Notes on Etale cohomology: 3

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This is the third chapter of the etale cohomology notes, where we have finally finished the prerequisites and could consider something interesting, i.e. the **etale fundamental groups**. We begin with some necessary backgrounds:

1 Finite Group Actions on a Scheme

Definition 1.1. Suppose that G is a finite group and X is a scheme, we say that G acts on X on the right if for each element $g \in G$, we have an automorphism of X, called $\rho_g : X \to X$ such that for $g_1, g_2 \in G$, we have:

$$\rho_{g_1 \circ g_2} = \rho_{g_2} \circ \rho_{g_1}$$

and for the identity element $e \in G$, $\rho_e = \mathrm{Id}_X$.

For example if $X = SpecA$, I claim that G acts on X on the right is equivalent to G acts on X on the left, i.e. for each $g \in G$, we are given a ring automorphism ρ_g such that $\rho_{g_1 \circ g_2} = \rho_{g_1} \circ \rho_{g_2}$ and $\rho_e = \mathrm{Id}_A$.

Let G be a finite group acting on X on the right, then we know that for any scheme Z, G acts on the left of the set $\mathrm{Hom}(X, Z)$. The definition is defined to be $g \cdot f$ is $f \circ \rho_g$. This is indeed a left action as we know that $(g_1 \cdot g_2)f = f \circ \rho_{g_1 \cdot g_2} = f \circ \rho_{g_2} \circ \rho_{g_1} = g_1 \cdot (g_2 \cdot f)$. We will use $\mathrm{Hom}(X, Z)^G$ to denote the morphisms that is invariant under G's action, i.e. those f such that $g \cdot f = f$ for all $g \in G$.

Definition 1.2. Let G be a finite group acting on a scheme X, a morphism $X \to Y$ invariant under G is called a **quotient of X by G** if the canonical map:

$$\mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)^G$$

is bijective for any scheme Z. In other words, Y represents the functor $Z \mapsto \mathrm{Hom}(X, Z)^G$. In this case, we will denote Y by X/G as it's unique up to a unique isomorphism.

Remark 1.3. What is the natural map in the above definition. Given a morphism $f : Y \to Z$, we get a morphism $X \to Z$ by pre-composing it with $X \to Y$. Then since $X \to Y$ is invariant under G, the composition is also invariant under G.

How should we think about this quotient and its universal property? This is actually very similar to what we saw in the category of topological spaces. Where if you have a finite group G acting on a topological space X, then $X \to Y = X/G$ is defined to be the quotient map and the topology on Y is defined by the quotient topology. Then the universal property is just a restatement of the universal property of quotient topological spaces in this special case. Given a G-invaraint morphism $X \to Z$, we know that the orbits of G will be mapped to the same point in Z, therefore it factors uniquely through $Y = X/G$. In the category of schemes, we are just trying to reproduce this situation, but since at this point, we don't have a direct quotient of X by G, we can only define $Y = X/G$ using the same universal property as in topological spaces. Later in this section, you will see how we can give a concrete description of Y, at least in the case when X is affine, which is the following proposition.

Proposition 1.4. Let A be a ring on which a finite group G acts on the left, let $B = A^G$, i.e. the subring that is fixed by the group actions. Let $X = SpecA, Y = SpecB$, and $\pi : X \to Y$ be the morphism corresponding to the inclusion of rings $A^G \hookrightarrow A$, then:

1. $A^G \hookrightarrow A$ is an integral extension of rings, or $\pi : X \to Y$ is an integral morphism.

2. $\pi : X \to Y$ is surjective, it's fibers are orbits of G and the topology on Y is the quotient topology induced from X.

3. Given $x \in X$ and let $y = \pi(x)$, if we denote:

$$G_x = \{g \in G : gx = x\}$$

by the stabilizer of x, then the residue field $k(x)$ is a normal algebraic extension of the residue field $k(y)$ and the canonical map $G_x \to \mathrm{Gal}(k(x)/k(y))$ is surjective.

4. The canonical morphism $\mathcal{O}_Y \to (\pi_* \mathcal{O}_X)^G$ is an isomorphism and Y is the quotient of X by G.

Proof. You probably have noticed that the above definitions makes sense even if this group G is not finite, however, in this proof, we will see that we do need finiteness of G.

First, we want to prove that each $a \in A$ is the roof of some monic polynomial in $B[t]$. Consider $\prod_{g \in G}(t - g \cdot a) \in A[t]$. Notice that if we apply g to this polynomial, it's coefficients remains the same, thus it's actually in $B[t]$, and we know that it's monic with a root a.

Then we know that integral field extension implies surjective morphism on the spectrum. Obviously, since $B \hookrightarrow A$ is an inclusion, we know that the image of $V(I)$ for an ideal $I \subset A$ is going to be $V(I \cap B)$, therefore it's a closed map, thus the topology on Y is indeed the quotient topology. Now let $q \subset B$ be a prime ideal and suppose that we have p, p' prime ideals in A such that $p' \cap B = q = p \cap B$. We notice that the element $\prod_{g \in G} ga \in p' \cap B = q \subset p$. Therefore we know that for some $g \in G$, $ga \in p$, thus $p' \subset \cup_{g \in G} gp$. Recall that gp, for all $g \in G$, are all prime ideals, then by the standard prime avoidance lemma, we conclude that $p' \subset gp$ for some $g \in G$. This implies that $p' = gp$ since:

$$gp \cap B = g(p \cap B) = p \cap B = p' \cap B$$

Then since $p' \subset gp$ and $p' \cap B = gp \cap B$, $p' = gp$ from the fact that $B \hookrightarrow A$ is integral.

Let p be a prime ideal of A corresponding to x and let $q = p \cap B$ be the prime corresponding to y. Consider the ring extension:

$$B_q \hookrightarrow A \otimes_B B_q$$

Recall that integrality is preserved by localization, so this is still an integral closure, and we know that G naturally acts on the localization (can you see why, maybe it's clearer if you think about it in the tensor terms) and it's easy to check that B_q in this is $(A \otimes_B B_q)^G$. Moreover, we know that this localization doesn't affect the residue field, therefore we can assume that B is a local ring with q the maximal ideal. I claim that p is this case is maximal, this is because q is maximal and $B \to A$ is integral again, then we know that for any element in $k(x)$, it comes from something in A, then we can use the same polynomial considered as a polynomial in $k(y)[t]$, and it splits in $k(x)$, therefore the field extension is normal and algebraic.

Now consider the separable closure of $k(y)$ in $k(x)$, which we denote by $k(y)_s$. We know that $k(y)_s/k(y)$ is separable and for any element $x \in k(y)^s$, it's minimal polynomial over $k(y)$ is at most of degree $|G|$, this implies that $k(y)_s/k(y)$ is in fact finite. To see why, suppose that K/L is a separable algebraic extension that is of infinite degree, then we can choose a sequence of elements $a_n \in K$ such that:

$$a_{n+1} \notin L(a_1, \cdots, a_n)$$

Therefore we know that $L(a_1, \cdots, a_n)/K$ is at least of degree 2^n. However, since each finite separable extension $L(a_1, \cdots, a_n)/L$ can be written as $L(\alpha_n)$ for some $\alpha_n \in K$, we know that the degree of the extension of the degree of the minimal polynomial of α_n, therefore a contradiction. Therefore $k(y)_s/k(y)$ is finite separable, therefore generated by one element, therefore degree is at most $|G|$.

If $g \in G_x$, since g is an automorphism of A that fixes p, it also gives an automorphism of $k(x)$. I claim that this automorphism fixes $k(y)$, thus gives the natural morphism $G_x \to \mathrm{Gal}(k(x)/k(y))$. This is obvious since we have the following diagram:

$$\begin{tikzcd}
& A^G \arrow[rr, "\mathrm{Id}"] \arrow[dl, hook] \arrow[d] & & A^G \arrow[dl, hook] \arrow[d] \\
A \arrow[rr, "g"'] \arrow[d] & k(y) \arrow[rr] \arrow[dl] & A \arrow[d] & k(y) \arrow[dl] \\
k(x) \arrow[rr, "g"] & & k(x)
\end{tikzcd}$$

I will let you check this diagram indeed commutes, the key is to remember A^G is fixed under g.

