Notes on Etale cohomology: 1

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In this whole notes series, we will basically be following Lei Fu's book on *é*tale cohomology theory which covers a large part of Grothendieck's SGA. For the notations, it will be consistent with the note series on Vakil's FOAG, so we don't specify them here.

This is the first chapter of the notes, and it basically deals with Descent Theory, which is the content in SGA 1 Chapter 4. Before we head to the real topic on descent theory, let's spend some time discuss some basic facts that we will need. We start with the notion of flatness. At first sight, this is just a notion related to the exactness of tensors. However, it turns out is just the right assumption we have to make if we want some nice geometric property to hold. For example, it's a natural question to ask what is the right assumption such that $SpecA \to SpecB$ maps the generic points of irreducible components to generics of irreducible components, or in the algebraic point of view, the inverse image of minimal primes are still minimal primes. It turns out that a lot of these good properties are facilitated by the notion of (faithfully) flatness.

1 Flat Modules

Let A be a ring, we say an A-module M is flat if the function $-\otimes_A M$ is exact on the category of A-modules. We also say that M is flat over A or M is A-flat. We will now state a few propositions on the basic properties of flat modules without proof, they are just meant to be a reminder:

Proposition 1.1. If $A \to B$ is a ring map, and M is flat over A, then $B \otimes_A M$ is flat over B.

The slogan here is: **Extension of flat modules over any ring map is flat.**

Proposition 1.2. If $A \to B$ is a ring map such that B is flat over A via this map, and N is flat over B, then N is flat over A.

The slogan here is: **Restriction of flat modules via a flat ring map is flat.**
The following are some equivalent characterizations of flat modules:

Proposition 1.3. Suppose A is a ring and M is an A-module, then the following are equivalent:

1. M is flat over A.
2. For any A-module N, $\mathrm{Tor}_i^A(M,N) = 0$ for $i > 0$.
3. For any finitely generated A-module N, $\mathrm{Tor}_i^A(M,N) = 0$ for $i > 0$.
4. For any A-module N, $\mathrm{Tor}_1^A(M,N) = 0$.
5. For any finitely generated A-module N, $\mathrm{Tor}_i^A(M,N) = 0$.
6. For any ideal I of A, $\mathrm{Tor}_1^A(M,A/I) = 0$.
7. For any finitely generated ideal I, $\mathrm{Tor}_1^A(M,A/I) = 0$.
8. For any ideal I of A, the canonical map:

$$I \otimes M \to M$$

is injective.

9. For any finitely generated ideal I of A, the canonical map:

$$I \otimes M \to M$$

is injective.

Recall that if M is flat over A, N is any A-module with submodules N', N'', since $-\otimes_A M$ is exact, it preserves injective maps, therefore we can view $M \otimes_A N', M \otimes_A N''$ as submodules of $M \otimes_A N$. We have the following two canonical isomorphisms:

Proposition 1.4.

$$M \otimes_A (N' \cap N'') \simeq (M \otimes_A N') \cap (M \otimes_A N'')$$
$$M \otimes_A (N' + N'') \simeq (M \otimes_A N') + (M \otimes_A N'')$$

where the intersection and summation is done inside $M \otimes_A N$.

The slogan here is: **For flat modules, tensor commutes with intersection and summation.** The proof of the second is quite easy and for the proof of the first isomorphism, we need to consider the following SES that express the intersection as a kernel:

$$0 \to N \cap N'' \to N' \oplus N'' \to N \to 0$$

I will let you figure out the details.

Proposition 1.5. 1. Let A be a ring and S be a multiplicative set, $S^{-1}A$ is flat over A and if M is flat over A, $S^{-1}M$ is flat over $S^{-1}A$.

2. Let $A \to B$ be a map of rings, let $S^{-1}A \to T^{-1}B$ be the induced map of localized rings, if N is a B-module such that it's flat over A, then $T^{-1}N$ is flat as A-module and then as an $S^{-1}A$-module.

3. Let $A \to B$ be a ring map and N be a B module, suppose that for every maximal ideal n of B N_n is flat over A, then N is flat over A.

Proof. The first assertion is our slogan that extension of a flat module over any ring map is flat.

The second assertion is checked by:

$$T^{-1}N \otimes_A M \simeq T^{-1}B \otimes_B (N \otimes_A M)$$

Then you see that this is just a composition of two exact functors, thus exact. Then by assertion 1, we know that $S^{-1}T^{-1}N$ is flat as an $S^{-1}A$-module. But $S^{-1}T^{-1}N \simeq T^{-1}N$ as we know that inverting S^{-1} after we invert T^{-1} is inverting something we already inverted, thus is doing nothing.

The last assertion is basically the statement that exactness can be checked at only maximal ideals. If we want to check the following sequence is exact:

$$0 \to M' \otimes_A N \to M \otimes_A N \to M'' \otimes_A N \to 0$$

Notice that this is actually a sequence of B-module maps, therefore to check it's exact, we only need to check it on each maximal ideal of B, then notice that $(M \otimes_A N)_n \simeq M \otimes_A (N_n)$, we are done. □

Remark 1.6. The second assertion in the above proposition has important geometric consequences. As the ring map between localizations $S^{-1}A \to T^{-1}B$ are always given by localize at a prime. For example for any prime q of B, we can consider it's inverse image p, then we have an induced map $A_p \to B_q$. Then we know that if N is a B-module that is flat over A, N_q is flat as both an A and A_p module. We will use this later.

Remark 1.7. One special case of the last assertion is $id : A \to A$, then it implies that to prove an A-module M is flat, it suffices to prove M_m is flat over A_m for all maximal ideal m in A.

Proposition 1.8. 1. If A is a ring and M is a flat A-module, if a is not a zerodivisor of A (not of M), then the canonical map:

$$\times a : M \to M$$

is injective. In particular, if A is an integral domain and M is flat, then M is torsion free.

2. Let A be an integral domain such that A_m is a DVR for every maximal ideal m, then an A-module if flat iff it's torsion free.

Proof. We need to be careful with the first assertion, as it states that a is not a zero divisor of A, not of M, therefore we don't directly get $\times a : M \to M$ is injective, the only thing we can conclude is:

$$\times a : A \to A$$

is injective, but recall that we can tensor with M and still get injective, we are done.

For the second assertion, if M is flat, then the first assertion shows that it has no torsion. Now suppose that it has no torsion, we want to prove it's flat, recall that it suffice to prove M_m if flat over A_m for every maximal ideal m. Let I be a nonzero (if it's zero then there is nothing to prove as $I \otimes M_m = 0$) ideal of A_m, since A_m is a DVR, I is principle, say it's generated by some element $r \in A$, then we have a canonical isomorphism of A_m-modules:

$$A_m \to I \qquad a \mapsto ar$$

This is obviously surjective, it's also injective because A_m is an integral domian an r is not 0. So we have an isomorphism:

$$M_m \simeq I \otimes_{A_m} M_m \qquad m \mapsto r \otimes x$$

Then we want to prove that:

$$I \otimes_{A_m} M_m \to M_m$$

is injective, but this is:

$$M_m \to M_m \qquad x \mapsto rx$$

this is injective as we assumed that M has no torsion and A is an integral domain. $\square$

2 Faithfully Flat Modules

Let's start with some general category theory:

Let $\mathcal{C}, \mathcal{D}$ be two categories and F is a functor between these two categories, we say that F is faithful if the induced map on the Hom sets is injective for all objects X, Y in $\mathcal{C}$:

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

If for example F is an additive functor between additive categories, this implies that $F(u) = 0$ implies $u = 0$ for $u \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. In this case $F(X) = 0$ implies $X = 0$ since we know that $F(id_X) = 0$ then $id_X = 0$.

The functor $F : \mathcal{C} \to \mathcal{D}$ is said to be fully faithful if:

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

is not only injective, but is an isomorphism for all X, Y. It's called essentially surjective if for any object Z in $\mathcal{D}$, there is some object X in $\mathcal{C}$ such that $F(X) \simeq Z$. We know that a functor is an equivalence of categories if it's fully faithful and essentially surjective.

Theorem 2.1. Let $\mathcal{C}, \mathcal{D}$ be abelian categories and let F be an additive functor, the following are equivalent:

1. F is exact and faithful.

2. A sequence:

$$M' \to M \to M''$$

is exact iff:

$$FM' \to FM \to FM''$$

is exact.

3. F is exact and the condition $F(X) = 0$ implies $X = 0$.

   If furthermore there exists a family of nonzero objects $\{Z_i\}$ in $\mathcal{C}$ such that for any nonzero object X in $\mathcal{C}$ there is some Z_i and some object Y in $\mathcal{C}$ such that there is a monomorphism $Y \to X$ and an epimorphism $Y \to Z_i$, then the above three conditions are equivalent to the following:

4. F is exact and $F(Z_i) \neq 0$ for all Z_i.

Proof. Let's first prove $1 \Rightarrow 2$. $FM' \to FM \to FM''$ is exact, then the composition is 0, therefore since F is faithful, we know that $M' \to M \to M''$ is 0, i.e. $\mathrm{Im} \subset \mathrm{Ker}$. Then:

$$F(\mathrm{Ker}/\mathrm{Im}) \simeq F\mathrm{Ker}/F\mathrm{Im} \simeq \mathrm{KerF}/\mathrm{ImF} = 0$$

Again because F is exact, then $F(\mathrm{Ker}/\mathrm{Im}) = 0$ implies ($\mathrm{Ker}/\mathrm{Im} = 0$ because F is faithful, then this implies that $M' \to M \to M''$ is exact.

Then for $2 \Rightarrow 3$, since $F(X) = 0$, then:

$$0 \to F(X) \to 0$$

is exact, then we know:

$$0 \to X \to 0$$

is exact, therefore $X = 0$.

To prove $3 \Rightarrow 1$, suppose that $F(u) = 0$ we want to prove $u = 0$. Notice that $\mathrm{Im}F(u) = 0$, then since F is exact $\mathrm{Im}F(u) = F(\mathrm{Im}u) = 0$, therefore $\mathrm{Im}u = 0$ therefore $u = 0$.

$3 \Rightarrow 4$ is obvious. To prove $4 \Rightarrow 3$, for each nonzero object, we need to prove $F(X) \neq 0$. Choose such a Y, then since F is exact, we know that:

$$F(Y) \to F(Z_i)$$

is still an epimorphism, therefore $F(Z_i) \neq 0$ implies $F(Y) \neq 0$, then $F(Y) \to F(X)$ is a monomorphism, since $F(Y) \neq 0$, then $F(X) \neq 0$. □

We discuss this abstract result because we want to apply it to modules:

Corollary 2.2. Let A be a ring an M is an A-module, the following conditions are equivalent:

1. The functor $N \mapsto M \otimes_A N$ on the category of A-modules is exact and faithful.

2. A sequence of A-modules is exact:

   $$N' \to N \to N''$$

   iff the following sequence is exact:

   $$N' \otimes_A M \to N \otimes_A M \to N'' \otimes_A M$$

3. M is flat an $M \otimes_A N = 0$ implies $N = 0$.

4. M is flat and $M \otimes_A A/m$ is not 0 for all maximal ideals of A.

Proof. Recall that for each nonzero A-module M, choose a nonzero element x in M, we take I to be the annihilator of x, then we have a monomorphism:

$$A/I \hookrightarrow M$$

Then consider a maximal ideal containing I, we have an epimorphism:

$$A/I \to A/m$$

we are done. □

Definition 2.3. If an A-module M satisfies the above equivalent conditions, we say that M is **faithfully flat**, which is saying that the corresponding tensor functor is faithful and is exact.

Corollary 2.4. Let $(A, m) \to (B, n)$ be a map of local rings and let M be a finitely generated B-module, then M is faithfully flat over A iff it's flat over A and nonzero.

Proof. We need to prove that $M \otimes_A A/m$ is not 0, but recall that m is mapped to n, therefore we only need to prove that:

$$M \otimes_B B/n \neq 0$$

But since M is finitely generated over B, we can use Nakayama's lemma, and we know this $M \otimes_B B/n$ is 0 iff $M = 0$. □

Remark 2.5. The above corollary is actually quite powerful, as it will usually used together with the notion of supports, because if you localize a finitely generated module at the support, you get a finitely generated nonzero module, and if it's flat, then it's automatically faithfully flat. We will see applications of this later.

Proposition 2.6. Let $A \to B$ be a homomorphism of rings, if there is a B-module that is faithfully flat over A, then $SpecB \to SpecA$ is onto.

Proof. Recall that it suffice to show that for any $p \in SpecA$, the fiber is not empty, or $B \otimes_A A_p/pA_p$ is nonzero.

Now if M is a B-module that is faithfully flat over A, then we know that:

$$M \otimes_A A_p/pA_p$$

is first, flat over A_p/pA_p (as this is an extension of rings), then it's also a fiathfully flat module over A_p/pA_p (this is a field, therefore it's equivalent to flat and nonzero), but notice that M if faithfully flat over A and A_p/pA_p is not 0, therefore $M \otimes_A A_p/pA_p \neq 0$, we are done.

Notice that $M \otimes_A A_p/pA_p$ is a $B \otimes_A A_p/pA_p$-module, therefore the ring is not 0. □

Remark 2.7. This proof shows that why the property on modules, which seems to have nothing to do with the ring morphisms, can affect the morphism of rings, and thus, morphisms of schemes.

Simply speaking, one simple information of rings that modules can tell us is whether a ring is 0 or not. To be more specific, if a ring is 0, then the modules over it are all 0-modules. Therefore, if you have a nonzero module over a ring, then the ring can not be 0. This can give you important geometric information, especially on the fibers, just as in the above proof.

Corollary 2.8. Let $\phi : A \to B$ be a ring map and let M be a finitely generated module over B that is flat over A. Suppose that $\mathrm{Supp}M = SpecB$, then for any $p \in SpecA$ and any prime ideal $q \in SpecB$ that is minimal among those lying above pB, we have $\phi^{-1}q = p$. In particular, if for any minimal prime q of B, $\phi^{-1}q$ is also a minimal prime ideal.

Proof. I will let you prove the assertion in the "in particular".

To prove the main assertion. We know that M_q is faithfully flat over $A_{\phi^{-1}q}$. This is because first it's nonzero (the support condition), then it's flat over $A_{\phi^{-1}q}$ by a previous remark, finally, it's also finitely generated over B_q. Then this is enough for M_q to be faithfully flat over $A_{\phi^{-1}q}$.

Then the map $SpecB_q \to SpecA_{\phi^{-1}q}$ is onto. Now we know that $pA_{\phi^{-1}q}$ is one prime ideal in $SpecA_{\phi^{-1}q}$, then the inverse image of $pA_{\phi^{-1}q}$ is nonempty. However, we know that q is already minimal over pB, therefore the preimage is a single prime ideal qB_q, then it follows that $\phi^{-1}q = p$. □

Remark 2.9. The support condition in the previous result is easily satisfied if $M = B$, then we know that if $A \to B$ is flat then in particular, we know the inverse image of a minimal prime is going to be a minimal prime.

Now we give some condition for a ring map $A \to B$ to be faithfully flat over A:

Proposition 2.10. Let $\phi : A \to B$ be a map of rings, the following conditions are equivalent:

1. B is faithfully flat over A.

2. B is flat over A and $SpecB \to SpecA$ is onto.

3. B is flat over A and for every maximal ideal m of A, there is a maximal ideal n of B such that $\phi^{-1}n = m$.

4. B is flat over A and for any A-module M, the canonical A-module map:

$$M \to M \otimes_A B \quad m \mapsto m \otimes 1$$

is injective.

5. For every ideal I of A:

$$I \otimes_A B \to B \quad x \otimes b \mapsto bx$$

is injective and $\phi^{-1}(IB) = I$.

6. ϕ is injective and cokerϕ is flat over A.

Proof. $1 \Rightarrow 2$ follows from B itself is a fully faithful module over A.

For $2 \Rightarrow 3$, let m be a maximal ideal of A, we know from $SpecB \to SpecA$ is onto that there is a prime of B such that the preimage is m, then take any maximal ideal that contains that prime, we are done.

For $3 \Rightarrow 1$, recall that we need to prove that for any maximal ideal m of A, B/mB is not zero. We know that there is a n maximal ideal of B such that $\phi^{-1}n = m$, then we know that B/n is a quotient of B/mB which is not zero, therefore B/mB is not 0. This proves the equivalence of the first three conditions.

Then let's prove $1 \Rightarrow 4$, since B is fully faithful over A, to show that:

$$M \to M \otimes_A B \quad m \mapsto m \otimes 1$$

is injective as an A-module map, we can base change to B and show that:

$$M \otimes_A B \to M \otimes_A B \otimes_A B \qquad m \otimes r \mapsto m \otimes 1 \otimes r$$

is injective. But the advantage of tensoring with B is that we can easily construct a left inverse:

$$M \otimes_A B \otimes_A B \to M \otimes_A B \qquad m \otimes b_1 \otimes b_2 \mapsto m \otimes b_1 b_2$$

This argument is pretty interesting as you notice that there is no natural way to define even a map $M \otimes_A B$ to M.

Then let's prove $4 \Rightarrow 5$. The injectivity of $I \otimes_A B \to B$ is just the fact that B is flat over A. Then let $M = A/I$, 4 implies that:

$$A/I \to A/I \otimes_B \quad \overline{a} \mapsto \overline{a} \otimes 1$$

is injective. This map is just $A/I \to B/IB$, therefore it's injective, this implies that the inverse image of IB is I (do you see why).

Then let's prove $5 \Rightarrow 3$. Again, B is flat follows from $I \otimes_A B \to B$ is injective for all ideals I of A. To prove the inverse image condition, notice that for any maximal ideal m of A, we consider mB, we know that it's inverse image is m, then take any maximal ideal containing mB, we are done.

Finally, let's prove that $4 \Rightarrow 6$ and $6 \Rightarrow 4$. Let's first do $4 \Rightarrow 6$. We need to prove ϕ is injective, then let's just take $M = A$, then we have a SES:

$$0 \to A \to B \to \text{coker} \to 0$$

We want to prove coker is flat, then for any A-module M, we need to prove $\text{Tor}_1^A(M, \text{coker}) = 1$. Consider the associated LES:

$$0 \to \text{Tor}_1^A(M, B) \to \text{Tor}_1^A(M, \text{coker}) \to M \to M \otimes_A B$$

Since B is flat over A, we know $\text{Tor}_1^A(M, B) = 0$, therefore $\text{Tor}_1^A(M, \text{coker})$ is the kernel of $M \to M \otimes_A B$, which is 0 since the map is injective.

Then we do $6 \Rightarrow 4$. We again consider the above associated LES, this times we know that $\text{Tor}_1^A(M, \text{coker}) = 0$, this proves that $\text{Tor}_1^A(M, B) = 0$ and $M \to M \otimes_A B$ is injective. ∎

For later use, we will need the following two standard results on completions. Here we don't want to spend too much time on completions but you can check the corresponding sections in the FOAG note series and there will be a write up following Mustumura's Commutative Algebra:

Proposition 2.11. Let A be a Noetherian ring and I an ideal of A, $\hat{A}$ the I-adic completion of A, then $\hat{A}$ is flat over A via the canonical ring map $A \to \hat{A}$. Moreover, it's faithfully flat over A iff I is contained in the radical of A.

Proof. See the write up on completions for the proof and more on the notion on completions of rings. □

Proposition 2.12. Let A be a ring, I an ideal of A and M an A-module. Suppose one of the two conditions holds:

1. I is nilpotent.
2. I is contained in the Jacobson radical of A and M is finitely generated.

Then the following conditions are equivalent:

1. M is free over A.
2. $M \otimes_A A/I$ is free over A/I and $\mathrm{Tor}_1^A(M, A/I) = 0$.
3. $M \otimes_A A/I$ is free over A/I and the canonical map:

$$M/IM \otimes_{A/I} (\oplus_{n=0}^{\infty} I^n/I^{n+1}) \to \oplus_{n=0}^{\infty}(I^n M/I^{n+1}M)$$

   is an isomorphism.

Proof. Let's first think a little bit about what is the canonical map. Recall that tensor product always commutes direct sums, therefore we are actually defining a morphism:

$$\oplus_{n=0}^{\infty}(M/IM \otimes_{A/I} I^n/I^{n+1}) \to \oplus_{n=0}^{\infty} I^n M/I^{n+1}M$$

This can be defined component by component, i.e. we need to define:

$$M/IM \otimes_{A/I} I^n/I^{n+1} \to I^n M/I^{n+1}M$$

we define it by $\overline{m} \otimes \overline{i} \to \overline{im}$, this is defined by the universal property but we still need to check it's well defined. Say $\overline{m} = \overline{m'}$, we know that the difference is in IM, then the multiplication is in $I^{n+1}M$, or if $\overline{i} = \overline{i'}$, we know that the difference is in I^{n+1}, then the multiplication is also in $I^{n+1}M$, we are done.

It's easy to notice that this morphism is always surjective, but it's not always an isomorphism. As an example, if $M = A$, we know that this is an isomorphism, as we have a canonical isomorphism $A/I \otimes_{A/I} I^n/I^{n+1} \simeq I^n/I^{n+1}$. But for arbitrary M, this isomorphsim is clearly not available, therefore there is no reason to expect this is an isomorphism for arbitrary M.

Then let's prove this proposition. $1 \Rightarrow 2$ is clear as free objects are flat. $1 \Rightarrow 3$ is also clear as if $M = \oplus_{j \in J} A$, then we only need to prove(there are some identification issues, but all not hard):

$$\oplus_{j \in J} A/I \otimes_{A/I} I^n/I^{n+1} \simeq \oplus_{j \in J} I^n/I^{n+1}$$

which is clear.

Then let's prove $2 \Rightarrow 1$. Let's first assume I is nilpotent and we do not assume any Noetherian condition on A or finite condition of M. Now let's assume $\{x_\lambda\}_{\lambda \in \Lambda}$ is a family of elements, such that there image in M/IM generate M/IM and is a basis. I claim that $\{x_\lambda\}_{\lambda \in \Lambda}$ also generates M. To see why, let N be the submodule generated by $\{x_\lambda\}_{\lambda \in \Lambda}$, then we consider $N + IM$, since we know that $\{x_\lambda\}_{\lambda \in \Lambda}$ generated M/IM, we know that $M = N + IM$. Now we want to consider:

$$Q = M/N = \frac{IM + N}{N} = I(\frac{M}{N}) = IQ$$

where the equality $\frac{IM+N}{N} \simeq I(\frac{M}{N})$ is done inside $\frac{M}{N}$, therefore we know that $Q = IQ$. Notice that $I^n = 0$ for some n, then we know that $Q = 0$. Now if we assume that M is finitely generated and I is in the Jacobson radical, we can apply the Nakayama's lemma and still conclude that $Q = 0$. In both case x_λ generate M.

Let L be the free A-module indexed by $\lambda \in \Lambda$. And let $L \to M$ be the obvious surjection, we take R to be it's kernel, therefore we have a SES:

$$0 \to R \to L \to M \to 0$$

Take the LES associated to $\mathrm{Tor}^A(-, A/I)$, we get:

$$0 \to R/IR \to L/IL \to M/IM \to 0$$

By definition, we know that $L/IL \to M/IM$ is actually an isomorphism, therefore $R/IR = 0$, i.e. $R = IR$, then either from I is nilpotenet, or from Nakayama's lemma, we conclude $R = 0$.

Finally, let's prove $3 \Rightarrow 1$. Let's keep the notation above, then we have a commutative diagram:

$$\begin{tikzcd}
L/IL \otimes_{A/I} (\oplus_{n=0}^{\infty} I^n/I^{n+1}) \arrow[r] \arrow[d] & \oplus_{n=0}^{\infty}(I^nL/I^{n+1}L) \arrow[d] \\
M/IM \otimes_{A/I} (\oplus_{n=0}^{\infty} I^n/I^{n+1}) \arrow[r] & \oplus_{n=0}^{\infty}(I^nM/I^{n+1}M)
\end{tikzcd}$$

I will let you find out why two horizontal arrows and the first vertical arrow are isomorphism, therefore the following is an isomorphism:

$$\oplus_{n=0}^{\infty}(I^nL/I^{n+1}L) \to \oplus_{n=0}^{\infty}(I^nM/I^{n+1}M)$$

I claim that this implies $R \subset \cap_{n=0}^{\infty} I^nL$. This is because let's first consider $L/IL \to M/IM$, which is an isomorphsim, then suppose $r \in R$, then we know that $\frac{R+IL}{IL}$ is in the kernel, this implies that:

$$R + IL \subset IL \Rightarrow R \subset IL$$

Then we consider $IL/I^2L \to IM/I^2M$, we know that since R is in IL, the $\frac{R+I^2L}{I^2L}$ is in the kernel, but the kernel is 0, therefore R is in I^2L, then you can keep going. Then either you assume I is nilpotent, which implies the intersection is 0. Or you can assume that if I is in the Jacobson, we know that $\bigcap_{n=0}^{\infty} I^nL = I\bigcap_{n=0}^{\infty} I^nL$, but then by Nakayama's lemma, $\bigcap_{n=0}^{\infty} I^nL = 0$, we are done. $\square$

The following is a corollary of the above result:

Corollary 2.13. Suppose that m is a maximal ideal of a ring A and M is an A-module. Suppose one of the following conditions holds:

1. m is nilpotent.
2. A is Noetherian local and M is finitely generated.

Then the following are equivalent:

1. M is free.
2. M is projective.
3. M is flat.
4. $\mathrm{Tor}_1^A(M, A/m) = 0$.
5. The canonical map:

   $$M/mM \otimes_{A/m} (\oplus_{n=0}^{\infty} m^n/m^{n+1}) \to \oplus_{n=0}^{\infty} m^nM/m^{n+1}M$$

   is an isomorphism.

   If (A, m) is a local Noetherian integral domain with residue field k and fraction field K, M is finitely generated, then the above conditions are equivalent to:

6. $\dim_K(M \otimes_A K) = \dim_k(M \otimes_A k)$

Proof. Since m is a maximal ideal, A/m is a field, then we know that M/mM is always free over A/m, therefore the freeness condition in the previous result can be ignored. Then from 1 to 5, the equivalence is just the previous result.

Then since $1 \Rightarrow 6$ is obvious even without the extra assumption, we only have to prove $6 \Rightarrow 1$. As you will see in the proof, the assumption that R is a domain is just for the existence of K, and we don't use particular properties of domains. Now suppose 6 holds, let x_λ be a collection of elements in M whose image in M/mM is a basis of M/mM, then by Nakayama or by m is nilpotent, we know that x_λ also generates M, we keep the notation of the above proposition and have a SES:

$$0 \to R \otimes_A K \to L \otimes_A K \to M \otimes_A K \to 0$$

Since $L \otimes_A K$ and $M \otimes_A K$ have the same dimension (the same cardinality even), then $R \otimes_A K = 0$. But since R is a submodule of L, it has no torsion, therefore $R \otimes_A K$ implies $R = 0$. □

Remark 2.14. The above proposition and corollary, especially the corollary is quite typical, and is what we will try to generalize that in the next section.

If you try to think about it, it says that under some conditions on the ideal, for example if you work with a local ring and you consider the maximal ideal, which is a local notion, then to prove M is free, you only need to prove a sequence of isomorphism. That is to say, the behavior at the maximal ideal determines if the module is free or not. Later, we will see a similar version of this kind of statement, but for flatness, not for freeness, this is why the title of next section is called the local criteria for flatness.

3 Local Criteria for Flatness

Proposition 3.1. Let A be a ring, I an ideal of A and M an A-module. If $\mathrm{Tor}_1^A(M, A/I^n) = 0$ for all $n \geqslant 0$, then the canonical map:

$$M/IM \otimes_{A/I} (\oplus_{n=0}^{\infty} I^n/I^{n+1}) \to \oplus_{n=0}^{\infty}(I^n M/I^{n+1}M)$$

is an isomorphism. The converse is true if I is nilpotent.

Proof. First suppose the Tor functor vanishes for all $n \geqslant 0$. Then for each n, we have a SES:

$$0 \to I^n/I^{n+1} \to A/I^{n+1} \to A/I^n \to 0$$

Then consider the associated LES of Tor, we know that we have:

$$0 \to M \otimes_A I^n/I^{n+1} \to M \otimes_A A/I^{n+1} \to M \otimes_A A/I^n \to 0$$

Notice that the last arrow is the map: $M/I^{n+1}M \to M/I^n M$ and the kernel of this map is $I^n M/I^{n+1}M$, then the canonical map to the kernel is an isomorphism, therefore we have the following:

$$M \otimes_A I^n/I^{n+1} \to I^n M/I^{n+1}M$$

Notice that $M \otimes_A I^n/I^{n+1} \simeq M/IM \otimes_{A/I} I^n/I^{n+1}$, this is because:

$$M \otimes_A (I^n/I^{n+1}) \simeq M \otimes_A (A/I \otimes_{A/I} I^n/I^{n+1})$$

Therefore, the canonical map $M/IM \otimes_{A/I} \otimes_{A/I} I^n/I^{n+1} \to I^n M/I^{n+1}M$ is an isomorphism for each n, we are done.

For the converse under the assumption that I is nilpotent. Notice that if we write out more terms in the LES, we get:

$$\mathrm{Tor}_1^A(M, A/I^{n+1}) \to \mathrm{Tor}_1^A(M, A/I^n) \to M \otimes_A I^n/I^{n+1} \to M \otimes_A A/I^{n+1} \to M \otimes_A A/I^n \to 0$$

Then if the canonical map is an isomorphism, then we know that $M \times_A (I^n/I^{n+1})$ will be the kernel, therefore the map is injective in particular, therefore $\mathrm{Tor}_1^A(M, A/I^{n+1}) \to \mathrm{Tor}_1^A(M, A/I^n)$ is onto for all n. Now since I is nilpotent, we know that A/I^{n+1} is A for large n, but notice that $\mathrm{Tor}_1^A(M, A) = 0$ for all M, then we are done. □

Proposition 3.2. Let $A \to B$ be a map of rings and let M be an A-module, the following conditions are equivalent:

1. For every B-module $\mathrm{Tor}_1^A(M, N) = 0$.
2. $M \otimes_A B$ is flat over B and $\mathrm{Tor}_1^A(M, B) = 0$.

Proof. Let's prove $1 \Rightarrow 2$ first. Take $N = B$, we get $\mathrm{Tor}_1^A(M, B) = 0$. Then let's assume that:

$$0 \to N' \to N \to N'' \to 0$$

is a SES of B-modules, since $\mathrm{Tor}_1^A(M, N'') = 0$, we know that tensor with M gives you an exact sequence:

$$0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

But this is:

$$0 \to (M \otimes_A B) \otimes_B N' \to (M \otimes_A B) \otimes_B N \to (M \otimes_A B) \otimes_B N'' \to 0$$

This shows $M \otimes_A B$ is flat over B.

Conversely, let $L \to B$ be a surjective map of B-modules where L is free over B and we still let R be the kernel, then we have a LES:

$$\mathrm{Tor}_1^A(M, L) \to \mathrm{Tor}_1^A(M, N) \to M \otimes_A R \to M \otimes_A L \to M \otimes_A N \to 0$$

Notice that the first three terms, can be thought as tensor with $M \otimes_A B$ over B, but $M \otimes_A B$ is flat, then we know that $M \otimes_A R \to M \otimes_A L$ is injective. Recall that Tor commutes with direct sums, then $\mathrm{Tor}_1^A(M, L) \simeq \oplus \mathrm{Tor}_A^1(M, B) = 0$. Combine all these, we get $\mathrm{Tor}_1^A(M, N) = 0$. $\square$

Corollary 3.3. Let A be a ring, I an ideal and M an A-module, then the following conditions are equivalent:

1. $\mathrm{Tor}_1^A(M, N) = 0$ for all A/I-modules N.
2. $M \times_A A/I$ is flat over A/i and $\mathrm{Tor}_1^A(M, A/I) = 0$.
3. $\mathrm{Tor}_1^A(M, N) = 0$ for any A-module N annihilated by some power of I.

If these conditions are satisfied the canonical map:

$$M/IM \otimes_{A/I} (\oplus_{n=0}^{\infty} I^n/I^{n+1}) \to \oplus_{n=0}^{\infty} (I^n M/I^{n+1}M)$$

is an isomorphism.

Proof. 1 and 2 are equivalent by the above result and $3 \Rightarrow 1$ is obvious.

To prove $1 \Rightarrow 3$, say N is some A-module, annihilated by some power of I. From 1 we know that:

$$\mathrm{Tor}_1^A(M, I^n N/I^{n+1}N) = 0$$

for all $n \geqslant n$. Then we know that first $\mathrm{Tor}_1^A(M, N/IN) = 0$. Then consider the following SES:

$$0 \to IN/I^2N \to N/I^2N \to N/IN \to 0$$

Then take the LES associated to $\mathrm{Tor}_1^A(M, -)$, we get $\mathrm{Tor}_1^A(M, N/I^2N) = 0$, then keep going and use induction we get the desired conclusion.

Finally, the last assertion follows from 3 because in particular 3 implies:

$$\mathrm{Tor}_1^A(M, A/I^n) = 0$$

$\square$

Now we are finally able to say something about flatness:

Proposition 3.4. Again, let A be a ring, I be an ideal and M is an A-module, then consider the following conditions:

1. M if flat over A.
2. $M \otimes_A A/I$ is flat over A/I and $\mathrm{Tor}_1^A(M, A/I) = 0$.
3. $M \otimes_A A/I$ is flat over A/I and the canonical homomorphism:
$$M/IM \otimes_{A/I} (\oplus_{n=0}^{\infty} I^n/I^{n+1}) \to \oplus_{n=0}^{\infty}(I^n M/I^{n+1}M)$$
is an isomorphism.
4. For any $n \geqslant n$, $M \otimes_A A/I^n$ is flat over A/I^n.

Without any extra assumption, we always have $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$. Moreover, if I is nilpotent or A is Noetherian and $M \otimes_A N$ is separated with respect to the I-adic topology any finitely generated A-module N, then $4 \Rightarrow 1$.

Proof. $1 \Rightarrow 2$ is just the fact that extension of scalar preserves flatness. $2 \Rightarrow 3$ is the above corollary. Now let's prove $3 \Rightarrow 4$. Consider the ring A/I^N, we know that in this ring, the ideal I/I^N is nilpotent. Also consider the module $N = M/I^N M$, we know that:

$$\frac{M/I^N M}{(I/I^N)M/I^N M} \simeq \frac{M/I^N M}{(IM + I^N M/I^N M)} \simeq M/(IM + I^N M) = M/IM$$

and

$$A/I^N/(I/I^N) \simeq A/I$$

Moreover:

$$\frac{(I^n/I^N)(M/I^N M)}{(I^{n+1}/I^N)(M/I^N M)} = \frac{(I^N M + I^n M)/(I^N M)}{(I^N M + I^{n+1}M)/(I^N M)} \simeq \frac{I^N M + I^n M}{I^N M + I^{n+1}M} \simeq \frac{I^n M}{I^{n+1}M \cap I^n M} \simeq \frac{I^n M}{I^{n+1}M}$$

Therefore, the canonical isomorphism:

$$M/IM \otimes_{A/I} (\oplus_{n=0}^{\infty} I^n/I^{n+1}) \to \oplus_{n=0}^{\infty}(I^n M/I^{n+1}M)$$

implies that we have a similar result for $M/I^N M \otimes_{(A/I^N)/(I/I^N)} \cdots$, then we can quote the first result in this section, since I/I^N is nilpotent, we know from this isomorphism that $\mathrm{Tor}^{A/I^N}(M/I^N M, A/I) = 0$.

Now again, we consider $M/I^N M$, $A/I^N/I/I^N$ and $\mathrm{Tor}_1^{A/I^N}(M/I^N M, A/I)$, I claim that condition 2 of the previous corollary is satisfied by these three objects $M/I^N M$, $A/I^N/I/I^N$ and $\mathrm{Tor}_1^{A/I^N}(M/I^N M, A/I)$ and thus we get condition 3, i.e.:
$$\mathrm{Tor}_1^{A/I^N}(M/I^N M, N) = 0$$
for any A/I^N-module N that is annihilated by some parts of I/I^N, but notice that I/I^N is nilpotent, then N is any A/I^N-module. But this is what you need for $M/I^N M$ to be flat over A/I^N, we are done.

Then let's try to prove $4 \Rightarrow 1$ in the assumption I is nilpotent or $M \otimes_A N$ is separated. The case for I is nilpotent is easy if you take n to be large enough. So the real issue is the separatedness condition. Recall that if M is an A-module, then it's separated with repsect to the I-adic topology. Recall that being separated in the I-adic topology can be interpreted as:

$$\bigcap_{n=0}^{\infty} I^n(M \otimes_A N) = 0$$

Now to prove M is flat over A, we need to prove that for any two finitely generated A-modules N, N' and an injective map of A-modules $i : N \hookrightarrow N'$, we get an injective map:

$$M \otimes_A N \hookrightarrow M \otimes_A N'$$

Since we know that $I^n(M \otimes_A N) = 0$, it's enough to show that the kernel is contained in each $I^n(M \otimes_A N)$, for all $n \geqslant 0$. Notice that $I^n(M \otimes_A N)$, as a submodule of $M \otimes_A N$, is he same as the image:

$$\mathrm{Im}(M \otimes_A I^n N \to M \otimes N)$$

Similarly, suppose that V is a collection of submodule of N such that it forms a basis at 0 with respect to the I-adic topology, then we consider:

$$W = \mathrm{Im}(M \otimes_A V \to M \otimes_A N) = \ker(M \otimes_A N \to M \otimes_A (N/V))$$

(The reason the kernel is the image is because tensor is right exact) Since V forms a basis, for each such V, we know that there exists a n_0 such that for all $N \geqslant n_0$ $I^n N \subset V$. Then notice that we know that the image of $M \otimes_A I^n N \subset M \otimes_A V$. Then if we can show that the kernel is in all of the images of $M \otimes_A V \to M \otimes_A N$, we are good.

