Phi|_U$ is 0.

This case by case discussion shows that 2 implies that Φ is 0 on $\mathbb{C}\backslash K$.

Then let $f \in \mathcal{O}(K)$ and let W be an bounded open set containing K and let $F \in \mathcal{H}(W)$ such a holomorphic function such that $F|_K = f$. Let's choose $\alpha \in C_0^\infty(W)$ such that $\alpha = 1$ on a neighborhood W_0 of K, then using the inhomogeneous Cauchy's integral formula, we have:

$$f(w) = F(w) = \frac{-1}{\pi} \iint\limits_W \frac{\partial \alpha}{\partial \overline{z}} F(z) \frac{1}{z-w} dxdy, \quad \forall w \in K$$

Then since we assumed that α is constant on W_0, we know that the above integral can be further written as:

$$f(w) = F(w) = \frac{-1}{\pi} \iint\limits_{W\backslash W_0} \frac{\partial \alpha}{\partial \overline{z}} F(z) \frac{1}{z-w} dxdy, \quad \forall w \in K$$

Now if we take $\{R_\nu\}$ to be a finite set of rectangles whose interiors are disjoint which cover $\mathrm{supp}(\alpha)\backslash W_0$, and if ζ_ν is a point in $R_\nu \cap \mathrm{supp}(\alpha)$, then we know that as the maximum of the diameter tends to 0, the sum:

$$\frac{-1}{\pi} \sum_\nu \frac{\partial \alpha}{\partial \overline{z}}(\zeta_\nu) F(\zeta_\nu) \frac{1}{\zeta_\nu - w} \mathrm{m}(R_\nu)$$

is going to converge to the integral where R_ν is the planar Lebesgue measure. Notice that this convergence is uniform in each $w \in K$.

Indeed, let's compute what is the difference:

$$\frac{1}{\pi} | \iint\limits_{W\backslash W_0} \frac{\partial \alpha}{\partial \overline{z}} F(z) \frac{1}{z-w} dxdy - \sum_\nu \frac{\partial \alpha}{\partial \overline{z}}(\zeta_\nu) F(\zeta_\nu) \frac{1}{\zeta_\nu - w} \mathrm{m}(R_\nu)|$$

Notice that for each ν, we know that:

$$\frac{\partial \alpha}{\partial \overline{z}}(\zeta_\nu) F(\zeta_\nu) \frac{1}{\zeta_\nu - w} \mathrm{m}(R_\nu) = \iint\limits_{R_\nu} \frac{\partial \alpha}{\partial \overline{z}}(\zeta_\nu) F(\zeta_\nu) \frac{1}{\zeta_\nu - w} dxdy$$

On the other hand, since R_ν covers all of $\mathrm{support}(\alpha)\backslash W_0$ such that $R_\nu \cap (\mathrm{support}(\alpha)\backslash W_0) \neq \varnothing$, we know that:

$$\iint\limits_{W\backslash W_0} \frac{\partial \alpha}{\partial \overline{z}} F(z) \frac{1}{z-w} dxdy = \sum_\nu \iint\limits_{R_\nu} \frac{\partial \alpha}{\partial \overline{z}} F(z) \frac{1}{z-w} dxdy$$

Therefore the absolute value of the difference is at most:

$$\frac{1}{\pi} \sum_\nu \iint\limits_{R_\nu} | \frac{\partial \alpha}{\partial \overline{z}} F(z) \frac{1}{z-w} - \frac{\partial \alpha}{\partial \overline{z}}(\zeta_\nu) F(\zeta_\nu) \frac{1}{\zeta_\nu - w})| dxdy$$

Now we know that since K is compact and $\mathrm{supp}\alpha\backslash W_0$ is also compact, then the distance between K and $\mathrm{supp}\alpha\backslash W_0$ is positive, that is to say that there is some $\delta > 0$ such that for all $x \in K$, $y \in \mathrm{supp}\alpha\backslash W_0$, $d(x,y) \geqslant \delta$. Then we know that if we set ν to be small enough, then we can assume that for all $x \in R_\nu$ and $w \in K$, the distance is at least positive value δ. Then if we consider the function:

$$g(z,w) = \frac{\partial \alpha}{\partial \overline{z}} F(z) \frac{1}{z-w}$$

We know that once we choose R_ν to be small enough, we can assume that each R_ν is contained in a closed subset S of $W\backslash W_0$ such that S is still has positive difference to K. Then we know that the function $g(z,w)$ is uniformly continuous on S, then we know that for any $\epsilon > 0$, we can find small enough R_ν such that:

$$| \frac{\partial \alpha}{\partial \overline{z}} F(z) \frac{1}{z-w} - \frac{\partial \alpha}{\partial \overline{z}}(\zeta_\nu) F(\zeta_\nu) \frac{1}{\zeta_\nu - w})| < \epsilon, \quad \forall z \in R_\nu, w \in K$$

Then we know that:
$$\iint_{R_\nu} |\frac{\partial\alpha}{\partial\overline{z}}F(z)\frac{1}{z-w} - \frac{\partial\alpha}{\partial\overline{z}}(\zeta_\nu)F(\zeta_\nu)\frac{1}{\zeta_\nu - w})|dxdy < \epsilon\mathrm{R}_\nu$$
Therefore the summation of all the integrations becomes at most:
$$\frac{1}{\pi}\sum_\nu \epsilon\mathrm{m}(R_\nu) \leqslant \frac{\epsilon}{\pi}\mathrm{m}(S) \to 0$$
Here we know that S is a compact subset thus it's Lebesgue measure is finite. This shows our claim of uniform convergence for all $w \in K$.

This uniform continuity implies if we define a function:
$$g_\nu(w) = \frac{-1}{\pi}\sum_\nu \frac{\partial\alpha}{\partial\overline{z}}(\zeta_\nu)F(\zeta_\nu)\frac{1}{\zeta_\nu - w}\mathrm{m}(R_\nu)$$
We know that it converges to $f(w)$ uniformly for $w \in K$, or in other words in $C(K)$, therefore we know that:
$$\lambda(f) = \lim_\nu \frac{-1}{\pi}\sum_\nu \frac{\partial\alpha}{\partial\overline{z}}(\zeta_\nu)F(\zeta_\nu)\lambda(\frac{1}{\zeta_\nu - w})\mathrm{m}(R_\nu)$$
Now we notice that the function defined on S by:
$$z \mapsto \lambda(w \mapsto \frac{1}{z-w})$$
Is exactly the function Φ we define before, which is continuous on S, then we find the limit of the Riemann sum above is just:
$$\iint_S \frac{\partial\alpha}{\partial\overline{z}}F(z)\Phi(z)dz$$
However, we already know that for $z \in W\backslash W_0 \subset \mathbb{C}\backslash K$, Φ is 0, therefore the integral is 0, therefore $\lambda(f) = 0$, we are done. This proves that if $\lambda|_E = 0$ then $\lambda|_{\mathcal{O}(K)} = 0$, thus we are done with $2 \Rightarrow 1$.

Then we deal with $3 \Rightarrow 2$, and $2 \Rightarrow 3$ which are easier. For $3 \Rightarrow 2$, let U be a connected component which is relatively compact in Ω, then again we know that $\partial U \subset K$, then using the maximum principle, since U is relatively compact, then for $a \in U$ and all $f \in \mathcal{H}(\Omega)$:
$$|f(a)| \leqslant \sup_{z\in\partial U}|f(z)| \leqslant |f|_K$$
which contradicts 3.

To show $2 \Rightarrow 3$, let $\Omega\backslash K = \cup_\nu U_\nu$ be the decomposition of $\Omega\backslash K$ into connected components. By assumption, we know that no $\overline{U_\nu}$ is compact contained in Ω.

Now let $a \in \Omega$ and $a \notin K$. Let μ be the value of ν for which $a \in U_\nu$. Then let $L = K \cup \{a\}$. Then we know:
$$\Omega\backslash L = \cup_{\nu\neq\mu}U_\nu \cup U_\mu\backslash\{a\}$$
Now since the closure of $U_\mu\backslash\{a\}$ is the same as that of U_μ, thereofre we know that L is again compact and the complement has no connected component that is relatively compact in Ω. Then since 2 is equivalent to 1, we know that:
$$\{f|_L : f \in \mathcal{H}(\Omega)\}$$
is dense in $\mathcal{O}(L)$. Now the function ϕ defined by $\phi = 0$ on K and $\phi(a) = 1$ is clearly $\mathcal{O}(L)$ since $a \notin K$. Then there is a $f \in \mathcal{H}(\Omega)$ such that:
$$|f - \phi|_K < \frac{1}{2}$$
which means that:
$$|f(a) - 1| < \frac{1}{2}$$