Now let $a \in A$ be an element whose image in $k(x)$ generate $k(y)_s$ over $k(y)$. By our definition o G_x, we know that for any $g \in G - G_x$, gp is a maximal ideal of A distinct from p, therefore all coprime to p, then we know that p is also coprime to $\prod_{g \in G - G_x} gp = J$, then we know that by the Chinese remainder theorem, we know that the morphism:

$$A \to A/p \times A/J$$

is surjective. Therefore there is some $a' \in A$ such that it's image in A/p is $\overline{a}$ and it's image in A/J is 0. In other words, we can find $a' \in A$ such that $a' - a \in p$ and $a' \notin gp$ for all $g \notin G_x$.

Consider the following polynomial in $k(y)[t]$:

$$f(t) = \prod_{g \in G} (t - \overline{ga'})$$

where $\overline{ga'}$ is the image of ga' in $k(x)$. We know that if $\tau \in \mathrm{Gal}(k(x)/k(y))$, we know that $\tau(\overline{a'})$ is still a root of $f(t)$, therefore $\tau(\overline{a'}) = \overline{ga'}$ for some $g \in G$. If $g \notin G_x$, we know that, we know that $g^{-1} \notin G_x$, therefore we know that $a' \in g^{-1}p$, therefore $ga' \in p$, therefore $\sigma(a') = 0$ which implies $\overline{a'} = 0$. Therefore we know that $k(y)_s = k(y)$ as it's generated over $k(y)$ by something already in $k(y)$. Therefore we know that $k(x)/k(y)$ is purely inseparable, recall that the Galios group of a purely inseparable field extension is trivial, therefore $G_x \to \mathrm{Gal}(k(x)/k(y))$ is trivially surjective. Recall that since $a' - a \in p$, we know that $\overline{a'} = \overline{a}$, therefore we know that:

$$\sigma(\overline{a}) = \overline{ga} = g \cdot \overline{a}$$

This implies that $g^{-1} \circ \sigma$ fixes $k(y)_s$ as well, therefore we know that it's in $\mathrm{Gal}(k(x)/k(y)_s)$ which is trivial, therefore $g = \sigma$.

Finally, let's first figure out what is $\mathcal{O}_Y \to (\pi_* \mathcal{O}_X)^G$. I claim that the group G naturally acts on $\pi_* \mathcal{O}_X$. This is because we have the following diagram:

$$\begin{tikzcd}
X \arrow[rr, "g"'] \arrow[dr, "\pi"'] & & X \arrow[dl, "\pi"] \\
& Y &
\end{tikzcd}$$

Then we define the action of g on $\pi^* \mathcal{O}_X(V)$ to be the action of g on $\mathcal{O}_X(\pi^{-1}V)$. I will check this is compatible with restrictions using the above diagram.

Then I define $(\pi^* \mathcal{O}_X)^G$ open by open and you chan check it's indeed a sheaf. Then you can check that the morphism $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$ uniquely factors through $(\pi_* \mathcal{O}_X)^G \hookrightarrow \pi_* \mathcal{O}_X$. To check this is indeed an isomorphism, we check for each $f \in B$ on $D(f)$, we know that the morphism is just:

$$\begin{tikzcd}
B_f \arrow[rr] \arrow[dr, hook] & & (A_f)^G \arrow[dl, hook] \\
& A_f &
\end{tikzcd}$$

where the top morphism is $\frac{b}{f^n} \mapsto \frac{b}{f^n}$. Then clearly this is injective as $B_f \to A_f$ is injective. I will let you check it's surjective, which is not hard.

We check Y is indeed the quotient of X by G. First, obvious by the definition of Y π is invariant under G. We then need to check $\mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, G)^G$ is bijective for all Z. We first notice that $\pi : X \to Y$ is an epimorphism in the category of schemes, since it's surjective and the morphism: $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$ is injective. Therefore suppose that if $p_1 \circ \pi = p_2 \circ \pi$, then $p_1 = p_2$, therefore the morphism is injective. To show it's surjective, we need to show that we can always find a morphism $Y \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Z \\ & Y \arrow[ur, dashed] & \end{tikzcd}$$

Recall that the fibers of π are orbits of G, and we know that since f is also invariant under G, then we know that orbits of G must be mapped to the same point in Z, therefore there is a unique map of topological spaces $g : Y \to Z$, we only need to upgrade it to a morphism of schemes. What we notice here is that since f is also invariant under G, then for any $V \subset Z$, we know that the morphism $\mathcal{O}_Z(V) \to \mathcal{O}_X(f^{-1}V)$ maps into those that is fixed by G, since we know that $f^{-1}V = \pi^{-1}(g^{-1}V)$ and $\mathcal{O}_Y(g^{-1}V)$ is the elements in $\mathcal{O}_X(f^{-1}V)$ that is fixed by G, therefore we have the factorization:

$$\begin{tikzcd} \mathcal{O}_Z(V) \arrow[rr, "f^\sharp"] \arrow[dr, dashed] & & \mathcal{O}_X(f^{-1}V) \\ & \mathcal{O}_Y(g^{-1}V) \arrow[ur, "\simeq"'] & \end{tikzcd}$$

This clearly is compatible with restrictions, and it makes this diagram commute, then the fact that π and f are both morphisms of locally ringed spaces makes g a morphism of locally ringed spaces. I will let you check. $\square$

Definition 1.5. Let X be a scheme with a right finite group action by G. We say this action is **admissible** if there is an affine morphism $\pi : X \to Y$ invariant under G such that $\mathcal{O}_Y = (\pi_* \mathcal{O}_X)^G$, this equality comes from the canonical morphism $\mathcal{O}_Y \to (\pi_* \mathcal{O}_X)^G$ being an isomorphism.

Remark 1.6. In this case, we can first choose an affine cover of Y, and by taking the inverse image, we get a bunch of morphisms between affine schemes say $\pi^{-1}U \to U$, which is still invariant under G-actions. Then suppose that this is $SpecB \to SpecA$, we know that the condition $\mathcal{O}_Y = (\pi_* \mathcal{O}_X)^G$ implies that B is just A^G, therefore we can apply what we have proved in the previous proposition and obtain a lot of information. For example, we know $\pi : X \to Y$ is surjective and the topology on Y is still the quotient topology coming from X. This even holds for every $\pi^{-1}U \to U$ for not necessarily affine U and we know that the orbit G is contained in some of the affine opens.

Moreover, recall that when we proved the quotient in the affine case, the fact we need is the quotient topology condition and the condition $\mathcal{O}_Y \to (\pi_* \mathcal{O}_X)^G$ is an isomorphism, without any usage of the affine condition. Therefore here we can prove that $\pi^{-1}U \to U$ is a quotient by G for each open subscheme U of Y. In particular $X \rightharpoonup Y$ is the quotient.

Proposition 1.7. Let X be a scheme with a finite group G acting on the right, the following conditions are equivalent:

1. The action is admissible.
2. X is a union of affine opens stable under the action of G.
3. Each orbit of G stays in an affine open subset.

Proof. Let $\pi : X \to X/G$ be the canonical morphism of the quotient from X being admissible. Then by definition this is an affine morphism, and we know that by choosing an affine open cover of X/G we can cover X by an affine opens. Moreover, suppose that we have a point $x \in X$, since each morphism $\pi^{-1}U \to U$ is invariant under G action, we know that each orbit is contained in one of these affine opens. This proves $1 \Rightarrow 3$.

To show $3 \Rightarrow 2$, let S be an orbit of G and let U be an affine open subset that contain S. Then we know that $S \subset \cap_{g\in G} gU \subset U$. Recall that since G is finite, S is finite and $\cap_{g\in G} gU$ is a nonempty open. Then I claim that there is an affine open V such that $S \subset V \subset \cap_{g\in G} gU$. To see why, suppose that $U = SpecA$ and $S = \{p_1, \cdots, p_n\}$, then we know that $\cap_{g\in G} gU = SpecA - V(a)$ for some ideal $a \subset A$. We know that no p_i contains a for all i, then we know that a is not contained in the union of p_i, i.e. by the prime avoidance lemma. Therefore we can take $f \in a - \cap_{g\in G} gU$, then we note that $V = D(f)$ suffices. Finally, we notice that for all $g \in G$, since $S \subset V \subset \cap_{g\in G} gU$, we know that $gV \subset \cap_{g\in G} gU \subset U$. Moreover, we know that since g induces an automorphism on U, V as an affine open in U, we know that gV is still affine. Recall that in a separated scheme (here being in an affine scheme), finite intersections of affine schemes is still affine. Therefore $\cap_{g\in G} gV$ is affine, and stable under G-actions, which contains S, therefore X is indeed a union of affine opens that is stable under G-action.