Recall that we assumed that N is finitely generated and A is Noetherian, we can use the Artin-Rees lemma to pick V to be $N' \cap I^n N$ for $n \geqslant 0$ since we have an injection $N \hookrightarrow N'$. Then we have the following commutative diagram:

$$\begin{tikzcd}
M \otimes_A N \arrow[r] \arrow[d] & M \otimes_A (N'/(N \cap I^n N')) \arrow[d] \\
M \otimes_A N' \arrow[r] & M \otimes_A (N'/I^n N')
\end{tikzcd}$$

We want this diagram because on the right hand side, the stuff you tensor is A/I^n-module, then we know that it's:

$$M/I^n M \otimes_{A/I^n} (N'/(N \cap I^n N')) \to M/I^n M \otimes_{A/I^n} N'/I^n N'$$

What is important is that $N/(N \cap I^n N') \to N'/I^n N'$ is injective, then we know that the tensor morphism is injective, therefore we know that the kernel of $M \otimes_A N \to M \otimes_A N'$ is contained in the kernel of $M \otimes_A N \to M \otimes_A (N'/(N \cap I^n N'))$, we are done. $\square$

Combine all the above results we have, we get one important result:

Theorem 3.5. Grothendieck's local Criteria for Flatness Let $A \to B$ be a map of Noetherian rings with I an ideal of A such that IB is contained in the Jacobson radical of B and M a finitely generated B-module, then the following are equivalent:

1. M if flat over A.
2. $M \otimes_A A/I$ is flat over A/I and $\mathrm{Tor}_1^A(M, A/I) = 0$.
3. $M \otimes_A A/I$ is flat over A/I and the canonical homomorphism:

   $$M/IM \otimes_{A/I} (\oplus_{n=0}^{\infty} I^n/I^{n+1}) \to \oplus_{n=0}^{\infty}(I^n M/I^{n+1}M)$$

   is an isomorphism.
4. For any $n \geqslant n$, $M \otimes_A A/I^n$ is flat over A/I^n.

Proof. All the role that B plays here is for us to check the condition that $M \otimes_A N$ is separated with respect to the I-adic topology for this I for any finitely generated A-module N. Notice that $M \otimes_A N$ is a finitely generated B-module, and IB is contained in the Jacobson radical of B, then $M \otimes_A N$ is separated with respect to the IB-adic topology, or more precisely:

$$\bigcap_{n=0}^{\infty} (I)^n (M \otimes_A B) \subset \bigcap_{n=0}^{\infty} (IB)^n (M \otimes_A B) = 0$$

We are done. $\square$

Remark 3.6. Since IB is contained in the Jacobson radical, we can use the Krull Intersection theorem and Nakayama's lemma to conclude that for any B-module X, X is separated with respect to the I-adic topology, I will leave the details as an exercise.

Proposition 3.7. Let $(A, m) \to (B, n)$ be a map of local rings, and $u : M \to M'$ a map of B-modules between finitely generated B-modules with M' flat over A. Then the following two conditions are equivalent:

1. u is injective and coker$u$ is flat over A.

2. $M \otimes_A A/m \to M' \otimes_A A/m$ is injective.

Proof. $1 \Rightarrow 2$ is to think about the SES:

$$0 \to M \to M' \to \mathrm{coker} u \to 0$$

Since coker$u$ is flat, then consider the LES of Tor, we get the desired injection.

Suppose that $M \otimes_A A/m \to M' \otimes_A A/m'$ is an injection. Consider the following diagram:

$$\begin{tikzcd} M/mM \otimes_{A/m} (\oplus_{n=0}^{\infty} m^n/m^{n+1}) \arrow[r, hook] \arrow[d] & M'/mM' \otimes_{A/m} (\oplus_{n=0}^{\infty} m^n/m^{n+1}) \arrow[d, dashed, "\simeq"] \\ \oplus_{n=0}^{\infty}(m^n M/m^{n+1}M) \arrow[r] & \oplus_{n=0}^{\infty}(m^n M'/m^{n+1}M') \end{tikzcd}$$

Since A/m is a field, then tensor with everything is exact, therefore the top horizontal map is injective, and since M' is flat over A, we know that the right vertical map is an isomorphism. Moreover, we know the left vertical map is surjective, which implies the bottom horizontal map is injective, therefore u is injective.

Therefore, we have a SES:

$$0 \to M \to M' \to \mathrm{coker} u \to 0$$

Consider the LES of $\mathrm{Tor}(-, A/m)$, you will be able to prove coker$u$ is flat. ☐

Proposition 3.8. Let $(C, l) \to (A, m) \to (B, n)$ be a sequence of maps of local rings and let M be a finitely generated B-module. Say A is flat over C, then the following conditions are equivalent:

1. M is flat over A.

2. M is flat over C and $M \otimes_C C/l$ is flat over $A \otimes_C C/l$.

Proof. $1 \Rightarrow 2$ is just our early slogan that restriction along a flat map and extension by scalar.

To show $2 \Rightarrow 1$. Suppose that M is flat over C and M/lM is flat over A/lA. By the local criteria for flatness, we know that it suffice to prove the following is an isomorphism:

$$M/lM \otimes_{A/lA} (\oplus_{n=0}^{\infty} l^n A/l^{n+1}A) \to \oplus_{n=0}^{\infty} l^n M/l^{n+1}M$$

Recall that A, M is flat over C, then we know that we have we have the following isomorphisms again by the local criteria:

$$A/lA \otimes_{C/l} (\oplus_{n=0}^{\infty} l^n/l^{n+1}) \to \oplus_{n=0}^{\infty} l^n A/l^{n+1}A$$

$$M/lm \otimes_{C/l} (\oplus_{n=0}^{\infty} l^n/l^{n+1}) \to \oplus_{n=0}^{\infty} l^n M/l^{n+1}M$$

Then consider:

$$M/lM \otimes_{A/lA} (A/lA \otimes_{C/l} (\oplus_{n=0}^{\infty} l^n/l^{n+1})) \to M/lM \otimes_{A/lA} (\oplus_{n=0}^{\infty} l^n A/l^{n+1}A) \to \oplus_{n=0}^{\infty} l^n M/l^{n+1}M$$

Notice that the first arrow is an isomorphism and the composition of two arrows is also an isomorphism, therefore the second arrow is also an isomorphism. ☐

4 Constructible Sets

We also do a little review on constructible sets in this section since we have already discussed it in the FOAG notes sereis, we don't use the notion of constructible sets a lot in the following study of étale cohomology besides some side topological applications in next section on flat morphisms. So this discussion is here just for completeness. A more important notion is called constructible sheaves, which we will discuss later.

First, suppose that X is a topological space, we say that a subset of X is **locally closed** if it's the intersection of an open and a closed subset. We say that a subset of X is constructible if it's the union of finitely many locally closed subsets. It's easy to prove that if X_1, X_2 are locally closed, then their union, intersection and $X_1 - X_2$ are all locally closed. Another common way to define locally closed subsets is to say that it's the algebra generated by all open subsets of X. I.e. the collection of subsets such that all open sets are in it, and it's closed under finite intersections and complements. (This is not a sigma algebra as sigma algebra is closed under countably many intersections.) We have already proved that these two definition are equivalent in the FOAG series. In this discussion, we will give the proof of Chevalley again, but with a different approach:

Proposition 4.1. Let X be an Noetherian topological space. A subset Y of X is constructible iff for any irreducible closed subset F of X such that $Y \cap F$ is dense in F, then the set $Y \cap F$ contains a nonempty open of F.

Proof. Here we assume we work with Noetherian topological spaces because we want to use Noetherian induction.

Suppose that Y is constructible, and $Y \cap F$ is dense in Y. Recall that $Y \cap F$ is also constructible, therefore:

$$Y \cap F = (U_1 \cap F_1) \cup \cdots \cup (U_n \cap F_n)$$

for open subsets U_i and closed subsets F_i. Notice that we can replace every F_i by $F_i \cap F$ and assume $F_i \subset F$. Since F is the closure of $Y \cap F$ and taking closure commutes with finite unions, therefore we know that in the end:

$$F = \overline{(U_1 \cap F_1)} \cup \cdots \cup \overline{(U_n \cap F_n)}$$

Recall that each $\overline{U_i \cap F_i}$ is a closed subset in F, which is irreducible, therefore $F = \overline{U_i \cap F_i}$ for some F_i, but we know that the closure of $U_i \cap F_i$ is inside F_i and $F_i \subset F$, we know that $F = F_i$, then we know that $Y \cap F = U_i \cap F$.

Conversely, suppose that $Y \cap F$ contains a nonempty open of F for all F. We will use Noetherian induction, therefore consider:

$$\mathcal{F} = \{Z \subset X : Z \neq \varnothing,\ Z \text{ is closed and } Z \cap Y \text{ is not constructible}\}$$

If we assume Y is not constructible, then $\mathcal{F}$ is nonempty since Y is inside $\mathcal{F}$. Since X is Noetherian, we know that $\mathcal{F}$ contains a miminal element, say Z_0, and if Z_0 is not irreducible, we know that it's a finite union of irreducible components, again because we work in Noetherian spaces. I claim that there is at least one of the components is in $\mathcal{F}$, which is a contradiction to the minimality of Z_0, therefore Z_0 is irreducible.

If $Z_0 \cap Y$ is not dense in Z_0, then by minimality of Z_0, we know that $Y \cap \overline{(Y \cap Z_0)}$ is constructible. However, notice that $Y \cap \overline{(Y \cap Z_0)}$ is just $Y \cap Z_0$. This contradicts $Z_0 \in \mathcal{F}$, therefore $Y \cap Z_0$ is dense, then by our assumption, there is a nonempty open subset of Z_0 contained in $Y \cap Z_0$. Again, by minimality of Z_0, we know that $Y \cap (Z_0 - U)$ is constructible and $Y \cap Z_0 = U \cup (Y \cap (Z_0 - U))$, but this implies that $Y \cap Z_0$ is constructible, a contradiction, therefore Y is constructible. ☐

Theorem 4.2. (Chevalley) Let f be a morphism of finite type between Noetherian schemes, then $f(X)$ is constructible.

Proof. Since f is finite type and X, Y are Noetherian, we can cover Y by finitely many affine opens V_i, and then cover $f^{-1}V_i$ with finitely many affine opens U_{ij}. We know that $X = \bigcup_{ij} f(U_{ij})$, therefore we only need to prove each $f(U_{ij})$ is constructible. Since $f(U_{ij})$ is in V_i, which is open, then we only need to show that the image in V_{ij} is constructible, therefore we can assume we deal with $X = SpecB$ and $Y = SpecA$.

Recall that any irreducible closed subset of $SpecA$ is of the form $V(p)$ for some prime p. Then we know that $f(SpecB) \cap V(p)$ can be though as the image of:

$$SpecB/pB \to SpecA/p$$

We know that if the image is dense in $SpecA/p$, the morphism $A/p \to B/pB$ is injective, therefore by the above proposition, to prove $f(SpecB)$ is constructible in $SpecA$, we need to prove that if $SpecB/pB \to SpecA/p$ is dense in $SpecA/p$, i.e. $A/p \to B/pB$ is injective, then the image contains an open subset, and we can take it to be a distinguished open. Therefore it can be translated to the algebraic problem: If $A \to B$ is an injective map of rings between Noetherian rings where A is a domain and B is a finitely generated A-algebra, then there is a nonzero $a \in A$ such that every prime in $D(a)$ is the inverse image of some prime in B.

Let's write $B = A[x_1, \cdots, x_n]$ where we can assume that the first r variables are algebraically independent over A and the remaining variables are algebraic over $A[x_1, \cdots, x_r]$, i.e. choose a maximal set of algebraically independent subset. Then we know that for each $r+1 \leqslant j \leqslant n$, we have a relation:

$$p_{j0}x_j^{d_j} + p_{j1}x_j^{d_j-1} + \cdots + p_{jd_j} = 0$$

where $p_{ji} \in A[x_1, \cdots, x_r]$ and $p_{j0} \neq 0$. Let's first assume that $r+1 \leqslant n$, i.e. there is at least one x_j such that it's algebraic over $A[x_1, \cdots, x_r]$.

Then consider the element $\prod_{j=r+1}^n p_{j0}$, i.e. the product of all the top coefficients of the relations of x_j for $j \geqslant r+1$. This is a nonzero polynomial in $A[x_1, \cdots, x_r]$, and we take a to be any of the nonzero coefficients of this polynomial. It turns out this a and $D(a)$ works. Suppose that $p \in D(a)$, consider $p' = p[x_1, \cdots, x_n]$, and this is a prime ideal of $A[x_1, \cdots, x_r]$ because the quotient is $A/p[x_1, \cdots, x_n]$, which is still a domain. The key here is that $\prod_{j=r+1}^n p_{j0} \notin p'$ since by definition, the coefficients of the polynomials in p' are in p. This consider $A[x_1, \cdots, x_r] \to B$, we localize both rings at p', then this time, the relations for x_j for $j \geqslant r+1$ can be made monic, then we know that $B_{p'}$ is monic over $A_{p'}$, then we can apply the lying over theorem saying that there is some prime q of $B_{p'}$ whose inverse image is p', then consider:

$$A \to A[x_1, \cdots, x_r]$$

since the inverse image of p', by definition is p, we know the inverse image of q in A is p, we are done. □

Remark 4.3. Perhaps what a lot of people think when they see constructible sets is that this is a notion that is not a clear as opens and closed subsets. Yes it is, but this is not something mysterious and you see by these two results above, at least for schemes, the way you prove something is constructible is to translate that to something you know, i.e. irreducible closed, density and affine opens. So, don't be scared by this notion of being constructible, they are actually not so bad.

Corollary 4.4. Let $f : X \to Y$ be as in the above theorem, if Z is constructible of X, then $f(Z)$ is constructible.

Proof. Recall that Z is a finite union of locally closed subsets, therefore we only have to prove this for locally closed subsets. But if it's locally closed, we can assume $Z = U \cap F$, and therefore, we can consider the canonical locally closed embedding composed with f:

$$Z \hookrightarrow U \hookrightarrow X \to Y$$

Recall that since X is Noetherian and any open embedding is locally of finite type, then the open embedding is finite type, then the composition of three finite type morphisms is finite type, then consider the image of Z under this map, we are done. □

Proposition 4.5. Suppose that X is a Noetherian topological space such that every irreducible closed subset has a unique generic point, let U be a constructible subset of X and let x be a point in U, then U contains an open neighborhood of x iff every generalization y of x is in U.

Proof. Suppose that U contains a neighborhood of x, then we know that for each generalization of x it must be in this neighborhood, therefore we are done.

Conversely, we want to prove that if every generalization of x is in U, then U contains a neighborhood of x. Again, we assume for the sake of contradiction that U doesn't contain any open neighborhood of x, then the following collection is not empty:

$$\mathcal{F} = \{x \in Z \subset X : Z \text{ is closed and } U \cap Z \text{ doesn't contain a neighborhood of } x \text{ in } Z\}$$

This is because we can take $Z = X$. Again since X is Noetherian, we pick a minimal element, say Z_0. I claim that Z_0 is irreducible, since if not, first assume $x \in Z_1 \cap Z_2$. By the minimality of Z_0, we know that for Z_1, Z_2, there are V_1, V_2 such that $V_i \cap Z_i = U \cap Z_i$, then we know that $(V_1 \cap V_2) \cap (Z_1 \cup Z_2) = ((V_1 \cap V_2) \cap Z_1) \cup ((V_1 \cap V_2) \cap Z_2) = ((U \cap Z_1) \cap V_2) \cup ((U \cap Z_2) \cap V_1)$, each of which is a neighborhood of x, therefor the intersection is also a neighborhood of x. Now suppose that $x \in Z_1 - Z_2$, again by the minimality of Z_0, there is a V_1 neighborhood of x such that $Z_1 \cap V_1 \subset U \cap Z_1$ then we know that $U \cap Z_0$ contains $(V_1 - Z_2) \cap Z_0$ of x, this is a contradiction. Similarly if it's in $Z_2 - Z_1$, we get a contradiction again. Therefore Z_0 must be irreducible. Let y be the generic point of y, then since $x \in Z$, we know that y is a generalization of x, therefore $y \in U$. Then $U \cap Z_0$ is dense in Z_0, therefore by our previous result, $U \cap Z_0$ contains a nonempty open V of Z_0, but since Z_0 is in $\mathcal{F}$, we must have $x \notin V$, then consider $Z_0 - V$, by minimality of Z_0, we know that there is a W neighborhood of x in X such that $W \cap (Z_0 - V) \subset U \cap (Z_0 - V)$. I claim this implies that $W \cap Z_0 \subset U \cap Z_0$ and it contains x, this contradict $Z_0 \in \mathcal{F}$, contradiction. □

Corollary 4.6. Let X be a Noetherian topological space such that every irreducible closed subset has a generic point. A subset U of X is open if and only if the following two conditions hold:

1. U contains generalizations of points in U.

2. For every $x \in U$, $U \cap \overline{x}$ contains a nonempty open subset of $\overline{x}$.

Proof. Again the only if part is clear. Then suppose the above two conditions hold, we will first prove that U is constructible. Suppose that F is an irreducible closed subset of X with generic point x, then if $U \cap F$ is dense in F, we know that $x \in U \cap F$, then condition 1 implies $x \in U$, then condition 2 implies that $U \cap F$ contains a nonempty open of F, then the above proposition tells us $U \cap F$ contains a nonempty open of F, so U is constructible.

Then to prove U is open, we need to prove for each $x \in U$, there is a neighborhood of x contained in U. But this is again true by 1 and the previous proposition. □

5 Flat Morphisms

Since we have already discussed a lot of flat morphisms in our discussion in the FOAG notes series, we only gather and prove, as a reminder, some results we will use later. Let's begin our discussion by a simple proposition:

Proposition 5.1. Let $f : X \to Y$ be a finite type morphism between Noetherian schemes. Let $x \in X$ and $y = f(x)$, then the following conditions are equivalent:

1. f maps every neighborhood of x to a neighborhood of y, i.e. f is open at x.

2. For every generalization y' of y there is a generalization x' of x such that $f(x') = y'$.

3. The morphism $Spec\mathcal{O}_{X,x} \to Spec\mathcal{O}_{Y,y}$ induced by f is surjective on the underlying topological space.

Proof. I will leave it for you to prove that 2 and 3 are equivalent, which should be obvious.

Then let's prove $1 \Rightarrow 2$. Assume that $y \in \overline{y'}$. Consider the inverse image $f^{-1}(\overline{y'})$, which is also a closed subset, therefore is a finite union of irreducible components because X is Noetherian. Let F be the union of irreducible components that don't contain x. Then $X - F$ is a neighborhood of x, then by our assumption on 1, we know $f(X - F)$ is a neighborhood of y, hence $y' \in f(X - F)$. This implies that there is some $x_1 \in X - F$ such that $f(x_1)$ is y'. Then x lives in one of the components of $f^{-1}\overline{\{y'\}}$ that contains x, say the component is C. Then let x' be the generic point of C, I claim $f(x') = y'$. We know that $x_1 \in \overline{\{x'\}}$, hence $f(x_1) = y' \in \overline{\{f(x')\}}$, but notice that by definition $x' \in f^{-1}\overline{\{y'\}}$, therefore $f(x') \in \overline{\{y'\}}$. Therefore $y' = f(x')$. We are done.

Conversely, let's prove $2 \Rightarrow 1$. Let U be a neighborhood of x, we know that $f(U)$ is constructible, then suppose that $y = f(x)$ is in $f(U)$, we know that for each generalization of y, say y', there is a generalization of x, say x' in U such that $f(x') = y'$. This implies $f(U)$ is open. □

Remark 5.2. This proposition is an excellent example of why we say that the stalk at a point isn't just about the information at that single point, but actually contains much more information on the surrounding geometry. For example, if you are willing to add the condition of finite type and Noetherainity, it tells you whether a morphism is open at a point or not.

The following theorem is key for flat morphisms:

Theorem 5.3. Suppose that $f : X \to Y$ is a morphism of finite type between Noetherian schemes and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module with Supp$\mathcal{F} = X$. Suppose that $\mathcal{F}_x$ is flat over $\mathcal{O}_{Y,f(x)}$ for all $x \in X$, then f is open.

Remark 5.4. Before we prove it, you should find that this is a quite surprising result. The reason for being surprised is very well grounded, since typically, a module over a ring can be quite arbitrary, and people in general don't think an arbitrary module have any important effect on the geometry. However, what you see here is that the notion of flatness, especially faithfully-flatness of modules plays very well with the geometry of morphisms. One special case is when $\mathcal{F} = \mathcal{O}_X$, and you will see later that we have a special name for it. This remark echos with one of the previous remark on why faithfully flatness is related to surjectivity.

Proof. Recall that since we have a morphism of local rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ for each $x \in X$. And we know that $\mathcal{F}_x$ is flat over $\mathcal{O}_{Y,f(x)}$ and is finitely generated. Then we know that since $\mathcal{F}_x$ is not 0, it's faithfully flat over $\mathcal{O}_{Y,f(x)}$, then we know that this implies that $Spec\mathcal{O}_{X,x} \to Spec\mathcal{O}_{Y,f(x)}$ is surjective on the underlying topological spaces. This plus the above proposition shows that f is open at $x \in X$ for all x, then f is open. □

The following lemma are useful, which is closed related to the **generic flatness**. We discussed that in the FOAG series, and you can see that note series for more information.

Lemma 5.5. Let A be a Noetherian integral domain and B is a finitely generated A-algebra. Suppose M is a finitely generated B-algebra then there is a nonzero element $f \in A$ such that M_f is free over A_f.

Proof. Let's assume that M is not 0, then we know that any nonzero finitely generated module over a Noetherian ring has a filtration:

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that $M_i/M_{i-1} = B/p_i$ for some $p_i \in SpecB$. Recall that if we have a SES:

$$0 \to M' \to M \to M'' \to 0$$

where M', M'' are free, then M is also free. Then using induction, if we can prove that there is some f_i such that $(M_i/M_{i-1})_f$ is free over A_{f_i} for each i, we take $f = \prod_i f_i$ and we are done.

Therefore, we can assume that M is some finitely generated A-algebra B over A such that it's also a domain, and we want to find some f such that B_f is free over A_f. Let's first assume that $A \to B$ has a nontrivial kernel, then we know that if we invert f, B_f is 0, then the assertion is trivial. So let's assume that $A \to B$ is injective, i.e. A is a subring of B. Let K be the field of fractions of A, then we have the following diagram:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, hook] & B \arrow[d, hook] \\ K \arrow[r, hook] & S^{-1}B \end{tikzcd}$$

where S is all the nonzero elements of A, and $S^{-1}B$ is contained in the field of fractions of B. Recall that the $S^{-1}B$ is a domain, and it's finitely generated as a K-algebra, then we know that $\dim S^{-1}B = \mathrm{trdeg}_K S^{-1}B < \infty$. Say $n = \dim S^{-1}B$. Let's first take a look at what happens if $n = 0$. We know that from

Noether normalization that $S^{-1}B$ is just a finite K-algebra, and assume that a basis consists of elements in B, say $b_r, \cdots, b_r$. Then consider:

$$A^r \to B$$

that maps e_i to b_i, tensor this with K we get an isomorphism, but notice that we know that tensor by K over A is flat, therefore we know that the kernel and cokernel of $A^r \to B$ is 0 after we tensor with K. This implies that the kernel and cokernel are both 0 after we invert at some element, this is because the kernel and cokernel are finitely generated. So we are done with the base case and we will use induction to handle the general case where $n \neq 0$.

Now assume that $n \neq 0$, by Noether normalization again, we know that there are n algebraically independent elements $y_1, \cdots, y_n \in S^{-1}B$, such that $S^{-1}B$ is algebraic over $K[y_1, \cdots, y_n]$, bu multiply y_i's with the numerators, we can assume that $y_i \in B$.

Now think about $A \hookrightarrow A[y_1, \cdots, y_n] \hookrightarrow B$, we know that $y_1, \cdots, y_n$ is a maximal set of algebraically independent elements of B over A since if there is one more, this contradictions $y_1, \cdots, y_n$ being a transcendence basis. Then since B is finitely generated over $A[y_1, \cdots, y_n]$, we can assume that they are $x_1 \cdots, x_m$. Then consider the following diagram:

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, hook] & A[y_1, \cdots, y_n] \arrow[r, hook] \arrow[d, hook] & B \arrow[d] \\
K \arrow[r, hook] & K[y_1, \cdots, y_n] \arrow[r, hook] & S^{-1}B
\end{tikzcd}$$

we consider the generators in B as elements in $S^{-1}B$, since we know that $S^{-1}B$ is finite over $K[y_1, \cdots, y_n]$, we know that x_i satisfy some monic polynomial over $K[y_1, \cdots, y_n]$:

$$x_i^{d_i} + a_{id_{i-1}} x_i^{d_i - 1} + \cdots + a_{i0} = 0$$

Then by localize at the product g of the denominators of the coefficients of a_{ij}'s for all i, we can assume that B_g is integral over $A_g[y_1, \cdots, y_n]$.

Then denote $C = A[y_1, \cdots, y_n]$, we can assume that B is finite over C. Since B is a finitely generated module over C, we can assume there is a maximal class of elements that is linearly independent, say they are $b_1, \cdots, b_k$, then we consider the SES:

$$0 \to C^k \to B \to B' \to 0$$

Where B' is a finitely algebra over C, which is torsion. Then we again split B' using the filtration, but since B' is torsion, we know that if we tensor the filtration with the fraction field of C, everything is 0, therefore we know that in the filtration no subquotient is C, they are all C/p for some nonzero prime of C. Now consider $K \otimes (C/p)$, which is $K[y_1, \cdots, y_n]/p \otimes_A K$, notice that $p \otimes_A K$ is not zero then we know that the dimension of $K[y_1, \cdots, y_n]/p \otimes_A K$ is less than n, then the induction hypothesis implies that there is a $f \in A$ such that B'_f is free over A_f, but notice that $C = A[y_1, \cdots, y_n]$, which is already free over A, then C_f is free over A_f, this implies that B_f is free over A_f. We are done. $\square$

Lemma 5.6. Let A be a Noetherian ring, B is a finitely generated A-algebra, q a prime ideal of B, p is the inverse image of q in A. Suppose that M_q is flat over A_p, there is a $g \in B - q$ such that $(M/pM)_g$ is flat over A/p and $\mathrm{Tor}_1^A(M, A/p)_g = 0$.

Proof. By the above result, we know that there is a $f \in A - p$ such that $(M/pM)_f$ is flat over $(A/p)_f$. Recall that M_q is flat over A_p, therefore it's flat over A as $A \to A_p$ is also flat. Then we consider $\mathrm{Tor}_1^A(M, A/p)_q$, since we know that Tor functor commutes with flat base change:

$$\mathrm{Tor}_1^A(M, A/p)_q \simeq \mathrm{Tor}_1^A(M \otimes_B B_q, A/p \otimes_B B_q) = \mathrm{Tor}_1^A(M_q, A/p) = 0$$

Then consider the SES:

$$0 \to p \to A \to A/p \to 0$$

consider the LES of Tor, we get:

$$0 \to \mathrm{Tor}_1^A(M, A/p) \to M \otimes_A p \to M \to\to M \otimes_A A/p \to 0$$

Since recall that $M \otimes_A p$ is also a finitely generated module of B, then $\mathrm{Tor}_1^A(M, A/p)$ is also a finitely generated B-module, then since $\mathrm{Tor}_1^A(M, A/p)_q = 0$ and $\mathrm{Tor}_1^A(M, A/p)$ is finitely generated, we have some $g' \in B - q$ such that $\mathrm{Tor}_1^A(M, A/p)_{g'} = 0$, then we can take $g = fg'$. ☐

Lemma 5.7. Under the condition of the above lemma, for every prime q' of B that contains q but not g, $M_{q'}$ is flat over A.

Proof. Apply the Grothendieck's local criteria to $A \to B_{q'}$ with $I = p$ and $M = M_{q'}$.

Then we know that if we want to prove $M_{q'}$ is flat over A, we need to show that $M_{q'} \otimes_A A/p$ is flat over A/p and $\mathrm{Tor}_1^A(M_{q'}, A/p) = 0$. But these are all just further localizations for the localization at g since q' doesn't contain g, we are done. ☐

Remark 5.8. You probably are confused why we need the above two lemma, and here is a partial answer. Notice that in this proposition, we are considering primes q' that are in $V(q)$, but not containing g, which means that it's in $D(g)$, i.e. we are looking at primes in $D(g) \cap \overline{x}$. For each prime q' in this subset $M_{q'}$ is flat over A, which implies that it's also flat over $A_{p'}$ where p' is the inverse image of q'. Then if we consider the primes q' of $SpecB$ where $M_{q'}$ is flat over $A_{p'}$ where p' is the inverse image of q'. Say this subset is called U, then we know that the above two lemma tells us if $q \in U$, then $U \cap \overline{q}$ contains a nonempty open of $\overline{x}$. And if you remember where we used this conditions, this together with the condition that U contains generalizations, can be used to show that U is open. And suppose that $q'' \subset q$, we know we have:

$$\begin{array}{cccc} & M & M_q & M_{q''} \\ \\ A \longrightarrow & B \longrightarrow & B_q \longrightarrow & B_{q''} \end{array}$$

I will let you figure out why this implies $M_{q''}$ is also flat over A (hint: $B_q \to B_{q''}$ is flat and you can see the proof of the following theorem for more information), then U obviously contains generalizations. This proves U is open.

We will need the following theorem later, which is always called the **generic flatness theorem**:

Theorem 5.9. Let $f : X \to Y$ be a finite type morphism between Noetherian schemes and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then the following hold:

1. If Y is integral, then there is a a nonempty open of Y, hence dense such that for all $x \in f^{-1}V$, $\mathcal{F}_x$ is flat over $\mathcal{O}_{Y,f(x)}$.
2. In general, the set U of points $x \in X$ such that $\mathcal{F}_x$ is flat over $\mathcal{O}_{Y,f(x)}$ is open.

Proof. Let's first prove 1. Pick any affine open $SpecA$ of Y, we know that A is a domain, and we know that we can cover the inverse image with finitely many $SpecB_i$ where B_i is a finitely generated A-algebra. Then we know that we can assume $\mathcal{F}|_{SpecB_i}$ is $\widetilde{M_i}$ for some finitely generated M_i over B_i. We know that there is a $f_i \in A$ nonzero, such that M_{f_i} is free over A_{f_i}, thus flat. We take $f = \prod f_i$, then consider the inverse image, which can be covered by $Spec(B_i)_f$ and $\mathcal{F}|_{Spec(B_i)_f}$ is $\widetilde{(M_i)_f}$, where $(M_i)_f$ is free over $SpecA_f$. Then we know that if we localize $(M_i)_f$ at q, which is a prime in $(B_i)_f$, we can do this in two steps. First, we can first localize at the image of $A_f - p$, i.e.:

$$(M_i)_f \otimes_{(B_i)_f} (B_i)_p$$

But we have a canonical isomorphism:

$$(M_i)_f \otimes_{(B_i)_f} (B_i)_p \simeq (M_i)_f \otimes_{A_f} A_p$$

Therefore, this is free over A_p, which is flat over A_p, then consider:

$$A_p \to B_p \to B_q$$

We know that $(M_i)_f \otimes_{(B_i)_f} (B_i)_p$ over M_p is flat over A_p, and $(M_i)_q$ is a further localization of this, i.e.:

$$(M_i)_q \simeq ((M_i)_f \otimes_{(B_i)_f} (B_i)_p) \otimes_{(B_i)_p} B_q$$

And we know that $B_p \to B_q$ is flat, then we know that $(M_i)_q$ is flat over A_p. This is because we have the following result if $A \to B \to C$ is a sequence of rings such that C is flat over B and M is a B-module flat over A, then $M \otimes_B C$ is flat over A. For example we know that:

$$- \otimes_A (M \otimes_B C) \simeq (- \otimes_A M) \otimes_B C$$

we know that this is a composition, where the first is exact and second is also exact, we are done.

Then let's prove 2, I will let you figure out why we can assume that $X = SpecB$ and $Y = SpecA$, then this is just the previous remark. $\square$

Definition 5.10. A morphism $f : X \to Y$ of schemes is called **flat at** $x \in X$ if $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,f(x)}$. It's called **flat** if it's flat at every $x \in X$.

It's called a faithfully flat morphism iff it's flat and surjective on the underlying space.

Example 5.11. For example, if $f : SpecB \to SpecA$ is a morphism between affine schemes, then it's faithfully flat iff B is a faithfully flat A-algebra.

First, let's assume that f is faithfully flat. Then we know that suppose q is a prime of B, whose inverse image is p in A, then the assumption tells you that $A_p \to B_q$ is flat, since $A \to A_p$ is flat, we know that B_q is flat over A for all primes of q, this tells you that B is flat over A. We also want to prove that B is faithfully flat over A, notice that $SpecB \to SpecA$ is surjective implies that for all maximal ideals of A, say m, the fiber, i.e. $Spec(B \otimes_A A/m)$ is not empty, this plus B is flat over A implies that B is faithully flat over A.

Then if B is faithfully flat over A, we know that $SpecB \to SpecA$ is surjective. We also know $A_p \to B_q$ is flat, then f is faithfully flat, we are done.

Remark 5.12. What we discussed in this section tells you that if f is a finite type morphism between Noetherian schemes, the locus where f is flat is open. And if we know that f is flat, then f is an open mapping.

6 Descent of Quasicoherent Sheaves

From this section, we can really touch on the real topic of this chapter, i.e. descent theory. In the later study of *é*tale cohomology, you will find that descent theory is a very powerful tool. Let's start from descent of quasicoherent sheaves first.

A morphism of schemes $f : X \to Y$ is called **quasicompact** if the inverse image of any quasicompact open in Y is still quasicompact and it's equivalent to the inverse image of any affine open in Y is quasicompact in X, or there is an affine cover of Y such that the inverse image of each affine open is quasicompact in X. For the notation we will use, suppose that $g_1, g_2 : B \to C$ are two maps of sets, we define:

$$\ker(B \underset{g_2}{\overset{g_1}{\rightrightarrows}} C) = \{b \in B : g_1(b) = g_2(b)\}$$

and if $f : A \to B$ is another map of set, we say that the sequence:

$$A \xrightarrow{f} B \underset{g_2}{\overset{g_1}{\rightrightarrows}} C$$

is exact if f is injective and $\mathrm{im} f = \ker(B \underset{g_2}{\overset{g_1}{\rightrightarrows}} C)$. Now we can state the most important theorem in this sections:

Theorem 6.1. Let $g : S' \to S$ be a quasicompact faithfully flat morphism and (We usually abbreviate this as qcfp where this "p" comes from the French word "plat" which is the English word "flat") let $S'' = S' \times_S S'$, for any $1 \leqslant i \leqslant 2$, let:

$$p_i : S'' = S' \times_S S' \to S'$$

be the projection to the i-th factor, and for all $1 \leqslant i < j \leqslant 3$, let:

$$p_{ij} : S' \times_S S' \times_S S' \to S' \times_S S' = S''$$

be the projection to the (i, j)-th factor, then:

1. Let $\mathcal{F}, \mathcal{G}$ be quasicoherent $\mathcal{O}_S$-modules and let $\mathcal{F}', \mathcal{G}'$ (resp. $\mathcal{F}'', \mathcal{G}''$) be their pullback on S' (resp. S''), then the sequence:

$$\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{F}, \mathcal{G}) \xrightarrow{g^*} \mathrm{Hom}_{\mathcal{O}_{S'}}(\mathcal{F}', \mathcal{G}') \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathrm{Hom}_{\mathcal{O}_{S''}}(\mathcal{F}'', \mathcal{G}'')$$

   is exact.

2. For any $1 \leqslant i \leqslant 3$, let $\pi_i : S' \times_S S' \times_S S' \to S'$ be the projection to the i-th factor. Let $\mathcal{F}'$ be a quasicoherent sheaf on S' such that there exists an isomorphism $\sigma : p_1^*\mathcal{F}' \xrightarrow{\simeq} p_2^*\mathcal{F}'$ satisfying:

$$p_{13}^*(\sigma) = p_{23}^*(\sigma) \circ p_{12}^*(\sigma)$$

   Here we will identify $p_{13}^*(\sigma)$ with the morphism:

$$\pi_1^*\mathcal{F}' \to \pi_3^*\mathcal{F}'$$

   Similarly identity $p_{23}^*\sigma$ as a morphism $\pi_2^*\mathcal{F}' \to \pi_3^*\mathcal{F}'$, thus the composition makes sense. Here the identification is done via the canonical transformations:

$$p_{13}^* \circ p_1^* \simeq \pi_1^*, \quad p_{13}^* \circ p_2^* \simeq \pi_3^*$$

$$p_{12}^* \circ p_1^* \simeq \pi_1^*, \quad p_{12}^* \circ p_2^* \simeq \pi_2^*$$

$$p_{23}^* \circ p_1^* \simeq \pi_2^*, \quad p_{23}^* \circ p_2^* \simeq \pi_3^*$$

   Then there is a quasicoherent $\mathcal{O}_S$ module $\mathcal{F}$ and an isomorphism $\tau : g^*\mathcal{F} \simeq \mathcal{F}'$ such that the diagram:

$$\begin{tikzcd} p_1^*g^*\mathcal{F} \arrow[r, "p_1^*\tau"] \arrow[d, "\simeq"'] & p_1^*\mathcal{F}' \arrow[d, "\sigma"] \\ p_2^*g^*\mathcal{F} \arrow[r, "p_2^*\tau"'] & p_2^*\mathcal{F}' \end{tikzcd}$$

   where the vertical isomorphism is the canonical isomorphism coming from $p_1 \circ g = p_2 \circ g$. Moreover $\mathcal{F}$ is unique up to a unique isomorphism.

We say that g satisfies **descent for morphisms** if the first assertion holds. We say f satisfies **effective descent** if both 1 and 2 holds. We also say that any isomorphism $\sigma : p_1^*\mathcal{F}' \to p_2^*\mathcal{F}'$ satisfying $p_{13}^*(\sigma) = p_{23}^*\sigma \circ p_{12}^*\sigma$ is a **descent datum** for $\mathcal{F}'$.

Remark 6.2. Let's first make sense of some of the statements above. First, consider this diagram:

$$\begin{tikzcd} S' \times S' \arrow[r, "p_1"] \arrow[d, "p_2"'] & S' \arrow[d, "g"] \\ S' \arrow[r, "g"'] & S \end{tikzcd}$$

since $p_1 \circ g = p_2 \circ g$, then we know that $\mathcal{F}'', \mathcal{G}''$ is independent of you use p_1 or p_2 to pullback $\mathcal{F}', \mathcal{G}'$.

