Notes on Etale cohomology: 2

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This is the second chapter of the notes series on étale cohomology. This chapter mainly focus on preliminaries we need before we discuss étale fundamental groups. The first section should have been on the sheaf of relative differentials, however, since this part of discussion is given in the FOAG series, we decide to omit it. We will provide a write-up latter following the treatment of the stack project, with a more general perspective.

1 Unramified Morphisms

Proposition 1.1. Let $f : X \to Y$ be a morphism locally of finite type and $x \in X$ is a point. We denote the image of x by $y = f(x)$, then the following condistions are equivalent:

1. $m_y\mathcal{O}_{X,x} = m_x$ and the residue field $k(x)$ is a finite separable extension of the residue field $k(y)$.
2. $(\Omega_{X/Y})_x = 0$.
3. The diagonal morphism $\Delta : X \to X \times_Y X$ is an open embedding in a neighborhood of x.

Proof. For $1 \Rightarrow 2$, recall that to compute $\Omega_{X/Y}$, we only need the local information of the morphism f, so we can assume that f is $SpecB \to SpecA$ where B is a finitely generated A-algebra. Then we know that $\Omega_{X/Y}$ restricted to $SpecB$ is $\Omega_{B/A}$ which is a finitely generated B-module, then we know that $(\Omega_{X/Y})_x$, say x correspond to the prime p in $SpecB$, then this stalk is $\Omega_{B/A} \otimes_B B_p$. By our assumption on the maximal ideal we know that:

$$\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} k(y) = k(x)$$

Then we know we have the following commutative diagram:

$$\begin{tikzcd} k(y) \arrow[d] & A_q \arrow[l] \arrow[d] & A \arrow[l] \arrow[d] \\ k(x) & B_p \arrow[l] & B \arrow[l] \end{tikzcd}$$

where the square on the right is a localization diagram and the square on the left is a cofibered product(because $m_y\mathcal{O}_{X,x} = m_x$). Then we know that $\Omega_{B/A}\otimes_B B_p \simeq \Omega_{B_p/A_q}$ and $\Omega_{B_p/A_q}\otimes_{B_p} k(x) \simeq \Omega_{k(x)/k(y)} = 0$ since $k(x)$ is a finite separable extension of $k(y)$. Therefore we have:

$$(\Omega_{X/Y})_x \otimes_{\mathcal{O}_{X,x}} k(x) = \Omega_{B/A} \otimes_B B_p \otimes_{B_p} k(x) \simeq \Omega_{k(x)/k(y)} = 0$$

Since $(\mathcal{O}_{X/Y})_x$ is finitely generated over $\mathcal{O}_{X,x}$, we can apply Nakayama's lemma and conclude it is 0.

Now assume 2, we know that the diagonal is always a locally closed embedding, i.e. we can factor the morphism as:

$$X \xhookrightarrow{\text{closed}} V \xhookrightarrow{\text{open}} X \times_Y X$$

Suppose we denote the closed embedding by i We know that $\Omega_{X/Y} \simeq i^*(\mathcal{I}/\mathcal{I}^2)$ where $\mathcal{I}$ is the quasicoherent ideal sheaf of $\mathcal{O}_V$ defining the closed embedding. Then recall that if we want to prove Δ is an open embedding in a neighborhood of x, we only need to prove $X \hookrightarrow V$ is an open embedding in a neighborhood of x. It suffices to prove $\mathcal{I}_{\Delta(x)} = 0$. Then we know from $(\Omega_{X/Y})_x = (i^*(\mathcal{I}/\mathcal{I}^2))_x \simeq \mathcal{I}_{\Delta(x)}/\mathcal{I}^2_{\Delta(x)} = 0$, i.e. $\mathcal{I}_{\Delta(x)} = \mathcal{I}^2_{\Delta(x)}$, now if we can use Nakayama's lemma, since $\mathcal{I}_{\Delta(x)}$ is contained in $m_{\Delta(x)}$, we conclude $\mathcal{I}_{\Delta(x)} = 0$. So we only need to prove that $\mathcal{I}_{\Delta(x)}$ is finitely generated over $\mathcal{O}_{X\times_Y X,\Delta(x)}$. Recall that by a local computation, the morphism Δ is just given by $B\otimes_A B \to B$ where B is a finitely generated A-algebra, then the kernel is I, and if B is generated by $a_1,\cdots,a_n$, we know that I is generated by $a_i\otimes 1, 1\otimes a_j$. This is because $B\otimes_A B$ is generated by $b\otimes 1 - 1\otimes b$, but we know that if $b = f(a_1,\cdots,a_n)$ where f is a polynomial with a coefficients, we know that $b\otimes 1 - 1\otimes b = f(a_1,\cdots,a_n)\otimes 1 - 1\otimes f(a_1,\cdots,a_n)$, then we are done.

Now assume 3, we want to prove 1. Recall that since what we want to prove are just the stalks at a point, we can just consider the fiber of y, i.e. we can consider:

$$\begin{tikzcd}
f^{-1}y \arrow[r] \arrow[d] & Speck(y) \arrow[d] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

and we only need to prove that $m_x = 0$. Moreover, after this base change, we can consider the new diagonal morphism, and we have a Cartesian diagram:

$$\begin{tikzcd}
& f^{-1}y \arrow[r] \arrow[dl] & f^{-1}y\times_k f^{-1}y \arrow[dl] \\
X \arrow[r] & X\times_Y X &
\end{tikzcd}$$

Moreover, since we know that after we restrict to an open neighborhood of X, $X\to X\times_Y X$ is an open embedding, we know that $f^{-1}y\to f^{-1}y\times_k f^{-1}y$ is also an open embedding after we restrict to an open of x in $f^{-1}y$.

Therefore, we can assume that $Y = Speck$ and Δ is an open embedding. Now if k is algebraically closed, then for any closed point t of X, we have a k-morphism $Speck\to X$ with image t, let Γ_t be the graph, then we have a Cartesian diagram:

$$\begin{tikzcd}
Speck \arrow[r, "\Gamma_t"] \arrow[d] & Speck\times_k X \arrow[d] \\
X \arrow[r, "\Delta"] & X\times_k X
\end{tikzcd}$$

Since Δ is an open embedding, Γ_i is also an open embedding, therefore $Speck\to Speck\times_k X\simeq X$ is t is an open embedding, this shows that every closed point of X is also open, therefore $\Gamma(t,\mathcal{O}_X) = k$, therefore we know that X is a disjoint union of copies of $Speck$ as every point of X is a closed point. To see why, suppose that $x\in X$, then we can pick an affine open $SpecA$ of x and A is finite type over k, therefore we know from Nullstellensatz that closed points in $SpecA$ are also closed points in X, then we know that the closure of x has a closed point, which is a contradiction since the closed point is open.

In general, we know that k is not necessarily algebraically closed, but we can consider base change to $\overline{k}$, i.e. after shrinking X, we have $X\to Speck$, now base change this to $X\times_k\overline{k}\to Spec\overline{k}$, the diagonal of this map is the base change of $X\to X\times_k X$, therefore this is also an open embedding, as shown in the following diagram:

$$\begin{tikzcd}
X_{\overline{k}} \arrow[r, "\Delta"] \arrow[d] & X_{\overline{k}}\times_{\overline{k}} X_{\overline{k}} \arrow[d] \\
X \arrow[r, "\Delta"] & X\times_k X
\end{tikzcd}$$

Then we know that by further restrict X, we can assume that $X_{\overline{k}}$ is actually a finit disjoint union of $Spec\overline{k}$, therefore, $X = SpecA$ is also a finite set and A is a finitely generated k-algebra. Then we know that A is actually finite dimensional over k, therefore if we tensor with $\overline{k}$, we get a ring isomorphism $A\otimes_k\overline{k}\simeq\overline{k}^{\oplus n}$(this isomorphism always exists as an isomorphism of $\overline{k}$ vector spaces, but not always as rings, here we have such an isomorphism of rings since $X_{\overline{k}}$ is a disjoint union of $Spec\overline{k}$), so $A\otimes_k\overline{k}$ is reduced, hence A is reduced. Then recall that $X = SpecA$ is discrete, then we can further restrict X such that X is a single point, then

$SpecA$ is a reduced ring with one single prime ideal, therefore A is a field, and $k \to A$ is a field extension. It's of course finite as we have seen that $A \otimes_k \overline{k}$ is reduced, therefore A is separable over k. □

To be able to define the object of this section, we need to introduce a new notion of morphism which we skipped in the FOAG series, that is locally of finite presentation:

Definition 1.2. A ring A is a finitely presented B-algebra, or $B \to A$ is finitely presented. If A has a finite number of generators and a finite number of relations over B, i.e. we have a surjective morphism of rings:

$$B[x_1, \cdots, x_n] \to A$$

such that the kernel is finitely generated over $B[x_1, \cdots, x_n]$.

If B is Noetherian, then this condition is equivalent to A is finitely generated as a B-algebra as the finite relations conditions comes for free.

Just as if a module is finitely presented, then it's always finitely presented, we can prove that if A is finitely presented, then it's always finitely presented:

Proposition 1.3. Suppose that $A = B[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$ suppose that $Y_1, \cdots, Y_m$ also generate A as an B-algebra, i.e. we have an surjection:

$$\phi : B[y_1, \cdots, y_m] \to A, \quad y_i \mapsto Y_i$$

Then the kernel is also finitely generated over $B[y_1, \cdots, y_m]$.

Proof. We can mimic what we did when we prove finite presentation for modules implies always finitely presented.

First, notice we have an surjection:

$$B[x_1, \cdots, x_n, y_1, \cdots, y_m] \to A, \quad x_i \mapsto x_i, y_j \mapsto Y_j$$

We know since we know that $B[x_1, \cdots, x_n] \to A$ is surjective, we can find lifts of Y_i, say $g_1, \cdots, g_m$, then we know that the kernel of the above surjection is:

$$(f_1, \cdots, f_r, y_1 - g_1, \cdots, y_m - g_m)$$

I.e. we have an isomorphism:

$$B[x_1, \cdots, x_n, y_1, \cdots, y_n]/(f_1, \cdots, f_r, y_1 - g_1, \cdots, y_m - g_m) \to A$$

Now since Y_i generates A, we know that there are $h_i \in B[y_1, \cdots, y_m]$ such that $\phi(h_i) = x_i$, i.e. a morphism $h : B[x_1, \cdots, x_n] \to B[y_1, \cdots, y_n]$. Then we know that we have an isomorphism:

$$B[x_1, \cdots, x_n, y_1, \cdots, y_n]/(f_1, \cdots, f_r, y_1 - g_1, \cdots, y_m - g_m, x_1 - h_1, \cdots, x_n - h_n) \to A$$

Then consider the morphism:

$$B[x_1, \cdots, x_n, y_1, \cdots, y_n]/(f_1, \cdots, f_r, y_1 - g_1, \cdots, y_m - g_m, x_1 - h_1, \cdots, x_n - h_n) \to$$

$$B[y_1, \cdots, y_m]/(h(f_1), \cdots, h(f_r), y_1 - h(g_1), \cdots, y_m - h(g_m))$$

where $y_i \mapsto y_i, x_j \mapsto h_j$. This is an isomorphism as we have an inverse defined by $y_i \mapsto y_i$, I will let you check this is well defined. □

Proof. The condition that A is a finitely presented B-algebra satisifes the affine communication lemma, i.e. suppose that we have a morphism $SpecA \to SpecB$ such that A is a finitely presented B-algebra, then for any $f \in A$ A_f is also a finitely presented B-algebra. Converse, if for all $f \in A$ A_f is a finitely presented B-algebra, then A is. □

Proof. The first half of the condition is easy, as suppose that $A = B[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$, then for any $f \in A$ we know that:

$$A_f = B[x_1, \cdots, x_n]_f/(f_1, \cdots, f_r)$$

Then we have a surjection:

$$B[x_1, \cdots, x_n, y] \to B[x_1, \cdots, x_n]_f/(f_1, \cdots, f_n), \quad x_i \mapsto x_i, y \mapsto \frac{1}{f}$$

What is the kernel of this map, I claim that it's $(f_1, \cdots, f_r, 1 - fy)$. To prove this is an isomorphism, we can construct an inverse and I will let you do it.

Then conversely, suppose that A is an B-algebra such that $(g_1, \cdots, g_m) = A$ and A_{g_i} are finitely presented over B. First, finitely presented implies finitely generated, then we already saw that A is finitely generated over B, and there is a surjective:

$$B[x_1, \cdots, x_n] \to A$$

We want to show the kernel I is finitely generated.

As $A = (g_1, \cdots, g_m)$, we may choose $h_j \in A$ such that $1 = h_1 g_1 + \cdots + h_m g_m$, choose lifts H_j, G_j of h_j, g_j in $B[x_1, \cdots, x_n]$ define $G_0 = 1 - H_1 G_1 - \cdots - H_m G_m$ so we know that $G_0 \in I$ and $(G_0, G_1, \cdots, G_m) = 1$ in $B[x_1, \cdots, x_n]$. Now, we only need to prove I_{G_i} is finitely generated over $B[x_1, \cdots, x_n]_{G_i}$ for all $G_0, \cdots, G_m$. First for G_0 we know that I_{G_0} is 1 as G_0 is in $G_o \in I$. Then for other G_j, we know that localization is exact, therefore:

$$B[x_1, \cdots, x_n]_{G_i} \to A_{G_i}$$

However, notice that $B[x_1, \cdots, x_n]_{G_i} = B[x_1, \cdots, x_n, y]/(1 - G_i y)$, therefore the kernel is finitely generated, we are done. ☐

Definition 1.4. We say that a B-scheme $X \to SpecB$ is locally finitely presented over B if there is an affine open cover $SpecA_i$ of X such that each A_i is finitely presented over B.

Then by the affine communication lemma and the previous proposition, it suffice to check any affine open cover of X.

Proposition 1.5. Suppose that $\pi : X \to Y$ is a morphism of schemes, then the conditions that $SpecB \subset Y$ satisfies the condition that $\pi^{-1}SpecB \to SpecB$ is locally finitely presented satisfies the affine communication lemma.

Proof. First, consider the following diagram:

$$\begin{tikzcd} SpecA_f \arrow[r, hook] \arrow[d, hook] & \pi^{-1}SpecB_f \arrow[r] \arrow[d, hook] & SpecB_f \arrow[d, hook] \\ SpecA \arrow[r, hook] & \pi^{-1}SpecB \arrow[r] & SpecB \end{tikzcd}$$

Then, we need to prove that if $B \to A$ describes A as a finitely presented B-algebra, then $B_f \to A_f$ describes A_f as a finitely presented B_f-algebra. This is quite easy since if:

$$A = B[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$$

Then obviously if we localize at an element in B, we get $B_f[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$.

Then suppose that $(g_1, \cdots, g_m) = B$ and each A_{g_i} is finitely presented over B_{g_i}, we want to prove A is finitely presented over B. Suppose that the generators are all from A, then we can define a morphism:

$$B[x_1, \cdots, x_n] \to A$$

where each x_i is mapped to the generators, I will let you prove this is surjective. And we only need to prove this kernel is finitely generated, again we localize at each g_j use the fact of always finitely presented. ☐

Definition 1.6. A morphism $X \to Y$ is locally finitely presented or locally of finite presentation if for any affine open $SpecB$ of Y $\pi^{-1}SpecB \to SpecB$ is locally finitely presented.

By the above proposition, we know that it suffice to check for an arbitrary affine cover of Y.

Remark 1.7. In particular combine the preceding two results, for any affine open $SpecB$ of Y and any affine open $SpecA$ such that the image of $SpecA$ is in $SpecB$, then A is finitely presented over B.

Proposition 1.8. Suppose that we have an affine open cover $SpecA_i$ of X and affine opens $SpecB_i$ of Y such that $\pi(SpecA_i) \subset SpecB_i$. Suppose that A_i is finitely presented over B_i, then π is locally finitely presented.

Proof. We need to check that there is an affine cover $SpecC_{ij}$ of $\pi^{-1}SpecB$ such that C_{ij} is finitely presented over B_i.

What is a little bit tricky is that we only know the image of $SpecA_i$ is in $SpecB_i$, but $SpecA_i$ may not cover $\pi^{-1}SpecB_i$. So we need something else. Now suppose that $x \in \pi^{-1}SpecA$, we know that the image in $SpecB_i$ is contained in some $SpecB_j \cap SpecB_i$, we can suppose that x is not in $SpecA_i$ then we know that $x \in SpecA_j$, then we can suppose that there is an distinguished open $(A_j)_{f_j}$ such that it's finitely presented over $(B_i)_{g_j}$, this is because we know that $(A_j)_{f_j}$ is finitely presented over B_j, therefore it's finitely presented over localizations if the morphism factors, i.e. if $B \to A$ describes A as a finitely presented B-algebra, then if this morphism factors through $B_f \to A$, then I claim A is finitely presented over B_f. Then we know that $(A_j)_{f_j}$ is finitely presented over $(B_i)_{g_j}$ and $(B_i)_{g_j}$ is finitely presented over B_i, thus the composition is also finitely presented. This is because if $C \to B \to A$ is a sequence of ring maps such that B is finitely presented over C and A is finitely presented over B, then A is finitely presented over C as:

$$B = C[x_1, \cdots, x_n]/(f_1, \cdots, f_r) \quad A = B[y_1, \cdots, y_m]/(g_1, \cdots, g_s)$$

Then $A = C[x_1, \cdots, x_n, y_1, \cdots, y_m]/(f_1, \cdots, f_r, g_1, \cdots, g_s)$, we are done. $\square$

Example 1.9. Open embeddings are locally of finite presentation as if $U \hookrightarrow X$ is an open embedding, for any $SpecA$ in X, we can cover the inverse image by distinguished opens, and it's trivial to show that localizations are finitely presented.