Finally, we show $2 \Rightarrow 1$. Let Y be the quotient topological space of X by G, and we have a canonical morphism of topological spaces:

$$\pi : X \to Y = X/G$$

Then consider the pushforward $\pi_* \mathcal{O}_X$ on Y. We know that G acts on $\pi_* \mathcal{O}_X$, which let us define a ringed spaces $(Y, (\pi_* \mathcal{O}_X)^G)$. I claim that this is a scheme, which means that we need to show that it's locally $SpecR$ for some ring R. Now suppose that X is the union of a bunch of $SpecA$ which is stable under the group action, then we know that we can describe Y as covered by $SpecA^G$. □

Corollary 1.8. Let X be a scheme on which a finite group G acts on the right, then:

1. If G acts admissibly on X, then so does any subgroup of G.

2. If there is an affine morphism $f : X \to Z$ invariant under G-actions, then G acts admissibly on X and X/G is Z-isomorphic to $Spec(f_* \mathcal{O}_X)^G$.

Proof. The first assertion is obvious using criterion 2 in the proposition.

To see the second assertion, we first know from the fact $f : X \to Z$ is affine and invariant under G-actions that G acts admissibly on X. Then consider the following diagram:

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "f"'] & & X/G \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

where $X/G \to Z$ is induced by the universal property. Then recall that if we consider the Z-scheme $Spec(f_* \mathcal{O}_X)^G$, we know that a Z-morphism from X/G to it correspond to a $\mathcal{O}_Z$-algebra morphism:

$$(f_* \mathcal{O}_X)^G \to g_* \mathcal{O}_{X/G} = g_*((\pi_* \mathcal{O}_X)^G) = (g_* \pi_* \mathcal{O}_X)^G = (f_* \mathcal{O}_X)^G$$

which is just the identity morphism. The second equality holds because if we compute:

$$g_*((\pi_* \mathcal{O}_X)^G)(V) = (\pi_* \mathcal{O}_X)^G (g^{-1} V) = \mathcal{O}_X(\pi^{-1} g^{-1} V)^G = (g_* \pi_* \mathcal{O}_X)^G (V)$$

Therefore we have a morphism $X/G \to \mathcal{S}pec(f_* \mathcal{O}_X)^G$. I claim that this is an isomorphism, which can be checked locally, and I will let you check this. □

Remark 1.9. These above results suggests that not like what happens in the topological spaces, X/G may not exist. We know from the above discussion the characterizations of the existence.

Remark 1.10. Let G be a finite group acting on X on the right, if we want to check that a morphism $X \to Y$ is the canonical quotient morphism, I claim that we can do it affine locally on the target. This is because either invariant under the group action and $\mathcal{O}_Y = (\pi_* \mathcal{O}_X)^G$ can be checked affine locally on the target. We will use this later.

Proposition 1.11. Let X be a scheme on which a finite group G acts, and let $X \to Y$ be a morphism invariant under G, for any morphism $Y' \to Y$, we will let G act on $X \times_Y Y'$ by base change.

1. If G acts admissibly on X with $Y = X/G$ and $Y' \to Y$ is flat then G acts admissibly on $X \times_Y Y'$ and $Y' = (X \times_Y Y')/G$.
2. If $Y' \to Y$ is quasi-compact and faithfully flat, G acts admissibly on $X \times_Y Y'$ and $Y' = (X \times_Y Y')/G$, then G acts admissibly on X and $Y = X/G$.

Proof. First, consider the morphism:

$$\begin{tikzcd} X \times_Y Y' \arrow[r] \arrow[d] & X \times_Y Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "g"] & X \arrow[r] & Y \end{tikzcd}$$

The morphism $X \times_Y Y' \to X \times_Y Y'$ is how we define the G-action.

Let's first prove the first assertion. I first claim that we can assume everything is affine due to the above above remark. Therefore we can assume that $X = SpecA, Y = SpecA^G$, and $Y' = SpecB'$ where B' is flat over B, then we need to show that:

$$B' \to A \otimes_{A^G} B'$$

describes B' as $(B' \to A)^G$.

Consider the following morphism of B-modules:

$$A \to \prod_{g \in G} A, \qquad a \mapsto (a - ga)$$

This is indeed a morphism of B-modules where we know that $b(a_1 + a_2) \mapsto (b(a_1 + a_2) - bg(a_1 + a_2)) = b((a_1 - ga_1) + (a_2 - ga_2))$. Notice that the kernel of this map is B, therefore we have a exact sequence:

$$0 \to B \to A \to \prod_{g \in G} A$$

Since B' is flat over B, then we have another exact sequence by tensoring B':

$$0 \to B' \to A \otimes_B B' \to \prod_{g \in G} A \otimes_B B'$$

where the morphism $A \otimes_B B' \to \prod_{g \in G} A \otimes_B B'$ maps $a \otimes b'$ to $((a - ga) \otimes b') = (a \otimes b' - g \cdot (a \otimes b'))$, therefore we know that the kernel is indeed $(A \otimes_B B')^G = B'$, we are done.

Recall that by our result in descent theory, for a faithfully flat quasicompact morphism $Y' \to Y$, the fact that $X \times_Y Y' \to Y'$ is affine implies that $X \to Y$ is also affine. Again, we can check if $X \to Y$ is the canonical quotient morphism by checking affine locally on affine, therefore we can assume that $X = SpecA, Y = SpecB$ and $Y' = SpecB'$ where B' is faithfully flat over B, from our assumption that the top morphism is the quotient morphism, we know:

$$0 \to B' \to A \otimes_B B' \to \prod_{g \in G} A \otimes_B B'$$

is exact. Now since B' is faithfully flat, we know that $0 \to B \to A \to \prod_{g \in G} A$ is exact, therefore B is A^G. $\square$

2 Etale Covering Spaces and Fundamental Groups

From this section, we begin to discuss etale theory in a serious way, let's begin with a few definitions:

Definition 2.1. Let X be a scheme, a **geometric point of** X is a morphism $\gamma : s \to X$ where s is the spectrum of a separably closed field. We often write $k(s)$ for this separably closed field and $s = Speck(s)$.

Remark 2.2. Recall that if $\gamma : s \to X$ is a geometric point with image $x \in X$. Then consider the corresponding morphism on stalks:

$$\mathcal{O}_{X,x} \to k(s)$$

we know that the maximal ideal of $\mathcal{O}_{X,x}$ is mapped to 0, therefore we actually have an extension. $k(x) \hookrightarrow k(s)$. Conversely, given such an extension, we can define a morphism $Speck(s) \to Speck(x) \to X$ where $Speck(x) \to X$ is the canonical morphism. These two constructions are actually inverses of each other and I will let you check it. Clearly, if $\gamma : s \to X$ is a geometric point, then for any morphism $f : X \to X'$, we have a geometric point $f \circ \gamma : s \to X'$.

Definition 2.3. A **pointed scheme** is a pair (X, γ) such that X is a scheme and $\gamma : s \to X$ is a geometric point.

A morphism between pointed schemes $(X, \gamma), (X', \gamma')$ is a morphism $f : X \to X'$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] & X' \\ s \arrow[u, "\gamma"] \arrow[r, "\mathrm{Id}"'] & s \arrow[u, "\gamma'"'] \end{tikzcd}$$

In particular, we need the base field of the two geometric points to be the same.

Definition 2.4. Let X be a scheme on which a finite group G acts on the right, and let $x \in X$, we will call the stabilizer subgroup:

$$G_x = \{g \in G : gx = x\}$$

the **decomposition subgroup at** x and is denoted by $G_d(x)$.

Remark 2.5. Clearly, since $g \in G_x$ fixes x we get a well-defined action of G_x on $k(x)$, the residue field at x. We will call the elements in $G_d(x)$ that is acting trivially on $k(x)$ is called the **inertia subgroup at** x and is denoted by $G_i(x)$.