Then we need to interpret what are these p_{ij}. Especially the projection to $(1,3)$, it's quite confusing if you think $S''' = S'' \times S'$ or $S' \times S''$. Just like p_1, p_2 are defined when we think of the two S' as the same object, we will want to define p_{ij} by thinking of S'' as the same object. What is good for the interpretation is to think of the triple fiber product, which is defined by the obvious universal property, i.e. it has three projections p_i to the three factors S' and any morphism to S''' is equivalent to three morphism to S', such that after you compose with g, they agree. Then using the projections p_1, p_2, p_3, and universal properties of S'', we know that p_i, p_j gives a morphism to S'', this morphism is called p_{ij}.

Remark 6.3. Let's give a remark here about why we want such a complicated theorem. First, the morphism is assumed to be quasicompact and faithfully flat, which is going to be satisfied by étale covers of S (we will make this term precise later, but you can see directly that the surjective condition is satisfied by the "covering" conditions), so we are really proving something more general in this theorem. This is the main motivation why we want this result.

By bringing something we haven't yet seen to this discussion, we can further justify this result. Later, suppose that you have this S, you will have something called the étale-site of S. And you will want to ask a question: how can we give a sheaf on this étale site, and it turns out this this result will be crucial. We will give me explanation when we discuss étale-sites. You will later see that this result actually gives you some functor is fully faithful.

Example 6.4. Before we prove this, let's look at this example, which very important because it's going to be used in the proof and because it serves as a very important motivation. We will prove, or review that any Zariski-open cover satisfies effective descent.

Suppose that S is any scheme, and suppose that we have an open cover U_i of S, then let's consider the map:

$$S' = \coprod_i U_i \to S$$

We know that this is may not be quasicompact, but nevertheless, we will see that this satisfies the effective descent. First, let's consider what is $S' \times_S S' \to S$, we see that it's just:

$$\begin{tikzcd} \coprod_{ij} U_i \cap U_i \arrow[r, "p_1"] \arrow[d, "p_2"'] & \coprod_j U_j \arrow[d, "g"] \\ \coprod_i U_i \arrow[r, "g"'] & S \end{tikzcd}$$

Then I will let you see that the first assertion is just the assertion that $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf. And the second assertion is just the fact that a collections of sheaves defined on an open cover glues to a global sheaf iff they are isomorphism on each intersection, and the isomorphism satisfies the cocycle conditions.

Now let's try to prove the general theorem:

Proof. This proof is divided to two parts: the first part is to reduce the proof to the case where $S' = SpecA', S = SpecA$ and A' is a faithfully flat A-algebra. Here we do not make any assumption on the finiteness of f or on Noetherianity of S, S', so it will be a quite general result and some of our result in the previous section fails, for example f can not be assumed to be open anymore. After we are done with the reduction, let's prove the result for affine opens.

Now let's begin the reduction. First, suppose that we cover S by $SpecA_i$ for $i \in I$. And we cover each $f^{-1}SpecA_i$ by $SpecA'_{ij}$ and since we assumed f is quasicompact, let's assume that for each i, we can take $j \in J_i$ where J_i is a finite set. The advantage of the fact that J_i is finite is we can consider the following morphism for each i:

$$f_i : \coprod_{j \in J_i} SpecA'_{ij} \to SpecA_i$$

Since j is finite, then we know that the disjoint union on the left hand side is still affine, which is not true if we consider an infinite disjoint union. Then we know that the top morphism is affine and faithfully flat.

Then we can consider the following commutative diagram:

$$\begin{tikzcd}
\coprod_{i,j} SpecA'_{i,j} \arrow[r] \arrow[d] & \coprod_i SpecA_i \arrow[d] \\
S' \arrow[r] & S
\end{tikzcd}$$

Notice that the two vertical maps are induced by Zariski open embeddings, which is a Zariski open cover. We already know from the preceding example that the two vertical morphisms satisfies effective descent. And we want to prove is that if we assume that $f_i : \prod_{j \in J_i} SpecA'_{ij} \to SpecA_i$ satisfies effective descent, then $S' \to S'$ satisfies effective descent. This step of reduction is given by the following lemma: $\square$

Lemma 6.5. Let $f : S \to T$ and $g : R \to S$ be two morphisms of schemes. Suppose that g satisfies descent for morphisms and suppose for any morphism $g' : R' \to S'$ obtained from g by base change with an arbitrary morphism $S' \to S$ and any pair of quasicoherent sheaves $\mathcal{E}', \mathcal{F}'$ on S', the map induced by $g' : \mathrm{Hom}(\mathcal{E}', \mathcal{F}') \to \mathrm{Hom}(g'^*\mathcal{E}', g'^*\mathcal{F}')$ is injective, then f satisfies descent for morphisms iff $f \circ g$ satisfies descent for morphisms. The diagram is given below:

$$\begin{tikzcd}
R' \arrow[r, "g'"] \arrow[d] & S' \arrow[d] & \\
R \arrow[r, "g"'] & S \arrow[r, "f"'] & T
\end{tikzcd}$$

Proof. To prove this, consider the following diagram:

$$\begin{tikzcd}
R \times_S R \arrow[r, "l"] \arrow[dr, shift left, "r_1"] \arrow[dr, shift right, "r_2"'] & R \times_T R \arrow[r, "k"] \arrow[d, shift right, "q_1"'] \arrow[d, shift left, "q_2"] & S \times_T S \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] & \\
& R \arrow[r, "g"] & S \arrow[r, "g"] & T
\end{tikzcd}$$

where l is given by the following diagram:

$$\begin{tikzcd}
R \times_S R \arrow[dr, dashed] \arrow[drr, "r_2"] \arrow[ddr, "r_1"'] & & \\
& R \times_T R \arrow[r, "q_2"'] \arrow[d, "q_1"] & R \arrow[d] \\
& R \arrow[r] & T
\end{tikzcd}$$

therefore you see that $q_i \circ l = r_i$. Similarly we have:

$$\begin{tikzcd}
R \times_T R \arrow[rr, "q_2"] \arrow[d, "q_1"'] \arrow[dr, dashed, "k"] & & R \arrow[d, "g"] \\
R \arrow[dr, "g"'] & S \times_T S \arrow[r, "p_2"'] \arrow[d, "p_1"] & S \arrow[d] \\
& S \arrow[r] & T
\end{tikzcd}$$

Then given quasicoherent sheaves $\mathcal{E}$ and $\mathcal{F}$ on T. Let's first assume that descent for morphisms holds for f, we want to prove this also holds for $f \circ g$ under the assumption that it holds for g. To prove it for $f \circ g$, we need to look at the following sequence:

$$\mathrm{Hom}_T(\mathcal{E}, \mathcal{F}) \to \mathrm{Hom}_R(g^* f^* \mathcal{E}, g^* f^* \mathcal{F}) \underset{q_2^*}{\overset{q_1^*}{\rightrightarrows}} \mathrm{Hom}_{R \times_T R}(\mathcal{E}'', \mathcal{F}'')$$

Then we need to prove that if we have a morphism $h'' : g^*f^*\mathcal{E} \to g^*f^*\mathcal{F}$ such that $q_1^*h'' = q_2^*h''$, then there is a unique morphism $h : \mathcal{E} \to \mathcal{F}$ such that $g^*f^*h = h''$. Since we assumed $q_1^*h'' = q_2^*h''$, then we know that:

$$r_1^*h'' = l^*q_1^*h'' = l^*q_2^*h'' = r_2^*h''$$

So by descent for morphisms of g, we know that there is a unique morphism $h' : f^*\mathcal{E} \to f^*\mathcal{F}$ such that $g^*h' = h''$. Now think about the morphism k, I claim that it factors as: $R \times_T R \to R \times_T R \to S \times_T S$, as shown in the following diagram:

$$\begin{tikzcd}
R \times_T R \arrow[r, "q_2"] \arrow[d, dashed, "q_1"'] \arrow[dr, dashed] & R \arrow[dr, "g"] & & \\
R \arrow[dr, "id"'] & R \times_T S \arrow[r] \arrow[d] \arrow[dr, dashed] & S \arrow[dr, "id"] & \\
& R \arrow[dr, "g"'] & S \times_T S \arrow[r, "p_2"'] \arrow[d, "p_1"] & S \arrow[d] \\
& & S \arrow[r] & T
\end{tikzcd}$$

and if you look at the two dotted arrows, you can put these two dotted arrows in:

$$\begin{tikzcd}
R \times_T S \arrow[r, dashed] \arrow[d] & S \times_T S \arrow[d, "p_1"] \\
R \arrow[r, "g"] & S
\end{tikzcd}
\qquad
\begin{tikzcd}
R \times_T R \arrow[r] \arrow[d, dashed] & R \arrow[d, dashed, "g"] \\
R \times_T S \arrow[r] & S
\end{tikzcd}$$

and you can check by universal property that these two squares are all Cartesian. Therefore, the two dotted arrows are all base change of g, and therefore by our assumption, should induce injective maps on the hom set upon pullback, then so is the composition, i.e. pullback by k. Then consider $k^*p_1^*h' = q_1^*g^*h' = q_1^*h'' = q_2^*h'' = q_2^*g^*h' = k^*p_2^*h'$, then this implies that $p_1^*h' = p_2^*h'$. Then since we assumed f satisfies descent for morphisms, there is a unique $h : \mathcal{E} \to \mathcal{F}$ such that $f^*h = h'$, hence this h will be the unique morphism such that $g^*f^* = h''$, thus descent for morphisms holds for $g \circ f$.

Conversely, if we assume $f \circ g$ satisfies descent for morphisms, if $h' : f^*\mathcal{E} \to f^*\mathcal{F}$ satisfies $p_1^*h' = p_2^*h'$, then I will let you show that $h'' = g^*h'$ satisfies $q_1^*h'' = q_2^*h''$, by the descent for morphisms of $g \circ f$, there is a unique $h : \mathcal{E} \to \mathcal{F}$ such that $g^*f^*h = h''$, since we also have descent of morphisms of g and $g^*h' = h''$, then $f^*h = h'$. Therefore f satisfies descent for morphisms. □

Lemma 6.6. Any affine faithfully flat morphisms of schemes satisfies descent for morphisms.

Proof. This is a result of the previous lemma. Suppose that $f : X \to Y$ is an affine morphism which is faithfully flat. Consider the following diagram:

$$\begin{tikzcd}
\coprod_i SpecB_i \arrow[r] \arrow[d] & \coprod_i SpecA_i \arrow[d] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

where $SpecB_i$ are affine opens in Y whose inverse image is $SpecA_i$. Then notice that the two vertical morphisms satisfies effective descent, and in particular descent for morphisms. Also, for each $SpecB_i \to SpecA_i$, we know from the affine case that descent for morphisms holds, and it also holds for the disjoint union by our lemma below. Then if we can prove $\coprod_i SpecB_i \to Y$ satisfies descent morphisms, since the base change of $\coprod_i SpecB_i \to X$ will be Zariski covers, thus satisfies descent for morphisms, we can conclude f satisfies descent for morphism by the composition satisfies descent for morphisms. Then we only need to show that any base change of $\coprod_i SpecB_i \to \coprod_i SpecA_i$ is injective when pulling back morphisms.

Notice that the base change is still an affine morphism which is faithfully flat. And we want to prove the pullback morphism is injective for all faithfully flat affine morphisms. Notice that we can consider the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{E},\mathcal{F}) \arrow[r] \arrow[d, hook] & \mathrm{Hom}(f^*\mathcal{E}, f^*\mathcal{F}) \arrow[d, hook] \\
\prod_i \mathrm{Hom}(\mathcal{E}|_{SpecA_i}, \mathcal{F}|_{SpecA_i}) \arrow[r, hook] & \prod_i \mathrm{Hom}(f^*\mathcal{E}|_{SpecB_i}, f^*\mathcal{F}|_{SpecB_i})
\end{tikzcd}$$

our the lower arrow is injective since it's injective on each component by our affine case assumption, and the two vertical maps are injective because $\mathcal{H}om$ is a sheaf. Then the top arrow is injective. We are done. □

Lemma 6.7. Any pullback of affine faithfully flat morphism of schemes is still affine faithfully flat.

Proof. We know that any pullback of affine morphism is affine. And I claim that an affine morphism is faithfully flat iff there is an affine open of the target such that the induced morphism from the inverse image gives you a faithfully flat ring map for each such affine opens in the cover. I will let you prove this easy assertion. Moreover, we know that surjectivity is preserved by pullback.

Then consider the pullback, which is locally of the form:

$$\begin{tikzcd}
A \otimes_B C & C \arrow[l] \\
A \arrow[u] & B \arrow[l] \arrow[u]
\end{tikzcd}$$

since surjectivity is preserved by pullback, we only need to show that $A \otimes_B C$ is flat over C, but this is trivial. □

In conclusion, we know that any base change of affine faithfully flat morphisms satisfies descent for morphisms.

Lemma 6.8. Let $f : S \to T$ and $g : R \to S$ be two morphisms of schemes. Suppose that g satisfies effective descent and suppose that any morphism obtained from g by base change satisfies descent for morphisms. Then f satisfies effective descent iff $f \circ g$ satisfies effective descent.

Proof. We again need the following diagram:

$$\begin{tikzcd}
R \times_S R \arrow[r, "l"] \arrow[dr, shift left, "r_1"] \arrow[dr, shift right, "r_2"'] & R \times_T R \arrow[r, "k"] \arrow[d, shift right, "q_1"'] \arrow[d, shift left, "q_2"] & S \times_T S \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] & \\
 & R \arrow[r, "g"] & S \arrow[r, "f"] & T
\end{tikzcd}$$

Now suppose that f satisfies effective descent, g satisfies effective descent and any base change of g satisfies descent for morphisms, then we want to show $g \circ f$ satisfies effective descent.

Suppose that we have a quasicoherent sheaf $\mathcal{E}''$ on R together with an isomorphism $\varphi' : q_1^*\mathcal{E}'' \to q_2^*\mathcal{E}''$ subject to the cocycle conditions:

$$\pi_{13}^*\varphi' = \pi_{23}^*\varphi \circ \pi_{12}^*\varphi'$$

with π_{ij} being the projections $R \times_T R \times_T R$ to the ij-th factor. Notice that we can pullback the cocycle condition along $R \times_S R \times_S R \to R \times_T R \times_T R$, and we get a cocycle condition for g. Let's try to break this down in details. COnsider the following diagram:

$$\begin{tikzcd}
R \times_S R \times_S R \arrow[r, "\tau_{13}^*"'] \arrow[d] & R \times_S R \arrow[d, "l"'] \\
R \times_T R \times_T R \arrow[r, "\pi_{13}^*"] & R \times_T R
\end{tikzcd}$$

i.e. after we pullback π_{13}^* by $R \times_S R \times_S R \to R \times_T R \times_T R$, we get $\tau_{13}^* l^* \varphi'$. Similarly, we will get $\tau_{23}^* l^* \varphi' \circ \tau_{12}^* l^* \varphi'$. Therefore this descent datum after we pullback is a quasicoherent sheaf $\mathcal{E}''$ on R, together with an isomorphism:

$$l^*\varphi' : r_1^*\mathcal{E}'' \to r_2^*\mathcal{E}''$$

such that we have cocycle conditions:

$$\tau_{13}^* l^* \varphi' = \tau_{23}^* l^* \varphi' \circ \tau_{12}^* l^* \varphi'$$

Then we know that by the effective descent for the morphism g, there is a sheaf $\mathcal{E}'$ on S together with an isomorphism $\lambda' : g^*\mathcal{E}' \to \mathcal{E}''$ such that $l^*\varphi' \circ r_1^*\lambda' = r_2^*\lambda'$. Let's put this aside first which will be used later.

Now I claim that there exists a morphism $\varphi : p_1^*\mathcal{E}' \to p_2^*\mathcal{E}'$ such that $q_2^*\lambda' \circ k^*\varphi = \varphi' \circ q_1^*\lambda'$. I.e. the following diagram commutes:

$$\begin{tikzcd}
q_1^* g^* \mathcal{E}' \arrow[r, "k^*\varphi"] \arrow[d, "q_1^*\lambda'"'] & q_2^* g^* \mathcal{E}' \arrow[d, "q_2^*\lambda'"] \\
q_1^*\mathcal{E}'' \arrow[r, "\varphi'"'] & q_2^*\mathcal{E}''
\end{tikzcd}$$

Recall that k is a composition of two pullbacks of g, each of which satisfies descent for morphism by our assumption. Moreover, since the base change of something which is a base change of g is still a base change of g, we know that the composition also satisfies descent for morphisms. Then if we consider the morphism:

$$(q_2^*\lambda')^{-1} \circ \varphi' \circ q_1^*\lambda' : q_1^* g^* \mathcal{E}' \to q_2^* g^* \mathcal{E}'$$

and for notational benifit, we will write it as $k^*\varphi$ to indicate this morphism is indeed what we use to find φ. You can also think of this is a morphism from $k^*p_1^*\mathcal{E}' \to k^*p_2^*\mathcal{E}'$. Since k satisfies descent for morphisms, if we want to prove this morphism indeed comes from something on $p_1^*\mathcal{E}' \to p_2^*\mathcal{E}'$, we only need to pull it back along two projections of:

$$\begin{tikzcd}
(R \times_T R) \times_{S \times_T S} (R \times_T R) \arrow[r] \arrow[d] & R \times_T R \arrow[d] \\
R \times_T R \arrow[r] & S \times_T S
\end{tikzcd}$$

and prove that we get the same element after pulling back. This is a place where a lot of identifications are being made, so please read a few more times if you find it confused.

The first step is to identify $(R \times_T R) \times_{S \times_T S} (R \times_T R) \simeq R \times_S R \times_T R \times_S R$, where a Z-valued point (w, x, y, z) is mapped to (w, y, z, x). To see why, what is important is how you think about these two fibered products. Since we want to represent everything as 4 Z-valued points, we want to represent the universal property of the two fibered products as 4-maps from Z to R satisfying some conditions. For example:

$$\mathrm{Hom}(Z, (R\times_T R)\times_{S\times_T S}(R\times_T R)) \simeq R\times_S R\times_T R\times_S R) = \{(\alpha, \beta) : \alpha \in \mathrm{Hom}(Z, R\times_T R), \beta \in \mathrm{Hom}(X, R\times_T R), k\circ\alpha = k\circ\beta\}$$

We know that $\alpha = (x_1, x_2)$ where $x_1, x_2 \in \mathrm{Hom}(Z, R)$ so is $\beta = (x_3, x_4)$, and $k \circ \alpha = (g \circ x_1, g \circ x_2)$, $k \circ \beta = (g \circ x_3, g \circ x_4)$, therefore we know that $\mathrm{Hom}(Z, (R \times_T R) \times_{S\times_T S} (R \times_T R)$ is (x_1, x_2, x_3, x_4) where all $x_i \in \mathrm{Hom}(Z, R)$ such that:

$$f \circ g \circ x_1 = f \circ g \circ x_2 \quad f \circ g \circ x_3 = f \circ g \circ x_4 \quad g \circ x_1 = g \circ x_3 \quad g \circ x_2 = g \circ x_4$$

Similarly we know that we can wirte $\mathrm{Hom}(Z, (R \times_S R) \times_T (R \times_S R))$ as (x_1, x_2, x_3, x_4) such that:

$$g \circ x_1 = g \circ x_2 \quad g \circ x_3 = g \circ x_4 \quad f \circ g \circ x_2 \circ f \circ g \circ x_3$$

Then the isomorphism we talked about is given by Yoneda's lemma where we map:

$$\mathrm{Hom}(Z, (R \times_T R) \times_{S\times_T S} (R \times_T R)) \to \mathrm{Hom}(Z, (R \times_S R) \times_T (R \times_S R))$$

by $(w, x, y, z) \mapsto (w, y, z, x)$. I will let you check this is indeed a functorial isomorphism by finding it;s inverse, then Yoneda gives you the isomorphism between the two fibered products. Again, let's put this isomorphism aside and we will use it later, but notice that $(R \times_S R) \times_T (R \times_S R)$ has projections π_i for $1 \leqslant i \leqslant 4$, which is defined by first project down to the $(R \times_S R)$ and then to R. This again gives the projections π_{ij} for $1 \leqslant i < j \leqslant 4$ to $R \times_T R$. Then I claim that pullback along the first factor of $(R \times_T R) \times_{S \times_T S} (R \times_T R)$ is identified as pulling back on π^{14} from $R \times_S R \times_T R \times_S R$ to $R \times_T R$. The reason is that if you consider:

$$R \times_S R \times_T R \times_S R \simeq (R \times_T R) \times_{S \times_T S} (R \times_T R) \to R \times_T R$$

and you want to prove it is π_{14}. This seems like a very confusing work, but luckily Yoneda is here to help again. Recall that if you have a morphism $f : X \to Y$ and you want to prove that f is actually g. Suppose that $g_* : \mathrm{Hom}(U, X) \to \mathrm{Hom}(U, Y)$ is the map induced by g, and $f_* : \mathrm{Hom}(U, X) \to \mathrm{Hom}(U, Y)$ is the map induced by f, we know that $f = g$ iff for every U, $f_* = g_*$. Then in our case, we have π_{14} and this composition. For the projection π_{14}, we know that:

$$\pi_{14*} : \mathrm{Hom}(U, R \times_S R \times_T R \times_S R) \to \mathrm{Hom}(U, R \times_T R)$$

we know that it's given by $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_4)$. On the other hand, if we assume the composition is f, then we know that what it does on the hom sets are:

$$\pi_{14*} : \mathrm{Hom}(U, R \times_S R \times_T R \times_S R) \to \mathrm{Hom}(U, R \times_T R) \quad (x_1, x_2, x_3, x_4) \mapsto (x_1, x_4, x_2, x_3) \mapsto (x_1, x_4)$$

Therefore we are done. Similarly, you can prove that pulling back along the second factor of $(R \times_T R) \times_{S \times_T S} (R \times_T R)$ is pulling back along π_{23}. Then let's consider:

$$\pi_{14}^*((q_2^*\lambda')^{-1} \circ \varphi' \circ q_1^*\lambda') \qquad \pi_{23}^*((q_2^*\lambda')^{-1} \circ \varphi' \circ q_1^*\lambda')$$

We know that $\pi_{14}^* q_2^* = \pi_4^*, \pi_{14}^* q_1^* = \pi_4^*$, similarly $\pi_{23}^* q_2^* = \pi_3^*, \pi_{23}^* q_1^* = \pi_2^*$, then we actually need to prove:

$$(\pi_4^*\lambda')^{-1} \circ \pi_{14}^*\varphi' \circ \pi_1^*\lambda' = (\pi_3^*\lambda')^{-1} \circ \pi_{23}^*\varphi' \circ \pi_2^*\lambda'$$

That is we need to prove:

$$\pi_{14}^*\varphi' = \pi_4^*\lambda' \circ (\pi_3^*\lambda')^{-1} \circ \pi_{23}^*\varphi' \circ \pi_2^*\lambda' \circ (\pi_1^*\lambda)^{-1}$$

Consider $\pi_4^*\lambda' \circ (\pi_3^*\lambda')^{-1}$, I claim that it's $\pi_{34}^*\varphi'$, this is because we actually have two projections from $R \times_S R \times_T R \times_S R$ to $R \times_S R$ that we denote by τ_{12} and τ_{34} (can you see why we don't have other indices) and this combined with our earlier identity:

$$r_2^*\lambda' = l^*\varphi' \circ r_1^*\lambda'$$

implies that after we pull it back using τ_{12}^* and τ_{34}, we get our assertion, therefore we actually need to prove:

$$\pi_{14}^*\varphi' = \pi_{34}^*\varphi' \circ \pi_{23}^*\varphi' \circ \pi_{12}^*\varphi'$$

To check this, notice that we have a morphism $R \times_S R \times_T R \times_S R \to R \times_T R \times_T R \times_T R$ defined by $(x_1, \cdots, x_4) \mapsto (x_1, \cdots, x_4)$. Sorry for the notation confusion, but if you recall that we have an identity:

$$\pi_{13}^*\varphi' = \pi_{23}^*\varphi' \circ \pi_{12}^*\varphi'$$

where here the projection is from $R \times_T \times_T R$ to $R \times_T R$. We know that we can pull this back along π_{123} and π_{134} to obtain that:

$$\pi_{14}^*\varphi' = \pi_{34}^*\varphi' \circ \pi_{13}^*\varphi' \quad \pi_{13}^*\varphi' = \pi_{23}^*\varphi' \circ \pi_{12}^*\varphi'$$

Plug the second term to the first one and then pull back along $R \times_S R \times_T R \times_S R \to R \times_T R \times_T R \times_T R$, we get the desired assertion. This completes the proof of the existence of φ. This morphism φ is actually an isomorphism. To see why, notice that we can apply the entire argument above to the inverse of $(q_2^*\lambda')^{-1} \circ \varphi' \circ q_1^*\lambda'$, and conclude that there is a $\psi : p_2^*\mathcal{E}' \to p_1^*\mathcal{E}'$ such that:

$$k^*\psi = ((q_2^*\lambda')^{-1} \circ \varphi' \circ q_1^*\lambda')^{-1}$$

Therefore we know that $k^*(\psi \circ \varphi) = id$ and $k^*(\varphi \circ \psi) = id$, but since k^* is injective, we know that since k^* always maps id to id, then $\psi \circ \varphi$ and $\varphi \circ \psi$ are identity.

Now I claim that φ satisfies the cocycle condition of f ,i.e. suppose that we now denote t_{ij} by the projections from $S \times_T S \times_T S$ to $S \times_T S$, then:

$$t_{13}^*\varphi = t_{23}^*\varphi \circ t_{12}^*\varphi$$

To prove this holds, consider the map

$$R \times_T R \times_T R \to S \times_T S \times_T S$$

If you pullback $t_{13}^*\varphi$ and $t_{23}^*\varphi \circ t_{12}^*\varphi$ along the above map, we get:

$$\pi_{13}^*k^*\varphi \quad \pi_{23}^*k^*\varphi \circ \pi_{12}^*k^*\varphi$$

Notice that $k^*\varphi = (q_2^*\lambda')^{-1} \circ \varphi' \circ q_1^*\lambda'$, then we need to consider, for example:

$$\pi_{13}^*((q_2^*\lambda')^{-1} \circ \varphi' \circ q_1^*\lambda')$$

Notice that $\pi_{13}^*q_2^* = \pi_3^*$, similarly, we need to prove:

$$\pi_3^*\lambda'^{-1} \circ \pi_{12}^*\varphi' \circ \pi_1^*\lambda' = \pi_3^*\lambda'^{-1} \circ \pi_{23}^*\varphi' \circ \pi_2^*\lambda' \circ \pi_2^*\lambda'^{-1} \circ \pi_{12}^*\varphi' \circ \pi_1^*\lambda'$$

Then after the cancellation, we just get:

$$\pi_{13}^*\varphi' = \pi_{23}^*\varphi' \circ \pi_{12}^*\varphi'$$

which is something we know. Then if we can prove the pullback by $R \times_T R \times_T R \to S \times_T S \times_T S$ is injective, we are done. In particular, if we can prove it is something that satisfies descent for morphisms, we are done. I claim that $R \times_T R \times_T R \to S \times_T S \times_T S$ can be written as the composition of three morphisms, each of which being a base change of g. To see why, consider:

$$R \times_T R \times_T R \longrightarrow R \times_T R \times_T S \longrightarrow R \times_T S \times_T S \longrightarrow S \times_T S \times_T S$$

For example, the arrow $R \times_T R \times_T R \to R \times_T R \times_T S$ can fit in the following diagram:

$$\begin{tikzcd}
R \times_T R \times_T R \arrow[r] \arrow[d] & R \times_T R \times_T S \arrow[d] \\
R \arrow[r, "g"'] & S
\end{tikzcd}$$

and it's easy to see this is indeed Cartesian by checking universal properties. We want to prove that the composition satisfies descent for morphisms, and we use the previous lemma to check further base change, but again, further base change is still a base change of g, therefore the composition satisfies descent for morphisms. Therefore, by our assumption that f satisfies effective descent, we have a quasicoherent sheaf $\mathcal{E}$ on T and an isomorphism: $\lambda : f^*\mathcal{E} \simeq \mathcal{E}'$ satisfying:

$$p_2^*\lambda = \varphi \circ p_1^*\lambda$$

Pull this back by k, we get:

$$q_1^*g^*\lambda = k^*\varphi \circ q_2^*g^*\lambda$$

Notice that $q_1^*g^*\lambda$ is from $g^*f^*\mathcal{E}$ to $g^*\mathcal{E}'$, then we compose it with λ' to get an isomorphism:

$$\lambda' \circ g^*\lambda : g^*f^*\mathcal{E} \to \mathcal{E}''$$

Then I will let you prove that $\lambda' \circ g^*\lambda, \varphi'$ and $\mathcal{E}$ gives the datum we need for an effective descent of $g \circ f$, we are done.

This is not the end of the story as we only prove half of the lemma, where we assume f satisfies effective descent and we want to prove $g \circ f$ satisfies effective descent. The converse will be way easier. Now conversely, suppose that given $\mathcal{E}'$ on S and an isomorphism $\varphi : p_1^*\mathcal{E}' \to p_2^*\mathcal{R}$ satisfying cocycle conditions, i.e. the cocycle condition on $S \times_T S \times_T S$. To be specific we have an identity:

$$p_{13}^*\varphi = p_{23}^*\varphi \circ p_{12}^*\varphi$$

Let's pullback this condition on along $R \times_T R \times_T R \to S \times_T S \times_T S$ then we know that by effective descent for $f \circ g$, we know that there is a sheaf $\mathcal{E}$ on T and an isomorphism $\lambda' : g^* f^* \mathcal{E} \to \mathcal{E}'$ such that $q_2^*\lambda' = k^*\varphi \circ q_1^*\lambda'$.

Now consider $l \circ k$, I claim it factors through the image of S in $S \times_T S$ via the diagonal morphism, i.e. the following diagram holds:

$$\begin{tikzcd} R \times_S R \arrow[r] \arrow[dr] & R \times_T R \arrow[r] & S \times_T S \\ & S \arrow[ur] & \end{tikzcd}$$

You can check this by check that both maps are induced by the same maps to both factors in $S \times_T S$. Then let's try to pullback $q_2^*\lambda' = k^*\varphi \circ q_1^*\lambda'$ via l, then we get:

$$l^* q_2^* \lambda' = l^* k^* \varphi \circ l^* q_1^* \lambda'$$

Since we know that l^*k^* can be written as $(R \times_S R \to S)^*\Delta^*$, and I claim $\Delta^*\varphi = id$. To see why, we need to think about the triple diagonal:

$$\Delta_3 : S \to S \times_T S \times_T S$$

Then if we pullback the cocycle condition on $S \times_T S \times_T S$ along Δ_3, we get:

$$\Delta_3^* p_{13}^* \varphi = \Delta_3^* p_{23}^* \varphi \circ \Delta_3^* p_{12}^* \varphi$$

But notice that $p_{ij} \circ \Delta_3 = \Delta$, therefore we just get $\Delta^*\varphi^* = (\Delta^*\varphi)^2$, but since $\Delta^*\varphi$ is an isomorphism, we know that $\Delta^*\varphi = id$. Then we know that we have:

$$r_2^*\lambda' = r_1^*\lambda' = id \circ r_1^*\lambda'$$

Then as we assumed that g satisfies effective descent, then in particular if satisifes descent for morphisms, therefore therewill be a unique $\lambda : f^*\mathcal{E} \to \mathcal{E}'$ such that $g^*\lambda = \lambda'$. This λ is again an isomorphism because we know that we have:

$$r_2^*\lambda'^{-1} = r - 2^*\lambda'^{-1}$$

Then there is a unique $\psi : \mathcal{E}' \to f^*\mathcal{E}$ such that $g^*\psi = \lambda'^{-1}$, then we use g^* is injective and it take identity to identity.

$$k^* p_2^* \lambda = q_2^* g^* \lambda = k^* \varphi \circ q_1 f^* \lambda = k^*(\varphi \circ p_1^* \lambda)$$

Since we already know k satisfies descent for morphisms, this implies:

$$p_2^*\lambda = \varphi \circ p_1^*\lambda$$

This shows that f satisfies effective descent. $\square$

Lemma 6.9. Suppose that $f_i : S_i' \to S_i$ satisfies effective descent, then:

$$\coprod_i f_i : \coprod_i S_i' \to \coprod_i S_i$$

also satisfies effective descent.

Proof. The key here is to notice that the fibered product:

$$\begin{tikzcd}
\coprod_i (S_i' \times_{S_i} S_i') \arrow[r, "\coprod_i p_2"] \arrow[d, "\coprod_i p_1"'] & \coprod_i S_i' \arrow[d, "\coprod_i f_i"] \\
\coprod_i S_i' \arrow[r, "\coprod_i f_i"'] & \prod_i S_i
\end{tikzcd}$$

Then we know from the disjoint condition that the descent for morphism condition is just the descent condition of each $S_i' \to S_i$ and we can then works factor by factor.

For effective descent, we know that the triple fibered product and π_i is just

$$\coprod_i (S_i' \times_{S_i} S_i' \times_{S_i} S_i') \to \coprod_i S_i$$

Then we know that we can still take advantage of disjoint union and use the effective descent on each factor. ☐

Lemma 6.10. Suppose that U_i covers a scheme X, and $f : Y \to X$ is any morphism of schemes, with $f^{-1}U_i = V_i$, then we have the following Cartesian diagram:

$$\begin{tikzcd}
\coprod_i V_i \arrow[r] \arrow[d] & \coprod_i U_i \arrow[d] \\
Y \arrow[r, "f"'] & X
\end{tikzcd}$$

Proof. We can check the universal property directly:

$$\begin{tikzcd}
Z \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \coprod_i V_i \arrow[r] \arrow[d] & \coprod_i U_i \arrow[d] \\
& Y \arrow[r, "f"'] & X
\end{tikzcd}$$

the key here is to notice that the inverse images of U_i under $Z \to \coprod_i U_i$ are all disjoint, therefore Z is also a disjoint union of open subschemes, and I will let you do the remaining details. ☐

Combine all these lemma above we can successfully reduce the proof to the affine cave:

Proof. Again, consider our commutative diagram:

$$\begin{tikzcd}
\coprod_{i,j} SpecA_{i,j}' \arrow[r] \arrow[d] & \coprod_i SpecA_i \arrow[d] \\
S' \arrow[r] & S
\end{tikzcd}$$

We already know that the two vertical morphisms satisfy effective descent. Then if we assume that the result holds for faithfully flat morphisms between affine schemes, we know that it should also hold for the disjoint unions, which is the top morphism. And we have already proved that it's affine and faithfully flat. Then we know that any base change of this morphism satisfies descent of morphisms. Then we know that the composition $\coprod_{i,j} SpecA_{i,j}' \to S$ satisfies effective descent iff $\coprod_{i,j} SpecA_{i,j}' \to \coprod_i SpecA_i$ satisfies effective descent, which is true under our assumption, then we know the composition satisfies effective descent.

Then notice that for any base change of $\coprod_{i,j} SpecA_{i,j}' \to S'$, it's still induced by Zariski open embedding and is a cover, therefore we know that the base change satisfies effective descent, therefore we know that $\coprod_{i,j} SpecA_{i,j}' \to S$ satisfies effective descent iff $S' \to S$ satisfies effective descent, we are done. ☐

Then we only need to prove this for the affine case:

Proof. Suppose that $S = SpecA, S' = SpecA'$ where A' is a faithfully flat A-algebra. Let's first deal with the descent for morphisms part. Suppose that M, N are A-modules such that $\mathcal{F} = \widetilde{M}, \mathcal{G} = \widetilde{N}$. We set:

$$M' = M \otimes_A A', \quad N' = N \otimes_A A', M'' = M \otimes_A A' \otimes_A A', N'' = N \otimes_A A' \otimes_A A'$$

Since we know that the fibered products $S'' = S' \times_S S'$ is just $SpecA' \otimes_A A'$ with projection to $SPecA'$ the usual ring map. We know how to pullback quasicoherent sheaves. For $\mathcal{F} = \widetilde{M}$, we know that it's pullback to $SpecA'$ is $\widetilde{M \otimes_A A'}$ and it's further pullback to $SpecA' \otimes_A A'$ is just $(M \otimes_A A') \otimes'_A (A' \otimes_A A') \simeq M \otimes_A A' \otimes_A A'$. In conclusion, the sequence of hom sets we are interested in is:

$$\mathrm{Hom}_A(M, N) \overset{\otimes_A A'}{\to} \mathrm{Hom}_{A'}(M \otimes_A A', N \otimes_A A') \rightrightarrows \mathrm{Hom}_{A' \otimes_A A'}(M \otimes_A A' \otimes_A A', N \otimes_A A' \otimes_A A')$$

Let's first prove the injectivity of the first map. Recall that since A' is faithfully flat over A, one characterization of faithfully flat is flat plus the canonical map $M \to M \otimes_A A'$ is injective for all A-modules M. Then consider the following diagram:

$$\begin{tikzcd}
M \arrow[r] \arrow[d, hook] & N \arrow[d, hook] \\
M \otimes_A A' \arrow[r] & N \otimes_A A'
\end{tikzcd}$$

We know that if $f, g : M \to N$ give you the same morphism after we tensor with A', then we know that the two morphisms $M \to N \to N \otimes_A A'$ are also the same, but since $N \hookrightarrow N \otimes_A A'$ is injective, i.e. is a monomorphism, we know that this implies $f = g$.