Also, if $f : X \to Y, g : Y \to Z$ are both locally finitely presented, then so is $g \circ f$, this is basically because if $C \to B \to A$ is a sequence of ring maps such that B is finitely presented over C and A is finitely presented over B, then A is finitely presented over C, as what we discussed before.

Proposition 1.10. Locally of finite presentation is preserved by base change by any morphism.

Proof. Consider the following Cartesian diagram:

$$\begin{tikzcd} Z \times_Y X \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

where f is locally of finite presentation. We know that by cover Y by affines and then cover the inverse images in X, Z by affines, we can reduce the problem to the following algebra question: if $B \to A$ is finitely presented over B, then is $C \to A \otimes_B C$ finitely presented, I will let you prove it. $\square$

Definition 1.11. A morphism is called **finitely presented** or (of finite presentation) if it's locally finitely presented and qcqs.

Remark 1.12. This definition violates our usual naming convention that if we add quasi-compactness, then we can drop "locally". Here we are requiring this quasi-separated condition, here we also want quasi-separated because this is the top finiteness condition we can assume for non-Noetherian schemes, it's not finitely generated, the relations are finite, moreover, the intersections are still finitely presented.

And, if you work with Noetherian schemes, this is just the same as finite type. And if you work with locally Noetherian schemes, locally of finite presentation is the same as locally of finite type. No matter what schemes you work with, locally of finite presentation implies locally of finite type.

Definition 1.13. Suppose that $f : X \to Y$ is a morphism locally of finite presentation, then it's called unramified at at x if any of the following three equivalent conditions we proved holds:

1. $m_y\mathcal{O}_{X,x} = m_x$ and the residue field $k(x)$ is a finite separable extension of the residue field $k(y)$.
2. $(\Omega_{X/Y})_x = 0$.
3. The diagonal morphism $\Delta : X \to X \times_Y X$ is an open embedding in a neighborhood of x.

where $y = f(x)$. We will say f is unramified if f is unramified at every point.

Remark 1.14. Since $\Omega_{X/Y}$ is finite type under the hypothesis $f : X \to Y$ is locally of finite type(or locally of finite presentation), we know from Geometric Nakayama that $(\Omega_{X/Y})_x$ is 0 is an open condition, therefore the locus where f is unramified is open.

The following are some basic properties of unramified morphisms:

Proposition 1.15. Let's assume $f : X \to Y$ is a morphism locally of finite presentation. Then:

1. Unramified morphisms are locally quasi-finite.
2. Open and closed embeddings are unramified.
3. Compositions of unramified morphisms are unramified.
4. Base change of unramified morphisms are unramified.
5. If the composition $X \to Y \to Z$ is unramified , then so is $X \to Y$.

Proof. The last property, i.e. 5 is a consequence of the previous 4 properties. Recall that if a class of morphisms is preservd by composition and base change, then we have the cancellation theorem, since the diagonal morphism is a locally closed embedding, then it's unramified, then this implies that if $X \to Y \to Z$ is unramified, then so is $X \to Y$. So now let's prove the first 4 properties.

For property 1, recall that locally quasi-finite means that locally finite type and quasi-finite at each point. Unramified morphisms are locally of finite presentation therefore locally of finite type. To prove that this is quasi-finite at each point $x \in X$, we need to prove that $\mathcal{O}_{X,x}$ is quasi-finite over $\mathcal{O}_{Y,f(x)=y}$. Recall that being unramified at a point x means that $m_x = \mathcal{O}_{X,x}m_y$, then we know that $\mathcal{O}_{X,x}/\mathcal{O}_{X,x}m_y = k(x)$, which is finite separable over $k(f(x)) = \mathcal{O}_{Y,y}/m_y$, then we are done.

For property 2, we know that open and closed embeddings are locally of finite presentation, and recall that for any embedding $X \hookrightarrow Y$, we know that $\mathcal{O}_{X,x}/m_y\mathcal{O}_{X,x} \simeq \mathcal{O}_{Y,y}/m_y$, then we are done.

For property 3, recall that compositions of locally finite presentation morphisms are locally finite presentations. Then consider:

$$\mathcal{O}_{Z,gf(x)} \to \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

Since $\mathcal{O}_{Y,f(x)}m_{gf(x)} = m_{f(x)}$ and $\mathcal{O}_{X,x}m_{f(x)} = m_x$, we know that:

$$\mathcal{O}_{X,x}m_{g(f(x))} = \mathcal{O}_{X,x}(\mathcal{O}_{Y,f(x)}m_{gf(x)}) = m_x$$

Moreover, we know that we have:

$$k(gf(x)) \to k(f(x)) \to k(x)$$

since composition of finite extensions is finite and composition of separable extensions is separable, we are done.

For property 4, again, locally of finite presentation is preserved by base change. Moreover, we know that if we consider the following pullback diagram:

$$\begin{tikzcd} Z \times_Y X \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

where f is unramified, we know that $\Omega_{Z\times_Y X/Z} \simeq g'^*\Omega_{X/Y}$. Then we know that for $w \in Z \times_Y X$, we want to compute:

$$(\Omega_{Z\times_Y X/Z})_w = (g'^*\Omega_{X/Y})_w = (\Omega_{X/Y})_{g'(w)} \otimes_{\mathcal{O}_{X,g'(w)}} \mathcal{O}_{Z\times_Y X,w} = 0$$

We are done. ∎

Proposition 1.16. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two morphisms and if $g : Y \to Z$ is unramified then the graph $\Gamma_f : X \to X \times_Z Y$ is an open embedding.

Proof. Consider the canonical Cartesian diagram:

$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] \arrow[d] & X \times_Z Y \arrow[d] \\ Y \arrow[r, "\Delta"'] & Y \times_Z Y \end{tikzcd}$$

This is indeed Cartesian as you can check the universal property. For example:

$$\mathrm{Hom}(W, X \times_Z Y) = \{(\alpha, \beta) : g \circ f \circ \alpha = g \circ \beta\}$$

Then we know that if we denote a morphism $\gamma : W \to Y$ and we consider (γ, α, β) such that $\Delta \circ \gamma = (X \times_Z Y \to Y \times_Z Y) \circ (\alpha, \beta)$, we know that this is (γ, α, β) such that $(\gamma, \gamma) = (f \circ \alpha, \beta)$ and $g \circ \beta = f \circ g \circ \alpha$. Therefore this is just α since after α is given $\gamma = f \circ \alpha$ and $\beta = \gamma$. Therefore this is indeed Cartesian. Then since g is unramified, Δ is an open embedding, then so is Γ_f. ☐

Proposition 1.17. Consider a Cartesian diagram:

$$\begin{tikzcd} X \times_Y Y' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

Suppose that f is locally of finite presentation, then if f' is unramified, then so is f.

Proof. We know that $g'\Omega_{X/Y} = 0$. But since g' is faithfully flat, we know that this implies $\Omega_{X/Y} = 0$, we are done. ☐

2 Etale Morphisms

let's begin with the definition of étale morphisms:

Definition 2.1. Let $f : X \to Y$ be a morphism of locally finite presentation, and $x \in X$ is a point. We say f is étale at x if f is flat and unramified at x.

If f is étale at every point, we say f is étale.

This definition is also given in the FOAG series, but we presented this in a different order. Here we choose to do this to get into the topic in a more swift way. In this section, we will discuss properties of étale morphisms that we will use later.

Proposition 2.2. In this proposition, all morphisms are of locally finite presentation:

1. Open embeddings are étale.

2. Compositions of étale morphisms are etale.

3. Base change of étale morphisms are étale.

4. Let $f : X \to Y$ and $g : Y \to Z$ are two morphisms such that $g \circ f$ is étale and g is unramified, then f is étale.

Proof. Let's first explain why general closed embeddings are not étale. We have seen that closed embeddings are locally of finite presentation and unramified. Therefore, the only reason closed embeddings are not always *étale* is that closed embeddings are not always flat. This is not easy to see why, as we know that for closed embeddings $X \hookrightarrow Y$, for $x \in X$, the induced ring map on the local rings is

$$\mathcal{O}_{Y,x} \to \mathcal{O}_{Y,x}/\mathcal{I}_x$$

where $\mathcal{I}$ is the quasicoherent ideal sheaf defining the closed embedding. In general quotient ring maps are not flat. One criterion for $A \to A/I$ to be flat is when I is pure, i.e. for all finitely generated ideal J of A, $J \cap I = IJ$. I will let you prove why. But clearly this is not a condition that is satisfied by every ideal I, thus the quotient ring map is in general not flat.

This also shows why open embeddings are *étale*, since $A \to A$ is flat for all rings.

Then since we know that unramified, locally presentation and flat morphisms are preserved by composition, we conclude *étale* morphisms are preserved by composition. Again, these properties are preserved by base change, therefore *étale* morphisms are preserved by base change.

The last property, i.e. property 4 is again a formal consequence of the first 3 as if g is unramified, the diagonal is an open embedding, which is *étale*. Then we conclude the desired assertion from cancellation theorem. □

Remark 2.3. The same argument above shows that if $g \circ f$ is flat and g is unramified, then f is flat.

One case that shows up particularly often is the following:

Example 2.4. Let A be a ring and $F(t) \in A[t]$ be a monic polynomial. Let $B = A[t]/F(t)$, then the morphism $SpecB \to SpecA$ is *étale* at a prime ideal $q \in SpecB$ iff q doesn't contain the image of $F'(t)$ in B. Hence if we want $SpecB \to SpecA$ to be *étale*, then for every prime ideal of q, it doesn't contain the image of $F'(t)$, i.e. it's a unit, i.e. the ideal generated by $F(t)$ and $F'(t)$ is $A[t]$.

It's not hard to see this is true. First, we know that the morphism is clearly locally finitely presented, almost by definition. Then we need to check flatness and unrafimied at each point. Recall that we have $A \to A[t]/(F(t))$, as $F(t)$ is monic, $A[t]/(F(t))$ is free over A with rank the degree of F. Therefore being free implies flat, therefore flatness is good. To check unramified, we need to show that:

$$\Omega_{A[t]/(F(t))/A} = 0$$

Recall that we know how to compute the sheaf of differential in this case and it's going to be:

$$A[t]/(F(t), F'(t))$$

Then we know that if we want $(A[t]/(F(t), F'(t)))_q = 0$, then we know that q is not in $V(F(t), F'(t))$, i.e. $F'(t)$ is not in q. And if this is true for all q, then we know that $F'(t)$ is not in any q, therefore $F'(t)$ is a unit in $A[t]/(F(t))$, therefore we are done.

This example turns out to be not special and we will see later that all *étale* morphisms are related to it:

Lemma 2.5. Let (A, m) be a local ring and B' is a finite A algebra, n' a maximal ideal of B', u an element of B' not in n' but in all the other maximal ideals of B' distinct from n'. Now let $B = A[u] \subset B'$ and $n = n' \cap B$ so we have $A \to B = A[u] \hookrightarrow B'$. Now assume $B'_{n'}/mB'_{n'}$ is generated by the image of u as an A/m-algebra then the canonical map $B_n \to B'_{n'}$ is an isomorphism.

Proof. This is a quite weird result at first sight, but we will use it later to characterize *étale* morphisms.

Let $S = B - n$ and $S' = B' - n'$, we wish to show that $S^{-1}B \to S'^{-1}B'$ is an isomorphism. Let's first do it by show $S^{-1}B' \to S'^{-1}B'$ is an isomorphism, i.e. we need to show that all the elements in S' are invertible in $S^{-1}B'$. Or equivalently, we only need to show that the images of S' in $S^{-1}B'$ do not lie in any maximal ideals. Recall that B' is fiinte over A, therefore it's finite over B', therefore $S^{-1}B'$ is finite over $S^{-1}B$, and in particular integral, therefore the inverse image of maximal ideals are still maximal, and we conclude that any maximal ideal of $S^{-1}B'$ lies above the maximal ideal of B_n, and in particular over the maximal ideal n (n is a maximal ideal again because $B \hookrightarrow B'$ is finite.) Again, from B' is finite over B and $B \hookrightarrow B'$ is an inclusion, we know that any maximal ideal of B comes from a maximal ideal of B'. So if we consider:

$$\begin{tikzcd} B \arrow[r] \arrow[d] & B' \arrow[d] \\ S^{-1}B = B_n \arrow[r] & S^{-1}B' \end{tikzcd}$$

this tells you that any maximal ideal of $S^{-1}B'$ lies over a maximal ideal of B', hence is of the form $S^{-1}n''$ for some maximal ideal n'' of B' such that $n'' \cap B = n$. Now if $u \in n''$, then $u \in n'' \cap B = n \subset n'$, a contradiction that $u \notin n'$. So $u \notin n''$, but we know that u lies in all maximal ideals distinct from n', we must have $n'' = n'$. Therefore we know that the only maximal ideal of $S^{-1}B'$ is $S^{-1}n'$. By definition $S' = B' - n'$, therefore the images of S' in $S^{-1}B'$ can not be in $S^{-1}n'$, we are done and $S^{-1}B' \to S'^{-1}B$ is an isomorphism.

Then we need to show that $S^{-1}B \to S^{-1}B'$ is an isomorphism. First, being a localization of an inclusion, it's injective, so we only need to show that it's surjective. Let's do this directly by showing that $S^{-1}B \to S'^{-1}B'$ is surjective. Recall that $B_n = S^{-1}B \to S'^{-1}B' = B'_{n'}$ is finite, then to prove it's surjective, we need to prove the cokernel is 0, we know that as a quotient of $B'_{n'}$, the cokernel is again finite over $S^{-1}B = B_n$, then we only need to prove that $\operatorname{coker} \otimes B_n/nB_n = 0$ by Nakayamas lemma. Recall that $mB_n \subset nB_n$ since $A \to B \to B'$ are all finite, then we know that n'' pulls back to n which pulls back to m, therefore it suffice to prove:

$$B_n/mB_n \to B'_{n'}/mB'_{n'}$$

is surjective. But if you consider:

$$A/m \to B_n/mB_n \to B'_{n'}/mB'_{n'}$$

By our assumption, $B'_{n'}/mB'_{n'}$ is generated by the image of u, therefore this is surjective. □

The following theorem due to Chevalley tells you how to characterize étale morphisms:

Theorem 2.6. Chevalley Let $f : X \to Y$ be a morphism of locally finite presentation, $x \in X$ a point, and $A = \mathcal{O}_{Y,f(x)}$. Then f is étale at x iff we can find a monic polynomial $F(t) \in A[t]$ and a maximal ideal n_0 of $C = A[t]/(F(t))$ such that n_0 doesn't contain the image of $F'(t)$ in C and such that $\mathcal{O}_{X,x}$ is A-isomorphic to C_{n_0}.

Proof. Let's first deal with the if part. To prove f is étale at x, we need to prove it's flat and unramified at x, i.e. $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat and $m_x = \mathcal{O}_{X,x}m_y$. Recall that we have:

$$A \to A[t]/(F(t)) = B$$

which is finite since $F(t)$ is monic, then we know that any maximal ideal of B pulls back to the maximal ideal m_y of A, then since n doesn't contain $F'(x)$, we know that $SpecB \to SpecA$ is étale at n, which means that $A \to B_n$ is flat and unrafimied, but we have an isomorphism:

$$\begin{tikzcd} A = \mathcal{O}_{Y,f(x)} \arrow[rr] \arrow[dr] & & B_n \arrow[dl, "\simeq"] \\ & \mathcal{O}_{X,x} & \end{tikzcd}$$

Then we know from $A \to B_n$ is flat and $nB_n = m_yB_n$ that $\mathcal{O}_{X,x}$ is flat over A and $m_x = m_y\mathcal{O}_{X,x}$.

The important and difficult part is the converse. First, let's assume that f is étale at x, all we care about is the local information of x and $f(x)$, thus by using the canonical morphism $SpecA \hookrightarrow Y$ where $SpecA$ is $Spec\mathcal{O}_{Y,f(x)}$, we can assume that $Y = SpecA$ and the morphism is $f : X \to SpecA$ with f being étale at x. To see why we can do this reduction, first base change preserves locally of finite presentation, then I will let you see that the base change doesn't change the map between the local rings, therefore the map after the base change is still étale at x and with the same map of local rings as that induced the morphism we started with.