This means that $|f(a)| > \frac{1}{2}$ but clearly for $z \in K$, we know that:

$$|f(z)| < \frac{1}{2}$$

Therefore:

$$|f(a)| > 1/2 > |f|_K$$

Thus f is the function we desire, thus we are done with $2 \Rightarrow 3$. $\square$

Remark 2.4. Let's make a remark about the above proof and in fact the picture we gave above will be even more instructive with a small modification:

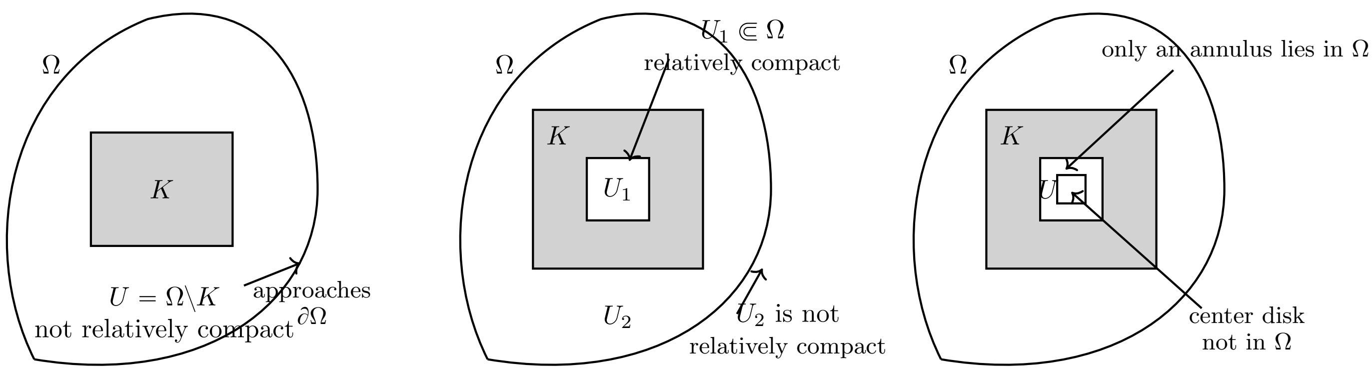

Figure 2: The third picture is different.

Notice that the first two pictures are still the same with what you saw above. However, if we declare that in the third picture, the hole in the center of Ω is not entirely in Ω, only the annulus is in Ω while the closed square in the center is not in Ω, then this time, the annulus is still not relatively compact as if you take the closure, the boundary in the inside is not going to lie in Ω. And in this case, our intuition doesn't call the connected component on the inner side a "hole" since it's just an annulus.

This sort of picture is actually very natural and the generally patter for when you can approximate functions in $\mathcal{O}(K)$ uniformly by restrictions of holomorphic functions of Ω is determined by if there is a "hole" after you take out K. Indeed, if there is a hole, you never can approximate uniformly with a very simple counter example:

$$\frac{1}{z-w}$$

where w is the point at the center of the hole. Indeed, if you can approximate this function by holomorphic functions f_n uniformly, we can integrate along the inner boundary:

$$0 = \lim_n \int_{\partial K} f_n dz \to \int_{\partial K} \frac{1}{z-w} dz = 2\pi i$$

this is a contradiction. I will let you think about why this issue doesn't exist for the third picture with the annulus.

This being said what the above proof shows that in fact $\frac{1}{z-w}$ is the key obstruction here for the uniform approximation.

This version of Runge's theorem is already enough to give the general solution of the equation $\frac{\partial u}{\partial \bar{z}} = f$. However, to give other versions of Runge's theorem, we still need to learn more about some topology. These will also help our later study.

3 Some Topological Results

In this section we will prove some topological results that are useful for other versions of Runge's Theorem.

Proposition 3.1. Let X be a locally compact Hausdorff space and let K be a connected component of X which is compact. Then K has a fundamental system of neighborhoods which are both open and closed in X.