The hard part is to show that it's exact in the middle. Suppose that $u' \in \ker(\mathrm{Hom}_{A'}(M', N') \to \mathrm{Hom}_{A''}(M'', N''))$. What this means is that suppose that we have:

$$u' : M \otimes_A A' \to N \otimes_A A'$$

then we get the same map after we tensor with $\otimes_{A'}(A' \otimes_A A')$ where we view $A' \otimes_A A'$ as an A'-algebra via A' maps to the first or second factor. Then we know that the difference between the two tensor products is that if we assume that A' maps to the first factor, then the image of u' is $u \otimes_A A'$, i.e. if we assume u' maps $m \otimes a' \mapsto n_1 \otimes a'_1 + \cdots + n_k \otimes a'_k$, then $u' \otimes_A A'$ maps $m \otimes a' \otimes a'' \mapsto (n_1 \otimes a'_1 + \cdots + n_k \otimes a'_k) \otimes a''$. But if we assume A' maps to the second factor, then we know that the image of u' maps $m \otimes a' \otimes a''$ to $\sum m_j \otimes a' \otimes a''_j$ where m_j, a''_j comes from the image of $m \otimes a''$ under u and these two images are not necessarily the same. We now assume that these two maps are the same and we want to show that if we assume this, then u' comes from something $M \to N$ by tensoring with A'. This is shown in the following diagram:

$$\begin{tikzcd}
 & M \otimes_A A' \arrow[r, "u"] \arrow[dl] \arrow[dr] & N \otimes_A A' \arrow[dl] \arrow[dr] & \\
(M \otimes_A A') \otimes_{A'} (A' \otimes_A A') \arrow[r] \arrow[dr] & (N \otimes_A A') \otimes_{A'} (A' \otimes_A A') \arrow[dr] & (M \otimes_A A') \otimes_{A'} (A' \otimes_A A') \arrow[r] \arrow[dl] & (N \otimes_A A') \otimes_{A'} (A' \otimes_A A') \arrow[dl] \\
 & M \otimes_A A' \otimes_A A' \arrow[r, shift left] \arrow[r, shift right] & N \otimes_A A' \otimes_A A' &
\end{tikzcd}$$

where we have two sets of tensors with $A' \otimes_A A'$ because the ring map $A' \to A' \otimes_A A'$ are different.

I claim that it suffices to prove that u' maps the A-submodule M of M' to the A-submodule N of N' (they are submodules again because A' is faithfully flat over A). This is because if u' comes from $M \to N$, then we know that $m \otimes a'$ is mapped to $n \otimes a'$ and the importance is that we get rid of the annoying linear combination and I will let you check that no matter A' maps to the first or second factor, we have $m \otimes a' \otimes a'' \mapsto n \otimes a' \otimes a''$, thus u' is indeed in the kernel. Then suppose that u' maps M to N, i.e. if you have something $m \otimes 1$, the image under u' is $n \otimes 1$ for some n. Then I can define $u : M \to N$ to be $m \mapsto n$ and it's clear that this is a map of A-modules such that $u \otimes_A A' = u'$.

Notice that since we assumed that u' is in the kernel, therefore, we know that any $m \in M$, the image $u'(x)$ is in the kerne of $N' \rightrightarrows N''$ as shown in the following diagram:

$$\begin{tikzcd}
M \otimes_A A' \arrow[rr, "u"] \arrow[d, shift left] \arrow[d, shift right] & & N \otimes_A A' \arrow[d, shift left] \arrow[d, shift right] \\
M \otimes_A A' \otimes_A A' \arrow[rr, shift left] \arrow[rr, shift right] & & N \otimes_A A' \otimes_A A'
\end{tikzcd}$$

Notice that the two arrows at the bottom are equal to each other by assumption, and since we are considering $m \otimes 1$, it's image in $M \otimes_A A' \otimes_A A'$ are all $m \otimes 1 \otimes 1$ no matter which map to M'' we are using.

So, to prove that $u'(x)$ is actually in N, it suffice to prove that:

$$N \to N' \to N''$$

is exact, because we have:

$$\begin{tikzcd}
M \arrow[r, hook] & M' \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & M'' \\
N \arrow[r, hook] & N' \arrow[r, shift left] \arrow[r, shift right] & N''
\end{tikzcd}$$

and if $N \to N' \rightrightarrows N''$ is exact and the image $u'(x)$ is in the kernel, then there is something n in N such that $u'(x) = n$. Therefore, with all the reductions we did above, the content of descent for morphisms is just: for any A-module N, $N \to N' \rightrightarrows N''$ is exact.

To prove this is indeed exact, consider $\psi : N' \to N''$ defined by $\psi(n \otimes a') \mapsto n \otimes a' \otimes 1 - n \otimes 1 \otimes a'$, then we only need to prove:

$$0 \to N \to N' \xrightarrow{\psi} N''$$

is exact. Here comes a really smart trick where since A' is faithfully flat over A, we can check exactness after tensor with A', i.e. we need to check the following is exact:

$$0 \to N \otimes_A A' \to N \otimes_A A' \otimes_A A' \xrightarrow{\psi \otimes 1} N \otimes_A A' \otimes_A A' \otimes_A A'$$

The advantage of this is that we can construct the following diagram and give a homotopy between identity and 0 quite easily:

$$\begin{tikzcd}
0 \arrow[r] & N \otimes_A A' \arrow[r] \arrow[d, "id"] \arrow[dl, "0"'] & N \otimes_A A' \otimes_A A' \arrow[r, "\psi \otimes id"] \arrow[d, dashed, "id"] \arrow[dl, "D_1"'] & N \otimes_A A' \otimes_A A' \otimes_A A' \arrow[d, dashed, "id"] \arrow[dl, "D_2"'] \\
0 \arrow[r] & N \otimes_A A' \arrow[r] & N \otimes_A A' \otimes_A A' \arrow[r, "\psi \otimes id"] & N \otimes_A A' \otimes_A A' \otimes_A A'
\end{tikzcd}$$

where D_1 is defined by $D_1(x \otimes a_1 \otimes a_2) = x \otimes a_1 a_2$ and D_2 is defined by $D_2(x \otimes a_1 \otimes a_2 \otimes a_3)$ is $x \otimes a_1 \otimes a_2 a_3$. To check this is indeed the homotopy we want, for example $x \otimes a_1 \otimes a_2$ is mapped first to $x \otimes a_1 \otimes 1 \otimes a_2 - x \otimes 1 \otimes a_1 \otimes a_2$, then mapped to $x \otimes a_1 \otimes a_2 - x \otimes 1 \otimes a_1 a_2$. On the other hand, we know that it's mapped to $x \otimes a_1 a_2$, then mapped to $x \otimes 1 \otimes a_1 a_2$.

Then let's prove this for effective descent. Let's first prove the existence of $\mathcal{F}$. Assume that $\mathcal{F}' = \widetilde{N'}$ for some A'-module N'. By assumption, we have an isomorphism $\sigma : p_1^* \mathcal{F} \to p_2^* \mathcal{F}$, which is given by an isomorphism of $A' \otimes_A A'$-modules:

$$\alpha : N' \otimes_{A'} (A' \otimes_A A') \to N' \otimes_{A'} (A' \otimes_A A')$$

where the two ring morphisms A' to $A' \otimes_A A'$ are defined by mapping to the first factor or the second factor. This notation is quite cumbersome, since we know that both are isomorphic to $N' \otimes_A A'$ as an $A' \otimes_A A'$-module, the only difference is they have different $a' \otimes_A A'$ actions. If $A' \to A' \otimes_A A'$ maps to the first factor, then we know that $a_1 \otimes a_2(n \otimes a) = a_1 n \otimes a_2 a$, but if it maps A' to the second factor, we know that $a_1 \otimes a_2(n \otimes a) = a_2 n \otimes a_1 a$. So for convenience, we can use the following notation to indicate the different $A' \otimes_A A'$ actions:

$$\alpha : N \otimes_A A' \to A' \otimes_A N$$

Then let's set:

$$N = \{x \in N' : \alpha(x \otimes 1) = 1 \otimes x\}$$

This N is an A-module as you can check that it's closed under addition and multiplication by elements in A. We know that this is not empty as at least 0 is in N.

Then consider the following exact sequence:

$$N \hookrightarrow N' \underset{\delta}{\overset{\gamma}{\rightrightarrows}} A' \otimes_A N'$$

where γ is given by $\gamma(x') = \alpha(x' \otimes 1)$, i.e. the composition of α with $N' \to N' \otimes_A A'$ and $\delta(x') = 1 \otimes x'$. And δ is defined by $N' \to A' \otimes_A N$. We know that since A' is faithfully flat over A, $N \hookrightarrow N'$ is indeed injective. And suppose that $\alpha(x' \otimes 1) = 1 \otimes x'$, then by definition, it comes from N, thus this is indeed exact. Since A' is fully faithfully over A, we again base change to A' and know that the following is exact:

$$N \otimes_A A' \to N' \otimes_A A' \to \underset{\delta \otimes id}{\overset{\gamma \otimes id}{\rightrightarrows}} A' \otimes_A N' \otimes_A A'$$

Now, I claim that $A' \to A' \otimes_A A'$ defined by $a' \mapsto 1 \otimes a'$ makes $A' \otimes_A A'$ a faithfully flat A'-algebra. This is because we know that any base change of a faithfully flat algebra is again faithfully flat. Then recall that in the proof of descent for morphisms we constructed an exact sequence:

$$N' \to N' \otimes_{A'} (A' \otimes_A A') \rightrightarrows N' \otimes_{A'} (A' \otimes_A A') \otimes_{A'} (A' \otimes_A A')$$

Simplify the notation using similar idea we know that the above sequence of $A' \otimes_A A'$-modules can be written as:

$$N' \to A' \otimes_A N' \rightrightarrows (A' \otimes_A N') \otimes_{A'} (A' \otimes_A A') \simeq A' \otimes_A A' \otimes_A N'$$

where $A' \otimes_A A'$ action on the last object is given by:

$$(a_1 \otimes a_2) \cdot (a' \otimes a'' \otimes n) = a_1 \cdot a' \otimes a'' \otimes a_2 n$$

Then we actually get a sequence:

$$N' \to A' \otimes_A N' \rightrightarrows A' \otimes_A A' \otimes_A N'$$

where the first arrow is given by $x \mapsto 1 \otimes x$ and the two arrows later are defined by $a \otimes x \mapsto 1 \otimes a \otimes x$ and $x \mapsto a \otimes 1 \otimes x$.

Then consider the following diagram:

$$\begin{tikzcd}
N \otimes_A A' \arrow[r] & N' \otimes_A A' \arrow[r, shift left, "\gamma \otimes id"] \arrow[r, shift right, "\delta \otimes id"'] \arrow[d, "\alpha"'] & A' \otimes_A N \otimes_A A' \arrow[d, dashed, "p_{23}^*(\alpha)"] \\
N' \arrow[r] & A' \otimes N' \arrow[r, shift left, "\xi"] \arrow[r, shift right, "\eta"'] & A' \otimes_A A' \otimes_A N
\end{tikzcd}$$

where ξ is $\xi(a' \otimes x') = a' \otimes 1 \otimes x$ and η is $\eta(a' \otimes x') \mapsto 1 \otimes a' \otimes x$. This $p_{23}^*(\alpha)$ is defined by tensor α with $A' \otimes_A A' \otimes_A A'$ as an $A' \otimes_A A'$ algebra where the ring morphism is defined by $a_1 \otimes a_2 \mapsto 1 \otimes a_1 \otimes a_2$. I will let you do the corresponding identifications, but in the end, this $p_{23}^*\alpha$ is $a_1 \otimes n \otimes a_2 \mapsto a_1 \otimes \alpha(x \otimes a_2)$.

Now I claim this square commutes. The reason is because we assume that:

$$p_{13}^*(\sigma) = p_{23}^*(\sigma) \circ p_{12}^*(\sigma)$$

Here since we use α to represent the module isomorphism corresponding to σ, let's see wht this equation means for α. We already saw what is $p_{23}^*\alpha$. I will let you prove that $p_{13}^*\alpha$ and $p_{12}^*\alpha$ can fit in the diagram:

$$N' \otimes_A A' \otimes_A A' \xrightarrow{p_{12}^*\alpha} A' \otimes_A N' \otimes_A A'$$

$$N' \otimes_A A' \otimes_A A' \xrightarrow{p_{13}^*\alpha} A' \otimes_A A' \otimes_A N$$

where $p_{12}^*\alpha(x'\otimes a_1\otimes a_2) = \alpha(x'\otimes a_1)\otimes a_2$ and $p_{13}^*\alpha(x'\otimes a_1\otimes a_2) = \sum a'_j\otimes a_1\otimes x'_j$ where $\sum a'_j\otimes x'_j = \alpha(x'\otimes a_2)$. Therefore our assumption is just the following two sequence is equal:

$$N' \otimes_A A' \otimes_A A' \xrightarrow{p_{12}^*\alpha} A' \otimes_A N' \otimes_A A' \xrightarrow{p_{23}^*\alpha} A' \otimes_A A' \otimes_A N$$

$$N' \otimes_A A' \otimes_A A' \xrightarrow{\qquad\qquad p_{13}^*\alpha \qquad\qquad} A' \otimes_A A' \otimes_A N$$

i.e. $a_1 \otimes \alpha(x \otimes a_2) = p_{23}^*\alpha(\alpha(x' \otimes a_1) \otimes a_2)$.

Then to prove community, for example if we want to prove the square with $\gamma \otimes id$ and ξ is commutative. We saw that $x' \otimes a' \mapsto \alpha(x' \otimes 1) \otimes a' = p_{12}^*\alpha(x' \otimes 1 \otimes a') \mapsto p_{23}^*\alpha(p_{12}^*\alpha(x' \otimes 1 \otimes a')) = p_{13}^*\alpha(x' \otimes 1 \otimes a') = \sum a'_j \otimes 1 \otimes x'_j$. On the other hand we know that $x' \otimes a' \mapsto \alpha(x' \otimes a') \mapsto \xi(\alpha(x' \otimes a'))$. Suppose that $\alpha(x' \otimes a') = \sum a'_j \otimes x'_j$, then we know that the result is $\sum_j a'_j \otimes 1 \otimes x_j$ To prove the other square is commutative, we know that $x' \otimes a'$ is mapped to $1 \otimes x' \otimes a'$ then it's mapped to $1 \otimes \alpha(x' \otimes a')$. On the other hand we know $x' \otimes a' \mapsto \alpha(x' \otimes a') \mapsto 1 \otimes \alpha(x' \otimes a')$. Any way, this is indeed commutative, then by the universal property of kernels, we have a induced map:

$$\begin{tikzcd}
N \otimes_A A' \arrow[r] \arrow[d, dashed, "\beta"] & N' \otimes_A A' \arrow[r, shift left, "\gamma\otimes id"] \arrow[r, shift right, "\delta\otimes id"'] \arrow[d, "\alpha"'] & A' \otimes_A N \otimes_A A' \arrow[d, dashed, "p_{23}^*(\alpha)"] \\
N' \arrow[r] & A' \otimes N' \arrow[r, shift left, "\xi"] \arrow[r, shift right, "\eta"'] & A' \otimes_A A' \otimes_A N
\end{tikzcd}$$

any 5-lemma tells you that β is an isomorphism from exactness of the two rows and the fact that $\alpha, p_{23}^*\alpha$ are isomorphisms. The corresponding isomorphism between $g^*\mathcal{F} \simeq \mathcal{F}'$ is the isomorphism τ we want, where $\mathcal{F} = \widetilde{N}$.

Finally, we need to check that we have $\sigma \circ p_1^*\tau = p_2^*\tau$. The key fact we need to check this is that $\beta(x \otimes 1) = x$ for any $x \in N$. To see why, first notice that $p_1^*\beta$ can be identified with:

$$N \otimes_A A' \otimes_A A' \to N' \otimes_A A' \quad x \otimes a_1 \otimes a_2 \mapsto a_1\beta(x \otimes 1) \otimes a_2 = (a_1 \otimes a_2)(\beta(x \otimes 1) \otimes 1) = (a_1 \otimes a_2) \cdot (x \otimes 1)$$

Then we know the composition is:

$$N \otimes_A A' \otimes_A A' \xrightarrow{p_1^*\beta} N' \otimes_A A' \xrightarrow{\alpha} A' \otimes_A N'$$

and we know that $x \otimes a_1 \otimes a_2$ is mapped to $(a_1 \otimes a_2) \cdot \alpha(x \otimes 1) = (a_1 \otimes a_2) \cdot 1 \otimes x$ On the other hand we know that $p_2^*\tau$ is identified with $N \otimes_A A' \otimes_A A' \to A' \otimes_A N', \quad x \otimes a_1 \otimes a_2 \mapsto a_1 \otimes a_2 \cdot (\beta(x \otimes 1)) = a_1 \otimes a_2 \cdot x$, thus indeed commutes.

Therefore in the end we only need to show that $\beta(x \otimes 1)$ is indeed x. To see this is true, we use the commutativity of the left square where we have:

$$1 \otimes \beta(x \otimes 1) = \alpha(x \otimes 1) = 1 \otimes x$$

But if you notice that the image of $1 \otimes \beta(x \otimes 1)$ and $1 \otimes x$ under the map $A' \otimes_A N' \to N'$ by multiplication x and $\beta(1 \otimes x)$ respectively, therefore we are done.

There is still the question of why $\mathcal{F}$ is unique up to a unique isomorphism. Suppose that $\mathcal{H}$ is another sheaf of $\mathcal{O}_S$-modules with an isomorphism $\pi : g^*\mathcal{H} \to \mathcal{F}'$ satisfying the desired conditions, then we have the following diagram:

$$\begin{tikzcd}
p_1^*g^*\mathcal{F} \arrow[r, "p_1^*\tau"] \arrow[d, dashed, "\simeq"] & p_1^*\mathcal{F}' \arrow[d, dashed, "\sigma"] & p_1^*g^*\mathcal{H} \arrow[l, "p_1^*\pi"'] \arrow[d, dashed, "\simeq"] \\
p_2^*g^*\mathcal{F} \arrow[r, "p_2^*\tau"'] & p_2^*\mathcal{F}' & p_2^*g^*\mathcal{H} \arrow[l, "p_2^*\pi"]
\end{tikzcd}$$

This implies that $p_1^*(\pi^{-1} \circ \tau) = p_2^*(\pi^{-1} \circ \tau)$, then we know from the descent of morphisms property that there is a unique morpihsm $f : \mathcal{F} \to \mathcal{G}$ such that $g^*f = \pi^{-1} \circ \tau$, and this f is an isomorphism since we can apply the same argument to $\tau^{-1} \circ \pi$, therefore we know done.

This concludes the proof of descent of quasicoherent sheaves. □

After we have done with such a long proof, let's see some corollaries of it:

Corollary 6.11. Let $g : S' \to S$ be a qcfp morphism and let $h : S' \times_S S' \to S$ be the canonical morphism, if $\mathcal{G}$ is a quasicoherent $\mathcal{O}_S$-module, then we have an exact sequence:

$$\mathcal{G} \to g_* g^* \mathcal{G} \rightrightarrows h_* h^* \mathcal{G}$$

where the two morphisms $g_* g^* \mathcal{G} \rightrightarrows h_* h^* \mathcal{G}$ are the canonical morphisms:

$$g_* g^* \mathcal{G} \to g_* p_{i*} p_i^* g^* \mathcal{G} \simeq h_* h^* \mathcal{G}$$

where p_i are the two canonical projections to S'.

Proof. Let's first explain how this morphism $g_* g^* \mathcal{G} \to g_* p_{i*} p_i^* g^* \mathcal{G}$ is defined. First, notice that we have an adjoint pair (p_i^*, p_{i*}), then we have a the canonical unit morphism:

$$g^* \mathcal{G} \to p_{i*} p_i^* g^*$$

Then apply g_* to this morphism, we are done.

To prove this sequence is exact, we can actually prove something stonger. Recall that exactness of a sequence of sheaves is equivalent to the corresponding morphism on each stalk is exact. Since we know that stalk is a filtered colimit, this preserves exactness, therefore we can actually prove that for each open U of S we have an exact sequence:

$$\mathcal{G}(U) \to (g_* g^* \mathcal{G})(U) \rightrightarrows (h_* h^* \mathcal{G})(U)$$

is exact, which is stronger than just saying the sequence is exact.

Recall that the morphism $g^{-1}U \to U$ is still qcfp for each open U of S. And we still have a Cartesian diagram:

$$\begin{tikzcd} h^{-1}U \arrow[r] \arrow[d] & g^{-1}U \arrow[d] \\ g^{-1}U \arrow[r] & U \end{tikzcd}$$

Therefore, we can apply descent for morphisms for each U and $\mathcal{O}_U, \mathcal{G}|_U$, i.e. for each U, we have an exact sequence:

$$\mathrm{Hom}_U(\mathcal{O}_U, \mathcal{G}|_U) \to \mathrm{Hom}(\mathcal{O}_{g^{-1}U}, g^* \mathcal{G}|_{g^{-1}U}) \rightrightarrows \mathrm{Hom}(\mathcal{O}_{h^{-1}U}, h^* \mathcal{G}|_{h^{-1}U})$$

But this is the sequence:

$$\mathcal{G}(U) \to g_* g^* \mathcal{G}(U) \rightrightarrows h_* h_* \mathcal{G}(U)$$

$\square$

Corollary 6.12. Let $g : S' \to S$ be a qcfp morphism and let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_S$-module. Let $\mathcal{F}', \mathcal{F}''$ be the pullback of $\mathcal{F}$ to S' and $S'' = S' \times_S S'$. Denote by $\mathrm{Quot}(\mathcal{F})$ (resp $\mathrm{Quot}(\mathcal{F}')$, $\mathrm{Quot}(\mathcal{F}'')$) the set of isomorphic classes of quasicoherent quotient sheaves of $\mathcal{F}$ (resp. $\mathcal{F}', \mathcal{F}''$), then the following sequence is exact:

$$\mathrm{Quot}(\mathcal{F}) \xrightarrow{g^*} \mathrm{Quot}(\mathcal{F}') \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathrm{Quot}(\mathcal{F}'')$$

Proof. Notice that $g : S' \to S$ is faithfully flat, then so are the two projections, as the pullback of g. In general, we know that the pullback functor is only right exact, but here I claim that pulling back along any flat morphism is actually an exact functor.

To prove this, we can actually work with arbitrary sheaf of $\mathcal{O}_S$-modules, not necessarily quasicoherent sheaves. We only need to prove that if $\mathcal{F} \hookrightarrow \mathcal{G}$ is an injective morphism of quasicoherent sheaves, $g^* \mathcal{F} \hookrightarrow g^* \mathcal{G}$ is also injective. We need to check that the morphism on the stalks is injective, i.e.:

$$(g^* \mathcal{F})_{s'} \hookrightarrow (g^* \mathcal{F}_{s'})$$

is injective for all $s' \in S'$. Recall that this can be identified with:

$$\mathcal{F}_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'} \to \mathcal{G}_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}$$

This is the morphism $\mathcal{F}_s \hookrightarrow \mathcal{G}_s$ tensor with $\mathcal{O}_{S,s}\mathcal{O}_{S',s'}$. Since we assumed that this morphism is flat, then $\mathcal{O}_{S',s}$ is flat over $\mathcal{O}_{S,s}$, we are done. Therefore, we know that the pullback of a quotient object is a quotient object and the pullback of a subobject is a subobject.

To prove this, suppose that $\mathcal{G}'$ is a quotient object of $\mathcal{F}'$ together a surjective morphism $\mathcal{F}' \to \mathcal{G}'$ (this surjective morphism is part of the definition of a quotient object!) such that $p_1^*\mathcal{G}' \simeq p_2^*\mathcal{G}'$, i.e. we are assuming $\mathcal{G}'$ is in the kernel. Suppose that the isomorphism is σ, then we will have the following diagram:

$$\begin{tikzcd}
p_1^*\mathcal{F}' \arrow[r, two heads] \arrow[d, dashed, "\simeq"] & p_1^*\mathcal{G}' \arrow[dd, "\sigma"'] \\
\mathcal{F}'' \arrow[d, dashed, "\simeq"] & \\
p_2^*\mathcal{F}' \arrow[r, two heads] & p_2^*\mathcal{G}'
\end{tikzcd}$$

This is because $p_1^*\mathcal{G}', p_2^*\mathcal{G}'$ represents the same isomorphic class of quotient objects of $\mathcal{F}$ means that there is a σ such that he following diagram commutes:

$$\begin{tikzcd}
p_1^*\mathcal{G}' \arrow[rr, "\sigma"'] & & p_2^*\mathcal{G}' \\
& \mathcal{F}'' \arrow[ul, two heads] \arrow[ur, two heads] &
\end{tikzcd}$$

The only thing we need to pay attention to is that the morphisms $p_i^*\mathcal{F}' \to p_i^*\mathcal{G}'$ are the pullback of the surjective morphism $\mathcal{F}' \to \mathcal{G}'$ we started with.

Now let $\mathcal{F}'''$ be the pullback of $\mathcal{F}$ to $S' \times_S S' \times_S S'$ via the canonical morphism to S. Then we can pullback the above square via p_{ij} for $1 \leqslant i < j \leqslant 3$ and obtain:

$$\begin{tikzcd}
p_{12}^*p_1^*\mathcal{F}' \arrow[r, two heads] \arrow[d, dashed, "\simeq"] & p_{12}^*p_1^*\mathcal{G}' \arrow[dd, "p_{12}^*(\sigma)"'] \\
p_{12}^*\mathcal{F}'' \arrow[d, dashed, "\simeq"] & \\
p_{12}^*p_2^*\mathcal{F}' \arrow[r, two heads] & p_{12}^*p_2^*\mathcal{G}'
\end{tikzcd}
\quad
\begin{tikzcd}
p_{23}^*p_1^*\mathcal{F}' \arrow[r] \arrow[d, dashed] & p_{23}^*p_1^*\mathcal{G}' \arrow[dd, "p_{23}^*(\sigma)"'] \\
p_{23}^*\mathcal{F}'' \arrow[d, dashed] & \\
p_{23}^*p_2^*\mathcal{F}' \arrow[r] & p_{23}^*p_2^*\mathcal{G}'
\end{tikzcd}
\quad
\begin{tikzcd}
p_{13}^*p_1^*\mathcal{F}' \arrow[r] \arrow[d, dashed] & p_{13}^*p_1^*\mathcal{F}' \arrow[dd, "p_{13}^*(\sigma)"'] \\
p_{13}^*\mathcal{F}'' \arrow[d, dashed] & \\
p_{13}^*p_2^*\mathcal{F}' \arrow[r] & p_{13}^*p_2^*\mathcal{G}'
\end{tikzcd}$$

This implies:

$$p_{13}^*(\sigma) = p_{23}^*(\sigma) \circ p_{12}^*(\sigma)$$

Then from effective descent, we know that there is a quasicoherent $\mathcal{O}_S$ module $\mathcal{G}$ with an isomorphism $\tau : g^*\mathcal{G} \simeq \mathcal{G}$ such that we have the following diagram:

$$\begin{tikzcd}
p_1^*g^*\mathcal{G} \arrow[r, "p_1^*\tau"'] \arrow[d, "\simeq"] & p_1^*\mathcal{G}' \arrow[d, "\sigma"'] \\
p_2g^*\mathcal{G} \arrow[r, "p_2^*\tau"] & p_2^*\mathcal{G}'
\end{tikzcd}$$

$$\begin{tikzcd}
p_1^*g^*\mathcal{G} \arrow[r, "p_1^*\tau"'] \arrow[d, "\simeq"] & p_1^*\mathcal{G}' \arrow[d, "\sigma"'] & p_1^*\mathcal{F}' \arrow[l, two heads] \arrow[d, dashed, "\simeq"] \\
p_2^*g^*\mathcal{G} \arrow[r, "p_2^*\tau"] & p_2^*\mathcal{G}' & p_2^*\mathcal{F}' \arrow[l, two heads]
\end{tikzcd}$$

This implies that $p_1^*(\tau^{-1} \circ f) = p_2^*(\tau^{-1} \circ f)$ where $f : \mathcal{F}' \to \mathcal{G}'$ is the surjective morphism we started with. Then effective descent implies that there is a morphism $h : \mathcal{F} \to \mathcal{G}$ inducing $\tau^{-1} \circ f$ after pullback using g. Since g is faithfully flat I claim that $\mathcal{F} \to \mathcal{G}$ is surjective (flat is not enough here, we must need faithfully flat, i.e. surjectivity on the underlying topological space). I will let you check this. But this implies $\mathcal{G}$ is indeed a quotient object of $\mathcal{F}$ such that it's pullback is $\mathcal{G}'$.

We still need to prove this is the unique isomorphic class. Suppose that $\mathcal{H}$ is also a quotient object of $\mathcal{F}$ with $q : \mathcal{F} \to \mathcal{H}$ such that we have the following diagram:

$$\begin{tikzcd} & \mathcal{F}' \arrow[dl, "g^*q"'] \arrow[d, "f"] \arrow[dr, "g^*h"] & \\ g^*\mathcal{H} \arrow[r, "\simeq"'] & \mathcal{G}' \arrow[r, "\simeq"'] & g^*\mathcal{G} \end{tikzcd}$$

Consider the composition of the two isomorphisms below, since we know that g^*h, g^*q pulls back to the same morphism via p_i and their pullback are still surjective morphisms, we know that the pullback of the composition of the two isomorphisms are equal, therefore it's g^*t for some isomorphism $t : \mathcal{H} \to \mathcal{G}$, then we know that $g^*(t \circ q = h)$ then again using descent of morphism we know that $t \circ q = h$, i.e. we have the following commutative diagram:

$$\begin{tikzcd} & \mathcal{F} \arrow[dl, "q"'] \arrow[dr, "h"] & \\ \mathcal{H} \arrow[rr, "t\simeq"'] & & \mathcal{G} \end{tikzcd}$$

therefore $\mathcal{H}$ and $\mathcal{G}$ are in the same isomorphic class, we are done with injectivity. □

Similarly, the same line of argument can be used to show the following, where quotient objects are replaced by subobjects:

Corollary 6.13. Let $g : S' \to S$ be a qcfp morphism and let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_S$-module. Let $\mathcal{F}', \mathcal{F}''$ be the pullback of $\mathcal{F}$ to S' and $S'' = S' \times_S S'$. Denote by $\mathrm{Sub}(\mathcal{F})$ (resp $\mathrm{Sub}(\mathcal{F}')$, $\mathrm{Sub}(\mathcal{F}'')$) the set of isomorphic classes of quasicoherent subsheaves of $\mathcal{F}$ (resp. $\mathcal{F}', \mathcal{F}''$), then the following sequence is exact:

$$\mathrm{Sub}(\mathcal{F}) \xrightarrow{g^*} \mathrm{Sub}(\mathcal{F}') \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathrm{Sub}(\mathcal{F}'')$$

Proof. Basically the same as the quotient case, the only thing to pay attention to is that here we will use the fact that if g is injective, then if:

$$g \circ f = g \circ h$$

it implies $f = h$. I will let you fill the details. □

Finally, we can also say something for closed subschemes:

Corollary 6.14. Let $g : S' \to S$ be a qcfp morphism, let $S'' = S' \times_S S'$. Denote by $\mathrm{Sub}(S)$ (resp. $\mathrm{Sub}(S'), \mathrm{Sub}(S'')$) the set of closed subschemes of S (recall that two closed embeddings give you the same closed subscheme iff there is an isomorphism that commutes with the two embeddings and this is true iff the two closed embeddings are defined by the same ideal sheaf), then the sequence:

$$\mathrm{Sub}(S) \xrightarrow{g^*} \mathrm{Sub}(S') \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathrm{Sub}(S'')$$

where g^*, p_i^* denote base change of closed embeddings.

Proof. Recall that by the previous result on subsheaves, we know that there is an exact sequence:

$$\mathrm{Sub}(\mathcal{O}_S) \xrightarrow{g^*} \mathrm{Sub}(\mathcal{O}_{S'}) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathrm{Sub}(\mathcal{O}_{S''})$$

Recall that subsheaves of $\mathcal{O}_S$ is in one-to-one correspondence with the closed subschemes, and if you consider the following diagram:

$$\begin{tikzcd} X \times_S S' \arrow[r, hook] \arrow[d] & S' \arrow[d] \\ X \arrow[r, hook] & S \end{tikzcd}$$

where $X \hookrightarrow S$ is defined by the ideal sheaf $\mathcal{I}$, i.e. there is an SES:

$$0 \to \mathcal{I} \to \mathcal{O}_S \to \mathcal{O}_X \to 0$$

Then I claim that the closed subscheme $X \times_S S'$ is defined by the following SES, or in other words, the pullback of the above SES:

$$0 \to g^*\mathcal{I} \to \mathcal{O}_{S'} \to \mathcal{O}_{X\times_S S'} \to 0$$

We already saw that pullback along a faithfully flat morphism is exact, therefore we only need to prove that $g^*\mathcal{I} \to \mathcal{O}_{S'}$ defines $X \times_S S' \hookrightarrow S'$. Let's check this locally, i.e. if we can find an affine open cover of S' such that on each affine open, the ideal sheaf $g^*\mathcal{I}$ is equal to the ideal sheaf defined by the closed embedding $X \times_S S' \to S'$, we are done. Recall that we can first cover X by affine opens, say $SpecA_i$, then in the inverse image of $SpecA_i$ in S', we cover each of them by $SpecB_{ij}$. I will let you find out why B_{ij} is flat over A_i. Then we know that the closed embedding is defined by:

$$\begin{tikzcd} SpecB_{ij}/I_iB_{ij} \arrow[r, hook] \arrow[d] & SpecB_{ij} \arrow[d] \\ SpecA_i/I_i \arrow[r, hook] & SpecA_i \end{tikzcd}$$

or in other words, we have a SES:

$$0 \to I_iB_{ij} \to B_{ij} \to B_{ij}/I_iB_{ij} \to 0$$

But since B_{ij} is flat over A_i, we can identify the above sequence with the following SES:

$$0 \to I_{ij} \otimes_{A_i} B_{ij} \to A_i \otimes_{A_i} B_{ij} \to (A/I_i) \otimes_{A_i} B_{ij} \to 0$$

But this is just the pullback $g^*\mathcal{I}$ on $SpecB_{ij}$, therefore the two ideal sheaves are equal to each other, we are done. Similarly, you can also pullback a closed subscheme along p_i and conclude that this is compatible with pulling back ideal sheaves along p_i. Therefore we have a commutative diagram:

$$\begin{tikzcd} \mathrm{Sub}(S) \arrow[r] \arrow[d, dashed, "\simeq"] & \mathrm{Sub}(S') \arrow[r, shift left] \arrow[r, shift right] \arrow[d, dashed, "\simeq"] & \mathrm{Sub}(S'') \arrow[d, dashed, "\simeq"] \\ \mathrm{Sub}(\mathcal{O}_S) \arrow[r] & \mathrm{Sub}(\mathcal{O}_{S'}) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Sub}(\mathcal{O}_{S''}) \end{tikzcd}$$

Then since the row down below is exact, the row up top is also exact. $\square$

Remark 6.15. We also have similar results for open embeddings, but we will have to wait for later results to prove it.

Proposition 6.16. Let $g : S' \to S$ be a qcfp morphism and let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_S$-module. Then $g^*\mathcal{F}$ is locally of finite type (resp. locally of finite presentation, resp. locally free of finite rank) iff $\mathcal{F}$ is so.

Proof. I will let you show that even if you don't assume quasicompactness or faithfully flatness, pullback of a finite type or finitely presented or locally free of finite rank sheaf is also of the same property. It's the only if part of the statement that needs quasicompact and faithfully flat.

Now assume that $g^*\mathcal{F}$ is finitely type. We want to prove that $\mathcal{F}$ is finite type, i.e. we need to show that there is an affine open cover of S such that $\mathcal{F}$ restricted on each of that is finitely generated. We again consider the usual diagram:

$$\begin{tikzcd} \coprod_{ij} SpecA'_{ij} \arrow[r, "g'"] \arrow[d, "\alpha"] & \coprod_i SpecA_i \arrow[d, "\beta"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

We know that the pullback of $\mathcal{F}$ to $\coprod_{ij} A'_{ij}$ is still finite type. And pulling back $\mathcal{F}$ to $\coprod_{ij} SpecA'_{ij}$ is first pulling back to $\coprod_i SpecA_i$ and then pulling back along $\coprod_{ij} SpecA'_{ij} \to \coprod_i SpecA_i$. Now we know that

$\beta^*\mathcal{F}$ on each $SpecA_i$ is just $\mathcal{F}|_{SpecA_i}$, then if we can prove $\beta^*\mathcal{F}|_{SpecA_i}$ is finite type, then $\mathcal{F}$ is finite type. Therefore, we have transferred the question downstairs to the questions upstairs.

Now notice that g' is actually $\coprod_i g_i$ where $g_i : \coprod_j SpecA'_{ij} \to SpecA_i$ is an affine morphism (since quasicompact therefore for a fixed i, we only have finitely many j) which is quasicompact and faithfully flat. Then we can actually prove that if $S' = SpecA', S = SpecA$ where A' is a faithfully flat A-algebra. Notice that $\alpha^*g^*\mathcal{F}$ is finitely generated on each $SpecA'_{ij}$, we know that on $\coprod_{ij} SpecA'_{ij}$, the restriction is still finitely generated.

To see why, if we have $X = \coprod_1^n SpecA_i$ and we have a quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$ such that on each $SpecA_i$, $\mathcal{F}$ is $\widetilde{M_i}$. Then I claim that $\mathcal{F}$ on X is $\widetilde{\coprod_1^n M_i}$. This is because we are considering disjoint unions, so there is no intersection overlapping issues, therefore we only need to prove that $\widetilde{\coprod_1^n M_i}$ restricted to each $SpecA_i$ is isomorphic to $\widetilde{M_i}$, but this is just the following algebraic statement:

$$\coprod_j M_j \otimes_{\coprod_j A_j} A_i \simeq \coprod_j (M_j \otimes_{\coprod_j A_j} A_i)$$

But notice that for $j \neq i$, $M_j \otimes_{\coprod_j A_j} A_i = 0$ as we know that for each $m \otimes a$, we can think of it as $(0, \cdots, a, \cdots, 0)m \otimes 1 = 0$. Therefore, the coproduct is just $M_i \otimes_{\coprod_j A_j} A_i$ which is isomorphic to M_i, I will let you prove it.