Recall that for morphisms locally of finite presentation, being unramified is an open condition, i.e. we can find an affine open of x such that $U \to SpecA$ is unrafimied. Recall that any morphism between two affine schemes is separated and unramified morphisms are locally quasi-finite, then by Zariski's main theorem we have a factorization:

$$U \hookrightarrow SpecB' \to SpecA$$

where the first morphism is an open embedding and the second morphism is finite, i.e. B' is a finite A-algebra. Let n' be the prime ideal of B' corresponding to x, recall that since B' is finite over A, in particular, it's integral, therefore the image of n' is still a maximal ideal, but the only maximla ideal of A is m. Recall that that $U \hookrightarrow SpecB' \to SpecA$ implies that $\mathcal{O}_{X,x}$ is A-isomorphic to $B'_{n'}$. Recall that all maximal ideals of B' lies over the unique maximal ideal of A and $SpecB \to SpecA$ being finite means that the inverse image of m is a finite set, therefore, we know that we only have finitely many maximal ideals in B. Say $n'_1, \cdots, n'_r$ are all the maximal ideals of B' that is distinct from n'. Recall that f is unramified at x, i.e. n', therefore $A/m \to B'/n'$ is a finite separable extension, therefore is generated by a single element, say u.

Now since $n'_1, \cdots, n'_r$ are distinct from n', by the Chinese Reminder theorem, we know we have a surjection:

$$B' \to B'/n' \times B'/n'_1 \times \cdots \times B'/n'_r$$

Consider $(\alpha, 0, \cdots, 0)$ in the product, since this ring map is surjective, there is a u in B' that is mapped to this $(\alpha, 0, \cdots, 0)$, which means $u \in n'_1 \cap \cdots \cap n'_r$ such that the image in B'/n' generates B'/n' as an A/m-algebra (by primitive element theorem). Now, let $B = A[u]$, and $n = n' \cap B$, we know from the previous lemma that $\mathcal{O}_{X,x}$ is A-isomorphic to B_n.

Now let $\overline{u} \in B/mB$, be the image of u in B/mB. Then the field extension $A/m \hookrightarrow B/mB \hookrightarrow B'/mB' = B'/n'$ is finite separable and B'/n' is generated as an A-algebra by u, therefore so is B/mB, then let $f(t)$ be the minimal polynomial of $\overline{u}$ in B/mB, then we know that we have an isomorphism:

$$(A/m)[t]/(f(t)) \simeq B/mB$$

Since $(A/m)[t]/(f(t))$ is free over A/m and is generated by t^i's, by Nakayama's lemma, since B is finite over A, we know that B is generated by $1, u, \cdots, u^{d-1}$ as an A-module where d is the degree of f. This implies that u^d can be written as an A-linear combination of $1, u, \cdots, u^{d-1}$, therefore, there is a degree d polynomial $F(t)$ in $A[t]$ such that $F(u) = 0$. Then notice that the image of $F(t)$ in $(A/m)[t]$ is degree d and satisfies $F(\overline{u}) = 0$, therefore the image of $F(t)$ in $(A/m)[t]$ is necessarily $f(t)$.

Now consider the morphism $SpecB/mB \to SpecA/m$, which is *é*tale at n/mB since this is first finite, then we know that $A/m \to B/n$ is flat (as A/m is a field) and it's unramified because base change preserved unramified points as well. Then we know from our characterization before that the image of $f'(t)$ in B/mB is not in n/mB, but the image is $f'(\overline{u})$, therefore $f'(\overline{u}) \notin n/mB$, this implies $F'(u) \notin n$.

Moreover since $f : U \hookrightarrow SpecB' \to SpecA$ is also flat at x, i.e. $B'_{n'}$ is flat over A, then we know that $\mathcal{O}_{X,x}$ is flat over A, but since $\mathcal{O}_{X,x}$ is A-isomorphic to B_n, we know that B_n is also flat over A. Then consider the following epimorphism:

$$A[t]/(F(t)) \to B, \quad t \mapsto u$$

Let n_0 be the inverse image of n, we know that since $F'(u) \notin n$, n_0 doesn't contain $F'(t)$, therefore we know that $SpecA[t]/(F(t)) \to SpecA$ is *é*tale at n_0, in particular unramified at a neighborhood of n_0, then consider the composition:

$$SpecB \to SpecA[t]/(F(t)) \to SpecA$$

I claim that this implies that $SpecB \to SpecA[t]/(F(t))$ is flat at n. This is because if:

$$X \to Y \to Z$$

is flat at x and $Y \to Z$ is unramified then this implies $X \to Y$ is flat at x. To prove this, consider the canonical morphism $Spec\mathcal{O}_{X,x} \to X \to Y$, I claim that the morphism $X \to Y$ is flat at x is equivalent to $Spec\mathcal{O}_{X,x} \to X \to Y$ being flat. It's clear that if $Spec\mathcal{O}_{X,x} \to X \to Y$ is flat, then the original morphism is flat at x. The converse follows from the image of $Spec\mathcal{O}_{X,x}$ in X can be describe in an affine open of x, and is the primes contained in x, then if $X \to Y$ is flat at x, then we know that $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat, suppose that the morphism is defined by $B \to A \to A_p$ we know that $B_q \to A_p$ is flat means that $B_{q'} \to A_{p'}$ is flat for p' contained in p, as we now that this is just a further localization of $B_q \to A_p$. Then we know from the cancellation theorem that we are done.

So $SpecB \to SpecA[t]/(F(t))$ is flat at n. So we know that $(A[t]/(F(t)))_{n_0} \to B_n$ is flat, and since these are local rings and the map is a map of local rings, therefore we know that this must be faithfully flat. Suppose that $A \to B$ is faithfully flat, it must be injective as this is $A \to A \otimes_A B \simeq B, \quad a \mapsto a \otimes a \otimes 1 \mapsto a$, which is injective. But on the other hand, we know $A[t]/(F(t)) \to B$ is surjective, therefore we know that every element in $B - n$ comes from something in $A[t]/(F(t)) - n_0$, therefore $A[t]/(F(t))_{n_0} \to B_n$ is also surjective, therefore B_n is A-isomorphic to $A[t]/(F(t))_{n_0}$, and since B_n is A-isomorphic to $\mathcal{O}_{X,x}$, we conclude $\mathcal{O}_{X,x}$ is A-isomorphic to $A[t]/(F(t))_{n_0}$ $\square$

As one corollary of this result, recall that we proved before that under the hypothesis of locally of finite presentation or locally of finite type, the unramified points are open. What about the étale-points? Your first thought might be to prove that flatness is open under our hypothesis, and it turns out we can do the following:

Corollary 2.7. Let $f : X \to Y$ be a morphism locally of finite presentation, then the set of points of X where f is étale is open.

Proof. Suppose that f is étale at X, then by Chevalley's result, we know that this is equivalent to there is a monic polynomial $F(t)$ in $\mathcal{O}_{Y,f(t)}[t]$ and a maximal ideal n of $B = \mathcal{O}_{Y,f(t)}[t]/(F(t))$ such that the image of $F'(t)$ is not in n and $\mathcal{O}_{X,x}$ is $\mathcal{O}_{Y,f(t)}$-isomorphic to B_n. We want to show this is an open condition.

Pick $V = SpecA$ an affine open of $f(x)$ in Y such that $F(t)$ is the image of a monic polynomial in $A[t]$ under the canonical morphism $A[t] \to \mathcal{O}_{Y,f(x)}[t]$, we need to pick this A carefully for example, we can first pick a $SpecA'$ which is an arbitrary affine open of $f(x)$, then we can shrink $SpecA'$ to distinguished affine opens where we are allowed to invert the denominators of the coefficients of $F(t)$. Then we denote such a monic polynomial in $A[t]$ also by $F[t]$. Then we have a morphism:

$$A[t]/(F(t)) \to \mathcal{O}_{Y,f(x)}[t]/(F(t))$$

and we denote by p the inverse image of n in $A[t]/(F(t))$. Then consider the morphism:

$$SpecA[t]/(F(t)) \to SpecA = V$$

We know that since $F'(t)$ is not in p, this morphism is etale at p, in particular, there is an open neighborhood W of p such that $W \to SpecA$ is unramified. Notice that $A[t]/(F(t)) \to SpecA$ is flat since $A[t]/(F(t))$ is free over A, then we know that $W \to SpecA$ is étale. I claim this implies $\mathcal{O}_{X,x}$ is $\mathcal{O}_{Y,f(x)}$-isomorphic to $(A[t]/(F(t)))_p$, this is because we have an isomorphism:

$$(A[t]/(F(t)))_p \simeq \mathcal{O}_{Y,f(x)}[t]/(F(t))_n$$

To see why, we know that $A[t]/(F(t)) \to \mathcal{O}_{Y,f(x)}[t]/(F(t))$ is actually $A[t]/(F(t)) \to A_m[t]/(F(t))$ where m is the prime ideal corresponding to $f(x)$. Then we know that $A[t]/(F(t)) \simeq \mathcal{O}_{Y,f(x)}[t]/(F(t))$ is a localization at $A - m$. Then there is a general fact, suppose that:

$$A \to S^{-1}A$$

is a localization, then for primes p of A such that $p \cap S = \varnothing$, we have a canonical isomorphism:

$$A_p \to (S^{-1}A)_{S^{-1}p}$$

I claim this is enough for the existence of an open subscheme W' of W containing p and an open subscheme U' of $f^{-1}SpecA$ containing X and an isomorphism $W' \to U'$ such that the following diagram commutes:

$$\begin{tikzcd} W' \arrow[rr, "\simeq"] \arrow[dr] & & U' \arrow[dl] \\ & SpecA & \end{tikzcd}$$

This is the consequence of the following lemma: ☐

Lemma 2.8.

3 Infinitesimal lifting of Morphisms

The main theorem of this section is the following:

Theorem 3.1. Let $f : X \to S$ be a morphism locally of finite presentation, then the following conditions are equivalent:

1. f is unramified (resp. etale, resp. smooth).
2. For any S-scheme $SpecA$ and any ideal I of A with the property $I^2 = 0$, the canonical map:

$$\mathrm{Hom}_S(SpecA, X) \to \mathrm{Hom}_S(SpecA/I, X)$$

is injective (resp. bijective, resp. surjective).

To prove this theorem, we will need the following lemma:

4 Direct Limits and Inverse Limits

5 Henselization

In this section, we will talk about this important algebraic construction of henselization that will be used repeatedly in the future. This will be a quite long and technical section, but it's good to see this results, new notions and proofs for at least once.

Lemma 5.1. 1. Let A be a ring, then the map

$$a \mapsto D(a)$$

gives a one-to-one correspondence between the set of idempotent elements in A and the set of both open and closed subsets of $SpecA$.

2. Let A be a ring, the map:

$$a \mapsto (\mathrm{Ann}(a), \mathrm{Ann}(1-a))$$

gives a one-to-one correspondence between the set of idempotent elements in A and the set of pairs (a_1, a_2) of ideals of A with properties:

$$a_1 \cap a_2 = 0, \qquad a_1 + a_2 = A$$

3. Let (R, m) be a local ring, and let $f(t) \in R[t]$ be a monic polynomial and let $A = R[t]/(f(t))$, then the map:

$$(g(t), h(t)) \mapsto (Ag(t), Ah(t))$$

gives a one-to-one correspondence between the set of pairs of monic polynomials $(g(t), h(t))$ in $R[t]$ with the properties:

$$f(t) = g(t)h(t), \qquad g(t)R[t] + h(t)R[t] = R[t]$$

and set of pairs of ideals (a_1, a_2) of A with the properties:

$$a_1 \cap a_2 = 0, \qquad a_1 + a_2 = A$$

Proof. Let's prove assertion (1): Since a is idempotent, we have:

$$a(1-a) = 0, \qquad a + (1-a) = 1$$

Then we know that $D(a) \cap D(1-a) = \varnothing, D(a) \cup D(1-a) = (V(a) \cap V(1-a))^c = V(a, 1-a)^c = SpecA$. Therefore $D(a)$ is both open and closed. Now let U be a subset of $SpecA$, both open and closed, then we know that there is an element in A that is equal to 1 and is equal to 0 on U^c (due to the property of the structure sheaf) and it's clear that a is idempotent as you can check again use the property of the structure sheaf. Moreover $D(a) = U$, therefore we know that the map in the statement is surjective. To show it's injective, suppose that a_1, a_2 are idempotent and $D(a_1) = D(a_2)$, we want to prove that $a_1 = a_2$. We know that it suffices to show that a_1, a_2 have the same image in A_p for each $p \in SpecA$. Recall that if A is a local ring, then $SpecA$ is connected since we know that any open and closed subset in $SpecA$ is $D(a)$ for some idempotent element in A, however in a local ring, only such element is 0 or 1. To see why, suppose that e is idempotent, then we know that $e(1-e) = 0$ and $e + 1 - e = 1$, therefore we know that either e or $1 - e$ is not in the maximal ideal, thus a unit, if e is a unit, then $1 - e = 0$ i.e. $e = 1$, if $1 - e$ is a unit, then $e = 0$. Then we know that the image of a_1, a_2 in A_p are both idempotent, thus they are 0 or 1. Recall that since $D(a_1) = D(a_2)$ in $SpecA$, then in $SpecA_p$, we still have $D(a_1) = D(a_2)$, therefore either they are both 0 or they are both 1 in A_p, therefore we are done.

Then let's prove assertion (2): Let a be an idempotent element, we first show that $\mathrm{Ann}(a), \mathrm{Ann}(1-a)$ satisfies the property. Since a is idemppotent, we know that $a \in \mathrm{Ann}(1-a)$ and $(1-a) \in \mathrm{Ann}(a)$, therefore $\mathrm{Ann}(1-a) + \mathrm{Ann}(a) = A$. Moreover, suppose that x is any element in A, if $x \in \mathrm{Ann}(1-a) \cap \mathrm{Ann}(a)$, then we know that:

$$x \cdot (a + 1 - a) = x = 0$$

thus $\mathrm{Ann}(1-a) \cap \mathrm{Ann}(a) = 0$.

Now we prove the morphism is injective. Suppose that a_1, a_2 are idempotent elements such that $\mathrm{Ann}(a_1) = \mathrm{Ann}(a_2)$, then since $(1-a_1) \in \mathrm{Ann}(a_1) = \mathrm{Ann}(a_2)$, then we know that $(1-a_1)a_2 = 0$, therefore $a_2 = a_1a_2$, similarly, we know that $a_1 = a_1a_2$, therefore we know that $a_1 = a_2$, thus injective. To show it's surjective, suppose that $a_1 \cap a_2 = 0$ and $a_1 + a_2 = A$, by Chinese Reminder Theorem, we know that:

$$A = A/a_1 \times A/a_2$$

Then we know that the element $(1,0), (0,1)$ are two idempotent element, and we know that the annihilator of $(1,0)$ is elements in a_1 and the annihilator of $(0,1)$ is elements in a_2, thus we know the morphism is surjective.

Finally let's prove the last assertion, which is the most technical one. Let $(g(t), h(t))$ be the pair such that $f(t) = g(t)h(t)$ and $g(t)R[t] + h(t)R[t] = R[t]$. Then we know that:

$$g(t)A + h(t)A = A$$

since A is just a quotient of $R[t]$. Let $a(t), b(t) \in R[t]$ be polynomials such that $a(t)g(t) + b(t)h(t) = 1$. Let $r(t) \in g(t)R[t] \cap h(t)R[t]$, then we have:

$$r(t) = g(t)p(t) = h(t)q(t)$$

Then we know that:

$$r(t) = a(t)g(t)h(t)q(t) + b(t)h(t)g(t)p(t) = f(t)(a(t)q(t) + b(t)p(t))$$

therefore we know that $r(t) = 0 \in A$. Therefore $Ag(t) \cap Ah(t) = 0$.

Then let's show this map is injective, suppose that $(g_1(t), h_1(t))$ is another pair of polynomials. Then if $g_1(t)A = g(t)A$, since A is just a quotient of $R[t]$, then $g_1(t)R[t] = g(t)R[t]$, then we can find:

$$g_1(t)\alpha(t) = g(t), g(t)\beta(t) = g_1(t)$$

Then I will let you see that since $g(t), g_1(t)$ are both monic, then this implies $g_1(t) = g(t)$. Similarly, $h(t) = h_1(t)$.

To prove this is surjective, again let the pair of ideals be (a_1, a_2), then if we let $A_1 = A/a_1, A_2 = A/a_2$, we know from the Chinese Remainder Theorem that:

$$R[t]/(f(t)) = A \simeq A_1 \times A_2$$

Let u be the image of t in A_1, then consider A_1/mA_1, which is a $k = R/m$-algebra, and is generated by the image of u in A_1/mA_1, denoted by $\overline{u}$. Consider we have a natural k-algebra morphism:

$$k[t] \to A_1/mA_1, \qquad t \mapsto \overline{u}$$

it's obviously surjective thus we know that there is a minimal polynomial $\overline{g}(t) \in k[t]$ for $\overline{u}$. Let $d = \deg\overline{g}(t)$, then we know that we have a basis $\{1, u, \cdots, u^{d-1}\}$ for A_1/mA_1 over k. Now recall that $R[t]/(f(t))$ is finitely generated as an R-module since $f(t)$ is monic. Then we know that A_1 is a direct summand of a $R[t]/(f(t))$, thus is also a quotient, thus A_1 is also finite over R. Therefore we know that we can use Nakayama's lemma, and conclude that $\{1, u, \cdots, u^{d-1}\}$ in A_1 generates A_1 as a R-module. Then I claim that there is a monic polynomial $g(t) \in R[t]$ of degree d such that $g(u) = 0$ in A_1.