One observation of this inertia subgroup is the following: if $\gamma : s \to X$ is a geometric point with image x. Then we know that G acts on $\mathrm{Hom}(s, X)$ where g acts by $\rho_g \circ \gamma$ and obviously G acts on the right since the action $(g_1 g_2)$ is $\rho_{g_1 g_2} \circ \gamma = \rho_{g_2} \circ \rho_{g_1} \circ \gamma = ((\gamma)g_1)g_2$. I claim that the inertia subgroup at x, i.e. $G_i(x)$, is exactly the stabilizer of $\gamma \in \mathrm{Hom}(s, X)$.

To see why, recall that we have a one to one correspondence between $\gamma : s \to X$ and extensions $k(x) \hookrightarrow k(s)$. Then you can check that after you apply g to γ, we get the geometric point which is obtained by $k(x) \xrightarrow{g} k(x) \hookrightarrow k(s)$. Therefore if g is the stabilizer, then we know that g acts trivially on $k(x)$, conversely, if g acts trivially, it's in the stabilizer of γ. Using this interpretation, we can prove:

Proposition 2.6. Consider a Cartesian diagram:

$$\begin{tikzcd} X \times_S S' \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

Suppose that a finite group G acts on the right, and suppose that $X \to S$ is invariant under G. Let G act on $X \times_S S'$ by base change. For any $x' \in X \times_S S'$ with image x in X, we have $G_i(x) = G_i(x')$.

Proof. Let $\gamma : s \to X \times_S S'$ be a geometric point with image x'. Then we know that $G_i(x')$ is the stabilizer of γ, that is:

$$\{g \in G : \gamma g = \gamma, \gamma : s \to X \times_S S'\}$$

Recall that by composing γ with $X \times_S S' \to X$, we get a geometric point of X. Then consider the following diagram:

$$\begin{tikzcd} & X \arrow[r, "g"] & X \arrow[r] & S \\ s \arrow[r, dashed, "\gamma"] & X \times_S S' \arrow[u] \arrow[r, "g"] & X \times_S S' \arrow[u] \arrow[r] & S' \arrow[u] \end{tikzcd}$$

Then we know that if g fixes γ, then obviously it fixes $s \to X \times_S S' \to X$. Conversely, if g fixes the geometric point to x, since g never plays with $s \to S'$, therefore we know that once g fixes the X factor, it fixes the geometric point to x'. Therefore the stabilizer interpretation implies the result. $\square$

Finally, let's give the first one of the most important definitions:

Definition 2.7. An **etale covering space** of a scheme S is a finite etale morphism $X \to S$. One observation that is good to make in the beginning is that: Recall that we have proved that etale morphism are open, and we also know that finite morphisms are closed. Therefore the image of $X \to S$ is both open and closed, thus if S is a connected (or irreducible), then this morphism is necessarily surjective, thus indeed matches our intuition that a covering map should be surjective.

Remark 2.8. Notice that if $X \to S$ is an etale covering, then the group $\mathrm{Aut}(X/S)$, i.e. the group of automorphism of X over S acts on X on the left. Consider the opposite group $G = \mathrm{Aut}(X/S)^\circ$, where the group law becomes $g_1 g_2 = g_2 \circ g_1$. Then this group acts on X on the right. If in particular $S = X/G$, then we say $X \to S$ is a **Galios etale covering space** with **Galios group** G.

Lemma 2.9. Let k be a field, and let A be a finite dimensional k-algebra. Then $A \simeq \prod_i L_i$ for finitely many fields L_i separable over k if and only if the following bilinear form is nondegenerate:

$$A \times A \to k \qquad ,(x, y) \mapsto \mathrm{Tr}_{A/k}(xy)$$

Proof. Let's first assume the form is nondegenerate, then if $x \in A$ is nilpotenet, we know that xy is also nilpotent. Recall that over any field, the trace of a nilpotent element is always 0, therefore the trace of xy is 0 for any y. Then since the bilinear form is nondegenerate, we know that $x = 0$, therefore A is reduced.

Now let K be any field extension of k, we consider another bilinear form:

$$A \otimes_k K \times A \otimes_k K \to K, \qquad (x, y) \mapsto \mathrm{Tr}_{A \otimes_k K/K}(xy)$$

I claim that this is still nondegenerate. To see why, we know that the first form is nondegenerate iff the associated morphism:

$$A \to A^\vee, x \mapsto B(x, -)$$

is an isomorphism, then we know that:

$$A \otimes_k K \to A^\vee \otimes_k K$$

is also an isomorphism where $x \otimes k \mapsto B(x, -) \otimes k$. Recall that $A^\vee \otimes_k K \simeq (A \otimes_k K)^\vee$ where:

$$\mathrm{Hom}_k(A, k) \otimes_k K \to \mathrm{Hom}_K(A \otimes_k K, K), \qquad f \otimes k \mapsto (x \otimes k' \mapsto k \cdot k' \cdot f(x))$$

you can check this using a basis. Then we know that the morphism $A \otimes_k K \to A^\vee \otimes_k K$ can be thought as:

$$A \otimes_k K \to (A \otimes_k K)^\vee$$

where $x \otimes k \mapsto B(x, -) \otimes k \mapsto (y \otimes k' \mapsto kk'B(x, y))$ and this is an isomorphism.

Notice that $A \otimes_k K \times A \otimes_k K \to K$ is defined by $(x \otimes k, y \otimes k') \mapsto kk'\mathrm{Tr}_{A \otimes_k K/K}(xy)$. Therefore we know that the new form is also nondegenerate. Therefore $A \otimes_k K$ is also reduced. Then recall that since A is a finite dimensional k-algebra, it's necessarily finite length, thus Artinian, therefore it decomposes to a finite product of local Artinian rings, say:

$$A = \prod_i A_i$$

reach of them is reduced, but we know that reduced local Artinian rings are fields.

Now suppose that $A = \prod_i L_i$ where L_i are all extensions of k. We know that $A \otimes_k K = \prod_i L_i \otimes_k K$, but since $A \otimes_k K$ is reduced, so is $L_i \otimes_k K$, therefore each L_i is separable.

Conversely, suppose that $A = \prod_i L_i$, we want to show that:

$$A \times A \to k$$

is nondegenerate. Recall that by breaking up A to L_i pieces, I will let you show that bilinear form is nondegenerate iff for every finite separable extension K of k, the following bilinear form is nondegenerate:

$$K \times K \to k, \qquad (x, y) \mapsto \mathrm{Tr}(xy)$$

Recall that every finite separable extension over k is generated by a single element by the primitive element theorem, then we choose this element to be x, and $f(t) = t^n + a_1 t^{n-1} + \cdots + a_n$ be the minimal polynomial of x. Say $x_1, \cdots, x_n$ are all the roots of f in an algebraic closure of k containing K. Again since K/k is separable, these are all distinct. Recall that we have a basis $\{1, x, \cdots, x^{n-1}\}$ as a basis of K over k, then to prove the form is nondegenerate, it suffices to show that the Gram matrix is invertible, i.e. the matrix $(\mathrm{Tr}(x^{i-1}x^{j-1}))_{ij}$ is invertible.

Recall that $\mathrm{Tr}(x^n) = \sum_{j=0}^{n-1} x_j^n$, therefore the matrix is the product VV^T where:

$$V = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{bmatrix}$$

Therefore we can compute it's determinant using the Vandermonde determinant formula it's

$$\prod_{i<j}(x_i - x_j)$$

since x_i's are all distinct, it's not 0. $\square$

Proposition 2.10. Let $f : X \to S$ be a finite flat morphism between Noetherian schemes, then $\mathcal{A} = f_*\mathcal{O}_X$ is locally free $\mathcal{O}_S$-module of finite rank, then f is etale iff the following bilinear form is nondegenerate:

$$\mathcal{A} \times \mathcal{A} \to \mathcal{O}_S, \qquad (x, y) \mapsto \mathrm{Tr}(xy)$$

Proof. Recall that when we work with Noetherian schemes, finite flat means that $f_*\mathcal{O}_X$ is locally free. Recall that we already know that the morphism is finite, flat, therefore we only need to prove it's unramified. This can be checked affine locally, thus we can assume that $X = SpecA \to SpecB = Y$ where A is a finitely generated flat B-module. We only need to prove that $\Omega_{A/B}$ is 0, that is for each $p \subset A$, $(\Omega_{A/B})_p = 0$. Now let q be the inverse image of p in B and consider the following diagram:

$$\begin{array}{ccc} A_q/qA_q & \longleftarrow & A \\ \uparrow & & \uparrow \\ B_q/qB_q & \longleftarrow & B \end{array}$$

We know that since $\Omega_{A/B}$ is finitely generated over A, $(\Omega_{A/B})_p$ is finitely generated over A_p, therefore by Nakayama's lemma, it's equivalent to $(\Omega_{A/B})_p = pA_p(\Omega_{A/B})_p$. Now is $(\Omega_{A/B}) \otimes A_q/qA_q = 0$, we know that $(\Omega_{A/B})_q = qA_q(\Omega_{A/B})_q$, then we know that by further localizing at p, we know that $(\Omega_{A/B})_p = qA_p(\Omega_{A/B})_p \subset pA_p(\Omega_{A/B})_p$, therefore $(\Omega_{A/B})_p = 0$, i.e. the morphism is etale. Conversely, if the morphism is etale $\Omega_{A/B} = 0$, then of course what tensor with it is 0.