Moreover if each M_i is finitely generated over A_i, then $\coprod_1^n M_i$ is finitely generated over $\coprod_i A_i$, and the generators can be given component-wise. Therefore, we have reduced the question to:

$$S' = SpecA' \to S = SpecA$$

where A' is a faithfully flat A-algebra and if we assume that M is an A-module such that $M \otimes_A A'$ is finitely generated over A', then we want to prove M is finitely generated over A. We know that any A-module M is a filtered colimit of it's finitely generated submodules, i.e. we have an canonical isomorphism:

$$M = \varinjlim_{\lambda} M_\lambda$$

where λ goes over all finitely generated submodules of M. Recall that tensor product commutes with filtered colimit, therefore:

$$M \otimes_A A' = \varinjlim_{\lambda} M_\lambda \otimes_A A'$$

Since we assumed that M is finitely generated over A', we can assume that these generators are $m_1 \otimes 1, \cdots, m_n \otimes 1$, therefore, we know that there are some λ such that the following map is surjective:

$$M_\lambda \otimes_A A' \to M \otimes_A A'$$

Since A' is faithfully flat over A, we know that $M_\lambda \to M$ must also be surjective, therefore M is finitely generated as well.

Now since finitely presented is also a affine local property, using the same argument as above, we again reduced to the case where $S' = SpecA' \to S = SpecA$ where A' is a faithfully flat A-module. Now if $M \otimes_A A'$ is finitely presented, it's finitely generated, therefore M is finitely generated, then we have a SES of A-modules:

$$0 \to R \to L \to M \to 0$$

where L is a free A-module of finite rank, then since A' is faithfully flat over A, we tensor A':

$$0 \to R \otimes_A A' \to L \otimes_A A' \to M \otimes_A A' \to 0$$

Recall that finitely presented module is always finitely presented, therefore $R \otimes_A A'$ is finitely generated, therefore R is finitely generated, we are done.

Finally to prove the statement for locally freeness, we need the following lemma: □

Lemma 6.17. Let M be an A-module, then the following statements are equivalent:

1. $\widetilde{M}$ is locally free of finite rank on $SpecA$.

2. M is a finitely presented projective A-module.

3. M is a flat A-module of finite presentation.

Proof. Let's first prove $1 \Rightarrow 2$. Since $\widetilde{M}$ is locally free of finite rank. There is a finite affine open cover $\{U_i\}$ of $SpecA$ such that $\widetilde{M}|_{U_i}$ is free of finite rank. Then consider the canonical morphism:

$$g : \coprod_i U_i \to SpecA$$

which is quasicompact and faithfully flat. It's obviously quasicompact, and we know that for each $x \in \coprod_i U_i$, the stalk of the structure sheaf at x is the same as the stalk of the structure sheaf of X at x, therefore obviously flat over itself. Then since this morphism is obviously surjective, it's indeed qcfp.

If you consider $g^*\widetilde{M}$, I claim that it's $\tilde{N}$ where N is finitely presented and more over it's free of finite rank. This is because N is actually a finite product of N_i where each N_i is free over A_i of finite rank. Then we know that M is finitely presented by the previous proposition. Since M is finitely presented, for any A-module N, we have a canonical isomorphism:

$$\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \tilde{N}) \simeq \widetilde{\mathrm{Hom}_A(M, N)}$$

This is not true if M is not finitely generated, the key here is that there is always a morphism:

$$(\mathrm{Hom}_A(M, N))_f \to \mathrm{Hom}_{A_f}(M_f, N_f)$$

which is defined by sending $g : M \to N$ to the localized map $g_f : M_f \to N_f$. This is not always an isomorphism, but when M is a finitely presented A-module, this is indeed an isomorphism, the reason is we have an exact sequence:

$$A^s \longrightarrow A^r \longrightarrow M \longrightarrow 0$$

and you notice that two terms in the sequence are free. For a free A-modulea M, you can check that for any A-module N, the morphism $(\mathrm{Hom}_A(M, N))_f \to \mathrm{Hom}_{A_f}(M_f, N_f)$ is indeed an isomorphism, which is easy to check. Then we can use five lemma to conclude our assertion.

Then since $\widetilde{M}$ is locally free, we know that $\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, -)$ is exact, therefore $\widetilde{\mathrm{Hom}(M, -)}$ is exact, therefore $\mathrm{Hom}(M, -)$ is exact. Therefore M is projective.

$2 \Rightarrow 3$ is easy, recall since M is finitely presented and projective, we know that $M \oplus R = A^{\oplus n}$ for some finite n and some finitely generated A-module A. This proves M is flat as $A^{\oplus n}$ if flat and tensor commutes with tensor.

Finally to prove $3 \Rightarrow 1$, we assume M is flat and finitely presented, we want to show $\widetilde{M}$ is locally free. For each $p \in SpecA$, we know that M_p is flat and finitely presented over A_p, this implies that M_p is actually free of finite rank. This is because we have:

$$0 \to R_p \to L_p \to M_p \to 0$$

where L_p is a finite rank free A_p-module. Since M_p is flat, we know that $\mathrm{Tor}^1_A(M_p, k(p))$ is 0, then we have a SES:

$$0 \to R_p/pA_pR_p \to L_p/pA_pL_p \to M_p/pA_pM_p \to 0$$

We can pick L such that $L_p/pA_pL_p \to M_p/pA_pM_p$ is an isomorphism, therefore $R_p = pA_pR_p$, therefore since R_p is finitely generated, $R_p = 0$.

This implies that there are some $x_1, \cdots, x_k \in M$ so their images in M_p forms a basis. Then consider the homomorphism:

$$\phi : A^k \to M, \quad e_i \mapsto x_i$$

We know that we have an exact sequence:

$$0 \to \ker\phi \to A^k \to M \to \mathrm{coker}\phi \to 0$$

We localize at p, we know that $\ker\phi_p = \mathrm{coker}\phi_p = 0$. Since we know that M is finitely generated, so is the cokerϕ, then we know that it vanishes at p implies that there is a distinguished open around p such that cokerϕ vanishes on that distinguished open ny geometric Nakayama's lemma. Replacing $SpecA$ by that distinguished open we can assume coker$\phi = 0$, therefore ϕ is an epimorphism, then since M restricted to that distinguished open is still finitely presented, we know that the kernel is also finitely generated, but we know that at p, the kernel vanishes, therefore there is a smaller distinguished open such that $\ker\phi$ also vanishes, we are done. ☐

Then we go back to the proof of the last assertion of the previous proposition:

Proof. We can again assume $S' = SpecA', S = SpecA$ such that A' is a faithfully flat A-algebra where M is an A-module such that $M \otimes_A A'$ is locally free of finite rank, therefore by the previous lemma, $M \otimes_A A'$ is flat over A' and finitely presented. Therefore we know that M is finitely presented over A and also flat over A since A' is faithfully flat over A, we are done. ☐

7 Descent Properties of Morphisms

Let's begin this section with a definition:

Definition 7.1. Let $f : X \to S$ be a morphism of schemes, we say that f is surjective (resp. injective) if f is surjective (resp. injective) on the underlying topological spaces.

f is called **radiciel** if f is **universally injective**, i.e. it remains injective upon any base change.

Proposition 7.2. Let $f :\to S$ be a morphism, the following conditions are equivalent:

1. f is radiciel.
2. For any algebracially closed field K the map $X(K) \to S(K)$ induced by f is injective where $X(K), S(K)$ are the K-valued points, i.e. $\mathrm{Hom}(SpecK, X), \mathrm{Hom}(SpecK, S)$ respectively.
3. f is injective and for any $x \in X$ the residue field $k(x)$ of X at x is a purely inseparable algebraic extension of the residue field $k(f(x))$ of Y at $f(x)$.
4. The diagonal morphism $\Delta : X \to X \times_S X$ is surjective.

In particular any radiciel morphism is separated.

Proof. This is going to be quite a long proof. Let's first start by $1 \Rightarrow 2$. Suppose that $t_1, t_2 : SpecK \to X$ such that $f \circ t_1 = f \circ t_2$. Consider the following diagram:

$$\begin{tikzcd}
SpecK \arrow[dr, dashed] \arrow[drr, "t_i"] \arrow[ddr] & & \\
& K \times_S X \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"'] \\
& SpecK \arrow[r, "f\circ t_i"'] & S
\end{tikzcd}$$

where the dotted arrow is the graph $\Gamma_{t_i} : SpecK \to X \times_S SpecK$ and f' is the base change of f. We want to prove $t_1 = t_2$, and it suffice to prove $\Gamma_{t_1} = \Gamma_{t_2}$. But notice that $f' \circ \Gamma_{t_i} = \mathrm{Id}_{SpecK}$. Since f' is injective on the underlying topological space, we know that $\Gamma_{t_1}, \Gamma_{t_2}$ have the same image in $K \times_S X$. Since $SpecK$ is just a single point, we know that the image is also a single point, say x'. Then we know that:

$$K \overset{(f')^\natural}{\to} k(x') \overset{\Gamma_{t_i}^\natural}{\to} K$$

are the identity. This implies that $\Gamma_{t_i}^\natural$ are isomorphisms and are inverse of $(f')^\natural$, thus they are the same, therefore $\Gamma_{t_1} = \Gamma_{t_2}$. We are done.

Then let's prove $2 \Rightarrow 1$. We need to prove that for any morphism $S' \to S$, the base change of f is injective. Consider the following diagram:

$$\begin{tikzcd} S' \times_S X \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, dashed, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

Suppose that x_1', x_2' are two points in $S' \times_S X$ such that their images is s' in S'. We know that the ring map $k(s') \to k(x_1')$ is always injective, then since $k(x_2)$ is flat over $k(x)$, we know that:

$$k(x_2) \to k(x_1) \otimes_{k(x)} k(x_2)$$

is also injective, therefore $k(x_1) \otimes_{k(x)} k(x_2)$ is not 0, which means it has a maximal ideal m, i.e. we can consider the algebraic closure K of $(k(x_1) \otimes_{k(s')} k(x_2))/m$, and we have:

$$\begin{tikzcd} k(s') \arrow[r] \arrow[d] & k(x_2') \arrow[d] \\ k(x_1') \arrow[r] & K \end{tikzcd}$$

But this implies that we have two morphisms $s_i : SpecK \to X \times_S S'$ with images x_i', we know that after we compose it with $S' \times_S X \to S'$, we get the same morphism, i.e. $f's_1 = f's_2$. Therefore:

$$fg's_1 = gf's_1 = gf's_2 = fg's_2$$

By our assumption, we know that $g's_1 = g's_2$. Also, since $f's_1 = f's_2$, we know from the universal property of fibered product that $s_1 = s_2$, so in particular, $x_1' = x_2'$, therefore f' is injective, we are done.

Then let's prove $1 \Leftrightarrow 3$. Before we proceed, we need be clear that purely inseparable extension only shows up in characteristic $p > 0$ as for $p = 0$, every field is perfect, therefore there is no nontrivial inseparable extension. And as you will see later, étale cohomology is most interesting when we deal with characteristic $p > 0$.

Suppose that f is universally injective first, then in particular f is injective,let $x \in X$ and s be it's image in S, then we know that $f^{-1}s$ only consist of one point in X, i.e. x. We again, consider the base change of $f : X \to S$ by $g : S' \to S$. Now for any point $s' \in S'$ whose image is $s \in S$, we consider the fiber $f^{'-1}(s')$, since the inverse image of s is only one point x, I claim that points in this fiber is in one to one correspondence with points in $Speck(x) \otimes_{k(s)} k(s')$. To see why, we know that since the image image of s is only x, the fibered product can be computed affine locally, i.e. Suppose that $SpecA$ is an affine neighborhood of s and $SpecB$ is an affine open in the inverse image of $SpecA$ that contains x, the fiber can be computed affine locally, i.e. suppose that $SpecA$ is an affine neighborhood of s, $SpecB$ is an affine neighborhood of x in the inverse image of $SpecA$, then we know the fiber is $Spec(B \otimes_A k(s'))$. Then consider:

$$B \otimes_A k(s') \simeq k(s') \otimes_{k(s)} (k(s) \otimes_A B)$$

Now I claim that the underlying topological space of $k(s') \otimes_{k(s)} (k(s) \otimes_A B)$ is in one to one correspondence with $Speck(s') \otimes_{k(s)} k(x)$. To see why, first notice that the ring $k(s) \otimes_A B$ only has one prime ideal. The reason is f is injective, thus the fiber of s is x. Then we know that the nilradical of $k(s) \otimes_A B$ is this prime ideal m. For notation simplicity let's denote $k(s) \otimes_A B$ by C, and denote the unique maximal ideal by m. Then we know that:

$$0 \to \mathrm{Rad}(k(s) \otimes_A B) \to k(s) \otimes_A B \to (k(s) \otimes_A B)_{red} = C/m \to 0$$

Then if $C/m \simeq k(x)$, we know that we tensor $\otimes_{k(s)} k(s')$, we know that the surjection $C \otimes_{k(s)} k(s') \to k(x) \otimes_{k(s)} k(s')$ has kernel that is contained in the nilradical (it's the nilradical tensor with something), thus the underlying topological space is the same. Therefore we only need to show C/m is indeed $k(x)$. First, we have such a diagram:

$$\begin{tikzcd}
k(x) & & \\
& B \otimes_A k(s) \arrow[ul, dashed] & B \arrow[l] \arrow[ull] \\
& k(s) \arrow[u] \arrow[uul] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

Therefore we know that the dotted arrow has kernel m since the kernel is a prime ideal and m is the unique prime, then $C/m \hookrightarrow k(x)$ is injective, and since C/m is a field, we know that it's generated by the image of C. However, we know that the image of C contains the image of B, which is enough to generate $k(x)$ as a field, therefore, we are done.

But this implies that if f is universally injective, for any field k' containing $k(s)$, the scheme $Spec(k(x) \otimes_{k(s)} k')$ only has one point. Conversely, if this is true, it's obviously that f is also universally injective. Therefore to prove $1 \Leftrightarrow 3$, we only need to prove that the condition that for any field k' containing $k(s)$, the scheme $Spec(k(x) \otimes_{k(s)} k')$ only has one point is equivalent to $k(x)$ is an algebraic purely inseparable extension of $k(s)$. For convenience, we will denote $k(s)$ by k and $k(x)$ by K.

First we assume $Spec(k(x) \otimes_{k(s)} k')$ only has one point, now if we assume K is not algebraic over k, there is an element $t \in K$ which is transcendental over k, which means that if we consider:

$$k(t) \otimes_k k[T] \simeq k(t)[T]$$

Let's consider $m = (T - t)$, since t is transcendental over k, we know that $m \cap k[T]$ this is because elements in $(T - t)$ are of the form $g(T - t) = gT - gt$ where g is a polynomial in $k(t)[T]$, then since t is transcendental over k, for any polynomial f with coefficients in k, $f(t) \neq 0$. I will let you prove that this implies $k[T] \cap m = 0$. Also we know that the 0 prime ideal in $k(t)[T]$ also intersect $k[t]$ at 0, therefore we know that if we consider the morphism:

$$Speck(t)[T] \to Speck[T]$$

There are at least 2 points in the fiber of the zero prime ideal. The fiber is the spectrum of:

$$k(t) \otimes_k k[T] \otimes_{k[T]} k(T) \simeq k(t) \otimes_k k(T)$$

Then consider the inclusion $k(t) \otimes_k k(T) \hookrightarrow K \otimes_k k(t)$, the morphism:

$$SpecK \otimes_k k(T) \to Speck(t) \otimes_k k(T)$$

is surjective, this is because $K \otimes_k k(T)$ is actually faithfully flat over $k(t) \otimes_k k(T)$. This is because we know that the field extension $k(t) \hookrightarrow K$ is faithfully flat, then consider the base change:

$$\begin{tikzcd}
SpecK \otimes_k k(T) \arrow[r] \arrow[d] & Speck(t) \otimes_k k(T) \arrow[d] \\
SpecK \arrow[r] & Speck(t)
\end{tikzcd}$$

and we know that base change of a faithfully flat morphism is always faithfully flat. Therefore, surjective, therefore $SpecK \otimes_k k(T)$ have at least 2 points, thus a contradiction.

Now suppose K is algebraic over k but not purely inseparable. We then know that there is a subfield extension of K/k, say F/k such that F is separable over k. Then let $\overline{k}$ be an algebraic closure of k, we know that:

$$Spec(F \otimes_k \overline{k})$$

contains at least 2 points. To see why, let's assume $F = k(\alpha)$ where α is separable over k with minimal polynomial f with degree at least 2. Then we know that $F = k[x]/f$ then $F \otimes_k \overline{k} = \overline{k}[x]/f$, we know that f

doesn't have repeated roots, therefore it splits to at least two distinct linear factors, therefore we know that we at least have two maximal ideals, i.e. at least two points. Then consider again:

$$SpecK \otimes_k \overline{k} \to SpecF \otimes_k \overline{k}$$

Again, consider:

$$\begin{tikzcd} SpecK \otimes_k \overline{k} \arrow[r] \arrow[d] & SpecF \otimes_k \overline{k} \arrow[d] \\ SpecK \arrow[r] & SpecF \end{tikzcd}$$

this tells you that $SpecK \otimes_k \overline{k} \to SpecF \otimes_k \overline{k}$ is surjective, therefore $SpecK \otimes_k \overline{k}$ at least has two points.

Now conversely, suppose that K/k is algebraic an purely inseparable, then for any $t \in K$ we have $t^{p^n} \in k$ for some large integer n, where p is the characteristic of k. Now let k' be a field containing k, consider $K \otimes_k k'$. Suppose that $t' \in K \otimes_k k'$, it's a finite linear combination of $\sum a_i \otimes b_i$ where $a_i \in K, b_i \in k'$, then if we consider t'^{p^n}, we know that since the characteristic is p, we get rid of the intersecting terms and know that for larger n t'^{p^n} is in k'. Notice that for any prime p in $K \otimes_k k'$, we must have $p \cap k' = 0$ as this is the inverse image of p in k', which is prime, therefore 0. Then suppose that $t' \in p$, then $t'^{p^n} \in k'$, therefore $t'^{p^n} = 0$, therefore $t' = 0$, therefore all the elements in p are nilpotenet. Therefore any prime ideal is contained in the nilradical, therefore this ring only has one prime ideal as the nilradical is the intersection of all primes.

Finally let's prove $2 \Leftrightarrow 4$. First assume 2, then suppose that $z \in X \times_S X$, let K be the algebraic closure of $k(z)$, we then have a morphism $t : SpecK \to X \times_S X$ with image z, then consider:

$$\begin{tikzcd} SpecK \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_S X \arrow[r, "p_2"'] \arrow[d, "p_1"] & X \arrow[d, "f"'] \\ & X \arrow[r, "f"] & S \end{tikzcd}$$

Then we know that $f \circ p_2 \circ t = f \circ p_1 \circ t$, so $p_2 \circ t = p_1 \circ t$. Let $t' = p_1 t = p_2 t$, then we consider:

$$p_i(\Delta t') = (p_i \Delta) t' = t' = p_i t$$

Then again by the universal property of fibered product, we know that:

$$t = \Delta t'$$

In particular z is in the image of Δ.

If we assume 4, let $t_1, t_2 : SpecK \to X$ be two morphisms such that $ft_1 = ft_2$, then by the universal property of fibered products, there is $t : SpecK \to X \times_S X$ such that $p_i t = t_i$. Let z be the image of t. Then since Δ is surjective, there is some $x \in X$ such that $\Delta(x) = z$, and since we know that Δ is surjective, it's a closed embedding, therefore $\Delta^\natural : k(z) \to k(x)$ is an isomorphism. So there is a morphism $t' : SpecK \to X$ such that $\Delta t' = t$, but then:

$$t_i = p_i t = p_i \Delta t' = t'$$

thus $t_1 = t_2$. □

Remark 7.3. Notice that in 2, we assumed for all algebraically closed field, but actually in the proof, we didn't use that the field is algebraically closed. In fact using the same argument, you can prove the same assertion with algebraically closed replaced by separably closed.

Now we officially start the descent property for morphisms, which is different from the descent for morphisms we discussed before. The key issue here is the following. When we discussed fibered products for the first time, we have seen that lots of properties of morphisms are preserved by base change. But there are essentially two problems towards this direction: First, some properties are not preserved by base change, thus we have created lots universal properties. Second, we know that if a morphism has some property, then it's pullback has this property, but we haven't discussed when can we deduce the morphism has such a property from the fact that it's pullback has that property. We will have partial answers to these questions in this section.

Proposition 7.4. Consider a Cartesian diagram:

$$
\begin{tikzcd}
S' \times_S X \arrow[r, "g'"'] \arrow[d, "f'"] & X \arrow[d, "f"'] \\
S' \arrow[r, "g"] & S
\end{tikzcd}
$$

where g is surjective, then:

1. f is surjective iff f' is.

2. If f' is injective, then so is f.

3. f is radiciel iff f' is.

Before we prove this, recall that injectivity is not something that is preserved by base change, therefore 2 in the above proposition is something we didn't expect. And usually, we say that if f is something then f' is also something. But here we have some if and only if statement, which is due to the surjectivity of g as you will see in the proof:

Proof. Let's first prove 1. For any $s' \in S'$, consider:

$$
\begin{tikzcd}
S' \times_S X \arrow[r] \arrow[dd] & X \arrow[dd] & \\
 & f'^{-1}s' \arrow[ul] \arrow[r, dashed] \arrow[dd] & f^{-1}(g(s')) \arrow[ul] \arrow[dd] \\
S' \arrow[r, "g"'] & S & \\
 & Speck(s') \arrow[ul] \arrow[r] & Speck(g(s')) \arrow[ul]
\end{tikzcd}
$$

where the dotted arrow is given by the universal property of fibered products, then I will let you prove that the square in the front face is also Cartesian, which is the result of the fact that if in the following diagram:

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & E \arrow[d] \\
B \arrow[r] & D \arrow[r] & F
\end{tikzcd}
$$

the larger rectangle is Cartesian and the right square is Cartesian, then the left square is also Cartesian. Then consider the morphism $f'^{-1}s' \to f^{-1}gs'$, as the base change of $Speck(s') \to Speck(g(s'))$, is surjective since $k(g(s'))$ is faithfully flat over $k(s')$. So we know that $f'^{-1}s'$ is not empty iff $f^{-1}(g(s'))$ is not empty, therefore f is surjective iff f' is surjective since g is surjective.

Then to prove 2 if f' is injective, then $f'^{-1}s'$ has at most one point, then since $f'^{-1}s' \to f^{-1}gs'$ is surjective, we know that $f^{-1}(g(s'))$ has at most one point, then since g is surjective, we know that $f^{-1}(s)$ for any $s \in S$ is at most one point, therefore f is injective.

For 3, if f is radiciel, we know that f' is also radiciel since each base change of f' is a base change of f. Then suppose that f' is radiciel, we want to prove f is also radiciel, consider:

$$\begin{tikzcd}
 & & X \times_S S' \times_S T \arrow[dll] \arrow[d] \arrow[drr] \arrow[ddd, shift left=2] & & \\
X \times_S S' \arrow[rr] \arrow[d, dashed, "f'"] & & X \arrow[d, dashed, "f"] & & T \times_S X \arrow[ll] \arrow[d] \\
S' \arrow[rr, "g"] & & S & & T \arrow[ll] \\
 & & S' \times_S T \arrow[ull] \arrow[urr, "g_T"'] & &
\end{tikzcd}$$

Then we know that all the four squares on the side are Cartesian by essentially the same argument as in the proof of 1, but here we also require the bottom to be Cartesian, in this case g_T is also surjective, then we know that $T \times_S X \to T$ is injective if $X \times_S S' \times_S T \to S' \times_S T$ is injective, but this as a base change of f', it's injective, we are done. $\square$

Proposition 7.5. Consider a Cartesian diagram:

$$\begin{tikzcd}
X \times_S S' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
S' \arrow[r, "g"'] & S
\end{tikzcd}$$

assume g is quasicompact and surjective, then f is quasicompact iff f' is.

Proof. We know that quasicompactness is preserved by base change no matter what kind of g we use. So the important direction here is how to use f' is quasicompact to show f is quasicompact. However, we still include the direction where we assume f is quasicompact.

First assume f is quasicompact. To prove f' is quasicompact, we know that we can do this by findding an affine cover of S' such that the inverse image under f' are all quasicompact. We can first cover S by affines, say $SpecA_i$, and cover the inverse images in S' by affines, say $SpecB_{ij}$, which gives us an affine cover of S'. Suppose that the inverse image of $SpecA_i$ in X is covered by $SpecC_{ik}$, then we know that the inverse image of $SpecB_{ij}$ can be covered by $Spec(B_{ij} \otimes_{A_i} C_{ik})$ where i, j are fixed and k can vary in a finite index set, thus we are done.

Conversely, suppose f' is quasicompact, let V be a quasicompact open of S, since g is surjective we know that $V = g(g^{-1}V)$ so:

$$f^{-1}V = f^{-1}(gg^{-1}V) = g'f'^{-1}(g^{-1}V)$$

to see why the last equality holds, we know that if g is surjective, then this is easy to show, however, if g is not surjective, we don't get this equality simply from this diagram is commutative, we do need to use this is a Cartesian diagram. I postpone the proof to a later proposition, here we first assume this.

Since g, f' are quasicompact, $f'^{-1}g^{-1}V$ are quasicompact, then we know that $g'(f'^{-1}g^{-1}V)$ is also quasicompact as g' is continuous, therefore we know that f is quasicompact. $\square$

Proposition 7.6. Consider a Cartesian diagram:

$$\begin{tikzcd}
X \times_S S' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
S' \arrow[r, "g"'] & S
\end{tikzcd}$$

Assume g is a quasicompact and faithfully flat, then f is of finite type if and only if f' is so.

Proof. Recall finite type is locally of finite type plus quasicompact. We already saw that if g is faithfully flat and quasicompact, then g is quasicomapct and surjective, therefore we know that quasicompactness of f', f is solved. We only have to think about how to deal with locally of finite type. We know that f being locally of finite type implies f' is locally of finite type, since pullback preserves locally of finite type. The proof is essentially the same as quasicompactness.

Now suppose f' is locally of finite type and quasicompact, we know that f is quasicompact and we want to prove it's locally of finite type. I claim that it suffice to assume that $S' = SpecA', S = SpecA, X = SpecB$ where A' is a faithfully flat A-algebra, then we need to prove that if $A' \otimes_A B$ is a finitely generated A'-algebra, then B is a finitely generated A-algebra. Let's first assume this claim is true. Recall that B is a filtered colimit of all finitely generated A-subalgebras:

$$B = \varinjlim_{\lambda} B_\lambda$$

where λ runs over the filtered system which indexes all finitely generated A-subalgebras. Then we know that $B \otimes_A A' \simeq \varinjlim_\lambda B_\lambda \otimes_A A'$, since we know that B is finitely generated, then we know that there is one λ such that:

$$B_\lambda \otimes_A A' \hookrightarrow B \otimes_A A'$$

is surjective, then we use A' is faithfully flat over A, we are done and conclude that $B_\lambda \hookrightarrow B$ is also surjective.

Then let's prove the claim. I claim that this is eventually will be the following algebra question. Suppose that $B_1, \cdots, B_n$ are finitely generated A-algebras, then $\coprod_1^n B_n$ is also a finitely generated A-algebra using the product algebra structure. Conversely if $\coprod_1^n B_n$ is a finitely generated A-algebra, then each B_i is a finitely generated A-algebra. I will let you prove this easy fact. Using this fact, notice that if we cover S by $SpecA_i$'s then from g, f are quasicompact, we know that we can cover the inverse images by $SpecB_{ij}, SpecC_{ik}$ respectively where for each fixed i, we can make j, k to be in a finite index set. Our job is to prove that C_{ik} for each k is a finitely generated A_i-algebra, by the above discussion, it's equivalent to prove that $\coprod_k C_{ik}$ is a finitely generated A_i-algebra. Then consider the following Cartesian diagram:

$$\begin{tikzcd}
\coprod_{ijk} SpecC_{ik} \otimes_{A_i} B_{ij} \arrow[r] \arrow[d] & \coprod_{ik} SpecC_{ik} \arrow[d, "\coprod_i f_i"'] \\
\coprod_{ij} SpecB_{ij} \arrow[r, "\coprod_i g_i"] & \coprod_i SpecA_i
\end{tikzcd}$$

This is Cartesian because fibered product always commutes with disjoint union. Then we know that our assumption is that for each fixed i and for all j, k, $C_{ik} \otimes_{A_i} B_{ij}$ is a finitely generated B_{ij}-algebra, then we can assume that for each fixed i $\coprod_{ijk} C_{ik} \otimes_{A_i} B_{ij}$ is a finitely generated $\coprod_{ij} B_{ij}$-algebra. Then we can fix i and consider the fibered product:

$$\begin{tikzcd}
\coprod_{jk} SpecC_{ik} \otimes_{A_i} B_{ij} \arrow[r] \arrow[d] & \coprod_{k} SpecC_{ik} \arrow[d, "f_i"'] \\
\coprod_{j} SpecB_{ij} \arrow[r, "g_i"] & SpecA_i
\end{tikzcd}$$

where $\coprod_j SpecB_{ij}$ is affine and is a faithfully flat A_i-algebra say is $SpecA'_i$, $\coprod_k SpecC_{ik}$ is also an affine scheme, say is $SpecB_i$, then the assumption is that the fiberd product is a finitely generated A'_i-algebra, we want to prove B_i is a finitely generated A_i-algebra, this is our claim, we are done. ☐

We then want to say more about the descent property of morphisms, especially topologically properties. We are interested because some important properties of morphisms, for example properness, involves topological properties, so we begin this discussion by the following results:

Proposition 7.7. Let $g : Y' \to Y$ be a flat morphism and let Z be a subset of Y. Assume there is a quasicompact morphism $f : X \to Y$ such that $Z = f(X)$, then $g^{-1}(\overline{Z}) = \overline{g^{-1}Z}$.

Proof. Clearly $g^{-1}\overline{Z}$ is a closed subset containing $g^{-1}Z$, therefore clearly $\overline{g^{-1}Z} \subset g^{-1}(\overline{Z})$.

Conversely, we need to show that if $y' \in g^{-1}(\overline{Z})$ then it's in $\overline{g^{-1}Z}$. Let's choose an affine open $SpecA' = U'$ of y' and an affine open $U = SpecA$ of $g(y') = y$ such that $g(U') \subset U$. We know that $y' \in U' \cap g^{-1}(U \cap \overline{Z \cap U}) = U' \cap g^{-1}(\overline{Z \cap U})$. This is because we have a standard fact from topology that for any subset Z and an open U, we know that $\overline{Z}^Y \cap U = \overline{Z \cap U}^U$ where the superscript indicates the closure is taken in U or in Y. Therefore we know that $\overline{U \cap Z} \cap U$ is actually $\overline{U \cap Z}^U$, i.e. the closure of $U \cap Z$ in U. Since we know that $y \in U \cap \overline{Z}^Y$, we know that $y \in \overline{U \cap Z}^U$, which is the same. To prove $y' \in \overline{g^{-1}Z}$, since y' is already in U', it suffices to show that $y' \in \overline{g^{-1}Z}^{U'} = \overline{g^{-1}Z \cap U} \cap U'$. Notice that $Z \cap U$ is the image of the quasicompact morphism $f^{-1}U \to U$ induced by f. We are therefore in the case where $SpecA' = Y', SpecA = Y$ and A' is a flat A-algebra. Since $f : X \to Y$ is quasicompact and Y is affine, we can cover X by finitely many affine opens and by replacing X by the disjoint union, we can also assume X is $SpecB$:

$$\begin{tikzcd}
 & SpecA = Y \\
X = SpecB \arrow[ur, "f"] & SpecA' = Y' \arrow[u, "g"']
\end{tikzcd}$$

What is $\overline{f(X)}$? Since it's induced by $A \to B$, we know that if we denote the kernel by I, I claim that $\overline{f(X)} = V(I)$. First, notice that $V(I)$ contains $f(X)$ since everything in $f(X)$ must contain the kernel. Now we need to prove that everything in $V(I)$ is in the closure, i.e. if you have a prime $p \supset I$, then any open that contains p we have that open intersect with $f(X)$. We can assume that this open is $D(f)$ where $f \notin p$. I want to show there is a prime q such that $f \notin q$ and q is the inverse image of some prime in B. Think about $A_f \to B_f$, we only need to show that B_f is not 0. If B_f is 0, then $f^n = 0$, i.e. $f \in I$, which means that $f \in p$, a contradiction, we are done.

Then we know that $g^{-1}(\overline{f(X)}) = g^{-1}V(I) = V(IA')$. Now consider the following Cartesian diagram:

$$\begin{tikzcd}
X = SpecB \arrow[r, "f"'] & SpecA = Y \\
X' = SpecB \otimes_A A' \arrow[u, "g'"'] \arrow[r, "f'"] & SpecA' = Y' \arrow[u, "g"]
\end{tikzcd}$$

since g is flat, we know that $A' \to B \otimes_A A'$ is the tensor product of $A \to B$, therefore the kernel of $A' \to B \otimes_A A'$ is IA'. Therefore $\overline{f'(X')} = V(IA')$ so we have:

$$\overline{g^{-1}Z} = \overline{g^{-1}f(X)} = \overline{f'(X')}$$

The key equality is $g^{-1}f(X) = f'g'^{-1}(X) = f'(X')$. We here run into a similar equality as what we saw in a previous proposition, here we prove this equality in the assumption that this is Cartesian. This affine case is general as the general case is reduced to the affine case. Suppose that p is a prime in B and q is a prime in A' such that their inverse images in A is x. I want to find a prime τ in $B \otimes_A A'$ such that it's inverse image in B is p in A' is q. To see how this is done, consider:

$$\begin{tikzcd}
 & & k(q) \arrow[dll] & \\
k(p) \otimes_{k(x)} k(q) & B \otimes_A A' \arrow[l, dashed] & A' \arrow[l] \arrow[u] & \\
 & B \arrow[u] \arrow[d] & A \arrow[l] \arrow[u] \arrow[r] & k(x) \arrow[uul] \arrow[dll] \\
 & k(p) \arrow[uul] & &
\end{tikzcd}$$

Then notice that $k(p) \otimes_{k(x)} k(q)$ is not 0, then it has a prime, say τ, then let the inverse image in $B \otimes_A A'$ be τ', then I will let you prove that this τ' is the prime we are looking for.

Then back to the proof, this shows $g^{-1}(\overline{Z}) = \overline{g^{-1}Z} = V(IA')$, we are one. $\square$

Remark 7.8. Note that the only place we used flatness is to prove that the kernel of $A' \to A' \otimes_A B$ is IA'.

This result has in fact a lot of useful corollaries:

Corollary 7.9. Let $g : Y' \to Y$ be a quasicompact flat morphism and let Z' be a closed subset of Y' satisfying $Z' = g^{-1}g(Z')$, then $Z' = g^{-1}(\overline{g(Z')})$. Moreover the subspace topology on $g(Y')$ induced from Y coincides with the quotient topology induced from Y'.

Proof. First, recall that the quotient topology on $g(Y')$ is defined by an subset of $g(Y')$ is closed iff it's inverse image if closed.

Let's prove the first assertion, notice that by the previous result, we know that if we define $Z = g(Z')$, then we can consider the canonical reduced closed embedding:

$$Z' \hookrightarrow Y' \to X$$

Recall that all closed embeddings are quasicompact, then the composition is quasicompact, with image Z. Therefore we can apply the previous proposition and deduce that:

$$g^{-1}(\overline{Z}) = g^{-1}(\overline{g(Z')}) = \overline{g^{-1}g(Z')} = \overline{Z'}$$

To prove the second assertion, we need to prove that any subset of $g(Y')$ that is closed under the subspace topology is closed under the quotient topology and vice versa. Notice that since g is continuous in the first place, suppose that a subset in $g(Y')$ is closed under the subspace topology, it's inverse image in Y' is closed, then it's closed under the quotient topology. Then suppose that now Z is closed under the quotient topology, then it's inverse image in Y' is closed. Now apply the first assertion to $Z' = g^{-1}Z$, we know that $g^{-1}Z = g^{-1}(\overline{gg^{-1}Z})$, i.e. $g^{-1}Z = g^{-1}\overline{Z}$. hence we know that $Z = \overline{Z} \cap g(Y')$, therefore Z is closed with respect to the subspace topology. □

Corollary 7.10. Assume $g : Y' \to Y$ be a quasicompact faithfully flat morphism then the topology on Y coincides with the quotient topology induced from Y'.

Proof. Since g is faithfully flat, it's surjective, therefore $g(Y') = Y$. □

Corollary 7.11. Suppose that $g : S' \to S$ is qcfp, then let $S'' = S' \times_S S'$ and let $\mathrm{O}(S)$ (resp. $\mathrm{O}(S'), \mathrm{O}(S'')$) be the set of open subsets of S (resp. S', S'') and let $\mathrm{F}(S)$ (resp. $\mathrm{F}(S'), \mathrm{F}(S'')$) be the set of closed subsets of S (resp. S', S''). Then the following sequences are exact:

$$\mathrm{O}(S) \to \mathcal{O}(S') \underset{p_2^{-1}}{\overset{p_1^{-1}}{\rightrightarrows}} \mathrm{O}(S'') \quad \mathrm{F}(S) \to \mathcal{F}(S') \underset{p_2^{-1}}{\overset{p_1^{-1}}{\rightrightarrows}} \mathrm{F}(S'')$$

Proof. Let's prove the assertion for closed subsets. First, injectivity, suppose that Z_1, Z_2 are closed subsets of S such that the inverse images coincide. Therefore let $Z' = g^{-1}Z_i$, then we know that $Z_i = gg^{-1}Z_i$ since g is surjective. Then we know that $Z_1 = Z_2$. Therefore the morphism is injective. Then suppose that Z' is closed in S' such that the inverse image under p_1, p_2 are the same, say is Z''. Then we know that if we denote $h = g \circ p_i$, we know that if we denote $Z = h(Z'')$, then $h^{-1}Z = Z''$, i.e. $h^{-1}h(Z'') = Z''$, then we know that this implies that:

$$Z'' = h^{-1}\overline{h(Z'')}$$

But $h^{-1}\overline{h(Z'')}$ is $p_i^{-1}g^{-1}\overline{h(Z'')}$. Consider $g^{-1}\overline{h(Z'')}$, I claim it's Z', since we know that $g^{-1}\overline{h(Z'')}$ contains Z' but if it strictly contains Z', we know that $p_i^{-1}g^{-1}\overline{h(Z'')}$ will be larger than Z'', which is a contradiction.