Recall that $R[t]/(f(t))$ is a free R-module, therefore we know that A_1, as a summand of a free R-module, is projective, and since R is local, we know that A_1 is also free. Then we know that $\{1, u, \cdots, u^{d-1}\}$ not only generates A_1 as an R-module, it's a basis of A_1 over R as we can consider:

$$0 \to \ker \to R^d \to A_1 \to 0$$

we know that after we quotient m, the kernel is 0, then we know that it lies insider mR^d, then since A_1 is projective, then this kernel is 0. Therefore is you consider the R-module homomorphism:

$$R[t]/(g(t)) \to A_1, \qquad t \mapsto u$$

This is an isomorphism indeed! Then consider the sujrective morphism $R[t]/(f(t)) = A \to A_1, \qquad t \mapsto u$ we know that the kernel is $(g(t))/(f(t))$, thus we know that a_1 is actually $Ag(t)$. Similarly, we know that there is a $h(t)$ such that $a_2 = Ah(t)$. Now we need to check $f(t) = g(t)h(t)$ and $g(t)R[t] + h(t)R[t] = R[t]$.

First, notice that we have an isomorphism:

$$R[t]/(f(t)) \to R[t]/(g(t)) \times R[t]/(h(t))$$

Then we know that $g(t)h(t)$ is mapped to 0, therefore we know that $g(t)h(t) \in (f(t))$, recall that since $f(t), g(t), h(t)$ are all monic, then we know that $\mathrm{Rank}(R[t]/(f(t))) = \mathrm{Rank}(R[t]/(g(t))) + \mathrm{Rank}(R[t]/(h(t)))$, therefore the degree of f is the degree of h plus the degree of g, then since they are all monic, then $f(t) = g(t)h(t)$.

Now we have an isomorphism:

$$R[t]/(g(t)h(t)) \to R[t]/(g(t)) \times R[t]/(h(t))$$

Now recall that the kernel of the two morphisms:

$$R[t]/(gh) \to R[t]/(g), \qquad R[t]/(gh) \to R[t]/(h)$$

has kernels that generates $R[t]/(gh)$, that is $(g)/(gh) + (h)/(gh) = R[t]/(gh)$. This implies g, h generate $R[t]$. I will let you check it. $\square$

Lemma 5.2. Let (R, m) be a local ring and let A be an R-algebra which is free of finite rank as an R-module, for any R-algebra A', let $\mathrm{Idem}(A' \otimes_R A)$ be the set of idempotent elements in $A' \otimes_R A$. Then there is an etale R-algebra B of finite presentation such that the functor $A' \to \mathrm{Idem}(A' \otimes_R A)$ is represented by B in the category of R-algebras, that is we have a one-to-one correspondence:

$$\mathrm{Idem}(A' \otimes_R A) \simeq \mathrm{Hom}_R(B, A')$$

that is functorial in A'.

Proof. Let $\{e_1, \cdots, e_n\}$ be a basis of A over R, then we know that:

$$e_i e_j = \sum_k a_{ijk} e_k$$

for some $a_{ijk} \in R$. Let:

$$P_k(t_1, \cdots, t_n) = \sum_{i,j} a_{ijk} t_i t_j - t_k$$

Now suppose that an element in A is $\sum_i a_i e_i$ and we know that it's idempotent iff:

$$\sum_i a_i e_i \cdot \sum_j a_j e_j - \sum_k a_k e_k = 0$$

We expand this and we know that the product is $(\sum_{i,j} a_i a_j a_{ijk})e_k$, therefore we know that $\sum_i a_i e_j$ is idempotent iff:

$$\sum_{i,j} a_i a_j a_{ijk} - a_k = 0$$

for all $k = 1, \cdots, n$. Therefore we know that it's idempotent iff $P_k(a_1, \cdots, a_n) = 0$ for all k.

Now since A is free over R with basis e_i, then we know that for any R-algebra A', we know that $A' \otimes_R A$ is free over A' with basis $\{1 \otimes e_1, \cdots, 1 \otimes e_n\}$. Then we know that an element in A' is of the form $\sum_i a'_i \otimes e_i$, and it's idempotent iff $P_k(a', \cdots, a'_n) = 0$ for all k. Therefore we have a bijection:

$$\mathrm{Idem}(A' \otimes_R A) \to \mathrm{Hom}_R(R[t_1, \cdots, t_n]/(P_1, \cdots, P_n), A')$$

I claim that this is functorial in A' and $R[T_1, \cdots, t_n]/(P - 1, \cdots, P_n)$ is etale over R. Let's first prove functoriality, i.e. we need to check the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Idem}(A' \otimes_R A) & \longrightarrow & \mathrm{Hom}_R(R[t_1, \cdots, t_n]/(P_1, \cdots, P_n), A') \\ \downarrow & & \downarrow \\ \mathrm{Idem}(A'' \otimes_R A) & \longrightarrow & \mathrm{Hom}_R(R[t_1, \cdots, t_n]/(P_1, \cdots, P_n), A'') \end{array}$$

I will let you check it. Then we only need to prove that $R[t_1, \cdots, t_n]/(P_1, \cdots, P_n)$ is etale over R. Recall that to prove this, we need to prove that for any R-algebra A' and any ideal I of A' that satisfies $I^2 = 0$, the canonical map:

$$\mathrm{Hom}_R(R[t_1, \cdots, t_n]/(P_1, \cdots, P_n), A') \to \mathrm{Hom}_R(R[t_1, \cdots, t_n, A'/I])$$

is bijective, that is we need to prove the natural map:

$$\mathrm{Idem}(A' \otimes_R A) \to \mathrm{Idem}(A'/I \otimes_R A)$$

is bijective. Recall that our previous proposition identifying idenpotents element of the open and closed subsets of the spectrum, we know that this holds because the following spectrum has the same underlying topological space:

$$Spec(A'/I \otimes_R A), \qquad Spec(A \otimes_R A)$$

To see why, consider the following sequence:

$$0 \to I \to A' \to A'/I \to 0$$

Let's tensor with $\otimes_R A$, we gave a sequence:

$$I \otimes_R A \to A' \otimes_R A \to A'/I \otimes_R A \to 0$$

We know that the kernel of $A' \otimes_R A \to A'/I \otimes_R A$ is the image of $I \otimes_R A \to A' \otimes_R A$, which means that the kernel is also nilpotent, then the usual morphism induced by the quotient is an isomorphism on the level of topological spaces, therefore we are done. □

Proposition 5.3. Let (R, m) be a local ring, $k = R/m$ be the residue field. Suppose that $S = SpecR$ and let s be the unique closed point of S, then the following conditions are equivalent:

1. Every finite R-algebra A is a direct product of local rings.
2. Condition 1 holds for $A = R[t]/(f(t))$ for any monic polynomial $f(t) \in R[t]$.
3. For every finite R-algebra A, the canonical homomorphism $A \to A/mA$ induces a one-to-one correspondence between the set of idempotent elements in A and the set of idempotent element in A/mA.
4. Condition 3 holds for $A = R[t]/(f(t))$ for every monic polynomial $f(t) \in R[t]$.

5. For any monic polynomial $f(t) \in R[t]$ and any factorization $\overline{f}(t) = \overline{g}(t)\overline{h}(t)$ where $\overline{f}(t)$ is the image of $f(t)$ in $k[t]$ and $\overline{g}(t), \overline{h}(t)$ are relatively prime monic polynomials in $k[t]$, then there is uniquely determined polynomials $g(t), h(t)$ in $R[t]$ such that $f(t) = g(t)h(t)$ and $\overline{g}(t), \overline{h}(t)$ are the images of $g(t), h(t)$ in $k[t]$. Moreover $g(t), h(t)$ generate $R[t]$.

6. Condition 5 holds for any factorization $\overline{f}(t) = (t - \overline{a})\overline{h}(t)$ such that $(t - \overline{a}), \overline{h}(t)$ are relatively prime monic polynomials in $k[t]$.

7. For any etale morphism $g : X \to S = SpecR$, any section of $g_s : X \otimes_R k \to Speck$ is induced by a section of g.

Definition 5.4. When a local ring (R, m) satisfies the above equivalent conditions, we say this local ring is **henselian**. If R is henselian and $k = R/m$ is separably closed, we say R is **strictly local** or **strictly henselian**.

Let's now prove these conditions are indeed all equivalent, which is going to be a rather long argument:

Proof. Let's first $1 \Rightarrow 3$: Let A be a finite R-algebra, then we know that in particular, it's integral, then we know that any maximal ideal of A lies over the maximal ideal m of R. Since A/mA is again finite over R/m, it's Artinian and thus has finitely many ideal, therefore A only has finitely many maximal ideals. For each maximal ideal n of A, consider the canonical morphism $SpecA_n \to SpecA$, we know that this is a embedding of topological spaces, and $SpecA_n$ is connected. Notice that as we run over all maximal ideals of A, the images of $SpecA_n \to SpecA$ cover $SpecA$, therefore we know that $SpecA$ only has finitely many connected components, as each image of $SpecA_n$ lives in some connected components and their image covers $SpecA$. Similarly, as a closed subset of $SpecA$, $SpecA/m$ also only has finitely many connected components.

Recall that if a topological space has finitely many connected components, then open and closed subsets are unions of connected components. Recall that if we want to prove that $A \to A/mA$ gives a bijection between idempotent elements, we need to show that first if $\overline{a}$ is an idempotent element in A/mA, then a in A is also idempotent and second, if a, a' are distinct idempotent element in A, $\overline{a}, \overline{a'}$ are also distinct idempotent elements in A/mA. Recall that we have a bijection between idempotent element and clopen subsets defined by $a \mapsto D(a)$. Then since the inverse image of $D(a)$ is $D(\overline{a})$ I will let you check that we only need to show that by taking inverse images under $SpecA/mA \to SpecA$, we get a bijection between clopen subsets. Now by our assumption:

$$A = \prod_{i=1}^{n} B_i$$

where each B_i is a local ring, I will even let you show that if n stands for all maximal ideals of A, then $A \simeq \prod_n A_n$, then we know that as a topological space, $SpecA$ is a disjoint union of $SpecA_n$, which is connected, therefore these are exactly the connected components. We also know that in this case $A/mA \simeq \prod_n A_n/mA_n$, and the morphism $SpecA/mA \to SpecA$ can be now written as:

$$\coprod_n SpecA_n/mA_n \to \coprod_n SpecA_n$$

Moreover, we see that for each $SpecA_n/mA_n$, since nA_n is the only prime lives over m, this is a one point space, therefore indeed the inverse image of $SpecA_n$ is $SpecA_n/mA_n$, therefore we have the desired bijection between clopen subsets, we are done.

Then let's prove $3 \Rightarrow 1$: Again, we can show that both $SpecA, SpecA/mA$ have finitely many connected components. Then the fact that $A \to mA$ defines a bijection between idempotent elements implies that by taking inverse image $SpecA/mA \to SpecA$ gives a bijection between clopen subsets, and thus the set of connected components of $SpecA$ and that of $SpecA/mA$. Notice that $SpecA/mA$ is discrete and it's connected components are $\{n/mA\}$ for maximal ideals n of A. Therefore we know that different maximal ideals live in different connected components, therefore $SpecA_{n_1}$ and $SpecA_{n_2}$ are disjoint as subsets of $SpecA$ for different maximal ideals n_1, n_2. Therefore we know that $SpecA$ as a set is a disjoint union of $SpecA_n$. I claim that these are exactly the connected components, since if C_i is a connected component that contain some point in $SpecA_{n_i}$, then it contains $SpecA_{n_i}$ as C_i and $SpecA_{n_i}$ are both connected. Then since different $SpecA_{n_i}$ live in different connected components and covers $SpecA$, $SpecA_{n_i}$ are connected components. Therefore clopen, therefore we know that $SpecA$ as a scheme is isomorphic to $\coprod_i SpecA_{n_i}$, thus we are done.

Then $1 \Rightarrow 2$ is trivial. And we know that $2 \Rightarrow 4$ is just the same as $1 \Rightarrow 3$ as they are just special cases of conditions 1 and 3. Let's prove $4 \to 3$, and $1, 2, 3, 4$ will be shown to be equivalent. Let's first prove that the map $A \to A/mA$ is injective when restricted to the set of idempotent elements in A. Suppose that e_1, e_2 are two idempotent elements in A with $e_1 - e_2 \in mA$. Consider:

$$(e_1 - e_2)^3 = e_1^3 - 3e_1^2 e_2 + 3e_1 e_2^2 - e_2^3 = e_1 - 3e_1 e_2 + 3e_1 e_2 - e_2 = e_1 - e_2$$

Therefore we know that:

$$(e_1 - e_2)((e_1 - e_2)^2 - 1) = 0$$

Since $e_1 - e_2 \in mA$, and every maximal ideal lies over mA, we know that $(e_1 - e_2)$ is in the Jacobson ideal of A, therefore so is $(e_1 - e_2)^2$, therefore $(e_2 - e_2)^2 - 1$ is a unit, therefore $e_1 = e_2$. So we only need to prove it's surjective. Suppose that $\overline{e} \in A/mA$ is an idempotent element, we want to show that it can be lifted to an idempotent element in A. Let a be an arbitrary lift of e, and consider the subring $A' = R[a]$ of A. Consider the following diagram:

$$\begin{tikzcd} A' \arrow[r, hook] \arrow[d] & A \arrow[d] \\ A'/mA' \arrow[r, hook] & A/mA \end{tikzcd}$$

I claim that $\overline{e}$ is the image of some idempotent element $\overline{e'}$ in A'/mA'. To see why, consider the morphism of schemes:

$$f : SpecA/mA \to SpecA'/mA'$$

Let $\overline{a}$ be the image of a in A'/mA', then we know that $f^{-1}(D(\overline{a})) = D(\overline{e})$. Recall that A' is also finite over R, this is because $R \to A'$ is integral and it's finitely generated, therefore finite. Then we conclude that A'/mA' is also discrete with finitely many closed points, therefore $D(\overline{a})$ is both open and closed, therefore we know that $D(\overline{a}) = D(\overline{e'})$ for some idempotent element $\overline{e'}$ in A'/mA'. Therefore $f^{-1}(D(\overline{e'})) = D(\overline{e})$, then we conclude from the bijection of idempotent elements and clopen subsets that the image of $\overline{e'}$ is $\overline{e}$. Then we know that if we can show that $\overline{e'}$ can be lifted to an idempotent element in A', we are done, therefore we can just replace A by A'.

Since a is integral over R, we know that there is an epimorphism of R-algebras:

$$R[t]/(f(t)) \to A', \qquad t \mapsto a$$

Then consider the following diagram:

$$\begin{tikzcd} R[t]/(f(t)) \arrow[r, two heads] \arrow[d] & A \arrow[d] \\ k[t]/(\overline{f}(t)) \arrow[r] & A/mA \end{tikzcd}$$

The morphism on the bottom is also surjective as it can be thought as the top surjective morphism tensor with $\otimes_R R/m$. Recall that since $\overline{f}(t)$ is a monic polynomial, we know that $k[t]/(\overline{f}(t))$ is finite dimensional over k, therefore it's spectrum is discrete as an Artinian ring. Then since the morphism $SpecA/mA \to Speck[t]/(\overline{f}(t))$ is a closed embedding, we know that the clopen subset $D(\overline{e})$ in $SpecA/mA$ has a closed and open image in $Speck[t]/(\overline{f}(t))$. Therefore there is an idempotent element $\overline{e''}$ in $k[t]/(f(t))$ such that $D(\overline{e'}) = D(\overline{e''})$, Therefore we know that the inverse image of $D(\overline{e''})$ in $SpecA/mA$ is $D(\overline{e'})$, or in other words, we know that the image of $\overline{e''}$ in A/mA is $\overline{e'}$. Then by condition 4, there is an idempotent element e'' in $R[t]/(f(t))$ lifting $\overline{e''}$, therefore it's image in A is the element we want.