Whatever, we need to prove $(\Omega_{A/B}) \otimes_A A_q/qA_q = 0$ which is the same as prove $\Omega_{(A_q/qA_q)/(B_q/qB_q)} = 0$ as module of differentials is compatible with base change. We will denote $k(q) = B_q/qB_q$, and A_q/qB_q is a finite dimensional over $k(q)$. Then we need to prove $\Omega_{A_q/qA_q/k(q)} = 0$, where A_q/qA_q is a finite $k(q)$-algebra. Then we know that this is equivalent to A_q/qA_q is a product of finite separable extensions of $k(q)$ by our characterization of finite k-algebras (see the remark below).

This condition, by the previous theorem, is equivalent to the following form being nondegenerate:

$$A_q/qA_q \times A_q/qA_q \to B_q/qB_q, \qquad (x, y) \mapsto \mathrm{Tr}(x, y)$$

Therefore we only need to prove that this condition is equivalent to the form in the statement being nondegenerate. I will let you check this locally. $\square$

Remark 2.11. In this remark, we give the proof of the characterization of finite etale algebras over a field k.

The claim is that if A is a finite k-algebra, then A is etale over k iff A is a finite product of finite separable extensions of k. First, let's assume that A is etale over k, then we first know from the fact that A is finite over k that A is a finite product of local rings, say A_m, then $SpecA \to Speck$ is etale iff $\Omega_{A/k}$ is 0, consider the following diagram:

$$\begin{tikzcd} A_m & A \arrow[l] \\ k \arrow[u] & k \arrow[l] \arrow[u] \end{tikzcd}$$

Then we know that $(\Omega_{A/k})_m = \Omega_{A_m/k}$, then we know that $\Omega_{A/k}$ is 0 iff $\Omega_{A_m/k} = 0$ for all m and still A_m/k is finite and flat. Now, we know that $\Omega_{A_m/k} = 0$ iff the maximal ideal of A_m is 0, i.e. it's a field and A_m/k is a finite separable extension (we proved this in the previous note chapter). The converse is easier to prove.

Now, once we established this there are other useful algebraic characterizations of a finite etale k-algebra, using the same notation as above:

1. A is a finite etale algebra over k.
2. There is some L extending k such that $A \otimes_k L$ is isomorphic to a finite product of L, i.e. $\prod_i L$.
3. $A \otimes_k L$ is reduced for all fields L containing k.

You can find the proof somewhere.

Proposition 2.12. Let (S, γ) be a pointed scheme, and let $(X_i, \alpha_i) \to (S, \gamma)$ be two morphisms of pointed schemes:

1. If X_1 is connected and X_2 is unramified and separated over S, then there exists **at most one** S-morphism from (X_1, α_1) to (X_2, α_2).
2. Suppose that X_1, X_2 are etale covering spaces of S, then there is a connected pointed etale covering space (X_3, α) of (S, γ) dominating (X_i, α_i), that is there is a S-morphism from (X_3, α_3) to (X_i, α_i).

Proof. Consider the following diagram:

$$\begin{tikzcd} s \arrow[r, dashed, "\alpha_1"] & X_1 \arrow[dr, "\Gamma_f"] \arrow[drr, "f"] \arrow[ddr, "\mathrm{Id}"'] & & \\ & & X_1 \times_S X_2 \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & X_2 \arrow[d] \\ & & X_1 \arrow[r] & S \end{tikzcd}$$

We know that any S-morphism $f : X_1 \to X_2$ is completely determined by it's graph in the sense that there is a one to one correspondence between f and Γ_f. Notice that the morphism Γ_f is a section of the projection π_1, notice that π_1 as the base change of $X_2 \to S$, it unramified and separated. Recall that we have proved before that: if $X \to Y$ is a morphism with f unramified and separated and Y connected, then there is a one to one correspondence between the set of sections of f and the set of connected components of X on which f induces and isomorphism. In particular, a section of f is completely determined by it's valued at one point of Y. Therefore Γ_f is completely determined by where α_1 is mapped to in $X_1 \times_S X_2$. Now if there is such a f, we know that α_1 is mapped to α_2 in X_2, thus we know that the image of α_1 in $X_1 \times_S X_2$ is completely determined by f, therefore Γ_f is unique, thus f is unique. This proves the first assertion.

For the second assertion, notice that (α_1, α_2) determines a point α_3 in $X_1 \times_S X_2$. Then let X_3 be the connected component of $X_1 \times_S X_2$ containing α_3, it suffices. $\square$

Proposition 2.13. Let X be a scheme on which a finite group G acts admissibly on the right and let $Y = X/G$ be the quotient of X by G, then:

1. Suppose that X is of finite presentation over Y, then if the inertia subgroup $G_i(x)$ is trivial for any $x \in X$, then X is etale over Y.

2. Suppose that X is connected, S a Noetherian scheme, $X \to S$ an etale covering space. Let G be a finite subgroup of $\mathrm{Aut}(X/S)^\circ$, then $G_i(x)$ is trivial for any $x \in X$. Moreover $X \to Y = X/G$ and $Y \to S$ are all etale covering spaces and in this case $G = \mathrm{Aut}(X/Y)^\circ$.

Proof. Let's first deal with 1. Recall that in the section of Jacobian criterion, we have the following result: if we have the following Cartesian diagram:

$$\begin{tikzcd} X \times_Y Y' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

Suppose that f is locally of finite presentation and $g : Y' \to Y$ is faithfully flat. If f' is unramified (resp. etale, resp. smooth), then so is f.

Back to our case: for any $y \in Y$, let $\widetilde{\mathcal{O}}_{Y,\overline{y}}$ be the strict henselization of $\mathcal{O}_{Y,y}$ with respect to a separable closure of $k(y)$. Then consider the diagram:

$$\begin{tikzcd} \coprod_{y\in Y} X \times_Y Spec\widetilde{\mathcal{O}}_{Y,\overline{y}} \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ \coprod_{y\in Y} Spec\widetilde{\mathcal{O}}_{Y,\overline{y}} \arrow[r, "g"'] & Y \end{tikzcd}$$

We know that $\coprod_{y\in Y} Spec\widetilde{\mathcal{O}}_{Y,\overline{y}} \to Y$ is faithfully flat, then if we can prove that f' is etale, then f will be etale. Since this is a disjoint union, we know that we only have to prove $X \times_Y Spec\widetilde{\mathcal{O}}_{Y,\overline{y}} \to Spec\widetilde{\mathcal{O}}_{Y,\overline{y}}$ is etale. So we can consider the diagram without the disjoint union:

$$\begin{tikzcd} X \times_Y Spec\widetilde{\mathcal{O}}_{Y,\overline{y}} \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Spec\widetilde{\mathcal{O}}_{Y,\overline{y}} \arrow[r, "g"'] & Y \end{tikzcd}$$

Here g is still flat, then since we assumed $Y = X/G$, we know that $Spec\widetilde{\mathcal{O}}_{Y,\overline{y}}$ will be $(X \times_Y Spec\widetilde{\mathcal{O}}_{Y,\overline{y}})/G$, therefore in particular we know that the morphism f' is affine finite presentation and our goal is to show this is etale. Therefore we can just assume that in the original $X \to Y$, $Y = SpecA$ for some strictly henselian local ring A.