Then for the assertion, we only need to use the fact that taking complements is injective, and g, p_i are surjective. □

Remark 7.12. Here the result for open subsets resolves the problem we had before when we discussed the exact sequences for closed embeddings. Since open embeddings are essentially just giving open subsets, therefore this results for open subsets implies the corresponding exact sequences for open embeddings.

Corollary 7.13. Assume $g : Y' \to Y$ is qcfp, let Z be a subset of Y such that Z is the image of a quasicompact morphism (for example Y is Noetherian and Z is constructible as it's a finite disjoint union of locally closed subsets), then Z is locally closed iff $g^{-1}Z$ is so.

Proof. First, assume that Z is locally closed, then no matter what g we use $g^{-1}Z$ is also locally closed. For the converse, consider the Cartesian diagram:

$$\begin{tikzcd} g^{-1}\overline{Z} \arrow[r] \arrow[d, hook] & \overline{Z} \arrow[d, hook] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Notice that $g^{-1}\overline{Z} \to \overline{Z}$ is also quasicompact and faithfully flat. Since we know that $g^{-1}\overline{Z} = \overline{g^{-1}Z}$, if we assume $g^{-1}Z$ is locally closed we know that $g^{-1}Z$ is $U \cap \overline{g^{-1}Z} = U \cap g^{-1}\overline{Z}$, therefore it's open in $g^{-1}\overline{Z}$, then we know that Z is open in $\overline{Z}$ since we know the topology in $\overline{Z}$ is the same as the quotient topology, then Z is locally closed. □

Corollary 7.14. Consider a Cartesian diagram:

$$\begin{tikzcd} S' \times_S X \arrow[r, "g'"'] \arrow[d, "f'"] & X \arrow[d, "f"'] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

such that g is qcfp, then if f' is an open mapping (resp. closed mapping, quasicompact locally closed embedding (i.e. locally closed embeddings with quasicompact images), homeomorphism), f is so.

Proof. Let's be clear with the terminology first, here by a quasicompact embedding, I just mean that f is a quasicompact morphism such that the underlying map on topological spaces is an embedding of topological spaces.

Let's first deal with the open or closed mapping stuff. Recall that pullbacks doesn't preserve open or closed mapping, this is why we don't put iff here. Recall that for any Cartesian diagram we know that $g^{-1}f(Z) = f'g'^{-1}Z$, then if Z is open (or closed) and f' is open (or closed), then we know that $f'g'^{-1}Z$ is open (or closed). So $g^{-1}f(Z)$ is open (or closed), recall that the quotient topology is the same as the subspace topology, therefore $f(Z)$ is open (or closed). Therefore f is open (or closed).

Then suppose that f' is a locally closed embedding, we know that first f is quasicompact and injective from previous results since g is surjective. Now to prove f is an embedding, we need to prove that $f : X \to f(X)$ is a homeomorphism where $f(X)$ is endowed with the subspace topology. Now I claim that if for any closed subset Z of X, we have $f^{-1}\overline{f(Z)} = Z$, then f is an embedding. To see why, we need to show that the inverse of f is continuous, i.e. we need to show that every closed subset Z in X, $f(Z)$ is closed in $f(X)$, i.e. $f(Z) = \overline{f(Z)} \cap f(X)$, since f is injective, this is equivalent to $f^{-1}\overline{f(Z)} = Z$.

Now suppose that Z is closed in X, then we know from f is quasicompact that

$$g^{-1}(\overline{f(Z)}) = \overline{g^{-1}f(Z)} = \overline{f'g'^{-1}Z}$$

Then since f' is an embedding, we know that:

$$f'^{-1}(\overline{f'g'^{-1}Z}) = g'^{-1}Z$$

Then we know that:

$$g'^{-1}f^{-1}(\overline{f(Z)}) = f'^{-1}g^{-1}\overline{f(Z)} = f'^{-1}(\overline{f'g'^{-1}Z}) = g'^{-1}Z$$

since g is surjective we know that $f^{-1}\overline{f(Z)} = Z$, we are done.

Finally, suppose that f' is a homeomorphism, we know that f is an embedding, it's also surjective since g is surjective, therefore f is a homeomorphism. □

Remark 7.15. Note that this corollary doesn't say any thing about open embeddings or closed embeddings or locally closed embeddings of schemes, everything here are merely topological statements and at this point there is no reason to generalize it to scheme embeddings using what we discussed before. However, there will be a discussion in the next section.

Corollary 7.16. Consider a Cartesian diagram:

$$\begin{tikzcd}
S' \times_S X \arrow[r, "g'"'] \arrow[d, "f'"] & X \arrow[d, "f"'] \\
S' \arrow[r, "g"] & S
\end{tikzcd}$$

such that g is qcfp. Then f is universally open, universally closed, universally a homeomorphism iff f' is.

Proof. The only if direction is easy. So let's assume f' is for example universally open, then we use our old diagram:

$$\begin{tikzcd}
 & X \times_S S' \times_S T \arrow[dl] \arrow[d] \arrow[dr] \arrow[ddd] & \\
X \times_S S' \arrow[r] \arrow[d, dashed, "f'"] & X \arrow[d, dashed, "f"] & T \times_S X \arrow[l] \arrow[d] \\
S' \arrow[r, "g"] & S & T \arrow[l] \\
 & S' \times_S T \arrow[ul] \arrow[ur, "g_T"] &
\end{tikzcd}$$

We then know that again g_T is qcfp, then $T \times_S X \to T$ is open iff $X \times_S S' \times_S T \to S' \times_S T$ is, but this is true since f' is universally open, we are done. The other assertions are proved similarly. □

Corollary 7.17. Consider a Cartesian diagram:

$$\begin{tikzcd}
S' \times_S X \arrow[r, "g'"'] \arrow[d, "f'"] & X \arrow[d, "f"'] \\
S' \arrow[r, "g"] & S
\end{tikzcd}$$

such that g is qcfp. Then f is separated or proper iff f' is.

Proof. Again, we know that pullbacks preserved separatedness and properness. Then let's first assume f' is separated, we have the following Cartesain diagram:

$$\begin{tikzcd}
X \times_S S' \arrow[r, "\Delta"] \arrow[d, "g'"] & (X \times_S S') \times_{S'} (X \times_S S') \arrow[d] \\
X \arrow[r, "\Delta"] & X \times_S X
\end{tikzcd}$$

You can check this is Cartesian by directly verifying the universal property: two morphisms a, b from Z to X and $(X \times_S S') \times_{S'} (X \times_S S')$ which are equal after we compose the morphism to $X \times_S X$ are first a morphism $a : Z \to X$ and two morphism $b_1 : Z \to X \times_S S', b_2 : Z \to X \times_S S'$ such that after you compose the projection to S', we get the same morphism, moreover, after we compose with the projection to X and then with $X \to S$, they are the same. Unpack these, you get essentially these are just two morphism a, b to $X \times_S S'$ which are given by a_1, a_2, b_1, b_2 such that $a_1 = b_1, a_2 = b_2$, i.e. just two morphisms a, b such that their composition to S are the same, therefore just a morphism to $X \times_S S'$, we are done.

Then notice g' is qcfp, then Δ at the bottom is a closed embedding if it's a closed mapping, if the top is a closed mapping, but the top Δ is a closed embedding of schemes, therefore is a closed mapping.

Then if f' is proper, then it's finite type, separated, universally closed, then we already saw f is so, we are done. □

8 Descent of Schemes

Recall that in the previous section, we discussed when open, closed mapping and quasicompact embeddings, but these are all topological properties, not embedding of schemes. Here in this section, our first goal is to study these schemes embeddings.

Proposition 8.1. Let $g : S' \to S$ be a morphism of schemes.

1. Suppose that g is surjective and the induced morphism of sheaves of rings $\mathcal{O}_S \to g_*\mathcal{O}_{S'}$ is injective, then g is an epimorphism in the category of schemes.

2. Suppose g is surjective and the topology on S is the quotient topology induced from S'. Let $S'' = S' \times_S S'$ and p_1, p_2 be the two projections to S', $h : S'' \to S$ be the canonical morphism. Assume

$$\mathcal{O}_S \to g_*\mathcal{O}_{S'} \underset{g_*p_2^\natural}{\overset{g_*p_1^\natural}{\rightrightarrows}} h_*\mathcal{O}_{S''}$$

is exact (i.e. the sequence we get using the difference $g_*p_1^\natural - g_*p_2^\natural$) is exact as a morphism of sheaves. Then for any scheme Z, the sequence:

$$\mathrm{Hom}(S, Z) \xrightarrow{g\circ} \mathrm{Hom}(S', Z) \underset{p_2\circ}{\overset{p_1\circ}{\rightrightarrows}} \mathrm{Hom}(S'', Z)$$

is exact where g^* maps a morphism $f : S \to Z$ to fg and p_i^* map a morphism $f : S' \to Z$ to $fp_i : S'' \to Z$.

Proof. To prove the first assertion, i.e. g is an epimorphism, we need to prove that if $f_1, f_2 : S \to X$ are two morphisms of schemes such that:

$$f_1 \circ g = f_2 \circ g$$

We want to prove $f_1 = f_2$. First we know that as a morphism of topological spaces, we know that since g is a surjective morphism of topological spaces, then $f_1 = f_2$ as morphism of topological spaces. Then we need to prove:

$$f_1^\natural : \mathcal{O}_X \to (f_1)_*\mathcal{O}_S = f_2^\natural : \mathcal{O}_X \to (f_2)_*\mathcal{O}_S$$

Since $f_1 \circ g = f_2 \circ g$, we know that:

$$\mathcal{O}_X \xrightarrow{f_1^\natural} (f_1)_*\mathcal{O}_S \xrightarrow{(f_1)_*g^\natural} (f_1)_*g_*\mathcal{O}_{S'} = \mathcal{O}_X \xrightarrow{f_2^\natural} (f_2)_*\mathcal{O}_S \xrightarrow{(f_2)_*g^\natural} (f_2)_*g_*\mathcal{O}_{S'}$$

Recall that pushforward is left exact, therefore pushforward of an injective morphism is still injective, i.e. still a monomorphism, therefore this implies $f_1^\natural = f_2^\natural$.

To prove the second assertion, first g is surjective and from our assumption on the sequence of sheaves, $\mathcal{O}_S \to g_*\mathcal{O}_{S'}$ is injective, therefore by 1, g is an epimorphism, therefore:

$$\mathrm{Hom}(S, Z) \xrightarrow{g*} \mathrm{Hom}(S', Z)$$

by precomposing with g is injective. Then let $f' : S' \to Z$ be a morphism of schemes such that $f'p_1 = f'p_2$. We want to find a morphism $f : S \to Z$ such that $fg = f'$. Let's first do this by finding f as a map of topological spaces which is why we want the topology on S' to be the quotient topology. Consider the following diagram:

$$\begin{tikzcd} S' \times_S S' \arrow[r, "p_2"'] \arrow[d, "p_1"] & S' \arrow[d, "g"'] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

Then suppose that s_1', s_2' are in S' such that $g(s_1') = g(s_2')$, then from the fact that this diagram is Cartesain, we know that there is a point $s'' \in S' \times_S S' = S''$ such that $p_i(s'') = s_i'$. Then in the following diagram:

$$S'' \underset{p_2}{\overset{p_1}{\rightrightarrows}} S' \xrightarrow[f']{} Z$$

We know that $f'(s'_1) = f'p_1(s'') = f'p_2(s'') = f'(s'_2)$, then consider the universal property of the quotient morphism in the category of topological spaces:

$$\begin{tikzcd} S' \arrow[r, "f'"] \arrow[d, "g"'] & Z \\ S \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

This implies that there is a unique continuous morphism of topological spaces f such that this diagram is commutative. Then using this f we have such a diagram:

$$\begin{tikzcd} & f'_*\mathcal{O}_{S'} \arrow[r] \arrow[d, "="] & f'_*p_1^*\mathcal{O}_{S''} \arrow[d, "="] \\ \mathcal{O}_Z \arrow[ur] \arrow[r] \arrow[dr] & f_*g_*\mathcal{O}_{S'} \arrow[r, shift left, "f_*g_*p_1^\natural"] \arrow[r, shift right, "f_*g_*p_2^\natural"'] \arrow[d, "="] & f_*h_*\mathcal{O}_{S''} \arrow[d, "="] \\ & f'_*\mathcal{O}_{S'} \arrow[r] & f'_*p_2^*\mathcal{O}_{S''} \end{tikzcd}$$

The two small triangles on the left are commutative be cause $f'_*\mathcal{O}_{S'}$ is canonically equal to $f_*g_*\mathcal{O}_{S'}$ and thus we can just define $\mathcal{O}_Z \to f_*g_*\mathcal{O}_{S'}$ to be the same as $\mathcal{O}_Z \to f'_*\mathcal{O}_{S'}$. Another two squares are commutative because we again have canonical identifications and the two horizontal morphisms on the top and bottom are just $f'_*p_i^\natural$. I claim that the row in the middle, obtained by using the difference $f_*g_*p_1^\natural - f_*g_*p_2^\natural$ composes to 0. Or in other words, we need to show that the two composition are equal:

$$\mathcal{O}_Z \to f_*g_*\mathcal{O}_{S'} \overset{f_*g_*p_1^\natural}{\to} f_*h_*\mathcal{O}_{S''} = \mathcal{O}_Z \to f_*g_*\mathcal{O}_{S'} \overset{f_*g_*p_2^\natural}{\to} f_*h_*\mathcal{O}_{S''}$$

Since we have this commutative diagram, this equality follows from $f'p_1 = f'p_2$. Then recall that f_* is a left exact functor, then we know that we have an exact sequence:

$$f_*\mathcal{O}_S \to f_*g_*\mathcal{O}_{S'} \rightrightarrows f_*h_*\mathcal{O}_{S''}$$

Then the universal property of kernels gives you a morphism $\mathcal{O}_Z \to f_*\mathcal{O}_S$, together with the topological map f, we get a map of ringed spaces f such that $f' = fg$, then we need to prove this is a morphism of schemes, which means that we need to prove it's a map of locally ringed spaces, i.e. the morphism on the stalks are all morphisms of local rings. For any $s' \in S'$, the composite of the homomorphisms:

$$\mathcal{O}_{Z,fgs'} \to \mathcal{O}_{S,gs'} \to \mathcal{O}_{S',s'}$$

is a morphism of local rings since f' is a morphism of schemes. But notice that $\mathcal{O}_{S,gs'} \to \mathcal{O}_{S',s'}$ is also a morphism of local rings. Then I claim that for the maximal ideal of $\mathcal{O}_{Z,fgs'}$, if it's not mapped to the maximal ideal of $\mathcal{O}_{S,gs'}$, it's mapped to an invertible element in $\mathcal{O}_{S',s'}$ which contradicts the fact that the composition is a local morphism. $\square$

Corollary 8.2. Let $g : S' \to S$ be qcfp and let $S'' = S' \times_S S'$, the for any scheme Z, the following sequence is exact:

$$\mathrm{Hom}(S, Z) \to \mathrm{Hom}(S', Z) \rightrightarrows \mathrm{Hom}(S'', Z)$$

Being faithfully flat means g is surjective, being quasicompact plus faithfully flat means that the topology of S is the same as the quotient topology induced from S'. Moreover, recall that we have prove that for qcfp g and any quasicoherent $\mathcal{O}_S$-module $\mathcal{G}$ we have an exact sequence:

$$\mathcal{G} \to g_*g^*\mathcal{G} \rightrightarrows h_*h^*\mathcal{G}$$

Now let $\mathcal{G} = \mathcal{O}_S$, we have an exact sequence:

$$\mathcal{O}_S \to g_*\mathcal{O}_{S'} \to h_*\mathcal{O}_{S''}$$

Where the first morphism is given by $\mathcal{O}_{S'} \xrightarrow{\text{Id}} \mathcal{O}_{S'}$ and the adjoint property, I will let you prove this is exactly the induced morphism of sheaves of rings. Then for $g_*\mathcal{O}_{S'} \to h_*\mathcal{O}_{S''}$ i claim that it's just $g_*(p_i^\natural)$. To see why, before we apply g_*, we know that this is defined by identity and adjoint property of $(p_i^*, (p_i)_*)$:

$$(p_i)^* g^* \mathcal{O}_S \to (p_i)^* g^* \mathcal{O}_S$$

which is just $\mathcal{O}_{S'} \to (p_i)_*(p_i)^*\mathcal{O}_S$ induced by the identity $p_i^*\mathcal{O}_{S'} \to p_i^*\mathcal{O}_{S'}$ and similarly this is $p_i^\natural$, then we apply g_*.

This discussion plus the previous result, we get the desired exact sequence on hom sets.

Corollary 8.3. Let $g : S' \to S$ be a qcfp morphism and let S'' again be $S' \times_S S'$. Suppose that X, Y are two S-schemes let $X' = X \times_S S', Y' = Y \times_S S'$ and $X'' = X \times_S S'', Y'' = Y \times_S S''$, then the sequence:

$$\operatorname{Hom}_S(X, Y) \to \operatorname{Hom}_{S'}(X', Y') \rightrightarrows \operatorname{Hom}_{S''}(X''.Y'')$$

is exact.

Proof. Let's first be clear about what are the morphisms between these hom sets. They are given by the following diagram:

$$\begin{tikzcd}
X'' \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & Y'' \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & S'' \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] \\
X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & S' \arrow[d] \\
X \arrow[r] & Y \arrow[r] & S
\end{tikzcd}$$

Since $S' \to S$ is qcfp, we know that $X' \to X$ is also qcfp. Then we know from the previous corollary that the following two sequence are exact:

$$\operatorname{Hom}(X, Y) \to \operatorname{Hom}(X', Y) \rightrightarrows \operatorname{Hom}(X'', Y)$$

$$\operatorname{Hom}(X, S) \to \operatorname{Hom}(X', S) \rightrightarrows \operatorname{Hom}(X'', S)$$

But these two exact sequences implies that:

$$\operatorname{Hom}_S(X, Y) \to \operatorname{Hom}_S(X', Y) \rightrightarrows \operatorname{Hom}_S(X'', Y)$$

is exact. This is because for example we first need to prove the first map is injective, but this is just a restricting $\operatorname{Hom}(X, Y) \to \operatorname{Hom}(X', Y)$ to $\operatorname{Hom}_S(X, Y)$ and $\operatorname{Hom}_S(X', Y)$, therefore is indeed injective. Then suppose that we have an S-morphism in $\operatorname{Hom}_S(X', Y)$ such that after we pull it back use two diffrent projections we get the same morphism, again, by thinking these are objects in $\operatorname{Hom}(X', Y), \operatorname{Hom}(X'', Y)$, we know that there is a morphism t in $\operatorname{Hom}(X, Y)$ such that after we compose with $X' \to X$, get the morphism we started. The only issue is that this morphism may not be an S-morphism. T prove this is an S-morphism, we need to prove that after we compose t with $Y \to S$, we get $X \to S$. I will let you prove it but the hint is to use the second exact sequence.

Note that the last sequence can be identified with the exact sequence we are interested in. For example we have an isomorphism:

$$\operatorname{Hom}_{S'}(X', Y') = \operatorname{Hom}_S(X', Y)$$

which is define by $X' \to Y' \to Y$ where X' is a S-scheme by $S' \to S$, I will let you prove this is indeed an isomorphism which is just the universal property of fibered products. Similarly we have:

$$\operatorname{Hom}_{S''}(X'', Y'') = \operatorname{Hom}_S(X'', Y)$$

the morphism is defined similarly. Then I will let you check after this identification, we get the sequence we are interested in. $\square$

Now we are able to prove our promise about open, closed embeddings or isomorphisms in the beginning of the section:

Corollary 8.4. Using the notation of the previous corollary let $f : X \to Y$ be an S-morphism and let $f' : X' \to Y'$ be the base change of f. Then f is an isomorphism iff f' is.

Proof. We still use the diagram from the previous corollary. Since f' is the base change of f, then if f is an isomorphism, f' is of course an isomorphism.

Conversely, suppose that f' is an isomorphism, let g' be it's inverse, which is still an S-morphism. Then consider the further base change to $Y'' \to X''$ along to projections, then I claim that we get the same morphism, since it's the inverse of f'', therefore we know that there is an S-morphism $g : Y \to X$ such that the pullback to X', Y' is g'. Since we know that $g'f' = \mathrm{Id}_{X'}$ and the morphism $\mathrm{Hom}_S(X, X) \to \mathrm{Hom}_{S'}(X', X')$ is injective, we conclude from the fact that base change commutes with composition that $g \circ f = \mathrm{Id}_X$, similarly we get $fg = \mathrm{Id}_Y$. So f is an isomorphism as we find it's inverse. $\square$

Corollary 8.5. Use the notation of the previous corollary, then f is a closed embedding, open embedding or a locally closed embedding iff f' is so.

Proof. We again still only need to assume f' is something and then prove f is also something. Let's first deal with closed embeddings, i.e. assume f' is a closed embedding. Recall that we have a SES of closed embeddings:

$$\mathrm{Sub}(Y) \hookrightarrow \mathrm{Sub}(Y') \rightrightarrows \mathrm{Sub}(Y'')$$

Recall $f' : X' \hookrightarrow Y'$ as a closed embedding of Y', we know that it's pullback along two projections are equal. Therefore, we know that by this exact sequence, there is a closed embedding $f_0 : X_0 \hookrightarrow Y$ such that it's pullback $f_0' : X_0' \hookrightarrow Y'$ is the same closed embedding as f', i.e. there is an isomorphism $\phi' : X' \to X_0'$ such that $f' = f_0' \circ \phi'$. This implies we have the following diagram:

$$\begin{tikzcd}
X_0'' \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & X'' \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & Y'' \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & S'' \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] \\
X_0' \arrow[r, "\phi'"'] \arrow[d] & X' \arrow[r, "f'"'] \arrow[d] & Y' \arrow[r] \arrow[d] & S' \arrow[d] \\
X_0 \arrow[rr, bend right, "f_0"'] & X \arrow[r, "f"] & Y \arrow[r] & S
\end{tikzcd}$$

Now use the fact that pullback commutes with composition of morphisms, we have:

$$f_0'' \circ p_1^*(\phi') = f'' = f_0'' p_2^*(\phi)$$

where p_i^* represent pullbacking along the two projections. Then since f_0'' is a closed embedding, then we know that $p_1^*\phi' = p_2^*\phi'$ because closed embeddings are monomorphisms. However being equal after pullbacked along p_i implies that there is a $\phi : X_0 \to X$ such that it's pullback to X_0', X' is ϕ', and I will let you check this is also an isomorphism using the exact sequence. Moreover, we also have $\circ f\phi = f_0$, therefore f is also a closed embedding. We are done.

Using similar arguments and the SES for open embeddings, we can prove if f' is an open embedding then f is also an open embedding.

Finally, the argument for locally closed embedding is just the fact that locally closed embeddings are compositions of a closed embedding with an open embedding. $\square$

Proposition 8.6. Let $g : S' \to S$ be a qcfp morphism, again let $S'' = S' \times_S S'$ and $S''' = S' \times_S S' \times_S S'$, let p_1, p_2 and p_{12}, p_{13}, p_{23} be the projections. Suppose that X' is a S'-scheme provided with an S''-isomorphism $\sigma : p_1^* X' \simeq p_2^* X'$ satisfying $p_{13}^*\sigma = p_{23}\sigma^* p_{12}^*\sigma$ where $p_i^* X'$ is as in the following Cartesian diagram:

$$\begin{tikzcd}
p_i^* X' \arrow[r] \arrow[d] & X' \arrow[d] \\
S'' \arrow[r, "p_i"] & S'
\end{tikzcd}$$

and $p_{ij}^*\sigma$ are defined by:

$$\begin{tikzcd}
p_j^*X' \arrow[r, "p_{ij}^*\sigma"] \arrow[d] & p_i^*X' \arrow[r] \arrow[d] & S''' \arrow[d, "p_{ij}"] \\
p_2^*X' \arrow[r, "\sigma"'] & p_1^*X' \arrow[r] & S''
\end{tikzcd}$$

If X' is affine over S', then there exists an S-scheme X and an S'-isomorphism $\tau : g^*X \simeq X'$ where g^*X is defined similarly such that the diagram commutes:

$$\begin{tikzcd}
p_1^*g^*X \arrow[r, "p_1^*\tau"] \arrow[d, "\simeq"'] & p_1^*X' \arrow[d, "\sigma"] \\
p_2^*g^*X \arrow[r, "p_2^*\tau"'] & p_2^*X'
\end{tikzcd}$$

This S-scheme X is unique up to a unique isomorphism and X is affine over S.

Any S''-isomorphism $\sigma : p_1^*X' \simeq p_2^*X'$ such that $p_{13}^*\sigma = p_{23}\sigma^* p_{12}^*\sigma$ is called a **descent datum** for the S'-scheme X'.

Proof. Let $f' : X' \to S'$ be the structure morphism and let $\mathcal{A}' = f'_*\mathcal{O}_{X'}$. Then we know that since f' is affine, then $\mathcal{A}'$ is a quasicoherent $\mathcal{O}_{S'}$-algebra. Now I claim that the descent datum of X' defines a descent datum for $\mathcal{A}'$ this is the key of this proof.

To see why this is true, consider for example:

$$\begin{tikzcd}
p_1^*X' \arrow[r, "f_1'"] \arrow[d] & S'' \arrow[d, "p_1"] \\
X' \arrow[r, "f'"] & S'
\end{tikzcd}$$

Since f' is affine, we know that $p_1^*X' \to S''$ is also affine, then recall that $\mathcal{A}'$ on S' is $f'_*\mathcal{O}_{X'}$ then we know that $X' \to S'$ is actually $\mathcal{S}pec\mathcal{A}' \to S'$ and since $\mathcal{S}pec$ commutes with base change, we know that $p_1^*X' \to S''$ is actually $\mathcal{S}pec p_1^*\mathcal{A}' \to S''$, therefore we have isomorphisms $\theta_1 : p_1^*\mathcal{A}' \simeq (f_1')_*\mathcal{O}_{p_1^*X'}$, then we know that we have isomorphism:

$$\varphi : p_1^*\mathcal{A}' \overset{\theta_1}{\simeq} (f_1')_*\mathcal{O}_{p_1^*X'} \overset{(f_1')_*\sigma^\natural}{\simeq} (f_1')_*\sigma_*\mathcal{O}_{p_2^*X'} = (f_2')_*\mathcal{O}_{p_2^*X'} \overset{\theta_2^{-1}}{\simeq} p_2^*\mathcal{A}'$$

Thus $\varphi = \theta_2^{-1} \circ (f_1')_*\sigma^\natural \circ \theta_1$. For notation simplicity, we will denote $(f_1')_*\sigma^\natural$ by $\sigma^\natural$, which is just the isomorphism of $\mathcal{O}_{S''}$-algebras induced by σ. One important property of this $\mathcal{S}pec$ is that it commutes with base changes, therefore we know that if you pull this back along p_{ij}, we just get the isomorphism induced by $p_{ij}^*\sigma$, since θ_i's also commute with base change, we know that pullback of these morphism indeed satisfies the cocycle condition.

Then we know from descent of quasicoherent sheaves that there is a quasicoherent $\mathcal{O}_S$-module and an isomorphism:

$$\tau : g^*\mathcal{A} \simeq \mathcal{A}'$$

such that the desired commutative diagram holds. Now we want to show that $\mathcal{A}$ is an $\mathcal{O}_S$-algebra, i.e. it has an algebra structure, which is a morphism:

$$\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A} \to \mathcal{A}$$

such that it satisfies some properties, for example, associative law, commutativity and distributive law. First, the morphism $\mathcal{O}_S \to \mathcal{A}$ comes from the descent of $\mathcal{O}_{S'} \to \mathcal{A}'$ and so does $\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A} \to \mathcal{A}$ descent from the morphism $\mathcal{A}' \otimes_{\mathcal{O}'_S} \mathcal{A}' \to \mathcal{A}'$. Finally to prove it has desired properties, we know that these are still just to prove some commutative diagrams, which still from the descent property. Then take $X = \mathcal{S}pec\mathcal{A}$, we can use the correspondence property between affine schemes and quasicoherent sheaves of algebras to check X satisfies the desired property. $\square$

Corollary 8.7. Let $g : S' \to S$ be qcfp, $S'' = S' \times_S S'$, if X, Y are two S-schemes with $X' = X \times_S S', Y' = Y \times_S S', X'' = X \times_S S'', Y'' = Y \times_S S''$. Suppose $f : X \to Y$ is a S-morphism, then if f' is the base change of f, then f' is affine iff f is.

Proof. Suppose f' is affine, then f' is quasicompact and separated, then we already know that f is also quasicompact and separated, so $f_*\mathcal{O}_X$ is quasicoherent over $\mathcal{O}_Y$. Since the base change is flat, we know that we have an isomorphism $\alpha^* f_*\mathcal{O}_X \simeq f'_*\mathcal{O}_{X'}$ by the flat base change theorem(see the FOAG notes series for further information):

$$\begin{tikzcd} X' \arrow[r, "f'"'] \arrow[d, "\beta"] & Y' \arrow[d, "\alpha"'] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Then we know that there is a canonical Y-morphism $X \to \mathcal{S}pec f_*\mathcal{O}_X$, and we can pullback it along $Y' \to Y$ which gives:

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "\beta"] & \mathcal{S}pec f'_*\mathcal{O}_{X'} \arrow[r] \arrow[d] & Y' \arrow[d, "\alpha"'] \\ X \arrow[r] & \mathcal{S}pec f_*\mathcal{O}_X \arrow[r] & Y \end{tikzcd}$$

since $\alpha^* f_*\mathcal{O}_X$ is isomorphic to $f'_*\mathcal{O}_{X'}$. Since f' is affine, we know that $X' \to \mathcal{S}pec f'_*\mathcal{O}_{X'}$ is an isomorphism. But this implies $X \to \mathcal{S}pec f_*\mathcal{O}_X$ is also an isomorphism since $Y' \to Y$ is qcfp. ☐

Corollary 8.8. Under the assumption and notation of the previous corollary, f is integral, finite or finite and locally free iff f' is so.

Proof. Let's prove the integral case first as the other cases are similar. For the terminology we used above, locally free means that $f_*\mathcal{O}_X$ is locally free over Y.

First, f' being an integral morphism means it's affine, therefore f is also affine, to prove f is integral we can assume $Y = SpecA, X = SpecB$. We can also assume Y' is affine $Y' = SpecA'$ such that A' is faithfully flat over A(again by taking disjoint unions). Then we know that $A' \otimes_A B$ is integral over A'. Now we know that $B = \varinjlim_i B_i$ where B_i are all the finitely generated A-subalgebras, then since tensor commutes with filtered colimit, we know:

$$A' \otimes_A B = \varinjlim_i A' \otimes_A B_i$$

since A' is flat over A, we know that $A' \otimes_A B_i \hookrightarrow A' \otimes_A B$ is injective. Then since each $A' \otimes_A B_i$ is finitely generated algebra over A' and integral over A', it's finitely generated as an A'-module. Then we know that B_i is finitely generated since A' is faithfully flat over A.

Then finite is similarly proved. Finally for locally closed, we need to prove that that B is flat and of finite presentation, but these are all similar to what we have seen before. ☐

Definition 8.9. A scheme is called **quasi-affine** if it's isomorphic to a quasicompact open subscheme of an affine subscheme. A morphism $f : X \to Y$ is called **quasi-affine** if there exists an affine open covering $\{V_i\}$ of Y such that each $f^{-1}V_i$ are quasiaffine.

One immediate property of quasi-affine morphism is that it's quasicompact. One thing that is not so easy to notice is that quasiaffine morphisms are also separated. To see why, we can assume that the target is affine, i.e. we have a morphism $X \to SpecA$ such that X is a quasicompact open subscheme of an affine open $SpecB$. Then we have the following diagram:

$$\begin{tikzcd} Spec\Gamma(X, \mathcal{O}_X) \arrow[rr] & & SpecA \\ & X \arrow[ul] \arrow[ur] & \end{tikzcd}$$

I claim that $X \to Spec\Gamma(X, \mathcal{O}_X)$ is an open immersion, which proves $X \to SpecA$ is separated as open embeddings are separated. To see why, suppose that $X \hookrightarrow SpecB$ is an open embedding and X is quasicompact, then we know that we can find $D(b_1), \cdots, D(b_n)$ such that $X = D(b_1), \cdots, D(b_n)$. Let $b'_i \in \Gamma(X, \mathcal{O}_X)$

be the images of b_i, let $X_{b_i'}$ be the open subscheme of X where b_i' doesn't vanish. Then if we consider a similar diagram:

$$\begin{tikzcd} Spec\Gamma(X,\mathcal{O}_X) \arrow[rr] & & SpecB \\ & X \arrow[ul] \arrow[ur, hook] & \end{tikzcd}$$

If we restrict this to $D(a_i)$, we get:

$$\begin{tikzcd} Spec\Gamma(X,\mathcal{O}_X)_{a_i'} \arrow[rr] & & SpecD(b_i) \\ & X_{a_i'} \arrow[ul] \arrow[ur, hook, "="'] & \end{tikzcd}$$

Since $X_{a_i'}$ if affine, we know that $X_{a_i'} \to \Gamma(X_{a_i'}, \mathcal{O}_X)$ induced by the ring map on global sections is an isomorphism. However, I claim that:

$$\Gamma(X_{a_i'}, \mathcal{O}_X) \leftarrow \Gamma(X, \mathcal{O}_X)_{a_i'}$$

is an isomorphism. First, this map should be obvious, induced by universal property of localization. It's also clear that this is an injective map. Suppose that $\frac{x}{a_i'^n}$ is mapped to 0, since a_i is invertible in the image, we know that x restricted to $X_{a_i'}$ is 0, this implies that $xa_i' = 0$ as we know that if x is a point on X where a_i' doesn't vanish, x vanishes, if a_i vanishes, we are done. The key is how to prove this is surjective: since X is quasicompact, we can use the second version of the qcqs lemma discussed in the FOAG series, I will let you check the details.

But this proves that $X_{a_i'} \to Spec\Gamma(X, \mathcal{O}_X)_{a_i'}$ is an isomorphism, which proves that $X \to \Gamma(X, \mathcal{O}_X)$ is an open embedding. We are done.

The following result, which is essentially a summary of the above discussion, gives you another characterization of quasi-affine morphisms:

Proposition 8.10. A morphism $f : X \to Y$ is quasi-affine if and only if it's quasicompact, separated, and the canonical Y-morphism $X \to \mathcal{S}pecf_*\mathcal{O}_X$ is an open embedding.

Proof. First the if part of the proof is easy and I will let you do it.

For the only if part, we can reduce to the affine case, which is just prove the canonical morphism $X \to Spec\Gamma(X, \mathcal{O}_X)$ is an open embedding, which is the above discussion. □

Corollary 8.11. Using the notation and assumption of the previous corollary, f is quasi-affine iff f' is quasiaffine.

Proof. If f is quasi-affine, then consider the following diagram:

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d] & \mathcal{S}pecf'_*\mathcal{O}_{X'} \arrow[r] \arrow[d] & Y' \arrow[d, "h"'] \\ X \arrow[r, hook] & \mathcal{S}pecf_*\mathcal{O}_X \arrow[r] & Y \end{tikzcd}$$

Then using the previous characterization of quasi-affine morphisms, we are done.

Now assume that f' is quasi-affine, we know that $X' \to \mathcal{S}pecf'_*\mathcal{O}_{X'}$ is open embedding, then since $Y' \to Y$ is qcpf, we know that $X \to \mathcal{S}pecf_*\mathcal{O}_X$ is also an open embedding, we are done. □

Corollary 8.12. The descent property of schemes holds as well with affine replaced by quasi-affine.

Proof. The diagram becomes:

$$\begin{tikzcd} \cdots \arrow[r] \arrow[d, hook] & X' \arrow[d, hook] \\ \cdots \arrow[r] \arrow[d] & \mathcal{S}pecf'_*\mathcal{O}_{X'} \arrow[d] \\ S'' \arrow[r, "p_i"'] & S' \end{tikzcd}$$

Then we can apply the affine case to $\mathcal{S}pecf'_*\mathcal{O}_{X'}$ and obtain:

$$\begin{tikzcd} \cdots \arrow[r] \arrow[d, hook] & X' \arrow[d, hook] & \\ \cdots \arrow[r] \arrow[d] & \mathcal{S}pecf'_*\mathcal{O}_{X'} \arrow[d] \arrow[r] & \mathcal{S}pec\mathcal{A} \arrow[d] \\ S'' \arrow[r, "p_i"'] & S' \arrow[r] & S \end{tikzcd}$$

then pullback X', we get the same open subscheme, then the descent property for open subsets gives the desired $X \hookrightarrow \mathcal{S}pec\mathcal{A}$. I will let you check the details. □

9 Quasi-finite Morphisms

In this section we spend a little bit of time discussing quasi-finite morphisms.

Definition 9.1. Let $(A, m) \to (B, n)$ be a map of local rings, we say B is quasi-finite over A if B/mB is finite dimensional over A/m.

Lemma 9.2. Let $f : X \to Y$ be a morphism of finite type and let $x \in X$, the following conditions are equivalent:

1. x is isolated in $f^{-1}f(x)$.
2. $\mathcal{O}_{X,x}$ is quasi-finite over $\mathcal{O}_{Y,f(x)}$.

If these conditions hold, we say that f is quasi-finite at x, in particular, being quasi-finite at a point means f is finite-type in the first place.

Proof. Consider the following diagram of fibers:

$$\begin{tikzcd} f^{-1}f(x) \arrow[r] \arrow[d] & Spec(k(f(x))) \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Let's compute the stalk of the structure sheaf of $f^{-1}f(x)$ at x, which can be done affine locally, and we can assume that this is $SpecB \to SpecA$ where q is mapped to p, and the fiber is $B \otimes_A k(p) = (B_p/pA_pB_p)$, then since taking quotients and localization commutes, we can conclude that the stalk at x is again B_q/pA_pB_q. Moreover, we know that in this case the fiber of an open neighborhood of x in the original fiber, therefore x is isolated in the original fiber iff x is isolated in this new fiber. Then we know that by considering $f^{-1}f(x) \to Speck(f(x))$, we have the same $\mathcal{O}_{X,x}/m_{f(x)}\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,f(x)}/m_{f(x)}\mathcal{O}_{Y,f(x)}$, so we can assume $Y = Speck$ for some k.