Then let's prove 4 implies 5. Given a monic polynomial $f(t) \in R[t]$, we first consider $A = R[t]/(f(t))$, then we know from condition 4 that the canonical morphism:

$$R[t]/(f(t)) \to k[t]/(\overline{f}(t))$$

induces a bijection between idempotent elements. Suppose we are also given the factorization $\overline{f}(t) = \overline{g}(t)\overline{h}(t)$. We want to show that we can uniquely lift $\overline{g}(t)$ and $\overline{h}(t)$ back to $R[t]$ such that $f(t) = g(t)h(t)$ and $g(t)R[t] + h(t)R[t] = R[t]$. Recall that since we assumed that $\overline{g}(t)$ and $\overline{h}(t)$ are relatively prime, we know that $(\overline{g}(t))/(\overline{f}(t)) \cap (\overline{h}(t))/(\overline{f}(t)) = (\overline{f}(t))/(\overline{f(t)}) = 0$. Moreover, we know that $(\overline{g}(t))/(\overline{f}(t)) + (\overline{h}(t))/(\overline{f}(t)) = k[t]/(\overline{f}(t))$. Therefore, we know that there is a uniquely determined idempotent element $\overline{e}$ in $k[t]/(\overline{f}(t))$ such that we have:

$$(\mathrm{Ann}(\overline{e}), \mathrm{Ann}(1-\overline{e})) = ((\overline{g}(t))/(\overline{f}(t)), (\overline{h}(t))/(\overline{f}(t)))$$

Since we also have an bijection of idempotent element, there is some $e \in R[t]/(f(t))$ whose image in $k[t]/(\overline{f}(t))$ is $\overline{e}$. We know that again:

$$(\mathrm{Ann}(e), \mathrm{Ann}(1-e))$$

gives a pair of ideals (a_1, a_2) with $a_1 \cap a_2 = 0, a_1 + a_2 = R[t]/(f(t))$, again we know from the first proposition we proved in this section that this correspond uniquely to a pair $(h(t), g(t))$ of polynomials in $R[t]$ such that $f(t) = g(t)h(t)$ $g(t)R[t] + h(t)R[t] = R[t]$. We only need to check that their image in $k[t]$ is $\overline{g}(t)$ and $\overline{h}(t)$. Recall that if we have a ring R and some idempotent element $x \in R$, suppose that $I = \mathrm{Ann}(x)$ and $J \subset I$, then $\mathrm{Ann}(\overline{x}) = I/J$ in R/J (notice that this doesn't hold for arbitrary x). Therefore we know that $((g(t)), (h(t)))$ in $R[t]$ is mapped to $(g(t)/f(t), h(t)/f(t)) = (\mathrm{Ann}(e), \mathrm{Ann}(1-e))$ in $R[t]/(f(t))$. It's again mapped to $((\overline{g}(t))/(\overline{f}(t)), (\overline{h}(t))/(\overline{f}(t)))$. Consider the following diagram:

$$\begin{tikzcd}
R[t] \arrow[r, two heads] \arrow[d, two heads] & R[t]/(f(t)) \arrow[d, two heads] \\
k[t] \arrow[r, two heads] & k[t]/(\overline{f}(t))
\end{tikzcd}$$

The above notice that the ideal $(\overline{g}(t))$ and $(\overline{h}(t))$ in $k[t]$ is mapped to $((\overline{g}(t))/(\overline{f}(t)), (\overline{h}(t))/(\overline{f}(t)))$ in $k[t]/(\overline{f}(t))$. Then we know that the image of $(g(t)), (h(t))$ in $k[t]$ is $(\overline{g}(t))$ and $(\overline{h}(t))$, then the fact that they are monic and of the same degree tells you they are equal.

Obviously $5 \Rightarrow 6$ is trivial, so let's prove $6 \Rightarrow 7$: Given a section of $g_s : X \otimes_R k \to k$, suppose that after we compose with the projection $X \otimes_R k$, the image is x in $X \otimes_R k$. Recall $g : X \to SpecR$ is etale, then if we consider the diagram:

$$\begin{tikzcd}
X \otimes_R k \arrow[r, "g_s"] \arrow[d] & Speck \arrow[l, shift left] \arrow[d] \\
X \arrow[r, "g"] & SpecR
\end{tikzcd}$$

We know that since the image of the point $Speck$ in $SpecR$ is the unique closed point, and we have a section, we know that x is mapped to the unique closed point of $SpecR$. Then by Chevalley's result on etale morphisms, we know that there is a monic polynomial $f(t) \in R[t]$ and a maximal ideal n of $R[t]/(f(t))$ not containing the image of $f'(t)$ in B such that $\mathcal{O}_{X,x}$ is R-isomorphic to B_n. Let $\overline{f}(t)$ be the image of $f(t)$ in $k[t]$. Then consider the k-morphism:

$$Speck \to Spec\mathcal{O}_{X,x}/m\mathcal{O}_{X,x}$$

induced by the section g_s. Then compose it with the canonical morphism:

$$Spec\mathcal{O}_{X,x}/m\mathcal{O}_{X,x} \simeq SpecB_n/mB_n \to SpecB/mB = Speck[t]/(\overline{f}(t))$$

we get a k-morphism:

$$Speck \to Speck[t]/(\overline{f}(t))$$

which correspond to a k-algebra homomorphism $k[t]/(\overline{f}(t)) \to k, \qquad t \mapsto \overline{a}$. Then we know that $\overline{f}(\overline{a}) = 0$ but we notice $\overline{f}'(\overline{a}) \neq 0$(as $f'(t)$ is a unit in B_n), therefore $\overline{a}$ is just a simple root of $\overline{f}$. Therefore we know that we have a decomposition:

$$\overline{f}(t) = (t - \overline{a})\overline{h}(t)$$

for some monic polynomial $\overline{h}(t)$ relatively prime to $t-\overline{a}$. Then by our hypothesis we can lift this factorization back to $R[t]$ and get:

$$f(t) = (t-a)h(t)$$

such that the ideal generated by $t-a$ and $h(t)$ is $R[t]$.

Now consider the R-algebra homomorphism:

$$B = R[t]/(f(t)) \to R, \qquad t \mapsto a$$

This induces another R-algebra homomorphism:

$$B_n = (R[t]/(f(t)))_n \to R$$

Hence an S-morphism:

$$S = SpecR \to SpecB_n \simeq Spec\mathcal{O}_{X,x}$$

We also have a canonical morphism $Spec\mathcal{O}_{X,x} \to X$, I claim that we get a section of g. To see why, consider the following diagram:

$$\begin{tikzcd} & X \otimes_R k \arrow[r, "g_s"] \arrow[d] & Speck \arrow[d] \\ SpecR \arrow[r] & X \arrow[r, "g"] & SpecR \end{tikzcd}$$

We want to prove $SpecR \to X \to SpecR$ is the identity. We know that this is the same as showing the corresponding morphism on the global sections is the identity. This is:

$$\begin{tikzcd} \mathcal{O}_{X,x} \arrow[r, "\simeq"] & B_n \arrow[r] & R \\ & R \arrow[ul, "g"] \arrow[u] \arrow[ur, "\mathrm{Id}"'] & \end{tikzcd}$$

Then we pullback this along $X \otimes_R k$ and get:

$$\begin{tikzcd} Speck \arrow[r] \arrow[d] & X \otimes_R k \arrow[r, "g_s"] \arrow[d] & Speck \arrow[d] \\ SpecR \arrow[r] & X \arrow[r, "g"] & SpecR \end{tikzcd}$$

Then the composition $Speck \to X \otimes_R j \xrightarrow{g_s} Speck$ is the identity, we are done.

Finally, let's prove that 7 implies 4 and completing the proof: Recall that if we use $A = R[t]/(f(t))$, it's a finite rank free R-module, then we know that there is an etale R-algebra B of finite presentation representing the functor $A' \to \mathrm{Idem}(A' \otimes_R A)$.

Our goal is to show that:

$$\mathrm{Idem}(A) \to \mathrm{Idem}(A/mA)$$

is an isomorphism and we have shown that it's always injective, therefore by the representability lemma, we only need to prove that the morphism:

$$\mathrm{Hom}_R(B, R) \to \mathrm{Hom}_R(B, R/m = k)$$

is surjective.

Notice that the two sets are identified with sections of $g : SpecB \to SpecR$ and $g_s : SpecB \otimes_R k \to Speck$ and the morphism between them is identified as pulling back. Since we know that g is etale, our hypothesis implies that the morphism is surjective. We are done. $\square$

Proposition 5.5. Suppose that (R, m) is a complete local Noetherain ring then R is Henselian.

Proof. The Noetherian hypothesis is crucial here as a lot of good properties of completion of rings happens only when we have Noetherianity. We give two proofs of this result. First, we can check that every finite R-algebra A is a finite product of local rings. This is because for any Notherian ring A and finitely generated A-module M, we have a canonical isomorphism:

$$\hat{A} \otimes_A M \to \hat{M}$$

Here R is A and we know that R itself is complete, therefore we know that the above morphism is:

$$A \otimes_A M \to \hat{M}$$

which is an isomorphism therefore any such finitely generated M is complete, therefore we know that A is m-adic complete. Recall that A has finitely many maximal ideals, say $m_1, \cdots, m_n$.

Now think about the nilpotent ideal of A/mA, which is $(m_1 \cap \cdots \cap m_n)/mA$. Since A is also Notherian, we know that there is an integer k such that:

$$(m_1 \cap \cdots \cap m_n)^k \subset mA \subset m_1 \cap \cdot \cap m_n$$

Therefore we know that in A, the filtration defined by $m_1 \cap \cdots \cap m_n$ and mA are cofinal, therefore they define the same completion, i.e. we have:

$$A \simeq \varprojlim_i A/m^i A \simeq \varprojlim_i A/(m_1 \cap \cdot \cap m_n)^i = \varprojlim_i A/m_1^i \cdots m_n^i$$

Recall that since m_i's are distinct maximal ideals, their powers are still coprime, therefore we can apply the Chinese Remainder Theorem and conclude that:

$$A \simeq \varprojlim_i (A/m_1^i \times \cdots \times A/m_n^i) \simeq \varprojlim_i A/m_1^i \times \cdots \varprojlim_i A/m_n^i = \hat{A}_{m_1} \times \cdots \times \hat{A}_{m_n}$$

I claim that each of these factors are local, this is because that we know that for any Noetherian ring, the completion of a maximal ideal, is the extension of the maximal ideal, and it's still going to be maximal. Moreover, it's contained in the intersection of all maximal ideals of the completion, thus if the Noetherian ring is local, then the completion at the maximal ideal is also local, with the unique maximal ideal being the completion of the unique maximal ideal of the original ring.

We can also check the property of lifting polynomials. Let $f(t) \in R[t]$ be a monic polynomial and let $\overline{f}(t) = (t - \overline{a})\overline{h}(t)$ be a factorization in $k[t]$ with $h(t)$ relatively prime to $\overline{h}(t)$. Now we will try to lift $\overline{a}$ to a root of $f(t)$ in $R[t]$ using a method somehow similar to Newton's method of find a root of a smooth function using limit, or in other words, $\mathbb{C}$ is complete. By our assumption, $\overline{a}$ is a simple root of $\overline{f}(t)$, thus we know:

$$\overline{f}(\overline{a}) = 0, \qquad \overline{f'}(\overline{a}) \neq 0$$

Let a_0 be an arbitrary lift of a, then we know that $f(a_0) \in m$ and $f'(a_0)$ is a unit. Now we will do induction, suppose that we have found a lift $a_i \in R$ of $\overline{a}$ such that $f(a_i) \in m^{i+1}$. Then since $f'(a_i)$ is a unit of R, we can define:

$$a_{i+1} = a_i - \frac{f(a_i)}{f'(a_i)}$$

Then we know that $a_{i+1} - a_i = -\frac{f(a_i)}{f'(a_i)} \in m^{i+1}$. Then I claim that a_{i+1} is also a lift of $\overline{a}$ since we know:

$$\overline{a_{i+1}} - \overline{a_i} = \overline{a_{i+1}} - \overline{a} = \overline{x}$$

for some $x \in m^{i+1}$.

Notice that for any $a, \delta \in R$, we can find $b \in R$ such that:

$$f(a + \delta) = f(a) + f'(a)\delta + b\delta^2$$

(You only have to compute what is $f(a + \delta) - (f(a) + f'(a)\delta)$) In particular, we can find b_i such that:

$$f(a_{i+1}) = f(a_i) + f'(a_i) \cdot (-\frac{f(a_i)}{f'(a_i)}) + b_i(-\frac{f(a_i)}{f'(a_i)})^2 = b_i(-\frac{f(a_i)}{f'(a_i)})^2 \in m^{2i+2} \subset m^{i+2}$$

Then we have constructed an element in the completion of R:

$$a = \lim_i a_i$$

as we know that $a_{i+1} - a_i \in m^{i+1}$, moreover since R is complete, this element is in R. If we compute $f(a)$, which is the limit of $f(a_i)$, we know that it's 0 as $f(a_{i+1}) \in m^{i+2}$. Since each a_i is a lift of a, the result a is also a lift of a. Therefore we have $f(t) = (t-a)h(t)$ for some monic polynomial $h(t)$ lifting $h(t)$. Notice that $R[t]/(f(t))$ is finitely generated over R, therefore since after we tensor with R/m it's 0, we know that it's already 0 by Nakayama's lemma, therefore we know that $(t-a), h(t)$ indeed generate $R[t]$. Therefore they are relatively prime, this also shows that if the lift of $\overline{a}$ to a is unique, then the factorization is also unique. So finally, let's show the lift of $\overline{a}$ is indeed unique.

Let's assume that a' be a roof of f and also a lift of a, then we know that there is a $b \in R$ such that:

$$f(a') = f(a) + f'(a)(a'-a) + b(a'-a)^2$$

Since $f(a) = 0 = f'(a')$, then we know that:

$$a' - a = -\frac{b}{f'(a)}(a'-a)^2$$

since $a' - a \in m$, then we know that $(a'-a) \in m^2$ and then it's in all m^i, therefore the difference is 0 since the ring is complete. We are done. ☐

Definition 5.6. Let $A \to A'$ be a local homomorphism of local rings, we say that A' is **essentially etale** over A if there is an etale A-algebra B and a prime ideal p of B lying above the maximal ideal of A such that A' is A-isomorphic to B_p. In a diagram, it's the following situation:

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & B \arrow[r] & B_p \\ & A' \arrow[ur, "\simeq"] & \end{tikzcd}$$

Remark 5.7. Recall that by the definition that B is etale over A, we know that $SpecB \to SpecA$ is finite presentation thus finite type. Moreover, since it's unramified, I will let you check that $SpecB \to SpecA$ is quasi-finite.

I claim knowing this implies that the maximal ideal of A' is the only prime of A' lying above the maximal ideal of A. Notice that if we consider the fiber of the closed point of $SpecA$ in $SpecB$, it's closed and is finite. I claim that there is no ideal in the inverse image that is contained in another ideal in the inverse image. To see why, we know that the inverse image is discrete, that is each of them is again closed, therefore it's also a closed point, thus a maximal element of B. Then from the above diagram that the inverse image of maximal ideal m' of A' in A is indeed the maximal ideal of A. Suppose that q is another such ideal, then we know that it's inverse image in B_p must be pB_q, therefore using the isomorphism $A' \simeq B_p$, we know that $q = m'$. Then we know that if A', A'' are two essentially etale A-algebras, the only possible A-homomorphism between A', A'' is local.

Definition 5.8. A local homomorphism $A' \to A'$ of local rings is called **strictly essentially etale** if it's essentially etale and the induced morphism on residue fields is an isomorphism.

**** This definition of essentially etale is probably related to the section of passage to the limit*

Proposition 5.9. If $A \to A'$ and $A' \to A''$ are (strictly) essentially etale, then so is the composition.

Proof. **** Need to first read the section on passage to limit* ☐

Lemma 5.10. Let A, A' be local rings, $A' \to A'$ a strictly essentially etale local homomorphism, and (R, m) a henselian local ring. Then the canonical map:

$$\text{loc.Hom}(A', R) \to \text{loc.Hom}(A, R)$$

between sets of local homomorphisms of local rings is a bijection.

Proof. We need to show that given a local morphism $A \to R$, we can uniquely extend it to a local morphism $A' \to R$.

Let's first show the existence: consider the morphism $SpecR \to SpecA$, we want to construct the following diagram:

$$\begin{tikzcd}
SpecA & & SpecA' \arrow[ll] \\
& SpecR \arrow[ul] \arrow[ur, dashed] &
\end{tikzcd}$$

such that the closed point of R is mapped to the closed point of A'.

Consider the fibered product:

$$\begin{tikzcd}
SpecA' \otimes_A R \arrow[r] \arrow[d] & SpecR \arrow[d] \arrow[dl, dashed] \\
SpecA' \arrow[r] & SpecA
\end{tikzcd}$$

We know that the existence of such a dotted morphism is equivalent to the existence of a section of $SpecA' \otimes_A R \to SpecR$ that maps the unique maximal ideal of R to the point in $SpecA' \otimes_A R$ above the closed point of $SpecA'$ and that of $SpecR$. I claim that the fact that $A \to A'$ induces an isomorphism on the residue field implies that there exists and only exists one point in $SpecA' \otimes_A R$ that maps to the closed point of R and $SpecA'$. To see why, consider the following diagram:

$$\begin{tikzcd}
Speck(x') \otimes_{k(x)} k(m) \arrow[rrr, "\simeq"] \arrow[dr] \arrow[ddd] & & & Speck(m) \arrow[dl] \arrow[ddd] \\
& SpecA' \otimes_A R \arrow[r] \arrow[d] & SpecR \arrow[d] \arrow[dl, dashed] & \\
& SpecA' \arrow[r] & SpecA & \\
Speck(x') \arrow[rrr, "\simeq"] \arrow[ur] & & & Speck(x) \arrow[ul]
\end{tikzcd}$$

where the morphism $Speck(x') \otimes_{k(x)} k(m) \to SpecA' \otimes_A R$ is induced by the universal property of the fibered product $SpecA' \otimes_A R$. Note that since $Speck(x') \to Speck(x)$ is an isomorphism, we know that $Speck(x') \otimes_{k(x)} k(m)$ is a single point. This proves existence, to see uniqueness, let $S = A' \otimes_A R$ and let $J = mS + x'S$. We know that any prime in S which pulls back to x', m must contain J, therefore we know that these primes are primes in S/J. Now what is S/J. Recall that for any A-algebra A' and an ideal $I' \subset A'$ and an A-module M, we have:

$$(A'/I') \otimes_A M \simeq (A' \otimes_A M)/I \cdot (A' \otimes_A M)$$

because tensor is right exact. Then we can apply this twice and:

$$A'/x' \otimes_A R/m \simeq (A' \otimes_A R/m)/(x')$$

Similarly $A' \otimes_A (R/m) \simeq (A' \otimes_A R)/(m)$, therefore we get the desired isomorphism. However, notice that $A'/x' \otimes_A R/m \simeq A'/x' \otimes_{A/x} R/m$, which is a field. Therefore J is in fact a maximal ideal, therefore it's unique. Moreover we know the residue field is $k(x') \otimes_{k(x)} k(m)$. This not only just holds for the affine case, you can prove this for any Cartesian diagram as long as you have the isomorphism of residue fields as you can always transform to this affine case.