Proof. See [NN01]. □

Proposition 3.2. Let Y be a locally compact Hausdorff space, let X be a closed subset of Y and let K be a connected component of X which is compact. Then there exists a fundamental system of neighborhoods U of K in Y such that:

$$(\partial U) \cap X = \varnothing$$

where (∂U) is the boundary of U in Y.

Proof. See [NN01]. □

The following definition fixes the possible "holes" $\Omega \backslash K$ might have:

Definition 3.3. Let Ω be an open subset in $\mathbb{C}$ and let A be a subset of $\mathbb{C}$, we define the **topological haul** of A with respect to Ω to be the union of A with the components of $\Omega \backslash A$ which are relatively compact in Ω. We will denote it by:

$$\hat{A} = \hat{A}_{\Omega}$$

We use the subscript if there are confusions about with respect to which open set.

Proposition 3.4. Let Ω be an open set in $\mathbb{C}$ and let K be a compact subset of Ω. Then $\hat{K}$ is still compact and there are only finitely many connected component of $\Omega \backslash \hat{K}$ and non of them is relatively compact.

Proof. See [NN01]. □

Below are some useful properties of the topological haul:

Proposition 3.5. If A, B are both subsets of Ω with $A \subset B$, then:

$$\hat{A} \subset \hat{B}$$

Proof. If $z \notin \hat{B}$, we know that z is in some component of $\Omega \backslash B$ which is not relatively compact in Ω. If z is in $\hat{A}$, then we know that z is either in A or is in some component of $\Omega \backslash A$ which is relatively compact. Since z is not in B, we know that z is not in A, then if V is a component of $\Omega \backslash A$ which is relatively compact, we know that since $z \in W$ and $z \in V$, then we know that $W \subset V$, however, this implies W is relatively compact. □

Proposition 3.6. If A, B are both compact subsets of Ω with $A \subset \mathring{B}$, then:

$$\hat{A} \subset \mathring{\hat{B}}$$

Proof. There are two cases: if $z \in A$, then we know that $z \in \mathring{B}$. Since $B \subset \hat{B}$, we know that:

$$\mathring{B} \subset \mathring{\hat{B}}$$

This implies $x \in \mathring{\hat{B}}$.

If $z \notin A$, we know that is in some component U of $\Omega \backslash A$ which is relatively compact. Since we are in $\mathbb{C}$, we know that U is also open, then we have:

$$U \subset \hat{A} \subset \hat{B}$$

This shows that $z \in U$ which means that z is in the interior of $\hat{B}$. □

Proposition 3.7. Let Ω be an open subset of $\mathbb{C}$, then there is a sequence of compact subset K_n such that:

1. $K_n \subset \mathring{K}_{n+1}$.
2. $\bigcup_n K_n = \Omega$.
3. $K_n = \hat{K_n}$.

Proof. It's standard to construct a sequence of K'_n with the first two conditions. Then we take $K_n = \hat{K'_n}$. □

4 Other Versions of Runge's Theorem

With the topological results we discussed in the previous section. Now we can proceed with the second form of Runge's theorem:

Definition 4.1. Let Ω, Ω' be open in $\mathbb{C}$ with $\Omega' \subset \Omega$. We call (Ω, Ω') a **Runge pair** if the restriction map:

$$r^{\Omega}_{\Omega'} : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega')$$

has dense image. We also say that Ω' is Runge in Ω in this case.

Theorem 4.2. (Runge's Theorem: Second Form) Let Ω, Ω' be open sets in $\mathbb{C}$, then (Ω, Ω') is a Runge pair iff no connected component of $\Omega \backslash \Omega'$ is compact.

Proof. Let's first assume there is no compact connected components of $\Omega \backslash \Omega'$, let's show (Ω, Ω') **** to be completed* ☐

Theorem 4.3. (Runge's Theorem: Classical Form) Let Ω be open in $\mathbb{C}$ and let $\mathbb{C} \backslash \Omega = \bigcup_{\alpha \in A} C_\alpha$ be the decomposition into connected components. The $A' \subset A$ be set where $A' = \{\alpha \in A : C_\alpha \text{ is compact.}\}$, and for each $\alpha \in A'$, choose a point $a_\alpha \in C_\alpha$.