Now let's assume $1 \Rightarrow 2$. As we only care about $\mathcal{O}_{X,x}$, we can replace X by an open neigborhood of x and assume that X is only one point, therefore form f being finite type, we know that $X = SpecB$ where B is a finitely generated k-algebra, which only has one prime ideal. This is because of the Nullstllensatz, and B/n is a finitely generated field extension of k, thus nit's necessarily finite dimensional over k. Also we know that n is nilpotent, say $n^n = 0$. Then since B is Noetherian, we know that we can consider the following filtration:

$$B \supset n \supset n^2 \supset \cdots n^n = 0$$

and each subquotient is finite dimensional over B/n, hence finite dimensional over k, hence B is finite dimensional over k.

Then let's prove $2 \Rightarrow 1$. We want to prove x is isolated, again, we can replace X by an affine niegborhood of x, say $X = SpecB$ where B is a finitely generated algebra over k. Suppose that q is the prime ideal corresponding to the point x, then we know that B_q is finite dimension over k, this implies that B_q is of finite length and Noetherian, therefore Artinian (in particular of Krull dimension 0), hence q is a minimal prime of B, therefore x is the generic point of some irreducible component of X. Moreover since B_q/qB_q is finite dimensional over k, we know that B/q is finite dimensional over k as $B/q \hookrightarrow B_q/qB_q$ is an injective k-linear map. Therefore B/q is a field, as it's a domain that is finite dimensional over a field. So q is a maximal ideal of B as well, therefore x is a closed point of B and the irreducible component of B consisting of x only has a single point x, therefore $\{x\}$ is open, therefore x is isolated. □

Lemma 9.3. Suppose that A if a finitely generated k-algebra where k is a field, then the following are equivalent:

1. $SpecA$ is discrete as a topological space.
2. $SpecA$ is finite as a set.
3. A has only finitely many maximal ideals.
4. A is a finite dimensional k-algebra.

Proof. $1 \Rightarrow 2$ is easy as any $SpecA$ is quasi-compact.

$2 \Rightarrow 3$ is trivial.

$3 \Rightarrow 1$: Let $m_1, \cdots, m_n$ be all the maximal ideals of A. Then since any finitely generated algebra over a field is Jacobson, i.e. for a prime p, $p = \cap_{m \supset p} m$, we know that we can take these to be finite and we know that p contains the product of these maximal ideals, then since all of these factors in the product are primes, p contains one of the factors, thus p is also maximal, we are done.

Then for $1 \Rightarrow 4$, we know that $SpecA$ is a finite discrete topological space, therefore:

$$SpecA = SpecA_{m_1} \coprod \cdots \coprod SpecA_{m_n}$$

Since we know that A is finitely generated, then A_{m_i} is finite dimensional over k, therefore $A = A_{m_1} \times \cdots \times A_{m_n}$, is finite dimensional over k.

Then $4 \Rightarrow 2$ if A is finite dimensional over k, then it's Artinian, therefore $SpecA$ is finite discrete. □

Proposition 9.4. Let $f : X \to Y$ be a morphism of finite type, then the following are equivalent:

1. For every $y \in Y$, $f^{-1}y$ is a discrete topological space.
2. For any $x \in X$ $\mathcal{O}_{X,x}$ is quasi-finite over $\mathcal{O}_{Y,f(x)}$.
3. For any $y \in Y$, $f^{-1}y$ is a finite set.

Proof. The fact that 1 and 2 are equivalent is from the first lemma and $1 \Leftrightarrow 3$ is from finite type plus the previous lemma. □

Definition 9.5. A finite type morphism $f : X \to Y$ satisfying the above conditions are called **quasi-finite**.

Proposition 9.6. Consider the following Cartesian diagram:

$$\begin{tikzcd} X \times_Y Y' \arrow[r, "g'"'] \arrow[d, "f'"] & X \arrow[d, "f"'] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

where g is qcfp, then f is quasi-finite if and only if f' is so.

Proof. Let's consider the following morphism:

$$\begin{tikzcd}
& g^{-1}y' \arrow[rr, dashed] \arrow[dd] & & f^{-1}(g(y')) \arrow[dd] \\
Speck(y') \arrow[ur] \arrow[rr] \arrow[dd] & & Speck(g(y')) \arrow[ur] \arrow[dd] & \\
& X \times_Y Y' \arrow[rr] \arrow[dl] & & X \arrow[dl, "f"] \\
Y' \arrow[rr, "g"] & & Y &
\end{tikzcd}$$

We know that if $g^{-1}y' \to f^{-1}(g(y'))$ is surjective, then if $g^{-1}y'$ is finite, then $f^{-1}(g(y'))$ is finite, which proves that if f' is quasi-finite, f is quasi-finite.

If f is quasi-finite. Then we still can use the above diagram, then we know that $g^{-1}y'$ is $f^{-1}(g(y')) \times_{k(g(y'))} k(y')$, however, we know that $f^{-1}g(y')$ is $\coprod_i A_{m_i}$ for finitely many A_{m_i} where each of A_{m_i} is a finite dimensional $k(g(y'))$ space, therefore the fiber $g^{-1}y' = \coprod_i A_{m_i} \otimes_{k(g(y'))} k(y')$, therefore $g^{-1}y'$ is just $Spec \coprod_i A_{m_i} \otimes_{k(g(y'))} k(y')$, therefore this is still a finitely generated and finite dimensional over $k(y')$, thus A is a finite set, we are done. □

Lemma 9.7. Let (A, m) be a local Noetherian ring, and let M be an A-module such that M/mM is finite dimensional over A/m, then $\hat{M} = \varprojlim_k M/m^k M$ is finitely generated over $\hat{A} = \varprojlim_k A/m^k$.

Proof. Let $\{x_1^{(0)}, \cdots, x_k^{(0)}\}$ be a family of generators for the A/m-module M/mM. Recall that we have morphisms:

$$M/m^{i+1}M \to M/m^i M$$

which is surjective, then we can choose by induction $x_1^{(i)}, \cdots, x_k^{(i)}$ in $M/m^{i+1}M$ such that their images in $M/m^i M$ are $x_1^{(i-1)}, \cdots, x_k^{(i-1)}$. Then for each i, consider the homomorphism:

$$(A/m^{i+1})^{\oplus k} \to M/m^{i+1}M, \quad (a_1, \cdots, a_k) \to \sum a_j x_j^{(i)}$$

I claim that this is a surjective morphism. To see why, we still can prove by induction, for example if we want to prove the case for $i = 1$, i.e.:

$$(A/m^2)^{\oplus k} \to M/m^2 M$$

is surjective. What we know is that $(A/m)^{\oplus k} \to M/mM$ is surjective, we know that:

$$(M/m^2M)/(mM/m^2M) = \frac{M}{mM + m^2M} = M/mM$$

Then we know from Nakayama's lemma that since m/m^2 is the unique maximal ideal of A/m^2, we know that $(A/m^2)^{\oplus k} \to M/m^2M$ is also surjective, let the kernel be R_i, we have the following exact sequence of all i:

$$0 \to R_i \to (A/m^{i+1})^{\oplus k} \to M/m^{i+1}M \to 0$$

Then use induction we can prove this for every i. What is important is that projective limit commutes with direct sums so if we can prove it's exact, then we have an exact sequence:

$$0 \to \varprojlim_i R_i \to \hat{A}^{\oplus k} \to \hat{M} \to 0$$

Then we know that $\hat{M}$ is finitely generated.

However, in general we only know that projective limit is left exact, not necessarily right exact. In our case, it's exact because we know that R_i satisfies the **Mittag-Leffler** condition (see the short write-up on Mittag-Leffler condition and inverse limit for more information). We are done □

Lemma 9.8. Let $(A, m) \to (B, n)$ be a map of local Noetherian rings, let $\hat{A} = \varprojlim_k A/m^k$ and let $\hat{B} = \varprojlim_k B/n^k$, then:

1. B is quasi-finite over A iff B/n is finite dimensional over A/m and there is a positive integer n such that $n^n \subset mB \subset n$.

2. If B is quasi-finite over A, $\hat{B}$ is finite over $\hat{A}$.

Proof. Suppose that B is quasi-finite over A, then B/mB is finite dimensional over A/m, then so is B/n as $mB \subset n$. Then we know that B/n is an Artinian ring, therefore n/mB is nilpotent, then there is a integer such that our assertion holds.

Conversely, suppose B/n is finite dimensional over A/m and there is n such that $n^n \subset mB \subset n$. We need to prove B/mB is finite dimensional over A/m. Since $mB \supset n^n$, we only need to prove that B/n^n is a finite length A-module (which implies B/mB, being a further quotient, is a finite length A-module, thus a finite dimensional A/m-vector space as it has a finite filtration with each subquotient A/m), then using the usual filtration we need to prove that n^{i-1}/n^i is a finite length A-module, but notice that it's finite dimensional over B/n as B is Noetherian, then it's also finite dimensional over A/m, thus we are done.

To prove the second assertion, notice that:

$$\hat{B} = \varprojlim_k B/n^k$$

However I claim that:

$$\varprojlim_k B/n^k \simeq \varprojlim_k B/(mB)^k$$

This is because the two filtrations $m^k B$ and n^k are cofinal and any two cofinal filtrations give isomorphic completions. I will let you figure this out.

But using this, we know that from the previous lemma that $\varprojlim_k B/m^k B$ is finitely generated over $\hat{A}$. We are done. $\square$

Proposition 9.9. Let A be a complete Noetherain local ring and $Y = SpecA$, let $f : X \to SpecA$ be separated and quasi-finite and $x \in X$ a point over the closed point of Y, then $\mathcal{O}_{X,x}$ is finite over A and the canonical morphism $Spec\mathcal{O}_{X,x} \to X$ is an open and closed immersion.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
X \arrow[rr, "f"] & & SpecA \\
& Spec\mathcal{O}_{X,x} \arrow[ul, "g"] \arrow[ur] &
\end{tikzcd}$$

Since A is complete, then the previous lemma tells us that:

$$A \to \mathcal{O}_{X,x} \to \hat{\mathcal{O}_{X,x}}$$

is finite as f is quasi-finite. Then we know that $A \to \mathcal{O}_{X,x}$ is also finite. To see why recall that $\mathcal{O}_{X,x} \to \hat{\mathcal{O}_{X,x}}$ has the kernel $\cap_i n^i$ where n is the maximal ideal, since the morphism is quasi-finite, then in particular, it's finite type, since A is Noetherian, $\mathcal{O}_{X,x}$ is Noetherian, then we know from Krull's interection theorem that $\cap_i n^i = 0$, thus $\mathcal{O}_{X,x} \to \hat{\mathcal{O}_{X,x}}$ is injective, thus as a submodule of a finite module over a Noetherian ring, it's also finite. Then we know that $f \circ g$ is finite with f being separated, then from the cancellation theorem of separated morphisms (as the diagonal is a closed embedding, in particular finite), then g is also finite, in particular closed, therefore $g(Spec\mathcal{O}_{X,x})$ is closed in X.

On the other hand, since $Spec\mathcal{O}_{X,x}$ induces an isomorphism between the local ring of the unique closed point and the local ring of x, we know that g is an open embedding when restricted to a sufficiently small neighborhood of the closed point in $Spec\mathcal{O}_{X,x}$. Something more general is true, suppose that $X \to Y$ is a finite morphism of schemes, such that the induced morphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an isomorphism, then there is an affine open of x such that if you restrict the morphism to this open, the morphism is an open embedding. The reason is just finite plus geometric Nakayama, I will let you figure out the details.

However, the only open neighborhood is $Spec\mathcal{O}_{X,x}$, thus g is an open embedding which is closed, therefore it's both open and closed embedding. $\square$

Besides the notion of quasi-finite, we have the notion of **locally quasi-finite**. As what you probably have expected, this notion is invented in case we have a morphism f which satisfies all the conditions of a quasi-finite morphism, except it's not quasicompact. Let's briefly discuss this condition as well, since we will still run into this notion later.

There are several definitions of locally quasi-finite morphisms, the one given in stack project is the following:

Definition 9.10. Suppose that $f : X \to Y$ is a morphism of schemes which is **locally of finite type**, then for $x \in X$, we say that f is **quasi-finite at** x if there is an affine open $V = SpecR$ of $f(x)$ and an affine open $U = SpecA$ of x such that the image of U is in V and the restricted morphism $SpecA \to SpecR$, which is finite type is quasi-finite at x in the sense of the definition given before.

We say f is **locally quasi-finite** if f is quasi-finite at every point of X.

Remark 9.11. Recall that to prove a point is isolated in a topological space, we can find an open cover of the topological space and prove the point is isolated in each of the open cover. Then we know that even if we assume f is locally finite type, the discussion we had for quasi-finite morphisms still apply and you are able to show that if $f : X \to Y$ is locally of finite type, then the following two conditions are equivalent:

1. For any $y \in Y$ $f^{-1}y$ is discrete.
2. For all $x \in X$ $\mathcal{O}_{X,x}$ is quasi-finite over $\mathcal{O}_{X,y}$.

However, since we loss quasi-compactness, we are not able to show that $f^{-1}y$ is finite. But, if it's quasi-compact, we can immediately conclude that this morphism is quasi-finite. Therefore, the main difference between quasi-finiteness and locally quasi-finiteness is whether the morphism is quasi-compact or not, as suggested before.

10 Passage to Limit

Throughout out this section, we will fix a scheme S_0, a direct set I and a direct system $(\mathcal{A}_\lambda, \phi_{\lambda\mu})$ where $\mathcal{A}_\lambda,\quad \lambda \in I$ are quasi-coherent $\mathcal{O}_{S_0}$-algebras and $\phi_{\lambda\mu} : \mathcal{A}_\lambda \to \mathcal{A}_\mu$ are morphisms of $\mathcal{O}_{S_0}$-algebras. One observation is that the direct limit $\mathcal{A} = \varinjlim_\lambda \mathcal{A}_\lambda$ exists and is quasicoherent. To show it's existence, we can define this sheaf to be the associated sheaf of the presheaf:

$$\mathcal{A}^{pre}(U) = \varinjlim_{\lambda} \mathcal{A}_\lambda(U)$$

This is obviously a presheaf and it even satisfies the identity axiom of a sheaf: suppose that the class $x \in \mathcal{A}_\lambda(U)$ on U restricts to 0 on a cover U_i, then we know that on each U_i, there is some μ, such that $\phi_{\lambda\mu}(x) = 0$, this implies that the class x is also 0 on U. Then we sheafify it and define it to be $\mathcal{A}$. I claim that this satisfies the universal property of the direct limit with canonial morphisms $\mathcal{A}_\lambda \to \mathcal{A}$ in $\mathcal{O}_{S_0}$-algebras. To see why, suppose that $\mathcal{B}$ is a $\mathcal{O}_{S_0}$-algebra with morphism $g_\lambda : \mathcal{A}_\lambda \to \mathcal{B}$ such that these commutes with $\phi_{\lambda\mu}$, then we know that there is a unique morphism from the presheaf to $\mathcal{B}$, thus a unique morphism from the associated sheaf to $\mathcal{B}$.

We can also show that this is quasi-coherent. To show this, we give a second definition of this direct limit which is obviously quasicoherent, and then we will show that this is isomorphic to the one we give above. We define it to be the associated sheaf of a sheaf on the distinguished affine basis. The key to show that it's quasicoherent is the following algebra result and the fact that quasicoherent $\mathcal{O}_{S_0}$-modules is equivalent to the data on each affine and it's distingsuihed opens such that the resitrction is localization: if $(M_\lambda, \phi_{\lambda\mu})$ is a direct system of A-modules, then we have a canonical isomorphism:

$$(\varinjlim_{\lambda} M_\lambda)_f \simeq \varinjlim_{\lambda} (M_\lambda)_f$$

In other words, we know that localization commutes with direct limits.

It's not hard to prove these two definitions are isomorphic, as we can construct a morphism from the sheaf defined by distinguished opens to the presheaf and show that the morphism induced on stalks are all identity.

Now we define $S_\lambda = \mathcal{S}pec\mathcal{A}_\lambda$ and $S = \mathcal{S}pec\mathcal{A}$, they are all S_0-schemes. By the universal property of relative spectrum, we know that we have morphisms of S_0-scheme:

$$u_{\lambda\mu} : S_\mu \to S_\lambda, \qquad u_\lambda : S \to S_\lambda$$

which are induced by $\phi_{\lambda\mu} : \mathcal{A}_\lambda \to \mathcal{A}_\mu, \phi_\lambda : \mathcal{A}_\lambda \to \mathcal{A}$. In this section, we will denote an object (for example, a sheaf $\mathcal{F}_0$) over S_0 by a symbol with a subscript 0, the corresponding object over S_λ using the same symbol with a subscript λ. If it's over S, we will use no subscript. For example if $\mathcal{F}$ is an $\mathcal{O}_{S_0}$-module, it's pullback to S_λ is $\mathcal{F}_\lambda$ and it's pullback to S is $\mathcal{F}$. If $f : \mathcal{F}_0 \to \mathcal{G}_0$ is a morphism of $\mathcal{O}_{S_0}$-modules, then it's pullback to S_λ, S are $f_\lambda : \mathcal{F}_\lambda \to \mathcal{G}_\lambda$ and $f : \mathcal{F} \to \mathcal{G}$. if we have a S_0-scheme X_0, we will call the pullback by $S_\lambda \to S_0$ by X_λ and $X = X_0 \times_{S_0} S$. Similarly, if $f_0 : X_0 \to Y_0$ is a morphism of S_0-schemes, we will use f_λ, f to denoted the corresponding morphism of X_λ, X-schemes.

In this section, we will prove result of the following flavor: if we have some object over S_0 which has some properties after we pullback to S, the it's pullback to S_λ has the same property for sufficiently large λ. These sort of result are often applied in the following situations:

1. Let A be a ring and let $\{A_\lambda\}$ be the direct system of subalgebras of A finitely generated by $\mathbb{Z}$, then we have $A = \varinjlim_\lambda A_\lambda$. Note that A_λ are Noetherian, therefore we can use results in this section to reduce the problem to Noetherian case.

2. Suppose that S_0 is an affine scheme, let $x \in S_0$ be a point. If $\{S_\lambda\}$ is an inverse system of of affine opens of x in S_0, then if we denote $S = Spec\mathcal{O}_{S_0,x}$, we have $\mathcal{O}_{S_0,x} = \varinjlim_\lambda \Gamma(S_\lambda, \mathcal{O}_{S_\lambda})$. Then using the result of this section, we can show that if the pullback to S has some property, then this property holds in an affine neighborhood of x.

The main theorem of this section is the following:

Theorem 10.1. Suppose that S_0 is quasi-compact and quasi-separated, let $f_0 : X_0 \to S_0$ be a morphism of finite presentation, then if f is a morphism of one of the types listed below, then for sufficiently large λ, f_λ is of the same type.

1. Open embedding.
2. Closed embedding.
3. Separated.
4. Finite.
5. Affine.
6. Surjective if we assume X_0, S_0 are Noetherian.
7. Radiciel if we assume X_0, S_0 are Noetherian.
8. Locally closed embedding if we assume X_0, S_0 are Noetherian.
9. Quasi-affine.
10. Quasi-finite if we assume S_0, X_0 are Noetherian.
11. Proper, if we assume S_0, X_0, X, S are Noetherian.

To prove this result, we will need some preparations:

Proposition 10.2. We will keep the notation of the preceding discussion:

1. S is the inverse limit of the inverse system $(S_\lambda, u_{\lambda\mu})$ in the category of schemes.

2. For any quasi-compact open subset U of S, there is a quasi-compact open U_λ of S_λ for some λ such that $u_\lambda^{-1}(U_\lambda) = U$.

3. The underlying topological space of S is the inverse limit of the inverse system $(S_\lambda, u_{\lambda\mu})$.

4. If S_0 is quasi-compact and $S = \varnothing$, then S_λ for sufficiently large λ.

5. Suppose that S_0, S_λ, S are Noetherian schemes, and let E_0 be a constructible subset of S_0 and let E_λ (resp. E) be the inverse images of E_0 in S_λ (resp. S). If $E = \varnothing$ then E_λ is $\varnothing$ for sufficient large λ.

Proof. To show 1, first, let's show that S is the inverse limit of $(S_\lambda, u_{\lambda\mu})$ in the category of S_0-scheme. Let $f : T \to S_0$ be an S_0-scheme, we need to show the canonical map:

$$\mathrm{Hom}_{S_0}(T, S) \to \varprojlim_{\lambda} \mathrm{Hom}_{S_0}(T, S_\lambda)$$

is bijective.

Since T is an S_0-scheme, we know that:

$$\mathrm{Hom}_{S_0}(T, S_\lambda) \simeq \mathrm{Hom}_{\mathcal{O}_{S_0}}(\mathcal{A}_\lambda, f_*\mathcal{O}_T), , \quad \mathrm{Hom}_{S_0}(T, S) \simeq \mathrm{Hom}_{\mathcal{O}_{S_0}}(\mathcal{A}, f_*\mathcal{O}_T)$$

However, since $\mathcal{A}$ is the direct limit of $\mathcal{A}_\lambda$, therefore we know that we have the isomorphism:

$$\mathrm{Hom}_{\mathcal{O}_{S_0}}(\mathcal{A}, f_*\mathcal{O}_T) \simeq \varprojlim_{\lambda} \mathrm{Hom}_{\mathcal{O}_{S_0}}(\mathcal{A}_\lambda, f_*\mathcal{O}_T)$$

You can check this identification is compatible with the morphism we care about, thus it's an isomorphism.

Then we need to show that it's the inverse limit not only in the category of S_0-schemes, but also in the category of all schemes, i.e. the following morphism is bijective:

$$\mathrm{Hom}(T, S) \to \varprojlim \mathrm{Hom}(T, S_\lambda)$$

Note that we have:

$$\mathrm{Hom}(T, S_\lambda) = \bigcup_{f \in \mathrm{Hom}(T, S_0)} \mathrm{Hom}_f(T, S_\lambda)$$

$$\mathrm{Hom}(T, S) = \bigcup_{f \in \mathrm{Hom}(T, S_0)} \mathrm{Hom}_f(T, S)$$

where Hom_f means morphisms of S_0-schemes when we view T as a S_0-scheme using $f : T \to S_0$. Recall that inverse limit of sets or groups is compatible with subsets or subgroups, i.e. we have:

$$\varprojlim \mathrm{Hom}(T, S_\lambda) = \varprojlim \bigcup_{f \in \mathrm{Hom}(T, S_0)} \mathrm{Hom}_f(T, S_\lambda) = \bigcup_{f \in \mathrm{Hom}(T, S_0)} \varprojlim \mathrm{Hom}_f(T, S_\lambda)$$

where the union is taken in $\varprojlim \mathrm{Hom}(T, S_\lambda)$, since inverse limit over an inverse system is always left exact. Therefore to show the above equality, we only need to show surjectiveness, and I will let you prove it. (Surjectivity may not always hold, union of subgroups is a special case. For an example, look at the remark following this proposition.) I will let you show that the morphism $\mathrm{Hom}(T, S) \to \varprojlim \mathrm{Hom}(T, S_\lambda)$ is compatible with the subgroup divisions, therefore to prove it's bijective it suffice to prove it for every $f \in \mathrm{Hom}(T, S_0)$, but we have prove it.

Now let's prove 2: it suffices to show that there is a basis of S of the form $\phi_\lambda^{-1} U_\lambda$ where U_λ are quasi-compact subsets of S_λ. Therefore we can assume that S_0 is affine, say it's $Spec A_0$, let A_λ be $\Gamma(S_0, \mathcal{A}_\lambda)$ and A be $\Gamma(S_0, \mathcal{A})$, then we know that we can assume that $S = Spec A$ and S_λ be $Spec A_\lambda$. Then we know that in $Spec A$, $D(f)$ forms a basis while any $f \in A$ is the image of some $f_\lambda \in A_\lambda$, we are done.

To prove 3: There are two ways to prove it, one is quite abstract and another is more concrete. First, we can show that the forgetful functor to topological spaces is right adjoint to the functor that gives a topological space the discrete scheme and right adjoint preserves limits, thus inverse limits. We will not adopt this approach as we will never use it elsewhere and it's a hard fact to establish.

Please note that even if we have an isomorphism in the category of scheme $S \to \varprojlim S_\lambda$, which means that there is a homeomorphism in the underlying topological spaces, this doesn't imply in the category of topological spaces, we still have this isomorphism since the at this point there is no reason to conclude underlying topological space of $\varprojlim S_\lambda$ is the inverse limit of S_λ in the category of topological spaces. So we still need to prove this carefully. Recall that the morphisms $u_\lambda : S \to S_\lambda$ gives a morphism of topological space $S \to \varprojlim S_\lambda$ where this inverse limit is in the topological spaces, not in the category of schemes. We know that locally this morphism is:

$$SpecA \to \varprojlim SpecA_\lambda$$

and we want to prove this is a homeomorphism of topological spaces. Recall that in the category of topological spaces, the underlying set of the inverse limit is defined as that in the category of sets, the topology is the subspace topology of the product space. Let's first prove this is a bijection of sets. Suppose that we have a system of prime ideals p_λ in A_λ such that $\phi_{\lambda\mu}^{-1}(p_\mu) = p_\lambda$, then we can consider the direct limit $p = \varinjlim p_\lambda$. I claim that this is the unique prime ideal in A such that $\phi_\lambda^{-1}(p) = p_\lambda$, thus proving it's a bijection. I will let you see this is indeed a prime ideal in A and it's not hard to see the inverse image is indeed p_λ. To show uniqueness, consider:

$$A_\lambda \hookrightarrow \coprod_\lambda A_\lambda \twoheadrightarrow \coprod_\lambda A_\lambda/\cdots$$

Suppose that there is an ideal that contracts to p_λ in each A_λ, then it first contract to $\coprod_\lambda p_\lambda$ in $\coprod_\lambda A_\lambda$, therefore we know from the correspondence between ideals in quotients and the original ring that the unique ideal in $\coprod_\lambda A_\lambda/\cdots$ is $\varinjlim p_\lambda$. Therefore $SpecA \to \varprojlim SpecA_\lambda$ is continuous and bijective. To show it's an isomorphism, recall that each $D(f)$ for some $f \in A$ comes from some $SpecA_\lambda$, therefore we have the following sequence:

$$SpecA \longrightarrow \varprojlim SpecA_\lambda \longrightarrow SpecA_\lambda$$

where the second map is induced by projection, and is still continuous. To prove a continuous bijection $SpecA \to \varprojlim SpecA_\lambda$ is a homeomorphism, we only need to show that there is a basis of $SpecA$ such that each open in the basis is the inverse image of some open in $\varprojlim SpecA_\lambda$. We take the basis of $SpecA$ to be the distinguished opens then there is a standard basis for $\varprojlim SpecA_\lambda$, which suffices.

Then let's prove 4: again, we can assume that S_0 is affine and is $SpecA_0$, then since $SpecA = S = \varnothing$, we know that in A, $1 = 0$. Then by definition of $A = \varinjlim A_\lambda$, we know that for sufficiently large A_λ, $0 = 1$, therefore $S_\lambda = 0$.

Finally, to prove 5: we know that a constructible subset in a Noetherian space is a union of locally closed subsets, then we can assume that $E_0 \subset S_0$ be $\cup_{i=1}^n (U_i \cap F_i)$ where U_i are open in S_0 and F_i are closed in S_0, therefore $U_i \cap F_i$ is closed in U_i. Let's give $U_i \cap F_i$ the reduced closed subscheme structure in U_i and we have locally closed embeddings $U_i \cap F_i \hookrightarrow U_i \hookrightarrow S_0$, then consider the morphism:

$$X_0 = \coprod_{i=1}^n U_i \cap F_i \to S_0$$

Then we know that E_λ and E, as subsets of S_λ and S are going to be the images of the morphisms in the following Cartesian diagram:

$$\begin{tikzcd} X_\lambda \arrow[r, hook] \arrow[d] & S_\lambda \arrow[d] \\ X_0 \arrow[r, hook] & S_0 \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, hook] \arrow[d] & S \arrow[d] \\ X_0 \arrow[r, hook] & S_0 \end{tikzcd}$$

Recall that the construction of relative spectrum commutes with pullbacks, therefore X_0, X_λ, X, just like S_0, S_λ, S is an inverse system. Then we can apply 4 and if $E = \varnothing$ we know $X = \varnothing$, therefore $X_\lambda = \varnothing$ for sufficiently large λ. The second part, i.e. when $E = S$ follows from apply the first part to the complement of E_0, which is still constructible. $\square$

Remark 10.3. To give you an example of the failure of right exactness of inverse limit consider the following diagram:

$$\begin{tikzcd}
 & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "2^3\times"] \arrow[d, "2\times"'] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{Id}"] & \mathbb{Z}/2^4 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "2^2\times"'] \arrow[d, "2\times"'] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{Id}" description] & \mathbb{Z}/2^2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "2\times"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0
\end{tikzcd}$$

After you take the inverse limit, we get:

$$0 \to 0 \to \mathbb{Z} \to \hat{\mathbb{Z}}_2 \to 0$$

Clearly, $\mathbb{Z} \to \hat{\mathbb{Z}}_2$ is not surjective.

Proposition 10.4. Assume S_0 is quasi-compact and quasi-separated, then:

1. Let $\mathcal{F}_0, \mathcal{G}_0$ be quasi-coherent sheaves on S_0. Suppose $\mathcal{F}_0$ is locally of finite presentation, then we have a bijection:
$$\varinjlim_{\lambda} \mathrm{Hom}_{\mathcal{O}_{S_\lambda}}(\mathcal{F}_\lambda, \mathcal{G}_\lambda) \to \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{F}, \mathcal{G})$$

2. $\mathcal{F}_0, \mathcal{G}_0$ be quasi-coherent sheaves on S_0 both locally of finite presentation. If there is an isomorphism $f : \mathcal{F} \to \mathcal{G}$, then for a sufficiently large λ, there is an isomorphism f_λ inducing f.

3. For any quasi-coherent $\mathcal{O}_S$-module $\mathcal{F}$ locally of finite presentation, there is a quasi-coherent $\mathcal{O}_{S_\lambda}$-module $\mathcal{F}_\lambda$ locally of finite presentation for a sufficiently large λ such that $\mathcal{F} \simeq \mu_\lambda^* \mathcal{F}_\lambda$.

Remark 10.5. Let's first think about what is the canonical morphism:

$$\varinjlim_{\lambda} \mathrm{Hom}_{\mathcal{O}_{S_\lambda}}(\mathcal{F}_\lambda, \mathcal{G}_\lambda) \to \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{F}, \mathcal{G})$$

We first need to see why $\mathrm{Hom}_{\mathcal{O}_{S_\lambda}}(\mathcal{F}_\lambda, \mathcal{G}_\lambda)$ form a direct system. Clearly, the morphisms between different λ is induced by pullbacks. You might worry about first pullback by a morphism of schemes and the pullback by another is not equal to, only canonically isomorphic to pullback by the composition thus one conditions of the direct system fails. To solve this problem consider, we need to show the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{O}_{S_\lambda}}(\mathcal{F}_\lambda, \mathcal{G}_\lambda) \arrow[d] \arrow[dr, dashed] & & \\
\mathrm{Hom}_{\mathcal{O}_{S_\lambda}}(\phi_{\lambda\mu}^*\mathcal{F}_\lambda, \phi_{\lambda\mu}^*\mathcal{G}_\lambda) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{O}_{S_\mu}}(\mathcal{F}_\mu, \mathcal{G}_\mu) \arrow[d] \arrow[dr, dashed] & \\
\mathrm{Hom}_{\mathcal{O}_{S_\lambda}}(\phi_{\mu\xi}^*\phi_{\lambda\mu}^*\mathcal{F}_\lambda, \phi_{\mu\xi}^*\phi_{\lambda\mu}^*\mathcal{G}_\lambda) \arrow[r] & \mathrm{Hom}_{\mathcal{O}_{S_\mu}}(\phi_{\mu\xi}^*\mathcal{F}_\mu, \phi_{\mu\xi}^*\mathcal{G}_\mu) \arrow[r] & \mathrm{Hom}_{\mathcal{O}_{S_\lambda}}(\mathcal{F}_\xi, \mathcal{G}_\xi)
\end{tikzcd}$$

We will define the dotted morphisms to be the morphisms between λ and the horizontal morphism:

$$\mathrm{Hom}_{\mathcal{O}_{S_\lambda}}(\phi_{\lambda\mu}^*\mathcal{F}_\lambda, \phi_{\lambda\mu}^*\mathcal{G}_\lambda) \to \mathrm{Hom}_{\mathcal{O}_{S_\mu}}(\mathcal{F}_\mu, \mathcal{G}_\mu)$$

is defined by the canonical isomorphism of functors we use to identity the composition of pullback functors first from S_0 to S_λ then along $\phi_{\lambda\mu}$ to S_μ and the pullback directly to S_μ from S_0. Then we only need to show the square at the left bottom corner commutes and I will let you do it. We will postpone the proof as it will be quite similar to one of the other proof we give later.

Corollary 10.6. Let S_0 be quasi-compact, quasi-separated, for any quasi-coherent $\mathcal{O}_{S_0}$-module $\mathcal{G}_0$, we have a bijection:

$$\varinjlim \Gamma(S_\lambda, \mathcal{G}_\lambda) \to \Gamma(S, \mathcal{G})$$

Proof. We just take $\mathcal{F}_0 = \mathcal{O}_{S_0}$. □

Recall that if A is a ring and B is an A-algebra, then B is of finite presentation if we have an A-algebra isomorphism: $A[t_1, \cdots, t_n]/I$ for some nonnegative integer n and I that is finitely generated over $A[t_1, \cdots, t_n]$. Please note that n could be 0, then in this case, I is just a finitely generated A-ideal. Moreover, if $s \in A$, then we know that $A_s \simeq A[t]/(st-1)$, therefore A_s is an A-algebra of finite presentation.

Lemma 10.7. Let A_0 be a ring, and $(A_\lambda, \phi_{\lambda\mu})$ a direct system of A_0-algebra and let $A = \varinjlim_\lambda A_\lambda$.

1. Let B_0 be an A_0-algebra, $B_\lambda = B_0 \otimes_{A_0} A_\lambda$ and $B = B_0 \otimes_{A_0} A$. Let (C_λ) be another direct system of A_0-algebras such that we have a morphism of direct systems $(A_\lambda) \to (C_\lambda)$. Let $C = \varinjlim C_\lambda$. If B_0 is an A_0-algebra of finite type (resp. finite presentation), then the canonical morphism:

$$\varinjlim \mathrm{Hom}_{A_\lambda}(B_\lambda, C_\lambda) \to Hom_A(B, C)$$

is injective (resp. bijective).

2. For any A-algebra B with finite presentation, there is an A_λ-algebra B_λ of finite presentation for a sufficiently large λ such that $B \simeq B_\lambda \otimes_{A_\lambda} A$.

Proof. Let's first prove 1: We need to figure out what is the canonical morphism. Notice that B_λ also forms a direct system and since it's defined by tensor, we have a morphism of direct systems $B_\lambda \to C_\lambda$ induced by that between A_λ and C_λ. Then check that given an element in $\varinjlim \mathrm{Hom}_{A_\lambda}(B_\lambda, C_\lambda)$, say represented by f_λ, then the result in $\mathrm{Hom}_A(B, C)$ is the map $f : B \to C$ such that $b_0 \otimes a_\lambda \mapsto f(b_0 \otimes a_\lambda)$. Notice that as B_λ and B are defined in terms of extension of scalars in A_0, we know that we have canonical isomorphisms:

$$\mathrm{Hom}_{A_\lambda}(B_\lambda, C_\lambda) \simeq \mathrm{Hom}_{A_0}(B_0, C_\lambda), \qquad \mathrm{Hom}_A(B, C) \simeq \mathrm{Hom}_{A_0}(B_0, C)$$

Then the morphism we care about can be identified as:

$$\varinjlim \mathrm{Hom}_{A_0}(B_0, C_\lambda) \to \mathrm{Hom}_{A_0}(B_0, C)$$

Assume B_0 is finite type over A_0, with generators $x_1, \cdots, x_n$, let's try to prove that the morphism is injective.

Suppose that we have $\alpha_1, \alpha_2 : B_0 \to C_\lambda$ are A_0-morphisms inducing the same morphism $B_0 \to C$. Then we know that $\alpha_1(x_i) = \alpha_2(x_i)$ in C for all i. Recall that $C = \varinjlim_{\mu \geqslant \lambda} C_\mu$. There is $\mu \geqslant \lambda$ such that $\alpha_1(x_i)$ and $\alpha_2(x_i)$ has the same image in C_μ. This proves after passing to C_μ, $\alpha_1 = \alpha_2$, we are done.

Then assume it's finitely presented, let's prove surjectivity. Since B_0 is finitely presented, we have a surjection of A_0-algebra:

$$\pi : A_0[t_1, \cdots, t_n] \to B_0$$

such that the kernel is generated by $f_j(t_1, \cdots, t_n), j = 1, \cdots, m$. Now suppose that $\alpha : B_0 \to C$ is an A_0-morphism, we have:

$$f_j(\alpha\pi(t_1), \cdots, \alpha\pi(t_n)) = \alpha\pi(f_j(t_1, \cdots, t_n)) = 0 \in C$$

since f_j is in the kernel. Then consider $\alpha\pi(t_i) \in C$, we can choose some large enough λ such that $\alpha\pi(t_i) \in C$ is the image of $y_{\lambda_i} \in C_\lambda$ for each i and $f_j()$. Therefore we know $f_j(y_{\lambda_1}, \cdots, y_{\lambda_n})$ in C is the same as $f_j(\alpha\pi(t_1), \cdots, \alpha\pi(t_n)) = 0$, therefore we know that passing to a larger λ, we get:

$$f_j(y_{\lambda_1}, \cdots, y_{\lambda_n}) = 0 \in C_\lambda, \quad \forall j$$

Therefore we can define:

$$A_0[t_1, \cdots, t_n] \to C_\lambda, t_i \mapsto y_{\lambda_i}$$

which factors through B_0, we are done.