Now let B be an etale A-algebra such that A' is A-isomorphic to B_p for some prime ideal p of B. Then let's consider a new Cartesian diagram:

$$\begin{tikzcd}
SpecB_p \otimes_A R \arrow[r] \arrow[d] & SpecB \otimes_A R \arrow[r] \arrow[d] & SpecR \arrow[d] \\
SpecB_p \arrow[r] & SpecB \arrow[r] & SpecA
\end{tikzcd}$$

We notice that the larger rectangle is isomorphic to the Cartesian square of $SpecA' \otimes_A R$. Then I claim that if we have section of $SpecB \otimes_A R \to SpecR$ that maps m to the unique point of $SpecB \otimes_A R$ that is above $p \in SpecB$ and m of $SpecR$, we know that the inverse image of m is over p in $SpecB$, therefore we have a unique morphism $B_p \to R$ by the universal property of the localization, thus the existence is proved. The uniqueness of the point in $SpecB \otimes_A R$ is essentially proved in a similar way as shown above.

Notice that $SpecB \to SpecA$ is etale, then $SpecB \otimes_A R \to SpecR$ is etale as well, then we know that we have a Cartesian diagram:

$$\begin{tikzcd} Speck(X) \arrow[r] \arrow[d] & Speck(m) \arrow[d] \\ SpecB \otimes_A R \arrow[r] & SpecR \end{tikzcd}$$

where X is the unique point in $SpecB \otimes_A R$ above m, x we talked about above. However, notice that since the residue field of X, i.e. $K(X)$ is exactly the same as that of m, i.e. $k(m)$, therefore we have a section of $Speck(X) \to Speck(m)$.

Then since R is henselian, we know that such a section of $Speck(X) \to Speck(m)$ is induced by a section from $SpecB \otimes_A R \to SpecR$. To see this section of $SpecB \otimes_A R \to SpecR$ is unique, notice that $SpecB \otimes_A R \to SpecR$ is separated and unramified and $SpecR$ is connected, then we know that a section of $SpecR$ is uniquely determined by the value of one point. Consider the unique closed point of $SpecR$, we know that this section can only map m to that unique point X of $SpecB \otimes_A R$, therefore it's unique. $\square$

Lemma 5.11. Let A be a local ring, and let A_1, A_2 be strictly essentially etale local A-algebras, then:

1. There is a strictly essentially etale A-algebra A_3 dominating A_1, A_2.
2. There is at most one A-homomorphism from A_1 to A_2.

Proof. To prove 1: by definition, we know that there are B_1, B_2, etale A-algebras and primes p_1, p_2 such that A_1, A_2 are A-isomorphic to $(B_1)_{p_1}, (B_2)_{p_2}$. Then consider the diagram:

$$\begin{tikzcd} SpecB_1 \otimes_A B_2 \arrow[r] \arrow[d] & SpecB_2 \arrow[d] \\ SpecB_1 \arrow[r] & SpecA \end{tikzcd}$$

Recall that A_1, A_2 are strictly essentially etale A-algebras, therefore the residue field of B_1, B_2 at p_1, p_2 is the same as that of m at A. Therefore we know that there exists and only exists one point p in $SpecB_1 \otimes_A B_2$ such that it lies over p_1, p_2. Then I claim that $A_3 = (B_1 \otimes_A B_2)_p$ suffices. This is essentially by definition as we know that $SpecB_1 \otimes_A B_2 \to SpecA$, as the composition of two etale morphisms is still etale, therefore A_3 is already the localization of an etale at a prime ideal and the prime is mapped to the desired prime in $SpecA$. Then we only need to check that the induced residue field morphism is in fact an isomorphism. This is essentially the same as we will be able to prove that the residue field of that point is $k(p_1) \otimes_{k(x)} k(p_2)$. But as $k(p_1) \simeq k(x) \simeq k(p_2)$, this is indeed isomorphic to $k(x)$, via the induced morphism of residue fields (this is easy to check here as we know that p_1, p_2 are in fact maximal ideals).

Now let's prove 2: keep the notation in 1, I will let you show that it suffices to show that there is at most one A-morphism from $SpecA_2$ to $SpecB_1$ such that the closed point of A_2 maps to p_1. Again, consider the Cartesian diagram:

$$\begin{tikzcd} SpecB_1 \otimes_A A_2 \arrow[r] \arrow[d] & SpecA_2 \arrow[d] \arrow[dl, dashed] \\ SpecB_1 \arrow[r] & SpecA \end{tikzcd}$$

therefore using a similar argument as above, we can see that such a dotted morphism is equivalent to a section of $SpecB_1 \otimes_A A_2 \to SpecA_2$ that maps the closed point of $SpecA_2$ to the unique point p of $SpecB_1 \otimes_A A_2$ that is over the closed point of $SpecA_2$ and p_1. Then if there is such a section, we know that the uniqueness follows from the fact that $SpecB_1 \otimes_A A_2 \to SpecA_2$ is unramified separated, and $SpecA_2$ is connected. $\square$

Now let A be a local ring, a local morphism $A \to A'$ is called **essentially of finite type** if there exists a finitely generated A-algebra B such that A' is isomorphic to to B_p for some prime ideal p of B. Notice that there is a set of local A-algebras essentially of finite type, such that each essentially finite type A-algebra is A-isomorphic to one of the elements in that set. For example, we can take the set to be:

$$\{(A[t_1, \cdots, t_n]/I)_p\}$$

where t ranges over all natural numbers and I ranges over all ideals of $A[t_1, \cdots, t_n]$ and p goes over all prime ideals of $A[t_1, \cdots, t_n]$ lying above m. Recall that each strictly essentially etale A-algebra is an essentially finite type A-algebra, therefore there is a set of strictly essentially etale A-algebra such that each such A-algebra is A-isomorphic to one of the members of the set.

Now let $\mathcal{I}$ be the category whose objects are strictly essentially etale A-algebras and whose morphisms are A-homomorphisms (essentially local, and rather the objects are isomorphism classes of those algebras, but this only matters when it comes to set-theoretic problems, so it's not that important for our discussion), then the previous discussion lemma and our discussion of direct limit shows that the following direct limit exists:

$$A^h = \varinjlim_{A' \in \mathrm{Ob}\mathcal{I}} A'$$

Definition 5.12. Let A be a local ring, then we define it's **henselization** to be:

$$A^h = \varinjlim_{A' \in \mathrm{Ob}\mathcal{I}} A'$$

Of course, this process is called henselization for some reasons. For example, A^h is going to be a henselian local ring and if A is itself henselian, $A^h \simeq A$. A^h will also be related to the original A in some ways. To state and prove these properties, we need some preparation:

Lemma 5.13. Let A be a ring and I a **finitely generated ideal** of A. Recall that we denote the completion of A at I by $\hat{A} = \varprojlim_n A/I^n$. If we write $J_n = \ker(\hat{A} \to A/I^n)$, then:

1. The canonical homomorphism $A \to \hat{A}$ induces isomorphisms $J_n = I^n\hat{A}$, $A/I^n \simeq \hat{A}/I^n\hat{A}$ and $I^n/I^{n+1} \simeq I^n\hat{A}/I^{n+1}\hat{A}$.
2. If A/I is Noetherian then $\hat{A}$ is Noetherian.

Proof. First, clearly that the canonical projection $\hat{A} \to A/I^n$ is surjective, therefore we know that we have an isomorphism $\hat{A}/J_n \to A/I^n$. Then we consider the following diagram of exact rows:

$$\begin{tikzcd}
0 \arrow[r] & I^n/I^{n+m} \arrow[r] \arrow[d] & A/I^{n+m} \arrow[r] \arrow[d] & A/I^n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J_n/J_{n+m} \arrow[r] \arrow[d] & \hat{A}/J_{m+n} \arrow[r] \arrow[d, dashed, "\simeq"] & \hat{A}/J_n \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & I^n/I^{n+m} \arrow[r] & A/I^{n+m} \arrow[r] & A/I^n \arrow[r] & 0
\end{tikzcd}$$

where the first vertical morphisms are induced by the canonical morphism $i : A \to \hat{A}$ and thus the composition of every vertical column is the identity morphism. Then we know that by the five-lemma, $J_n/J_{n+m} \to I^n/I^{m+n}$ is an isomorphism for all m, n, therefore $I^n/I^{n+m} \to J_n/J_{n+m}$ is also an isomorphism for all m, n. Notice that this also implies that $A/I^{m+n} \simeq \hat{A}/J_{m+n}$ for all m, n, however, it's not enough for us to conclude that $I^n\hat{A} \simeq J_n$. Consider the following case, if we have a ring map $f : A \to B$ such that the induced morphism $A/I \to B/J$ is an isomorphism, then notice that we can not conclude $IB = J$. Clearly that $f^{-1}J = I$, but this only implies that $IB \subset J$. A concrete example is:

$$k \to k[x]$$

We know that we have an isomorphism:

$$k/(0) \to k[x]/(x)$$

But clearly the extension of (0) is still 0, not (x).

Therefore we still need to show that $J_n \subset I^n\hat{A}$. Consider the isomorphism:

$$I^n/I^{n+m} \to J_n/J_{n+m}$$

This implies that $i(I^n) + J_{n+m} = J_n$. Let $m = 1$, then we know that $i(I^n) + J_{n+1} = J_n$. Suppose that I in A is generated by $x_1, \cdots, x_n$, then we know that for any $z \in J_n$, we can write it as:

$$i(\sum_{i_1+\cdots+i_k=n} a^0_{i_1\cdots i_k} x_1^{i_1}\cdots x_k^{i_k}) + z_{n+1}$$

where $z_{n+1} \in J_{n+1}$ and $a^0_{i_1\cdots i_k} \in A$. We know that $J_{n+1} = i(I^{n+1}) + J_{n+2}$, therefore we can conclude that for all m:

$$z = i(\sum_{i_1+\cdots+i_k=n} (a^0_{i_1\cdots i_k} + a^m_{i_1\cdots i_k}) x_1^{i_1}\cdots x_k^{i_k}) + z_{n+m+1}$$

for some $z_{n+m+1} \in J_{n+m+1}$ and $a^1_{i_1\cdots i_k} \in I$. Now we can take limit for m for both sides (can you figure what this means here?) and conclude that there is some $\hat{a}_{i_i\cdots i_k}$ in $\hat{A}$ such that:

$$z = \sum_{i_1+\cdots+i_k=n} \hat{a}_{i_i\cdots i_k} i(x_1^{i_1}\cdots x_k^{i_k})$$

Therefore $z \in I^n\hat{A}$, we are done. That is to say we take $\hat{a}_{i_i\cdots i_k}$ to be the infinite tuple where on the m-th slot we put $\sum_{q=0}^{m-1} a^q_{i_1\cdots i_k}$. To check these two are indeed the same, consider their difference and we conclude that the difference is in J_n for all n, and we know that if an element is in J_n for all n, it's 0. Then since $I^n/I^{n+m} \simeq J_n/J_{n+m}$ we have $I^n/I^{n+m} \simeq I^n\hat{A}/I^{n+m}\hat{A}$ for all m, n. In particular we have $A/I^n \simeq \hat{A}/I^n\hat{A}$ and $I^n/I^{n+1} \simeq I^n\hat{A}/I^{n+1}\hat{A}$.

To prove 2, we use the associated graded ring argument, and we need to prove the following ring is Noetherian:

$$\oplus_{n=0}^{\infty} I^n\hat{A}/I^{n+1}\hat{A} \simeq \oplus_{n=0}^{\infty} I^n/I^{n+1}$$

Since A is Noetherian, we know that $\oplus_{n=0}^{\infty} I^n/I^{n+1}$ being a quotient of $(A/I)[x_1, \cdots, x_n]$, is also Noetherian. ☐

Proposition 5.14. Let $(A_\lambda, \phi_{\lambda\mu})$ is a direct system of local rings with $\phi_{\lambda\mu}$ being local morphisms. For each λ, let m_λ be the maximal ideal of A_λ and let $k_\lambda = A_\lambda/m_\lambda$ be the residue field:

1. $A = \varinjlim_\lambda A_\lambda$ is a local ring with maximal ideal $m = \varinjlim_\lambda m_\lambda$ and it's residue field A/m is isomorphic to $k = \varinjlim_\lambda k_\lambda$.
2. If $m_\mu = m_\lambda A_\mu$ for any pair $\mu \geqslant \lambda$, then $m = m_\lambda A$ for all λ.
3. If A_μ is flat over A_λ for any pair $\mu \geqslant \lambda$, then A is flat over A_λ for all λ.
4. If $m_\mu = m_\lambda A_\mu$ and A_μ is flat over A_λ for any pair $\mu \geqslant \lambda$ and if A_λ if Noetherian for all λ, A is Noetherian.

Proof. Recall that $\varinjlim_\lambda$ is exact, therefore if we consider $m = \varinjlim m_\lambda$, it's still a subset of $\varinjlim A_\lambda$ and it's easy to check that this is an ideal of A, then consider the SES:

$$0 \to \varinjlim m_\lambda \to \varinjlim A_\lambda \to \varinjlim k_\lambda \to 0$$

Clearly direct limit of fields is still a field, therefore m is a maximal ideal of A. To check A is local with m being it unique ideal. We need to prove that $A - m$ are all units. Again, since $\phi_{\lambda\mu}$ are local morphisms, it induces an inclusion:

$$A - m = \varinjlim(A_\lambda - m_\lambda) \hookrightarrow A$$

Since $A_\lambda - m_\lambda$ are all units, we know that $A - m$ are also units.

Now if $m_\mu = m_\lambda A_\mu$, then we have:

$$m = \varinjlim_{\mu \geqslant \lambda} m_\lambda A_\mu \hookrightarrow \varinjlim_{\mu \geqslant \lambda} A_\mu$$

Clearly this inclusion lands in $m_\lambda A$, and given something in $m_\lambda A$, it's $x_\lambda a$ for some $x_\lambda \in m_\lambda$ and $a \in A$ is represented by a_μ for some $\mu \geqslant \lambda$, then we know that $\phi_{\lambda\mu}(x_\lambda)a_\mu$ suffices.

We want to show that $\otimes_{A_\lambda} A$ is exact, recall that tensor commutes with direct limits, therefore we know that $\otimes_{A_\lambda} A \simeq \varinjlim_{\mu \geqslant \lambda}(\otimes_{A_\lambda A_\mu})$. Since A_μ is flat over A_λ, the functor $\otimes_{A_\lambda A_\mu}$ is exact, and we also know that direct limit is exact, thus the composition of two exact functors is still exact.

Finally, we want to show that A is Noetherian. We will show that any ascending chain of ideals in A stabilize. Since each A_λ is Noetherian, we know that m_λ is finitely generated, therefore $m = m_\lambda A$ is also generated, since A/m is a field, thus Noetherian, then we know that $\hat{A} = \varprojlim_n A/m^n$ is Noetherian and $\hat{A}/m^n\hat{A} \simeq A/m^n$. Since

$$A/m^n = A/m_\lambda^n A \simeq A_\lambda/m_\lambda^n \otimes_{A_\lambda} A$$

and A is flat over A_λ, we know that:

$$\otimes_{A_\lambda/m_\lambda^n} A/m^n \simeq \otimes_{A_\lambda} A$$

There fore we know that A/m^n is flat over A_λ/m_λ^n. Then since we have the isomorphism $A/m^n \simeq \hat{A}/m^n\hat{A}$, we know that $\hat{A}/m^n\hat{A} = \hat{A}/m_\lambda^n\hat{A}$ is also flat over A_λ/m_λ^n. Now we consider the morphism between Noetherian rings:

$$A_\lambda \to \hat{A}$$

we know that after we tensor with $\otimes_{A_\lambda} A_\lambda/m_\lambda^n$ for all n, we get a flat morphism, then this implies that $\hat{A}$ is flat over A_λ.

Now let I be a finitely generated ideal of A, then by the definition of A, we have some λ and a finitely generated ideal $I_\lambda \subset A_\lambda$ such that $I = I_\lambda A$, but notice that A is flat over A_λ, therefore we know that: $I_\lambda \otimes A \to A_\lambda \otimes A$ is injective, therefore we know that $I \simeq I_\lambda \otimes A$.