Recall that X is affine, of finite presentation and integral over Y, therefore X is actually finite over $Y = SpecA$, then $X = SpecB$ for some finite A-algebra B, and we know that $B = B_{n_1} \times \cdots \times B_{n_k}$ where $n_1, \cdots, n_k$ are all the maximal ideals of B. Let $G_d(n_i), G_i(n_i)$ be the decomposition and inertia subgroup at n_i. We know that $A = B^G$, and we know that all the maximal ideals of B lies over the maximal ideal of A, therefore is the fiber of the closed point, therefore is one of the orbit of G. Therefore G acts transitively on $\{n_1, \cdots, n_k\}$.

Now think about the following diagram:

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "\simeq"'] & B \arrow[d, "\simeq"] \\ B_{n_1} \times \cdots \times B_{n_k} \arrow[r, "g"'] & B_{n_1} \times \cdots \times B_{n_k} \end{tikzcd}$$

where g on the bottom is the induced action of $B_{n_1} \times \cdots \times B_{n_k}$, and we want to figure out what is the action on this product. Since the vertical morphism is an isomorphism, we can assume that the element down blow is $(\frac{b}{1})$, then we know that after we apply g, we get $(\frac{gb}{1})$. Now if we want to write elements in the product as $(\frac{a_i}{b_i})$, then after we apply g, we also would like to see what happens.

Since $\frac{a_i}{b_i} = \frac{b}{1}$, then we know that for each i, there will be some $c_i \notin n_i$ such that $a_i c_i = b_i b c_i$. Then after we apply g, we know that $ga_i gc = gb_i gbgc_i$. Notice that if g is moves n_i to n_j, then we know that $\frac{gb}{1} = \frac{ga_i}{gb_i}$ in B_{n_j}, therefore we can sort of think that if g moves n_i to n_j, then $\frac{a_i}{b_i}$ is first moved to the j-th slot in the product, not the i-th slot and then to a_i, b_i, we apply g just as in B. So this action involves both swapping slots and after swapping slots, apply the action.

There is one way to think about this action, which will make it easier to compute B^G. To see what is this, let fix n_i. First, we know that each B_{n_j} is isomorphic to B_{n_i} via a chosen element $g_j \in G$. We can think of this g_j as an element that takes n_j to n_i. If for the time being, we write $H = G_d(n_i)$, we know that each each of these g_j will be in a different left coset in G/H with one coset corresponding to a different coset, therefore we can use them as the coset representative. Then consider the composition of isomorphisms:

$$B \to B_{n_1} \times \cdots \times B_{n_k} \to \prod_{g_j \in G/H} B_{n_i}$$

Then let's think about what is the G action on this $\prod_{g_j \in G/H} B_{n_i}$. Suppose that we have $(x)_{G/H} \in \prod_{g_j \in G/H} B_{n_i}$, consider the diagram:

$$\begin{tikzcd}
B_{n_1} \times \cdots \times B_{n_k} \arrow[r, "g"'] \arrow[d, "\prod g_j"] & B_{n_1} \times \cdots \times B_{n_k} \arrow[d, "\prod g_j"'] \\
\prod_{g_j \in G/H} B_{n_i} \arrow[r, "g"] & \prod_{g_j \in G/H} B_{n_i}
\end{tikzcd}$$

we know that this $(x)_{g_j}$ is just $(g_j^{-1}x) \in B_{n_1} \times \cdots \times B_{n_k}$, then we apply g, which is first swap and then apply the action of g. I will let you check that in the end, the consequence is that is g takes n_j to n_{jk}, then we know that the x in the g_j's slot is mapped to g_{jk}'s slot and x is acted by a unique element in H, that is $g_{jk} g g_j^{-1} \in H$. Then if we want to think about what is the subring fixed by G. I claim that it's $(B_{n_i})^{G_d(n)j)}$ for all j, in particular it A and for all j, $(B_{n_i})^{G_d(n)j)}$ are isomorphic to each other. To see why, we first notice that there is a natural map:

$$B_{n_i}^{G_d(n_i)} \to B^G$$

defined by $a \mapsto (a)$. I will let you check this is indeed an isomorphism by checking that the inverse is given by:

$$B^G \to B_{n_i}^{G_d(n_i)}$$

where the tuple is projected down to the component corresponding to g_i and you can take $g_i = 1$ for simplicity.

Recall that the residue field of A is separably closed, then if we consider $G_d(n_j)$, we find a canonical map:

$$\{e\} = G_i(n_j) \hookrightarrow G_d(n_j) \to \mathrm{Aut}_{k(A)}(k(n_j))$$

Notice that $k(A) \hookrightarrow k(n_j)$ is is a normal algebraic extension. Recall that any algebraic extension over a separably closed field has trivial Galois group, therefore $\mathrm{Aut}_{k(A)}(k(n_j))$ is trivial. This forces $G_d(n_j)$ to be trivial as well.

Therefore we know that $A \simeq B_{n_i}$ for all n_j, therefore we know that $SpecB \to SpecA$ is just $\coprod SpecA \to SpecA$, therefore is etale.

Then let's prove 2: consider

$$x \xrightarrow{\alpha} X \longrightarrow S$$

where α is a geometric point of X with image x and $X \to S$ finite etale. We want to show that $G_i(x)$ is $\{e\}$ for all x. Suppose that $g \in G_i(x)$, we know that we have the following diagram:

$$\begin{tikzcd}
Spec\overline{k} \arrow[rrr, "\mathrm{Id}"] \arrow[d] & & & Spec\overline{k} \arrow[d] \\
Speck(x) \arrow[r] & X \arrow[r, "g"] \arrow[dr] & X \arrow[d] & Speck(x) \arrow[l] \\
& & S &
\end{tikzcd}$$

We know that g induces an S-automorphism $(X, \alpha) \to (X, \alpha)$, this implies that g is identity, since we know that X is connected and is unramified and separated over S.

Recall that $X \to S$ is invariant under the group action, then we know that we have a canonical decomposition:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl, dashed] \\ & S & \end{tikzcd}$$

I claim that $X \to Y$ and $Y \to S$ are both finite. To prove this, we can assume that $S = SpecC$ is affine, then we know that $X = SpecB$ for some finite C-algebra B and $Y = SpecB^G$. Recall that S is Noetherian, therefore we can assume C is Noetherian, therefore we know that B^G, as a subring of B which contains the image of C, is going to be finite over C. Therefore $Y \to S$ is finite, therefore so is $X \to Y$.

Now since $X \to Y$ is finite, it's of finite presentation. Then since the inertia group is trivial for all $x \in X$, we conclude that $X \to Y$ is etale from 1. Recall that we know if X, Y are over S of finite presentation with X etale over S and $X \to Y$ is etale surjective, then $Y \to S$ is etale **** see section of Jacobian criterion*.

Finally to prove $G = \mathrm{Aut}(X/Y)^\circ$, we let σ be a Y automorphism of X, for $x \in X$, we know that $\sigma(x)$ is in the fiber of the image of x in X. As G acts transitive on the fibers, we know that there is $g_1 \in G$ such that $g_1 x = \sigma(x)$. Let y be the image of x in Y, then we know that σ and g_1 induces two $k(y)$-isomorphisms of $k(x), k(\sigma(x))$, which are denoted by $g_1^\sharp, \sigma^\sharp$ respectively. Notice that $\sigma^\sharp \circ g_1^{\sharp -1} \in \mathrm{Gal}(k(x)/k(y))$. Recall that the morphism $G_x(x) \to \mathrm{Gal}(k(x)/k(y))$ is surjective, there will be a $g_2 \in G_d(x)$ inducing $\sigma^\sharp \circ g_1^{\sharp -1}$. In other words, $\sigma^\sharp = g_1^\sharp \circ g_2^\sharp$, or $\sigma^\sharp - 1 \circ g_1^\sharp \circ g_2^\sharp$ is the trivial in the Galois group. Therefore $\sigma^{-1} \circ g_1 \circ g_2$ is in the inertial group of x, therefore inducing a S-morphism from $(X, \alpha) \to (X, \alpha)$, which is the identity by the uniqueness result. Therefore $\sigma = g_1 \circ g_2 = g_2 g_1 \in G$. Notice that there is a natural injection:

$$G \hookrightarrow \mathrm{Aut}(X/Y)^\circ$$

defined by for each $g \in G$, by definition of Y, g obvious is an automorphism of Y. The above discussion every automorphism also comes from such a product, we are done. $\square$

Corollary 2.14. Let S be a Noetherian scheme and $X \to S$ an etale covering space. Let G be a group acting on X on the right such that $X \to S$ is invariant under G. Then G acts admissibly on X and the quotient X/G is etale over S.