Now let's prove 2: by assumption we have:

$$A[t_1, \cdots, t_n] \to B$$

and the kernel is generated by $f_j(t_1, \cdots, t_n)$ for $j = 1, \cdots, m$. Consider the coefficients of these polynomials, since we only have finitely of them we can choose large enough λ such that we have $f_{\lambda_j} \in A_\lambda[t_1, \cdot, t_n]$ whose images in $A[t_1, \cdots, t_n]$ are f_j. Then we know that if we let $B_\lambda = A_\lambda[t_1, \cdots, t_n]/(f_{\lambda_1}, \cdots, f_{\lambda_m})$:

$$B \simeq B_\lambda \otimes_{A_\lambda} A$$

$\square$

Recall that a morphism $f : X \to Y$ is said to be locally of finite presentation if for every $x \in X$ that is an affine open $V = SpecB$ of Y around $f(x)$ such that there is an affine open $SpecC$ in $f^{-1}V$ around x with C being finitely presented over B.

Proposition 10.8.

1. Local isomorphisms are locally of finite presentation.
2. If two morphisms $f : X \to Y$ and $g : Y \to Z$ are locally of finite presentation, then so is their composition.
3. If $f : X \to Y$ is locally of finite presentation, then it's preserved under any base change.
4. If $g \circ f$ is locally of finite presentation and g is locally of finite presentation, then so is f.

Proof. The first three are easy to prove. To prove 4, we use the cancellation theorem: therefore we only need to prove that if $X \to Y$ is locally of finite presentation then so is the diagonal morphism $X \to X \times_Y X$.

Indeed we don't even need locally of finite presentation, since locally finite type suffices. Recall that locally the diagonal is:

$$B \otimes_A B \to B$$

where B is finitely generated over A, say by $b_1, \cdots, b_n$. Since we know that the kernel is generated by elements of the form $b \otimes 1 - 1 \otimes b$, then it's generated by $b_i \otimes 1, 1 \otimes b_i$. So the kernel is indeed finitely generated. $\square$

Lemma 10.9. Let C be a ring and I an ideal of C, suppose C/I is a C-algebra of finite presentation, then I is a finitely generated ideal of C.

Proof. By assumption, we have a surjection:

$$u : C[t_1, \cdots, t_n] \to C/I$$

such that the kernel is finitely generated. Let $c_i \in C$, $i = 1, \cdots, n$ such that $u(t_i)$ in C/I are the images of c_i in C/I. Then if we define:

$$v : C[t_1, \cdots, t_n] \to C$$

where $t_i \mapsto c_i$ and let $p : C \to C/I$ be the projection, then we have $p \circ v = u$. Therefore we know that the kernel of u is $v^{-1}I$. Notice that v is surjective, therefore we know that $I = vv^{-1}I$, therefore since $\ker u$ is finitely generated, it's extension is also finitely generated. $\square$

Remark 10.10. This is a special case of one result we proved in the FOAG series, that is if B is a finitely presented A-algebra, then it's always finitely presented, no matter what sort of surjection we choose, the kernel is always finitely generated.

Therefore if $SpecB \to SpecA$ is locally finitely presented, B is a finitely presented A-algebra and for any epimorphism $u : A[t_1, \cdots, t_n] \to B$, the kernel is finitely generated.

Proposition 10.11. Let A be a ring, and B a finite A-algebra. Then B is a finitely presented A-algebra iff B is a finitely presented A-module.

Proof. First assume that B is an A-module of finite presentation. Let $x_1, \cdots, x_n$ be generators in B as an A-module. Recall that finite implies integral, therefore we have $f_1(t), \cdots, f_n(t) \in A[t]$ monic polynomials such that $f_i(x_i) = 0$. Then consider the surjection:

$$A[t_1, \cdots, t_n]/(f_1(t_1), \cdots, f_n(t)_n) \to B$$

Notice that

$$A[t_1, \cdots, t_n]/(f_1(t_1), \cdots, f_n(t)_n) \simeq A[t_1]/(f_1(t)) \otimes_A A[t_2]/(f_2(t)) \otimes \cdots \otimes_A A[t_n]/(f_n(t))$$

Notice that each factor in the tensor is free of finite rank, then $A[t_1, \cdots, t_n]/(f_1(t_1), \cdots, f_n(t)_n)$ is in fact a finite rank free A-module. Then we know that the kernel is a finitely generated A-module, say by $g_1, \cdots, g_n$, then we know that:

$$B \simeq A[t_1, \cdots, t_n]/(f_1(t_1), \cdots, f_n(t)_n, g_1, \cdots, g_n)$$

Therefore B is a finitely presented A-algebra.

Conversely, suppose that B is a finitely presented A-algebra, then we know that the kernel is also finitely generated, but we know that $A[t_1, \cdots, t_n]/(f_1(t_1), \cdots, f_n(t)_n)$ is already a finitely generated A-module, therefore the ideal is also finitely generated as an A-module. □

Recall that a morphism $f : X \to Y$ is **of finite presentation** if it's locally of finite presentation and quasi-compact quasi-separated.

Proposition 10.12. Assume S_0 is a quasi-compact, quasi-separated scheme:

1. Let X_0, Y_0 be S_0-scheme, $X_0 \to S_0$ be quasi-compact quasi-separated and $Y_0 \to S_0$ be of locally finite presentation, then the canonical map:

$$\varinjlim_{\lambda} \mathrm{Hom}_{S_\lambda}(X_\lambda, Y_\lambda) \to \mathrm{Hom}_S(X, Y)$$

   is bijective.

2. Suppose X_0, Y_0 are S_0-schemes of finite presentation, if there is an S-isomorphism $f : X \to Y$, then for sufficiently large λ, there is a S_λ-isomorphism $f_\lambda : X_\lambda \to Y_\lambda$ inducing f.

3. For any S-scheme X of finite presentation, there is a S_λ-scheme X_λ of finite presentation for a sufficiently large λ such that $X \simeq X_\lambda \times_{S_\lambda} S$.

Proof. To prove 1: again, we first explain what is the morphism $\varinjlim_\lambda \mathrm{Hom}_{S_\lambda}(X_\lambda, Y_\lambda) \to \mathrm{Hom}_S(X, Y)$. As explained before, the direct system $\mathrm{Hom}_{S_\lambda}(X_\lambda, Y_\lambda)$ is defined by pullbacks, and we know that given an morphism $X_\lambda \to Y_\lambda$, we can pullback by $S \to S_\lambda$ to get a morphis in $\mathrm{Hom}_S(X, Y)$. This is well defined and explained in the diargam below:

$$\begin{tikzcd}
S \arrow[r] & S_\mu \arrow[r] & S_\lambda \arrow[r] & S_0 \\
Y \arrow[r] \arrow[u] & Y_\mu \arrow[r] \arrow[u] & Y_\lambda \arrow[r] \arrow[u] & Y_0 \arrow[u] \\
X \arrow[r] \arrow[u] & X_\mu \arrow[r] \arrow[u] & X_\lambda \arrow[r] \arrow[u] & X_0 \arrow[u]
\end{tikzcd}$$

Before we start our proof, we notice that we have functorial isomorphisms:

$$\mathrm{Hom}_{S_\lambda}(X_\lambda, Y_\lambda) \simeq \mathrm{Hom}_{X_\lambda}(X_\lambda, X_\lambda \times_{S_\lambda} Y_\lambda)$$

$$\mathrm{Hom}_S(X, Y) \simeq \mathrm{Hom}_X(X, X \times_S Y)$$

Then we know it suffices to prove the following isomorphism:

$$\varinjlim \mathrm{Hom}_{X_\lambda}(X_\lambda, X_\lambda \times_{S_\lambda} Y_\lambda) \to \mathrm{Hom}_X(X, X \times_S Y)$$

We can further simplify it so that we only need to prove:

$$\varinjlim \mathrm{Hom}_{X_\lambda}(X_\lambda, X_\lambda \times_{S_0} Y_0) \to \mathrm{Hom}_X(X, X \times_{S_0} Y_0)$$

Notice under these identifications, the morphism is still induced by pullbacks in the following diagram:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X \times_{S_0} Y_0 \arrow[r] \arrow[d] & X \arrow[d] \\
X_\mu \arrow[r] \arrow[d] & X_\mu \times_{S_0} Y_0 \arrow[r] \arrow[d] & X_\mu \arrow[d] \\
X_\lambda \arrow[r] & X_\lambda \times_{S_0} Y_0 \arrow[r] & X_\lambda
\end{tikzcd}$$

I will let you check the details, but notice that in this case we can assume that the case X_0 is just S_0 (notice that since S_0 is qcqs and $X_0 \to S_0$ is qcqs then X_0 is qcqs) and we only need to prove the following map is a bijection:

$$\varinjlim_{\lambda} \mathrm{Hom}_{S_\lambda}(S_\lambda, Y_\lambda) \to \mathrm{Hom}_S(S, Y)$$

Let's first show it's injective. Since morphisms of schemes are determined by it's local behaviors, we know that we can assume S_0 is affine. Then let $f_\lambda, f'_\lambda : S_\lambda \to Y_\lambda$ be two S_λ-morphisms such that after we base change by $S \to S_\lambda$, we get the same S-morphism. It's quite painful to write down the proof, not because it's particularly hard, but because of complication that occurs when you try to index different objects. Let's give a explanation of how this works and you can check the entire proof on Lei-Fu's book. First choose an affine open cover in Y_λ such that it covers the image of f_λ, f'_λ, then we consider the inverse images of these open covers and then their intersections, we can cover each intersection with again affine open covers, which covers S_λ. Then we consider their inverse images in S, which covers S, which implies that for some $\mu \geqslant \lambda$, their images in S_μ also covers S_μ. Since S_μ is also affine, we know that we can choose a finite subcovering. Since we know that $S_\mu \to S_\lambda$ is affine, then so is $Y_\mu \to Y_\lambda$, then we can assume that there is finitely many affine opens $V^i_\mu, i = 1, \cdots, n$ in Y_μ and finitely many affine opens $U^i_\mu, i = 1, \cdots, n$ in S_μ such that these opens cover S_μ and:

$$U^i_\mu \subset f_\mu^{-1}(V^i_\mu) \cap (f'_\mu)^{-1}(V^i_\mu)$$

Then to prove the desired morphism is injective, we only need to show that f_μ, f'_μ restricted to $U^i_\mu \to V^i_\mu$ are equal for all i.

Consider all $\nu \geqslant \mu$:

$$\mathrm{Hom}_{S_\nu}(U^i_\nu, V^i_\nu) = \mathrm{Hom}_{\Gamma(S_\nu, \mathcal{O})}(\Gamma(V^i_\nu, \mathcal{O}), \Gamma(U^i_\nu, \mathcal{O}))$$

$$\mathrm{Hom}_S(U^i, V^i) = \mathrm{Hom}_{\Gamma(S, \mathcal{O})}(\Gamma(V^i, \mathcal{O}), \Gamma(U^i, \mathcal{O}))$$

We want to prove that if two elements in $\mathrm{Hom}_{\Gamma(S_\nu, \mathcal{O})}(\Gamma(V^i_\nu, \mathcal{O}), \Gamma(U^i_\nu, \mathcal{O}))$ induce the same element under the map:

$$\varinjlim_{\nu \geqslant \mu}(\mathrm{Hom}_{\Gamma(S_\nu, \mathcal{O})}(\Gamma(V^i_\nu, \mathcal{O}), \Gamma(U^i_\nu, \mathcal{O}))) \to \mathrm{Hom}_{\Gamma(S, \mathcal{O})}(\Gamma(V^i, \mathcal{O}), \Gamma(U^i, \mathcal{O}))$$

they are equal, or in other words the above map is injective. Notice that $\Gamma(V^i, \mathcal{O}) = \Gamma(V^i_\mu, \mathcal{O}) \otimes_{\Gamma(S_\nu, \mathcal{O})} \Gamma(S, \mathcal{O})$ and we know that since U_ν is quasi-compact and quasi-separated $\Gamma(U^i, \mathcal{O}) = \varinjlim_{\nu \geqslant \mu} \Gamma(U^i_\nu, \mathcal{O})$, then we know that this is indeed injective since we know that $Y_\nu \to S_\nu$ is locally of finite presentation, thus $\Gamma(V^i_\nu, \mathcal{O})$ is finite type over $\Gamma(S_\nu, \mathcal{O})$.

Now let's prove surjectivity: suppose that $f : S \to Y$ is an S-morphism. Recall that we have $Y_0 \to S_0$ as the structure morphism of Y_0. Then the following diagram helps to clarify what is going on in the book:

$$\begin{tikzcd}
s \in S \arrow[r, "f"'] \arrow[d] & V_s \subset Y \arrow[r] \arrow[d] & S \arrow[d] \\
W_{s,0} \subset S_0 \arrow[r] & V_{0,s} \subset Y_0 \arrow[r] & S_0
\end{tikzcd}$$

Anyway, in the end, quasi-compactness tells us there are finitely many affines $V_0^i, i = 1, \cdots, n$ in Y_0 and finitely many affines $W_0^i, i = 1, \cdots, n$ in S_0, such that the images of V_0^i is contained in W_0^i. Moreover, quasi-compact opens U^i of S such that they cover S and $f(U^i) \subset V^i$. Since U^i are quasi-compact, then we know that there will be quasi-compact opens U_λ^i contained in W_λ^i such that $U^i = u_\lambda^i(U_\lambda^i)$ and U_λ^i forms a covering of S_λ. Again we use the fact:

$$\Gamma(U^i, \mathcal{O}) = \varinjlim_{\nu \geqslant \mu} \Gamma(U_\nu^i, \mathcal{O})$$

and the finitely presented algebra, we know that for each i, there is a $\mu \geqslant \lambda$ and a W_μ^i-morphism $f_\mu^i : U_\mu^i \to V_\mu^i$ such that it's base change gives $f : U^i \to V^i$. Now by the injectivity we just showed, for different pairs of (i, j), we know that on the intersection, if we pass to a large enough ν, the two morphisms will be equal. Since we only have finitely many i, we know that there is a large enough ν such that f_ν^i for different i glues on the intersection, therefore we are done.

Now we prove 2: First, since X_0, Y_0 are all finitely presented over S_0, we know that, the structure morphism to S_0 are both qcqs, then consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[d] & Y \arrow[l, shift left, "g"] \arrow[r] \arrow[d] & S \arrow[d] \\
X_0 \arrow[r, shift left, dashed] & Y_0 \arrow[l, shift left, dashed] \arrow[r] & S_0
\end{tikzcd}$$

We know that once we pass to large enough λ, there will be f_λ, g_λ inducing f, g. Since we know that $f_\lambda \circ g_\lambda$ induce $f \circ g = \mathrm{Id}$, and the pullback is bijection, we know $f_\lambda \circ g_\lambda$ must also be the identity, so is $g_\lambda \circ f_\lambda$.

Finally to prove 3, we have already proved the case when everything is affine. Therefore the only problem is how to assure that we can glue. First, we cover S_0 by finitely many affines, which gives a finite affine open cover of S, cover X by finitely many affine opens U^i such that their image in S is contained in one of the affines covering S. Now we know that for each i, there will be a S_λ-scheme U_λ^i such that $U_\lambda^i \times_{S_\lambda} S$ is S-isomorphic to U^i, since we have only finitely many i, let's take λ large enough such that it works for all i.

Then notice that $U_i \cap U^i$ in S is again quasi-compact, therefore we know that there will be quasi-compact U_λ^{ij} in U_λ^i such that $U_\lambda^{ij} \times_{S_\lambda} S$ is S-isomorphic to $U^i \cap U^j$. Then use 2, we know that if we further enlarge λ, we can assume that there are isomorphism $\phi_\lambda^{ij} : U_\lambda^{ij} \to U_\lambda^{ji}$ that satisfies the gluing cocycle condition, therefore glues to a S_λ-scheme X_λ, and $X_\lambda \times_{S_\lambda} S$ is S-isomorphic to X. □

Remark 10.13. This will also be a good place to prove Prop 10.4 that we skipped before. You will find that the proof is quite similar to what we did above.

Proof. Let's first prove 1: to show that it's injective, we can work affine locally as we know that morphisms of sheaves is determined locally. Therefore we are reduced to the algebra question that if M is an A_0-algebra of finite presentation, then the morphism:

$$\varinjlim \mathrm{Hom}_{A_\lambda}(M \otimes_{A_0} A_\lambda, N \otimes_{A_0} A_\lambda) \to \mathrm{Hom}_A(M \otimes_{A_0} A, N \otimes_{A_0} A)$$

is injective. Let's first prove that $N \otimes_{A_0} A$ is the direct limit of $N \otimes_{A_0} A_\lambda$ as an A-module. First, let's give the direct limit $\varinjlim N \otimes_{A_0} A_\lambda$ an A-module structure, or rather a map of abelian groups:

$$A \otimes_{\mathbb{Z}} \varinjlim N \otimes_{A_0} A_\lambda \to \varinjlim N \otimes_{A_0} A_\lambda$$

you can check the details, but obviously if we have some $a \in A$ and $n \otimes a'$, the action is defined to be $n \otimes aa'$ where we put a, a' in the same A_λ. To define a morphism $N \otimes_{A_0} A \to \varinjlim N \otimes_{A_0} A_\lambda$, we know that we just define a map $N \to \varinjlim N \otimes_{A_0} A_\lambda$ obviously. Conversely, we define it by first on the level of abelian groups using universal property of direct limits, then check it's A-linear, then you check these are inverses of each other. Then the proof of injectivity only uses the fact that M is finitely generated, I will let you prove it.

Then let's prove surjectivity, which use injectivity. Again, let's first check what happens in the affine case. We can again identify the morphism by:

$$\varinjlim \mathrm{Hom}_{A_0}(M, N \otimes_{A_0} A_\lambda) \to \mathrm{Hom}_{A_0}(M, \varinjlim N \otimes_{A_0} A_\lambda)$$

Since M is finitely presented, we have:
$$A_0^m \to A_0^n \to M \to 0$$
Then we know that if we have an A_0-map $M \to \varinjlim N \otimes_{A_0} A_\lambda$, then we know that the images of A_0^m are all mapped to 0, in particular, $e_1, \cdots, e_m$ are all mapped to 0. Then I will let you check that the map $A_0^m \to N \otimes_{A_0} A_\lambda$ defined by the images of $f_1, \cdots, f_n$ suffices when we pass to large enough λ since we also want the images of $e_1, \cdots, e_m$ to vanish.

Now we can cover S_0 by affines to reduce the global situation first to affines. Then we need to show that these locally defined morphisms glue, this is where we use injectivity and finiteness.

We omit $2, 3$ as it's basically the same as the proof of the previous proposition. $\square$

We briefly recall the notion of quasi-affine morphisms as we will prove something related to it below. Recall that we say that a scheme X is **quasi-affine** if we have a quasi-compact open embedding of X to an affine scheme. We can prove quite easily that X is quasi-affine iff the canonical morphism $X \to Spec\Gamma(X, \mathcal{O}_X)$ is an open embedding. We can easily show that if X is quasi-affine, then it's quasi-compact and separated. We say that a morphism of schemes $\pi : X \to Y$ is **quasi-affine** if the inverse image of every affine open subset of Y is a quasi-affine scheme. It's easy to see that this is equivalent to the $X \to \mathcal{S}pec\pi_*\mathcal{O}_X$ over Y is a quasi-compact open embedding. This implies that the notion is affine local on the target. Quasi-affineness is also preserved by base change as if we have:

$$\begin{tikzcd}
X \times_Y W \arrow[r, hook] \arrow[d] & Z \times_Y W \arrow[r] \arrow[d] & W \arrow[d] \\
X \arrow[r, hook] & Z \arrow[r] & Y
\end{tikzcd}$$

then we know that if Z is affine over Y and $X \to Z$ is quasi-compact open embedding, then $Z \times_Y W \to W$ is affine and $X \times_Y W \to Z \times_Y W$ is also quasi-compact affine. Finally, we readily check it's preserved by composition as we have:

$$\begin{tikzcd}
 & & U' \arrow[drr] & & & \\
X \arrow[r, hook, "i"] & U \arrow[ur, hook] \arrow[r] & Y \arrow[rr, hook, "j"] & & V \arrow[r] & S
\end{tikzcd}$$

where $U' \to V$ is affine. To see why, I define U' to be $\mathcal{S}pecj_*f_*\mathcal{O}_U$ and the morphism $U \to U'$ is given by the identity morphism on $j_*f_*\mathcal{O}_U$. I will let you check why $U \to U'$ is an open embedding.

With these results, we can prove the main theorem of this section:

Proof. 1 open embeddings: suppose that $f : X \to S$ is an open embedding, then we know that since f is still quasi-compact, we know that X itself is quasicompact, then by topological result, we know that there is a large λ and a quasi-compact open U_λ in S_λ such that U is X in S. Then consider $f_\lambda : X_\lambda \to S_\lambda$, we know that it factors through U_λ, then we consider:

$$\begin{tikzcd}
X \arrow[r, "\simeq"] \arrow[d] & X \arrow[r, hook] \arrow[d] & S \arrow[d] \\
X_\lambda \arrow[r] & U_\lambda \arrow[r, hook] & S_\lambda
\end{tikzcd}$$

I claim that $X_\lambda \to U_\lambda$ is an isomorphism, thus proving that desired result. Notice that all open embeddings are locally of finite presentation and all U_μ are quasi-compact, then we know that we can use the result of the above proposition. We know that there is a morphism $g : U_\lambda \to X_\lambda$ such that it;s pullback is the inverse of $X \to X$. Then we know that $f \circ g$ and $g \circ f$ is pulled back to identity, therefore there is some $\mu \geqslant \lambda$ such taht f_μ, g_μ are inverses of each other, we are done.

2 closed embeddings: suppose that $X \hookrightarrow S$ is a closed embedding, then in particular, it's qcqs, therefore $f_*\mathcal{O}_X$ is a quasicoherent $\mathcal{O}_S$-algebra. Moreover, we know that since $f_0 : X_0 \to S$ is of finite presentation, so is this closed embedding, therefore we know that this ideal is finitely generated, therefore we know that this quasicoherent $\mathcal{O}_S$-algebra is finitely presented as a quasicoherent $\mathcal{O}_S$-module. Then we know that there is a large enough λ together with a quasicoherent $\mathcal{O}_{S_\lambda}$-module $\mathcal{B}_\lambda$ such that $u_\mu^*\mathcal{B}_\lambda \simeq f_*\mathcal{O}_X$.

Then consider the $\mathcal{O}_S$-algebra structure, which is:

$$f_*\mathcal{O}_X \otimes_{\mathcal{O}_S} f_*\mathcal{O}_X \to f_*\mathcal{O}_X$$

which satisfies commutative, associative and distributive laws.

Then we know that the above morphism comes from pulling back:

$$\mathcal{B}_\lambda \otimes_{\mathcal{O}_{S_\lambda}} \mathcal{B}_\lambda \to \mathcal{B}_\lambda$$

Then since we know that if we pullback the commutative, associative and distributive laws of $\mathcal{B}_\lambda$, we get the commutative, associative and distributive laws of $f_*\mathcal{O}_X$, therefore by passing to a larger λ, we can assume these diagrams also hold for $\mathcal{B}_\lambda$. Moreover, if we consider $\mathcal{O}_S \to f_*\mathcal{O}_X$, there is also a λ such that it's the pullback of $\mathcal{O}_{S_\lambda} \to \mathcal{B}_\lambda$ and the axiom of the unit also holds for this morphism. Therefore we can assume that $\mathcal{B}_\lambda$ is actually an $\mathcal{O}_{S_\lambda}$-morphism.

Consider:

$$\mathcal{O}_{S_\lambda} \to \mathcal{B}_\lambda \to \text{coker} \to 0$$

We know that since u_λ^* is right exact, therefore we know that it's pullback is also exacts, which is:

$$\mathcal{O}_S \to f_*\mathcal{O}_X \to u_\lambda^*\text{coker} \to 0$$

However since $\mathcal{O}_S \to f_*\mathcal{O}_X$ is surjective, we know that $u_\lambda^*\text{coker} = 0$, then we know that by passing to a larger λ, $\text{coker} = 0$, therefore $\mathcal{O}_{S_\lambda} \to \mathcal{B}_\lambda$ is also surjective. Then we know that the morphism:

$$\mathcal{S}pec\mathcal{B}_\lambda \to S_\lambda$$

is a closed embedding. Moreover, since relative spec commutes with pullbacks, we know that it's pullback is $X \to S$. Therefore we know that it can be identified with $f_\lambda : X_\lambda \to S_\lambda$ for sufficiently large λ, we are done.

3 separated morphisms: if $X \hookrightarrow S$ is separated, we know that $X \to X \times_S X$ is a closed embedding. Then we have the following diagram:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X \times_S X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & S \arrow[d] \\
X_\lambda \arrow[r] & X_\lambda \times_{S_\lambda} X_\lambda \arrow[r] & X_\lambda \arrow[r] & S_\lambda
\end{tikzcd}$$

Notice that first $X_0 \times_{S_0} \to X_0$ is qcqs, and $X_0 \to X_0 \times_{S_0} \to X_0$ is locally of finite presentation since $X_0 \to S_0$ is. Then since $X_0, X_0 \times_{S_0} \to X_0$ are qcqs, it's qcqs, therefore the morphism is qcqs. Then we know that the system $X_\lambda \times_{S_\lambda} X_\lambda$ is just like the original system S_λ, the key algebra reason is that the direct limit of $M \otimes_{A_0} A_\lambda$ is $M \otimes_{A_0} A$. I will let you check the details is you want. But this implies we can apply 2 to $X \hookrightarrow X \times_S X$, we are done. This also proves that for 5 affineness.

4 finite morphisms: $f : X \to S$ is a finite, then again consider $f_*\mathcal{O}_X$ we know that it's finite presented as f is finitely presented, then we can again find $\mathcal{B}_\lambda$ and we can again identify f_λ with $\mathcal{S}pec\mathcal{B}_\lambda \to S_\lambda$. We want to prove this is finite for large λ. This problem is affine local, therefore we assume that S_0 is affine. That is we want to prove that if: $A_\lambda \to B_\lambda$ is a ring map such that after we tensor with $\otimes_{A_\lambda} A$, it becomes finite, then it's finite. We know that since $B_\lambda \otimes_{A_\lambda} A$ is finite, we have an exact sequence:

$$A^n \to B_\lambda \otimes_{A_\lambda} A \to 0 \to 0$$

We know that $A^n \to B_\lambda \otimes_{A_\lambda} A$ is induced by some:

$$A_\lambda^n \to B_\lambda$$

the we know that the cokernel is 0 after pullsback to A, therefore it's 0. We are done.

6 surjective morphisms for Noetherian schemes S_0, X_0: First we notice that $S_\lambda, X_\lambda, S, X$ are all Noetherian. Recall that finitely presented morphisms are of finite type, therefore we can apply Chevalley's theorem and conclude that if we denote $f_\lambda(X_\lambda)$ by E_λ and $f(X)$ by E. We know that $E = X$. Notice that $f_0(X) = E_0$

is also constructible, and we know that the its inverse images in S_λ and S are E_λ, E since we know that we have a Cartesian diagram:

$$\begin{tikzcd}
S \arrow[r] & S_\lambda \arrow[r] & S_0 \\
X \arrow[u] \arrow[r] & X_\lambda \arrow[u] \arrow[r] & X_0 \arrow[u]
\end{tikzcd}$$

and in a Cartesian diagram, images and inverse images commute. Then we know that if E is S, for large enough λ, E_λ is S_λ.

7 radiciel morphisms: we know that a morphism is radiciel iff it's diagonal is surjective, then just like how we prove separatedness, we apply the surjective argument to the diagonal morphism.

8 locally closed embeddings: recall that a locally closed embedding can be factored to a closed embedding followed by an open embedding. Under the Noetherian conditions, they are quasicompact and quasiseparaed, then consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, hook, "\text{clo.}"] \arrow[d] & U \arrow[r, hook, "\text{open}"] \arrow[d] & S \arrow[d] \\
X_\lambda \arrow[r] \arrow[d] & U_\lambda \arrow[r, hook] & S_\lambda \arrow[d] \\
X_0 \arrow[rr] & & S_0
\end{tikzcd}$$

and we know that $\{U_{\mu \geqslant \lambda}\}$ is also an inverse just like that of $\{S_\lambda\}$. Moreover we know that the square of $X, X_\lambda, U, U_\lambda$ is again Cartesian, then we know that since the pullback of $X_\lambda \to U_\lambda$ is a closed embedding, then we can find a larger $\mu \geqslant \lambda$ such that it's a closed embedding, we are done.

9 quasi-affine morphisms: recall that quasi-affineness can be checked affine locally on the target, therefore we know we can assume S_0 is affine. Here we again need the fact that S_0 is quasi-compact, which will be used later. Now by our assumption, we know that $X \to S$ is quasi-affine, since S is itself is affine, we know that X is a quasi-affine scheme, thus we know that if we denote $\Gamma(X, \mathcal{O}_X)$ by B, then the following canonical morphism is an open embedding:

$$X \to SpecB$$

Since X is quasi-compact, we know there are $b_1, \cdots, b_n \in B$, such that $X = D(b_1) \cup \cdots \cup D(b_n)$. Recall that $X \to S = SpecA$ is finitely presented, then we know that for any affine open in X, the global section is a finitely generated A-algebra.

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr, hook] & & SpecA \\
& SpecB \arrow[ur] &
\end{tikzcd}$$

Then we know that B_{b_i} is a finitely generated A-algebra, say generated by generators $\frac{b_{i1}}{b_i^{k_{i1}}}, \cdots, \frac{b_{in_i}}{b_i^{k_{in_i}}}$. Let C be the sub-A-algebra of B generated by b_i, b_{ij}, then it's obvious that $C_{b_i} \simeq B_{B_i}$. Then if we consider $g : X \to SpecC$ defined by $X \hookrightarrow SpecB \to SpecC$. We know that the image of g is contained in $D(b_1) \cup \cdots \cup D(b_n)$. Then if we consider $D(b_i)$ in C, we know that g restricts to an isomorphism, therefore $g : X \to SpecC$ is an open embedding. Notice that C is a finitely generated A-algebra, therefore there is a S-closed embedding:

$$\begin{tikzcd}
C \arrow[rr, hook] \arrow[dr] & & \mathbb{A}^n_S \arrow[dl] \\
X \arrow[u, hook, "g"] \arrow[r] & S = SpecA &
\end{tikzcd}$$

Then consider $h : X \hookrightarrow C \hookrightarrow \mathbb{A}^n_S$, which is the composition of an open embedding and then a closed embedding. Recall that we know that X in $\mathbb{A}^n_S$ is $C \cap U$ where U is an open subscheme of $\mathbb{A}^n_S$, then we have the following diagram:

$$\begin{tikzcd}
X \arrow[r, dashed] \arrow[d, hook] & U \arrow[d, hook] \\
SpecC \arrow[r, hook] & \mathbb{A}^n_S
\end{tikzcd}$$

where the morphism $X \to U$ is induced by the fact that the image of $X \to \mathbb{A}^n_S$ is contained in U. I will let you check that $X \to U$ is a closed embedding. Therefore we know that $h : X \to \mathbb{A}^n_S$ is a locally closed embedding. Then we know that the morphism h is induced by some $h_\lambda : X_\lambda \to \mathbb{A}^n_{S_\lambda}$ and it's a locally closed embedding. Or if we draw everything in a diagram:

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d] & \mathbb{A}^n_S \arrow[r] \arrow[d] & S \arrow[d] \\
X_0 \arrow[r] & \mathbb{A}^n_{S_0} \arrow[r] & S_0
\end{tikzcd}$$

Then we know that we can apply the conclusion of locally closed embeddings to $X \hookrightarrow \mathbb{A}^n_S$ and conclude that there is a large λ such that $X_\lambda \to \mathbb{A}^n_{S_\lambda}$ is a locally closed embedding, therefore X_λ is an open subscheme of an affine scheme. This implies that f_λ, restricted to the inverse image of one of the affine covers of S_0 is quasi-affine, then since we only have finitely many affines covering S_0, we can find a common λ.

10 quasi-finite morphisms: again, this question is affine local target since quasi-finite is affine local on the target and S_0 is quasi-compact. Then consider $X \to S$, which is quasi-finite and affine. Recall that every affine morphism of finite type is quasi-projective. To see why, if $SpecA \to SpecB$ is the affine morphism and $B \to A$ is finite type, then we know that there is a closed embedding:

$$SpecA \hookrightarrow \mathbb{A}^n_B$$

Then we know that there is an open embedding $\mathbb{A}^n_B \hookrightarrow \mathbb{P}^n_B$, thus we have a locally closed embedding $SpecA \hookrightarrow \mathbb{P}^n_B$. Notice that this is quasi-compact as we know that we can cover $\mathbb{P}^n_B$ by $\mathbb{A}^n_B$'s such that their intersection of each other are still affine. Then we know that a quasi-compact locally closed embedding can always be factored into a open embedding composed with a closed embedding as we know that the scheme theoretic closure can be computed affine locally, i.e. in the following diagram:

$$\begin{tikzcd}
X \arrow[r, dashed] \arrow[d, hook] & \overline{X} \arrow[d, hook] \\
U \arrow[r, hook] & Y
\end{tikzcd}$$

$\overline{X}$ is the scheme theoretic closure, we want to show that $X \to \overline{X}$ is an open embedding. We know that the image of X in $\overline{X}$ is contained in $U \cap X$, therefore we only have to show that for every affine $SpecA$ in X that is also contained in U, we know that the morphism is:

$$SpecA/I \to SpecA \to SpecA$$

therefore $\overline{X}$ is also defined by $SpecA/I$, therefore the morphism $X \to \overline{X}$ is $SpecA/I \to SpecA/I$ which is isomorphism. Therefore we know that $SpecB \hookrightarrow \mathbb{P}^n_B$ can be factored to:

$$SpecA \xhookrightarrow{\text{open}} V \xhookrightarrow{\text{closed}} \mathbb{P}^n_B$$

and we know that V is a projective B-scheme.

Then recall that by Zariski's main theorem, if we have a quasi-projective quasi-affine morphism $X \to Y$, then it factors as an open embedding $j : X \hookrightarrow \overline{X}$ followed by a finite morphism $\overline{f} : \overline{X} \to X$. Then consider the diagram:

$$\begin{tikzcd}
X \arrow[r, hook, "j"] \arrow[d] & \overline{X} \arrow[r, "\overline{f}"] & S \arrow[d] \\
X_0 \arrow[rr] & & S_0
\end{tikzcd}$$

Recall that we assume that S is Noetherian, then we know that $\overline{X}$ is also Noetherian and $\overline{f}$ being finite implies it's finitely presented, then we know that we have a λ and a S_λ-scheme $\overline{X}_\lambda \to S_\lambda$ such that it pulls back to $\overline{f} : \overline{X} \to S$. Then we know that, by considering the system defined by pulling back $\overline{X}_\lambda$, passing to a larger λ if needed, we can also find a $j_\lambda : X_\lambda \to \overline{X}_\lambda$ inducing j. Then we know that the result of finite and open embedding shows that $X_\lambda \to \overline{X}_\lambda$ is open embedding and $\overline{X}_\lambda \to S_\lambda$ is finite. Moreover, since their composition induce f, we know that they are f_λ by passing to perhaps a larger λ, therefore we know that f_λ is the composition of an open embedding and a finite morphism, therefore quasi-finite.

Finally 11 proper morphisms: recall that proper morphisms are also affine local on the target, therefore we know that we can assume $S_0 = SpecA_0$ $S = SpecA$ for a Noetherian rings A, A_0. Then we know that by Chow's lemma, there is a diagram:

$$\begin{tikzcd} X' \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & SpecA & \end{tikzcd}$$

such that $g : X' \to X$ is surjective projective and the composition $X' \to SpecA$ is also projective.

Then we know that there will be large λ and $f_\lambda : X_\lambda \to S_\lambda$ and $g_\lambda : X'_\lambda \to X_\lambda$ such that $f \circ g$ is induced by $f_\lambda \circ g_\lambda$. Since we know that $X' \to SpecA$ is projective, we know that there will be a closed embedding of A-schemes:

$$X \hookrightarrow \mathbb{P}^n_A \to SpecA$$

Then we know that this is given by some closed embedding:

$$X_\lambda \hookrightarrow \mathbb{P}^n_{A_\lambda} \to SpecA_\lambda$$

and the composition is $f_\lambda \circ g_\lambda$ for a large enough λ. This implies that $f_\lambda \circ g_\lambda$ is proper since it's the composition of a closed embedding and a projective morphism.

Similarly, we can prove that since g is projective and induced by g_λ, we can show that g_λ is projective for large λ, therefore proper. Morepver we can assume g_λ is surjective by the result on surjectivity. Recall that proper means finite type, separated and universally closed. Finite type is always true and separatedness can be obtained by going to even a larger λ by our previous result. Therefore only universally closed need to be checked. Consider the following diagram:

$$\begin{tikzcd} \cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & Z \arrow[d] \\ X'_\lambda \arrow[r, "g_\lambda"] & X_\lambda \arrow[r, "f_\lambda"] & S_\lambda \end{tikzcd}$$

we know that the top composition is closed since $g_\lambda \circ f_\lambda$ is proper, and we know that since g_λ is surjective the second morphism on the top row is surjective, then this implies $\cdots \to Z$ is closed. $\square$

Another big theorem we need to use later is the following version of Zariski's main theorem:

Theorem 10.14. Zariski Let S be a Noetherian scheme, $f : X \to S$ a separated quasi-finite morphism, then there is a finite morphism $\overline{f} : \overline{X} \to S$ and an open embedding $j : X \hookrightarrow \overline{X}$ such that $f = \overline{f}j$.

Moreover, if Y is an affine scheme, say $SpecA$, we don't have to impose Noetherian conditions.

This is not the version of the Zariski's main theorem we discussed in the FOAG series, but by Zariski's main theorem, people could refer to different results, recall that we know the following version of the theorem:

Theorem 10.15. Zariski Let $f : X \to Y$ be a quasi-projective quasi-finite morphism of Noetherian schemes, then we can factorize f as:

$$X \overset{j}{\hookrightarrow} \overline{X} \xrightarrow{\overline{f}} Y$$

such that j is an open embedding and $\overline{f}$ is finite.

The first version of the Zariski's main theorem follows from the second version and what we discussed above:

Proof. $\square$

References