Since $\hat{A}$ is also flat over A_λ, we know:

$$I_\lambda \otimes \hat{A} \to \hat{A}$$

is injective, thus $I_\lambda \otimes_{A_\lambda} A \otimes_A \hat{A} \to \hat{A}$ is injective, then the canonical morphism $I \otimes_A \hat{A} \to \hat{A}$ is injective for all finitely generated ideal of A, therefore $\hat{A}$ is flat over A. I claim that $\hat{A}$ is actually faithfully flat over A, therefore is we have a chain of ideals in A that is not stabilizing, after we tensor with $\hat{A}$, we get another chain that is not stabilizing, a contradiction. To show faithfully flatness, consider the ring map:

$$A \to \hat{A}$$

we know that this is local since we know that the completion of a local ring is still local, this is because $A/m \simeq \hat{A}/m\hat{A}$ thus a field, thus $m\hat{A}$ is a maximal ideal. And for any $x \in \hat{A} - m\hat{A}$, the image in $\varprojlim A/m^n \to A/m^n$ is not in m/m^n thus a unit, thus x is also a unit in $\hat{A}$. Then since $\hat{A}$ is flat over A and they are both local with a local morphism then automatically $\hat{A}$ is faithfully flat over A. We are done. $\square$

Using this lemma, we can prove several properties of the henselization:

Proposition 5.15. Let (A, m) be a local ring.

1. A^h is henselian local and the canonical morphism $A \to A^h$ is local and faithfully flat, mA^h is the unique maximal ideal, and the homomorphism $A/m \to A^h/mA^h$ is an isomorphism.

2. For every henselian local ring R, the canonical morphism:

$$\text{loc.Hom}(A^h, R) \to \text{loc.Hom}(A, R)$$

is bijective.

3. If A is already henselian then the canonical map $A \to A^h$ is an isomorphism.

4. The canonical morphism $\hat{A} \to \hat{A^h}$ is an isomorphism.

5. If A is Noetherian then so is A^h.

Proof. Suppose that $A \to A'$ is a strictly essentially etale. I claim that this is map is local and faithfully flat. Since both are local rings, once we prove the map is local, we only need flatness to conclude faithfully flatness. Recall that $A \to A'$ is $A \to B \to B_p$. Since $A \to B$ is flat and $B \to B_p$ is flat then we know that B_p is flat over A, therefore so is A'. We have saw that any morphism of essentially etale A-algebras is local and we notice $A \to A'$ is itself such a morphism, thus local. Recall that since $A \to B$ is etale, it's unramified, therefore we know that $A \to B_p$ makes pB_p the extension of m, then this also holds for A', this also tells you that $A/m \to A'/mA'$ is an isomorphism. Suppose that $A' \to A''$ is an A-morphism of essentially etale morphisms, we know that we have a diagram:

$$\begin{tikzcd} A' \arrow[rr] & & A'' \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

We want to prove that $A' \to A''$ is also flat, which implies it's faithfully flat. To see why, let's consider the following diagram:

$$\begin{tikzcd} A' = B_p \arrow[rr] & & C_q = A'' \\ B \arrow[u] & & C \arrow[u] \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Notice that B over A is finitely presented, then we know that $B \to C_p$ actually factors through C_t for some $t \notin p$, i.e. we have:

$$\begin{tikzcd} A' = B_p \arrow[rr] & & C_q = A'' \\ B \arrow[u] \arrow[r] & C_t \arrow[ur] & C \arrow[u] \arrow[l] \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Then since we know that $A \to B$ and $A \to C$ are both etale, then $A \to C_t$ is also etale since $C \to C_t$ is first finitely presented and also flat and unramified. Therefore $B \to C_t$ is also etale, then in particular flat. This implies that $B_p \to C_q$ is flat as flatness is preserved by localization. Finally since we work with strictly essentially etale A-algebras, the isomorphism on residue fields are all isomorphisms.

Then we apply the previous result and conclude that $A/m \to A^h/mA^h$ is an isomorphism, and $A \to A^h$ is local with mA^h the maximal ideal and $A \to A^h$ is faithfully flat. Therefore we only need to check A^h is indeed henselian. To check this, we will check that for any etale morphism $g : X \to SpecA^h$, any section of $g_s : X \otimes_{A^h} k \to Speck$ is induced by a section of g.

Let's suppose that we have a section $s : SpecA^h/mA^h \to X \otimes_{A^h} A^h/mA^h$, I claim that this can be lifted to a section of g $SpecA^h \to X$. Consider the diagram:

$$\begin{tikzcd} SpecA^h/mA^h \arrow[r, "s"] \arrow[d] & X \otimes_{A^h} A^h/mA^h \arrow[r] \arrow[d] & SpecA^h/mA^h \arrow[d] \\ SpecA^h \arrow[r, dashed, "?"'] & X \arrow[r] & SpecA^h \end{tikzcd}$$

Notice that we can shirnking X and assume that $X \to SpecA^h$ is finitely presented. To see why, we notice that the image of s in $X \otimes_{A^h} A^h/mA^h$ is juts one point, then we can choose an affine open U of the image in X and consider:

$$\begin{tikzcd} SpecA^h/mA^h \arrow[r, "s"] \arrow[d] & U \otimes_{A^h} A^h/mA^h \arrow[r] \arrow[d] & SpecA^h/mA^h \arrow[d] \\ SpecA^h \arrow[r, dashed, "?"'] & U \arrow[r] & SpecA^h \end{tikzcd}$$

Then if there is such a section of $U \to SpecA^h$, I claim that $SpecA^h \to U \hookrightarrow X$ is the section we are looking for. As we have:

$$\begin{tikzcd} SpecA^h/mA^h \arrow[r] \arrow[d] & U \otimes_{A^h} A^h/mA^h \arrow[r, hook] \arrow[d] & X \otimes_{A^h} A^h/mA^h \arrow[r] \arrow[d] & SpecA^h/mA^h \arrow[d] \\ SpecA^h \arrow[r, dashed] & U \arrow[r, hook] & X \arrow[r] & SpecA^h \end{tikzcd}$$

So by shirnking X, we can assume that $X \to SpecA^h$ is affine and finitely presented. Then we can use the results we discussed in the section of passage to limits, and conclude that we can find a strictly essentially etale local A-algebra A' and an etale morphism $g' : X' \to SpecA'$ inducing g after base change to $SpecA^h$. Let $x' \in X'$ be the image of the composite:

$$SpecA^h/mA^h \xrightarrow{s} X \otimes_{A^h} A^h/mA^h \to X \to X'$$

Then we notice that $\mathcal{O}_{X',x'}$ is a strictly essentially etale A-algebra since we have $A \to A' \to \mathcal{O}_{X',x'}$ where we know that $A \to A'$ is strictly essentially etale and $A' \to \mathcal{O}_{X',x'}$ is also strictly essentially. Then we know that by the definition of A^h, there is an A'-algebra homomorphism $\mathcal{O}_{X',x'} \to A^h$, which gives $SpecA^h \to Spec\mathcal{O}_{X',x'} \to X'$. Finally, notice that we have a morphism $SpecA^h \to X$ which can be thought as the graph of $SpecA^h \to X'$. I claim that this morphism is a section of g that induce s. The fact that it's a section of g comes from our interpretation as this as the graph. To check it recall that we have assumed that X is affine, say $SpecB$. Then we know that the morphism $X \otimes_{A^h} A^h/mA^h \to SpecA^h/mA^h$ is also etale and separated, therefore we know that it's section is again determined by where the unique point of $SpecA^h/mA^h$ goes. Notice that the image of $SpecA^h \to X$ lies above the maximal ideal of A^h and $x' \in X'$, and since $A' \to A^h$ induce an isomorphism on the residue field, we know that this point in $SpecB$ is the only point lying about the two points. However we know that the image of $SpecA^h \to X$ is also such a point, thus we know that these are the same point. Then we consider it's pullback by $SpecA^h/mA^h$, we know that the image is going to be the unique point lying above the point of $SpecA^h/mA^h$ and the image

$$SpecA^h \to X$$

, therefore we are done.

To prove 2, first recall that if A' is a strictly essentially etale A-algebra, we have a bijection:

$$\text{loc.Hom}(A', R) \to \text{loc.Hom}(A, R)$$

Now, consider the morphism:

$$\text{loc.Hom}(A^h, R) \to \text{loc.Hom}(A, R)$$

Let's first prove that it's injective. Recall that a map $A^h \to R$ is the same as a system of compatible maps $A' \to R$ that is compatible under $A' \to A''$, then if we assume that after we precompose $A \to A' \to R$ we get the same morphism, we know that each of the maps in the system are equal, therefore the final morphisms are equal. Surjectivity is similar, I will leave it here.

To prove 3, if A is already henselian, then we know that if A' is a strictly essentially etale A-algebra, consider: the following diagram:

$$\begin{tikzcd} Speck(x') \arrow[r] \arrow[d] & Speck(x') \arrow[r] \arrow[d] & Speck(x) \arrow[d] \\ SpecB_p \arrow[r] & SpecB \arrow[r] & SpecA \end{tikzcd}$$

where A' is B_p for an etale A-algebra B. Notice that $Speck(x') \to Speck(x)$ is an isomorphism by assumption. Then we know that it's generated by a section of $SpecB \to SpecA$. Notice that $SpecB \to SpecA$ is etale, then this section $SpecA \to SpecB$ correspond to an isomorphism of $SpecA$ with an open connected component of $SpecB$ since we know that the image of the closed point of A is p in B, we know that the image is contained in $SpecB_p$ and is surjective on $SpecB_p$ since we know that $SpecA \to SpecB$ is also going to be flat, therefore $SpecA \to SpecB_p$ is also flat and thus faithfully flat, therefore surjective, then we know that $SpecB_p \to SpecB$ is actually open and $SpecA \to SpecB_p$ is actually an isomorphism by our correspondence between sections of etale morphisms and isomorphism on connected components.

Therefore we know that $A \to A'$ is an isomorphism for any strictly essentially etale A-algebra, then A^h as the filtered colimit we know that the canonical map $A \to A^h$ is also an isomorphism.

To prove 4, we want to show an isomorphism of the inverse system:

$$A/m^n \to A^h/m^n A^h$$

To show this, recall that A^h is flat over A, therefore we have canonical isomorphisms $m^n \otimes_A A^h \simeq m^n A^h$ and thus $m^n/m^{n+1} \otimes_A A^h \simeq m^n A^h/m^{n+1}A^h$. Notice that $m^n/m^{n+1} \otimes_A A^h \simeq m^n/m^{n+1} \otimes_{A/m} A/m \otimes_A A^h \simeq m^n/m^{n+1} \otimes_{A/m} A^h/mA^h$. Since $A^h/mA^h \simeq A/m$ we know that:

$$m^n A^h/m^{n+1}A^h \simeq m^n/m^{n+1} \otimes_{A/m} A/m = m^n/m^{n+1}$$

When $n = 0$, we know $A/m \simeq A^h/mA^h$ and notice that for A/m^n, we have a filtration:

$$m^n/m^n \subset m^{n-1}/m^n \subset \cdots \subset m/m^n \subset A/m^n$$

Notice that the successive quotient is m^i/m^{i+1}. We do the same for $A^h/m^n A^h$ and use induction to prove the isomorphism:

$$A/m^n \to A^h/m^n A^h$$

Finally to prove 5, we notice that if A is Noetherian, then A' is Noetherian for any strictly essentially etale A-algebra A', then we apply the previous result to conclude that A^h is Noetherian. $\square$

Proposition 5.16. Let A be a local Noetherian ring, then A is reduced (resp. regular, resp. normal) iff A^h is so.

Proof. **** to be completed later* $\square$

Proposition 5.17. Let $(R_\lambda, \phi_{\lambda\mu})$ be a direct system of local rings such that $\phi_{\lambda\mu}$ are local homomorphisms.

1. If each R_λ is henselian, then so is $\varinjlim_\lambda R_\lambda$.

2. In general, we have $(\varinjlim_\lambda R_\lambda)^h \simeq \varinjlim_\lambda R_\lambda^h$

Proof. Recall that if we use R to denote the direct limit, then the maximal ideal is $m = \varinjlim m_\lambda$. Let's consider $A = R[t]/(f(t))$ where $f(t)$ is a monic polynomial. Then to show that R is indeed henselian, we need to show that $A \to A/mA$ induces an one-to-one correspondence between idempotents elements of A and A/mA. We already know that this is always injective when restricted to idempotent elements, therefore we only need to show it's surjective.

Let $\bar{e} \in A/mA$ be an idempotent element, notice that $R[t]$ is the filtered colimit of $R_\lambda[t]$, then we know that there will be a monic polynomial $f_\lambda(t) \in R_\lambda[t]$, whose image in $R[t]$ is $f(t)$. Then we define $A_\lambda = R_\lambda[t]/(f_\lambda(t))$ and we can consider $A_\lambda/m_\lambda A$. We notice that $\varinjlim(f_\lambda(t)) = (f(t))$ and $mA = \varinjlim m_\lambda A_\lambda$, then again using the fact that filtered colimit is exact we know that:

$$\varinjlim A_\lambda/m_\lambda A = A/mA$$

Then we know that for this idempotent element $\bar{e}$ in A/mA, there will be some $\bar{e}_\lambda$ in $A_\lambda/m_\lambda A_\lambda$ that is also idempotent whose image in A/mA is $\bar{e}$. Then since R_λ is henselian, we know that there will an idempotent element e_λ in A_λ lifting $\bar{e}_\lambda$. Consider the image of e in A, you can check it lifts $\bar{e}$.

Then let's prove 2: notice that for each R_λ, we have a canonical morphism $R_\lambda \to \varinjlim R_\lambda$. Consider the following sequence:

$$R_\lambda \longrightarrow \varinjlim R_\lambda \longrightarrow (\varinjlim R_\lambda)^h$$

Notice that this is a local homomorphism and $(\varinjlim R_\lambda)^h$ is henselian local. Then we know that this uniquely factors through the henselization of R_λ, i.e. we have the diagram:

$$\begin{tikzcd}
R_\lambda \arrow[r] \arrow[dr] & \varinjlim R_\lambda \arrow[r] & (\varinjlim R_\lambda)^h \\
& (R_\lambda)^h \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

Notice that for a pair $\mu \geqslant \lambda$, we have the diagram:

$$\begin{tikzcd}
R_\lambda \arrow[r] \arrow[dr] & R_\mu \arrow[r] \arrow[dr] & \varinjlim R_\lambda \arrow[r] & (\varinjlim R_\lambda)^h \\
& (R_\lambda)^h \arrow[r, dashed] & (R_\mu)^h \arrow[ur, dashed] &
\end{tikzcd}$$

The existence and uniqueness of the factorization shows that there is a local homomorphism from $(R_\lambda)^h \to (R_\mu)^h$ and the composition with $(R_\mu)^h \to (\varinjlim R_\lambda)^h$ is the morphism $(R_\lambda)^h \to (\varinjlim R_\lambda)^h$.

Then by the universal property of filtered colimits, we have a homomorphism:

$$f : \varinjlim R_\lambda^h \to (\varinjlim R_\lambda)^h$$

Notice that $\varinjlim R_\lambda^h$ is henselian as each of the factors is henselian, therefore we know that the morphism:

$$\varinjlim R_\lambda \to \varinjlim R_\lambda^h$$

induces a unique morphism:

$$g : (\varinjlim R_\lambda)^h \to \varinjlim R_\lambda^h$$

I claim that f, g are inverses of each other. To check this, for example to check:

$$\varinjlim R_\lambda^h \to (\varinjlim R_\lambda)^h \to \varinjlim R_\lambda^h$$

is the identity, we only need to check how each R_λ^h is mapped to $\varinjlim R_\lambda^h$. Therefore we need the following diagram:

$$\begin{tikzcd}
& & (R_\mu)^h \arrow[r] & \varinjlim R_\lambda^h \\
R_\lambda \arrow[r] \arrow[dr] & R_\mu \arrow[r] \arrow[dr] \arrow[ur] & \varinjlim R_\lambda \arrow[r] \arrow[ur] & (\varinjlim R_\lambda)^h \arrow[u, dashed] \\
& (R_\lambda)^h \arrow[r, dashed] & (R_\mu)^h \arrow[ur, dashed] &
\end{tikzcd}$$

We know that the top horizontal morphism $(R_\mu)^h \to \varinjlim R_\lambda^h$ is the factorization of $R_\mu \to \varinjlim R_\lambda \to \varinjlim R_\lambda^h$. We notice that the composition $(R_\mu)^h \to (\varinjlim R_\lambda)^h \to \varinjlim R_\lambda^h$ is also a factorization, therefore by the uniqueness, they are equal. This suffices. The other direction is via similar argument, I will let you do it. $\square$

Proposition 5.18. Let (A, m) be a local ring, and B a finite A-algebra. We know that B only has finitely many maximal ideals, say $n_1, \cdots, n_k$ are all of them. Then we have:

$$B \otimes_A A^h \simeq (B_{n_1})^h \times \cdots \times (B_{n_k})^h$$

Proof. First we know that $A \to A^h$ induces an isomorphism of the residue field, then consider:

$$\begin{tikzcd}
& SpecB \otimes_A A^h \otimes_{A^h} k \arrow[rr] \arrow[dl] \arrow[dd] & & Speck \arrow[dl] \arrow[dd] \\
SpecB \otimes_A k \arrow[rr] \arrow[dd] & & Speck \arrow[dd] & \\
& SpecB \otimes_A A^h \arrow[rr] \arrow[dl] & & SpecA^h \arrow[dl] \\
SpecB \arrow[rr] & & SpecA &
\end{tikzcd}$$

That is, the fiber of $SpecB \to SpecA$ and $SpecB \otimes_A A^h \to SpecA^h$ over the closed points are isomorphic. Since we know that $A^h \to B \otimes_A A^h$ is also finite, we know that the fiber over the closed point is also the maximal ideals of $B \otimes_A A^h$, therefore we have a one to one correspondence between the maximal ideals of B and $B \otimes_A A^h$. Let $n'_1, \cdots, n'_k$ be the distinct maximal ideals of $B \otimes_A A^h$ lying above $n_1, \cdots, n_k$ via $B \to B \otimes_A A^h$.