Proof. Recall that X is also Noetherian in this case, thus only with finitely many connected components. Then we know that G acts on the set of the connected components of X by permuting them. We also know that if some g takes one connected component to another, this should be an isomorphism. Now consider the orbits of G in the set of connected components, we take one from each orbit, say $X_1, \cdots, X_n$, then we know that any connected components is of the form gX_j for some $g \in G$ and a unique X_j. Let G_j be the stabilizer of the group permuting X_j, i.e. $G_j = \{g \in G : gX_j = X_j\}$.

Then I claim that $X/G \simeq \coprod_j X_j/G_j$, this is very plausible just by our intuition as we know that the quotient should identify all the components isomorphic to X_j to X_j and then doing further identification in X_j. To prove this rigorously, notice that given a morphism $X \to S$ is the same as a morphism from each connected components. Then notice that g acts transitively on each orbit, therefore we know that if the morphism is invariant under G, morphisms from components that are in the same orbit is determined by the morphism from one of them, then we can use X_j in each of the components and conclude that this is equivalent to give a morphism from X_j to S that is invariant under G_j. That is to say there is a natural bijection:

$$\mathrm{Hom}(X, S)^G \to \prod_j \mathrm{Hom}(X_j, S)^{G_i}$$

defined to be restricting to each X_j. I will let you figure out why this is an isomorphism, but this implies we have:

$$\begin{tikzcd} \mathrm{Hom}(X,S)^G \arrow[r, "\simeq"] & \prod_j \mathrm{Hom}(X_j, S)^{G_i} \\ \mathrm{Hom}(X/G, S) \arrow[u, "\simeq"] \arrow[r, dashed] & \prod_j \mathrm{Hom}(X_j/G_j, S) = \mathrm{Hom}(\coprod_j X_j/G_j, S) \arrow[u, "\simeq"'] \end{tikzcd}$$

that is there is a unique isomorphism $X/G \simeq \coprod_j X_j/G_j$. Moreover, since we have:

$$\begin{tikzcd}
\mathrm{Hom}(X,S) \arrow[rr] & & \prod_j \mathrm{Hom}(X_j,S) \\
\mathrm{Hom}(X,S)^G \arrow[u, hook] \arrow[rr, "\simeq"] & & \prod_j \mathrm{Hom}(X_j,S)^{G_i} \arrow[u, hook] \\
\mathrm{Hom}(X/G,S) \arrow[u, "\simeq"] \arrow[r, dashed] & \prod_j \mathrm{Hom}(X_j/G_j,S) = \mathrm{Hom}(\coprod_j X_j/G_j,S) \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

We can conclude by Yoneda that we have the following commutative diagram:

$$\begin{tikzcd}
\coprod_j X_j \arrow[r] \arrow[d, hook] & \coprod_j X_j/G_j \arrow[d, dashed] \\
X \arrow[r] & X/G
\end{tikzcd}$$

Therefore we have:

$$\begin{tikzcd}
& \coprod_j X_j \arrow[dl] \arrow[dr] & \\
X \arrow[r] \arrow[dr] & X/G \arrow[r] \arrow[d] & \coprod_j X_j/G_j \arrow[dl] \\
& S &
\end{tikzcd}$$

Therefore we may assume $X = X_j$ and $G = G_j$. Therefore we are back to the previous situation and X/G is indeed etale over S. $\square$

Proposition 2.15. Let (S, γ) be a pointed connected Noetherian scheme, X_1, X_2 two etale covering spaces of S. Let $u : X_1 \to X_2$ be an S-morphism and $X_i(\gamma)$ be the sets of geometric points of X_i lying over γ. If the map $X_1(\gamma) \to X_2(\gamma)$ induced by u is bijective, u is an isomorphism.

Proof. Notice that etale morphisms are open, finite morphisms are closed. Since X_2 is Noetherian, it only has finitely many connected components and each of them is both open and closed. Then we know that the image in S is also both open and closed, thus a union of connected components since S is also Noetherian. Since S is connected, the morphism $X_2 \to S$ is surjective. Then by replacing X_2 by one of its component and by X_1 by the inverse image of that component, we can assume that X_2 is connected. Note that $X_1 \to X_2$ is finite etale and our goal is to prove this is an isomorphism. Notice that this is equivalent to prove $\mathcal{O}_{X_2} \to u_*\mathcal{O}_{X_1}$ is an isomorphism.

First, recall that since u is a finite etale morphism between Noetherian schemes, we can conclude $u_*\mathcal{O}_{X_1}$ is locally free of finite rank as we know that flat module of finite presentation is locally free of finite rank. Then since X_2 is connected, we can assume it's of constant finite rank. Let $s \in S$ be the image of γ and since $X_2 \to S$ is surjective, we know that there is a $x_2 \in X_2$ lying above s, then consider:

$$\begin{tikzcd}
X_1 \times_{X_2} Spec\mathcal{O}_{X_2,x_2} \arrow[r] \arrow[d] & Spec\mathcal{O}_{X_2,x_2} \arrow[d] \\
X_1 \arrow[r, "u"] & X_2
\end{tikzcd}$$

Notice that this fibered product only depends on a neighborhood of X_2, then we can assume that X_2 is an affine open such that X_2 is also an affine open which is finite free over X_2, therefore we know that $X_1 \times_{X_2} SPec\mathcal{O}_{X_2,x_2}$ is just $SpecA$ where A is finite free over $\mathcal{O}_{X_2,x_2}$. Recall that $X_2 \to S$ is etale, therefore the induced morphism on the residue field is finite separable, then consider:

$$\begin{tikzcd} & k(\gamma) & \\ k(s) \arrow[ur, hook] \arrow[rr, hook] & & k(x_1) \arrow[ul, hook, dashed] \end{tikzcd}$$

we know that there is a factorization of $k(x_1) \hookrightarrow k(\gamma)$. For example we know that $k(x_1)$ is $k(s)[t]/(f(t))$ for a irreducible separable polynomial. Then since we know that f has a root in $k(\gamma)$, then we can define the $k(s)$-extension by map t to the root of f in $k(\gamma)$.

Therefore we know that x_1 is in $X_2(\gamma)$, then we know that since $X_1(\gamma) \to X_2(\gamma)$ is bijective, there is only one point $x_1 \in X_1$ lying over x_2 since we again know that $X_1 \to X_2$ is etale. This implies that A is a local ring. Again since u is etale, the base change $SpecA \to Spec\mathcal{O}_{X_2,x_2}$ is also etale, therefore $m_{x_2}A$ is the maximal ideal of A. We also know $A/m_{x_2}A$ is finite separably over $\mathcal{O}_{X_2,x_2}$. Notice that if this extension is not an isomorphism, say then it's degree is at least 2, therefore since finite separable extension is again $k(x_2)[t]/(f(t))$ for some separable irreducible polynomial $f(t)$, then it has distinct roots in $k(\gamma)$, therefore there will be at least two distinct $k(x_2)$-embedding of $k(x_1)$ to $k(\gamma)$, therefore contradicting $X_1(\gamma) \to X_2(\gamma)$ is bijective.