Then for any $f \in \mathcal{H}(\Omega)$, it can be approximated uniformly on any compact subset of Ω by rational functions all of whose poles are contained in the set $\{a_\alpha\}_{\alpha \in A'}$.

Proof. **** to be completed* ☐

Corollary 4.4. Let Ω be open in $\mathbb{C}$. Then the set of restrictions to Ω of polynomials in z is dense in $\mathcal{H}(\Omega)$ iff $\mathbb{C} \backslash \Omega$ has no compact connected component.

Proof. **** to be completed* ☐

5 General Solutions of $\frac{\partial u}{\partial \bar{z}} = f$

Using the above discussion of Runge's theorem, we now solve the d-bar equation.

Theorem 5.1. (Grothendieck) Let Ω be an open subset of $\mathbb{C}$ and let $f \in C^\infty(\Omega)$. Then there is a $u \in C^\infty(\Omega)$ such that:

$$\frac{\partial u}{\partial \bar{z}} = f$$

Remark 5.2. It's very interesting to see that this result is actually due to Grothendieck.

Proof. Let $\{K_n\}$ be a sequence of compact subsets of Ω such that $K_n \subset \mathring{K}_{n+1}$ $\cup K_n = \Omega$ and $K_n = \hat{K_n}$.

We know that for each n we consider K_n. We can choose a $\alpha_n \in C_0^\infty(\Omega)$ such that it's 1 in a neighborhood of K_n, then we consider g_n defined by $\alpha_n(z)f(z)$ if $z \in \Omega$ and 0 if otherwise. Then we actually see that now $g_n(z)$ is in $C_0^\infty(\mathbb{C})$ and thus we have u_n such that:

$$\frac{\partial u_n}{\partial \bar{z}} = g_n(z)$$

Notice that since $g_n(z) = f(z)$ on a neighborhood U_n of K_n, we know that if we are thinking of u_n as a smooth function on Ω, then it solves the d-bar equation in a neighborhood of K_n. Similarly, we have u_{n+1} solving the equation in a neighborhood U_{n+1} of K_{n+1}, therefore we know that on U_n:

$$\frac{\partial}{\partial \bar{z}}(u_{n+1} - u_n) = f - f = 0$$

Therefore we know that $u_{n+1} - u_n$ is actually in $\mathcal{H}(U_n)$ and we know that it's in $\mathcal{O}(K_n)$. Then by the first version of Runge's theorem we have h_n holomorphic in Ω such that:

$$|u_{n+1} - u_n - h_n|_K < \frac{1}{2^n}, \quad \forall n \geqslant 1$$

Now consider:

$$U_n = u_n + \sum_{l=n}^{\infty}(u_{l+1} - u_l - h_l) - h_1 - \cdots - h_{n-1}$$

Notice that the series converges uniformly on K_n, therefore we know that at least for points on K_n, the function is well defined. Similarly, we can define $U_{n+1}, \cdots$ and so on.

However, notice that this definition actually doesn't depend on n since for example:

$$U_n = u_n + \sum_{l=n}^{\infty}(u_{l+1} - u_l - h_l) - h_1 - \cdots - h_{n-1}$$

$$= u_n + (u_{n+1} - u_n - h_n) + \sum_{l \geqslant n+1}(u_{l+1} - u_l - h_l) - h_1 - \cdots - h_{n-1} = u_{n+1} + \sum_{l \geqslant n+1}(u_{l+1} - u_l - h_l) - h_1 - \cdots - h_n$$

Note that the last term is actually U_{n+1}. Then we know that we can just call this function U and it's well defined on every K_n, thus on Ω.

Moreover, we know that, for each n, this series converges uniformly on K_n, therefore we know $U - u_n$ is actually holomorphic on $\mathring{K}_n$. Notice that since each u_n and h_n are smooth on Ω and the limit is uniform, the limit function U is also smooth on Ω. Then we know that:

$$\frac{\partial U - u_n}{\partial \overline{z}} = 0 \Rightarrow \frac{\partial U}{\partial \overline{z}} = \frac{\partial u_n}{\partial \overline{z}} = f$$

Take $u = U$, we are done. □

References

[NN01] Raghavan Narasimhan and Yves Nievergelt, *Complex analysis in one variable*, 2 ed., Birkhäuser, Boston, 2001.