Then since A^h is henselian, we know that $B \otimes_A A^h$ being finite over A^h is going to be a finite product of local rings, and we know that this product can be written as:

$$B \otimes_A A^h \simeq (B \otimes_A A^h)_{n'_1} \times \cdots \times (B \otimes_A A^h)_{n'_k}$$

Therefore, to prove our assertion, we only need to prove that $(B \otimes_A A^h)_{n'_i} \simeq (B_{n_i})^h$.

Notice that we know that filtered colimit commutes with tensors, therefore we have:

$$B \otimes_A A^h \simeq \varinjlim_{A' \in \mathrm{Obj}\mathcal{I}} B \otimes_A A'$$

where $\mathcal{I}$ is the old category of strictly essentially etale A-algebras.

Then let $e_i \in B \otimes_A A^h$ be the element that is projected down to 1 in $(B \otimes_A A^h)_{n_i}$ and 0 in $(B \otimes_A A^h)_{n_j}$ for $j \neq i$. Then we know that $e_i e_j = 0$ and $e_i^2 = e_i$ and $e_1 + \cdots + e_k = 1$. Notice that since $B \otimes_A A^h$ is also a filtered colimit, we know that there is a strictly essentially etale A-algebra A' such that $e_1, \cdots, e_k$ are the images of some elements of $e'_1, \cdots, e'_k \in B \otimes_A A'$ with the same properties.

Therefore we know:

$$B \otimes_A A' \simeq (B \otimes_A A')e'_1 \times \cdots \times (B \otimes_A A')e'_k$$

If we tensor $B \otimes_A A'$ with $\otimes_{A'} A^h$, we get:

$$B \otimes_A A^h \simeq (B \otimes_A A^h)e_1 \times \cdots \times (B \otimes_A A^h)e_1 \simeq (B \otimes_A A^h)_{n'_1} \times \cdots \times (B \otimes_A A^h)_{n'_k}$$

Again here this holds since we know that A^h is flat over A'.

Consider the morphisms:

$$Spec(B \otimes_A A')e'_i \to SpecA', \qquad Spec(B \otimes_A A')e'_i \otimes_{A'} A^h \to SpecA^h$$

Just like the case with $A \to A^h$, we know that the residue field of A' is isomorphic to that of A^h, therefore we again conclude that the morphism:

$$Spec(B \otimes_A A')e'_i \otimes_{A'} A^h \to Spec(B \otimes_A A')e'_i$$

gives you a bijection between maximal ideals, therefore since $(B \otimes_A A')e'_i \otimes_{A'} A^h$ is local, we conclude that $(B \otimes_A A')e'_i$ is also local and under the morphism:

$$B \to (B \otimes_A A')e'_i$$

it's maximal ideal lives over n_i. This is because we have a commutative diagram:

$$\begin{tikzcd} Spec(B \otimes_A A')e'_i \otimes_{A'} A^h \arrow[r] \arrow[d] & Spec(B \otimes_A A')e'_i \arrow[d] \\ SpecB \otimes_A A^h \arrow[r] & SpecB \end{tikzcd}$$

Notice that this implies that we have a morphism $B_{n_i} \to (B \otimes_A A')e'_i$ and I claim this is strictly essentially etale. To see why, recall that A' is strictly essentially etale over A, therefore there will be an etale A-algebra C such that $A' \simeq C_p$, then we know that:

$$B_{n_i} \otimes_A A' \simeq B_{n_i} \otimes_A C_p$$

Then we consider the sequence:

$$B_{n_i} \to B_{n_i} \otimes_A C \to B_{n_i} \otimes_A C_p \to (B_{n_i} \otimes_A C_p)e''_i$$

where e''_i is the element in $B_{n_i} \otimes_A C_p$ corresponding to e'_i in $B_{n_i} \otimes_A A'$. Notice that we have:

$$B \otimes_A A' = (B \otimes_A A')e'_1 \times \cdots \times (B \otimes_A A')e'_k$$

we know that since $B_{n_i} \otimes_A A'$ is just some localization of $B \otimes_A A'$, it commutes with products and we get:

$$B_{n_i} \otimes_A A' \simeq (B_{n_i} \otimes_A A')e_i \times \cdots \times (B_{n_i} \otimes_A A')e'_k$$

And we know that the morphsim $B \otimes_A A' \to B_{n_i} \times_A A'$ is component wise. I will let you show that this localization contains e_i in the multiplicative system, therefore after the localization, only $(B_{n_i} \otimes_A C_p)e''_i$ survives and you can check the above composition $B_{n_i} \to (B_{n_i} \otimes_A C_p)e''_i$ is exactly $B_{n_i} \to (B \otimes_A A')e'_i$. Probably one useful tool you will use is the fact that if you have a finite product of local rings, say $A_1 \times \cdots \times A_n$, then we know exactly the maximal ideal of this product, then I will let you show that projection to the A_i factor is the same as localizing at the i-th maximal ideal.

Therefore we know that:

$$(B_{n_i})^h = ((B \otimes_A A')e'_i)^h$$

We also know that $A^h = (A')^h$. Therefore our goal becomes to prove that:

$$((B \otimes_A A')e'_i)^h \simeq (B \otimes_A A^h)_{n'_i} = (B \otimes_A A' \otimes_{A'} (A')^h)_{n'_i}$$

Then we can assume that B is actually $B \otimes_A A'$ and we know that the projection $B \otimes_A A' \to (B \otimes_A A')e'_i$ is actually the localization, we can assume that B is in fact a product of local rings over A.

We only need to show the case when B is a local ring, that is finite over A. In this case, I claim that $B \otimes_A A^h$ is local since we know that $SpecB \otimes_A A^h \to SpecB$ gives a bijection of maximal ideals. I claim it's also henselian. To check this, suppose that C is a finite $B \otimes_A A^h$ algebra, then since B is finite over A, we know that $B \otimes_A A^h$ is finite over A^h, then C is finite over A^h, therefore it's indeed a finite product of local rings. Moreover we know that:

$$B \otimes_A A^h = \varinjlim B \otimes_A A'$$

and we know that $B \otimes_A A'$ is strictly essentially etale over B. This gives us a B-morphism $B \otimes_A A^h \to B^h$, consider:

$$\begin{tikzcd} B^h \arrow[r, "\mathrm{Id}"] & B^h \\ B \arrow[u] \arrow[r, "\mathrm{Id}"'] & B \arrow[u] \end{tikzcd}$$

We know that since the identity extend $B \to B^h$ it's the unique local morphism extending $B \to B^h$ to $B^h \to B^h$. Then consider:

$$\begin{tikzcd} B^h \arrow[r, dashed] & B \otimes_A A^h \arrow[r] & B^h \\ B \arrow[u] \arrow[r, "\mathrm{Id}"'] & B \arrow[u] \arrow[r, "\mathrm{Id}"'] & B \arrow[u] \end{tikzcd}$$

We know that the composition must be the identity. Now recall that when we prove the fact about unique factorization through the henselization, we only use the fact that we have a direct system and the fact that each object in the system is strictly essentially etale, then we know that $B \otimes_A A^h$ also has the unique factorization property. Then consider:

$$\begin{tikzcd} B \otimes_A A^h \arrow[r, dashed] & B^h \arrow[r] & B \otimes_A A^h \\ B \arrow[u] \arrow[r] & B \arrow[u] \arrow[r, "\mathrm{Id}"] & B \arrow[u] \end{tikzcd}$$

where $B^h \to B \otimes_A A^h$ exists because $B \otimes_A A^h$ is henselian, then we know that the composition is still the identity, we are done. $\square$

Remark 5.19. Let's make a side remark on why if $A \to B$ is a local morphism and $A \to A'$ is a strictly essentially etale algebra, then $B \to B \otimes_A A'$ will also be strictly essentially etale.

First, since $A \to A'$ gives an isomorphism betweem the residue fields, we know that $B \otimes_A A'$ is indeed local and the morphism is a local morphism. To show we indeed have an isomorphism of residue fields, since $A \to A'$ is strictly essentially etale, we consider the following diagram:

$$\begin{tikzcd} SpecB \otimes_A C_p \arrow[r] \arrow[d] & SpecC_p \arrow[d] \\ SpecB \otimes_A C \arrow[r] \arrow[d] & SpecC \arrow[d] \\ SpecB \arrow[r] & SpecA \end{tikzcd}$$

We have proved that in this case, since C_p and A has the same residue field, then we know that the residue field of $B \otimes_A C_p$ will be $k(p) \otimes_{k(A)} k(C_p) \simeq k(p)$.

The notice that $B \to B \otimes_A C$ is etale as the base change of an etale morphism. Moreover, $B \otimes_A C \to B \otimes_A C_p$ can be interpreted as a localization and $B \otimes_A C_p$ is a local ring, therefore this is equivalent as localizing at a prime ideal, we are done.

Recall that we proved before that if R is a henselian local ring, then sections of the fiber over the closed point of $SpecR$ can all be induced by a section on the base if the morphism of the base is etale. In fact, we don't need etaleness, and rather it can be replaced by a smooth morphism of any relative dimension:

Proposition 5.20. Let R be a henselian local ring, $S = SpecR$, and s the closed point of S. For any smooth morphism $g : X \to S$, any section of $g_s : X_s = X \otimes_R k(s)$ can be lifted to a section of g.

Proof. Let $h_s : Speck(s) \to X$ be a section of g_s and let $x \in X_s \subset X$ be the image of h_s. Recall that since $g : X \to SpecR$ is smooth, there will be an open neighborhood U of X and an etale R-morphism:

$$j : U \to SpecR[t_1, \cdots, t_n]$$

Let $y = j(x)$. We know that as x is the image of the fiber, we conclude that $k(x) \simeq k(s)$ since the composite of the morphisms on the residue field is the identity. To prove that $k(x) = k(y)$, consider the following diagram of fiber over s:

$$\begin{tikzcd} U_s \arrow[r] \arrow[d] & A^n_{k(s)} \arrow[r] \arrow[d] & Speck(s) \arrow[d] \arrow[ll, bend right, "g_s"'] \\ U \arrow[r] & \mathbb{A}^n_R \arrow[r] & R \end{tikzcd}$$

We know that the morphism $Speck(s) \to U_s \to \mathbb{A}^n_{k(s)}$ makes the residue field of y $k(s)$ as we know that $y \in \mathbb{A}^n_{k(s)}$ is in fact a $k(s)$-point, thus the residue field is $k(s)$, then since we know that $U \to \mathbb{A}^n_R$ is etale, then we know that the residue field is a finite extension over $k(s)$, but we know that the residue field of y is actually the same as that of s, x, therefore this is in fact a trivial extension.

Then we know that the morphism $\mathcal{O}_{\mathbb{A}^n_R,y} \to \mathcal{O}_{X,x}$ induces an isomorphism of the residue field. Moreover I claim that this is essentially etale, thus strictly essentially etale. To see why, let's assume that $U = SpecB$ is just affine, then we know that B is going to be finitely presented over $R[t_1, \cdots, t_n]$ and the morphism is unramified and flat. Let the ring map be $R[t_1, \cdots, t_n] \to B$, then say the point y correspond to a prime ideal p in $R[t_1, \cdots, t_n]$, then we can consider:

$$\mathcal{O}_{\mathbb{A}^n_R,y} = R[t_1, \cdots, t_n]_p \to B_p$$

we know this is still finitely presented and flatness is preserved under localization. Finally, we know that sheaf of differentials is also compatible with localization, therefore this is still etale, then we know that $\mathcal{O}_{\mathbb{A}^n_R,y} \to \mathcal{O}_{X,x}$ expresses $\mathcal{O}_{X,x}$ as a localization of an etale algebra of $\mathcal{O}_{\mathbb{A}^n_R,y}$, thus is indeed essentially etale. This in particular implies that the following sequence is both the henselization of $\mathcal{O}_{X,x}$ and the henselization of $\mathcal{O}_{\mathbb{A}^n_R,y}$:

$$\mathcal{O}_{\mathbb{A}^n_R,y} \to \mathcal{O}_{X,x} \to \mathcal{O}^h_{X,x} = A$$

Then consider the following diagram:

$$\begin{tikzcd}
 & \mathbb{A}^n_R \arrow[dr] & \\
SpecA \arrow[ur] \arrow[r] & U \arrow[u, "j"] \arrow[r, "g|_U"] & S = SpecR
\end{tikzcd}$$

Then if we can find a section $SpecR \to SpecA$ of $SpecA \to SpecR$, then we know that we can compose it with $SpecA \to U \hookrightarrow X$, then conclude that $SpecR \to SpecA \to U \hookrightarrow X \to SpecR = \mathrm{Id}_{SpecR}$.

We already know that the y in the fiber of s in $\mathbb{A}^n_s = Speck(s)[t_1, \cdots, t_n]$ corresponds to a maximal ideal $(t_1 - \overline{a_1}, \cdots, t_n - \overline{a_n})$ where $\overline{a_i} \in k(s)$. Let $a_i \in R$ be elements such that their image in $k(s)$ are $\overline{a_i}$. Then we know that y in $SpecR[t_1, \cdots, t_n]$ is the maximal ideal we got by taking the inverse image of maximal ideal $(t_1 - \overline{a_1}, \cdots, t_n - \overline{a_n})$. Then if we consider R-algebra map:

$$R[t_1, \cdots, t_n] \to R, \quad t_i \mapsto a_i$$

I claim $R[t_1, \cdots, t_n] \to R[t_1, \cdots, t_n]_y \to \mathcal{O}^h_{X,x}$ gives a unique factorization from the henselization to R, which gives a morphism $SpecR \to SpecA$. The reason is that we know that we have the diagram:

$$\begin{tikzcd}
R[t_1, \cdots, t_n] \arrow[r, "t_i \mapsto a_i"] \arrow[d] & R \arrow[d] \\
k(s)[t_1, \cdots, t_n] \arrow[r, "t_i \mapsto a_i"'] & k(s)
\end{tikzcd}$$

which means that the inverse image of s in $R[t_1, \cdots, t_n]$ contains x, therefore we have a unique factorization:

$$R[t_1, \cdots, t_n] \to R[t_1, \cdots, t_n]_y \to R$$

which is local and then we know that by the universal property of henselization, we get a unique factorization from A to R, which thus gives $SpecR \to SpecA$.

I claim this is the section we want, which means we want to prove the following composition is the identity:

$$R \to R[t_1, \cdots, t_n]_y \to A \to R$$

This is summarized in the following diagram and I will let you check it:

$$\begin{tikzcd}
R \arrow[d] & R \\
R[t_1, \cdots, t_n]_y \arrow[ur] \arrow[r] & A \arrow[u, dashed]
\end{tikzcd}$$

□

Proposition 5.21. Let (R, m) be a local ring and $k = R/m$ the residue field, $S = SpecR$ and s the closed point of S. The following conditions are equivalent:

1. R is strictly henselian.

2. R is henselian and any finite etale S-scheme X is isomorphic to a disjoint union of S.

3. For any etale morphism $g : X \to S$ and any point $x \in X$ lying above s, there is a section $h : S \to X$ of g such that $h(s) = x$.

Proof. Let's prove $1 \to 3$: since R is strictly henselian, we know that the residue field $k(s)$ is separably closed. Then consider an etale morphism:

$$g : X \to S$$

and $x \in X$ lies above s. We know that the morphism on the residue field is a finite separable extension since g is etale morphism, but we know that $k(s)$ is separably closed, thus the extension is the trivial extension. Then consider the following diagram:

$$\begin{tikzcd} X_s \arrow[r] \arrow[d] & Speck(s) \arrow[d] \\ X \arrow[r] & S \end{tikzcd}$$

We know that the residue field of x, computed in X or in X_s is the same. Notice that a morphism $Speck(s) \to X_s$ with the image x is just a extension of of the residue field to $k(s)$, but we know that the residue field of x is $k(s)$. Therefore we know that the induced morphism of $k(s) \to k(x)$ is an isomorphism therefore by using the inverse we get a section of $X_s \to Speck(s)$ with image at x. This is summarized in the following diagram:

$$\begin{tikzcd} k(s) \arrow[r] \arrow[rr, bend left, "g"] & \mathcal{O}_{X,x} \arrow[r] & k(x) \arrow[r] & k(s) \end{tikzcd}$$

Then we know that since R is henselian, it's given by a section $h : S \to X$. We can show it in the diagram:

$$\begin{tikzcd} Speck(s) \arrow[r] \arrow[d] & X_s \arrow[r] \arrow[d] & Speck(s) \arrow[d] \\ S \arrow[r] & X \arrow[r] & S \end{tikzcd}$$

We are done. $\square$

References