Notice that the rank of $(u_*\mathcal{O}_{X_1})$ is the dimension of $A/m_{x_2}A$ as a $\mathcal{O}_{X_2,x_2}/m_{x_2}$ vector space, therefore it's rank is 0. Then let x_2 be an arbitrary point of X_2, we know that $X_1 \times_{X_2} Spec\mathcal{O}_{X_2,x'_2} \simeq SpecA'$, and since the rank $u_*\mathcal{O}_X$ is of rank 1, we know that A is a free $\mathcal{O}_{X_2,x'_2}$-module of rank 1. Then we know that $\mathcal{O}_{X_2,x'_2}/m_{x'_2} \to A'/m_{x'_2}A'$ is a nonzero morphism between 1-dimension $\mathcal{O}_{X_2,x'_2}/m_{x'_2}$-vector. Therefore this must be an isomorphism, in particular surjective. Recall that every ring here is Noetherian, then we know that if we consider the cokernel of $\mathcal{O}_{X_2,x'_2} \to A$, it's a finitely generated $\mathcal{O}_{X_2,x'_2}$-module, therefore since we know it vanishes after we quotient the maximal ideal, it's 0 in the first place, therefore $\mathcal{O}_{X_2,x'_2} \to A$ is surjective. I claim it's also injective, since if we consider:

$$\begin{tikzcd} & \mathcal{O}_{X_1,x'_1} & \\ \mathcal{O}_{X_2,x'_2} \arrow[ur] \arrow[rr] & & A' \arrow[ul] \end{tikzcd}$$

this is commutative as $\mathcal{O}_{X_1,x'_1}$ is just a further localization of A' where x'_1 is any point lying above x'_2 (there is such a point since $X_1 \to X_2$ is also surjective). This also proves that there is only one maximal ideal of A' lying over $m_{x'_2}$. In particular we know that A' is local. We know $\mathcal{O}_{X_2,x'_2} \to \mathcal{O}_{X_1,x'_1}$ is faithfully flat and $A' \to \mathcal{O}_{X_1,x'_1}$ is also faithfully flat (it's flat because it's a localization, it's faithfully flat because it's a local morphism between local rings), therefore $\mathcal{O}_{X_2,x'_2}$ is also faithfully flat, in particular injective since we know that:

$$\mathcal{O}_{X_2,x'_2} \to \mathcal{O}_{X_2,x'_2} \otimes_{\mathcal{O}_{X_2,x'_2}} A', \qquad x \mapsto x \otimes 1$$

is injective from faithfully flatness. Therefore $\mathcal{O}_{X_2,x'_2} \to A'$ is an isomorphism for all $x'_2 \in X_2$, therefore $\mathcal{O}_{X_2} \to (u_*\mathcal{O}_{X_1})$ is an isomorphism, we are done. □

Proposition 2.16. Let (S,γ) be a pointed connected Noetherian scheme and X a connected etale covering over S. Let $X(\gamma)$ be the set of geometric points of X lying over γ and let $G = \mathrm{Aut}(X/S)^\circ$, we first claim that G is a finite group and the following conditions are equivalent:

1. $X/G \simeq S$, that is X is an Galois covering over S.
2. G acts transitively on $X(\gamma)$.
3. G and $X(\gamma)$ has the same number of elements.

Proof. Let's first show that G is finite. We first notice that $X(\gamma)$ is finite as $X \to S$ is etale thus quasi-finite. It's clear how G acts on $X(\gamma)$, i.e. if α is a geometric point over γ, then $g\alpha = g \circ \alpha$. Then let α be a geometric of X lying above γ, then recall that since X is etale finite over S, if we assume $g_1\alpha = g_2\alpha$, then

$g_1 = g_2$, therefore we know that $\#G$ is just the number of element in the orbit of α in $X(\gamma)$. Since $X(\gamma)$ is itself a finite set, we conclude that G is finite. Then it's easy to see that 2 and 3 are equivalent.

Let's prove that $1 \Rightarrow 2$ first. Recall that we have the canonical factorization:

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] & & X/G \arrow[dl] \\
& S &
\end{tikzcd}$$

Our assumption in 1 is that $X/G \to S$ is an isomorphism, then let $\beta : s \to X/G$ be a geometric point in X/G lying above γ in S. If α_1, α_2 are two geometric points of X lying over γ, then we know that they also lies over β. Let x_1, x_2 be the image of α_1, α_2 in X, recall that this implies that x_1, x_2 is in the fiber of the image y of β. Therefore we know that there is an element $g_1 \in G$ such that $g_1 x_1 = x_2$. Then we will have the following diagram:

$$\begin{tikzcd}
k(y) \arrow[r] \arrow[d] \arrow[dr, "\beta^\sharp"] & k(x_2) \arrow[d, "\alpha_2^\sharp"] \\
k(x_1) \arrow[r, "\alpha_1^\sharp"'] & k(s)
\end{tikzcd}$$

i.e. the diagram showing α_1, α_2 as over β and $k(x_1) \to k(x_2)$ is the field extension induced by g_1. Recall that $k(x_1)$ is a normal algebraic extension, then we know that the two $k(y)$-extension of $k(x_1)$ to $k(s)$, i.e. $\alpha_1^\sharp$ and $\alpha_2^\sharp \circ g_1^\sharp$ have the same image in $k(s)$. Therefore there is a $\gamma \in \mathrm{Gal}(k(x_2)/k(y))$ such that $\alpha_2^\sharp \circ g_1^\sharp \circ \gamma = \alpha_1^\sharp$.

$$\begin{tikzcd}
k(y) \arrow[r] \arrow[d] \arrow[dr, "\beta^\sharp"] & k(x_2) \arrow[d, "\alpha_2^\sharp"] \\
k(x_1) \arrow[r, "\alpha_1^\sharp"'] \arrow[ur] & k(s)
\end{tikzcd}$$

Now recall that there is some $g_2 \in G_{x_2}$ such that $\alpha_2^\sharp \circ g_1^\sharp \circ g_2^\sharp = \alpha_1^\sharp$, therfore $g_1 \circ g_2$ takes α_2 to α_1, therefore G acts transitively on the set of geometric points of X over β. However since $X/G \to S$ is an isomorphism and β is mapped to γ, we know that this set is $X(\gamma)$, therefore we are done.

Now let's assume 2 and prove that the morphism $X/G \to S$ is an isomorphism. Recall that we proved that in this case $X \to X/G$ and $X/G \to S$ are both etale covering spaces. Let' first consider $X/G(\gamma)$ and I claim that there is only one element in this set. To see why, suppose that we have two distinct elements in $X/G(\gamma)$, then first notice that they can be lifted to $X(\gamma)$ as $X \to X/G$ is etale surjective, thus the induced morphism on residue fields finite separable. I claim that these two liftings are not in the same orbit of G.

Suppose that α_1, α_2 are elements in $X(\gamma)$ that is in the same orbit. Then I claim that their image in X/G will be the same. Suppose that there is g such that $g\alpha_1 = \alpha$, then we have the following diagram:

$$\begin{tikzcd}
& k(y) \arrow[dl, hook] \arrow[dr, hook] & \\
k(x_1) \arrow[rr, "g^\sharp"] \arrow[dr, hook] & & k(x_2) \arrow[dl, hook] \\
& k(\alpha_1) = k(\alpha_2) &
\end{tikzcd}$$

The commutivity of the bottom triangle is the fact that g take α_1 to α_2. The commutivity of the top triangle is the fact that x_2, x_2 are in the same orbit and over y. This contradicts our assumption that G acts transitively on $X(\gamma)$.

Then consider:

$$\begin{tikzcd} X/G \arrow[rr] \arrow[dr] & & S \arrow[dl, "\mathrm{Id}"] \\ & S & \end{tikzcd}$$

we know that $S(\gamma)$ only has oe point, so is $X/G(\gamma)$, therefore the map $X/G \to S$ induces an isomorphism, since $X/G, S$ are all connected Noetherian and $X/G \to S$ are etale coverings. $\square$

Proposition 2.17. Let S be a connected Noetherian scheme and let X be a connected etale covering of S. Then any S-morphism $u : X \to X$ is an isomorphism.

Proof. Any such u is finite etale, therefore $u(X)$ is both open and closed, then since X is connected we know that $u(X) = X$. Let γ be a geometric point of S and let α be a geometric point of X lying above γ. We will let x be the image of α and there will be $x' \in X$ such that $u(x') = x$. Since u is etale, the corresponding residue field extension is finite and separable, therefore there will be a geometric point α' of X, such that $u \circ \alpha' = \alpha$. Therefore we know that $X(\gamma) \to X(\gamma)$ is surjective. Note that $X(\gamma)$ is finite, therefore this morphism is injective, thus bijective. $\square$

Proposition 2.18. Let (S, γ) be a pointed connected Noetherian scheme, and let (Y, β) be a pointed etale covering space of (S, γ). Then there is a pointed Galois etale covering space (X, α) of (S, γ) dominating (Y, β), i.e. there is a S-morphism $(X, \alpha) \to (Y, \beta)$.

Proof. Recall that there are only finitely many points above the image of γ, then since the field extension is finite separable, then there are only finitely many embeddings to the separable closure, therefore there are only finitely many geometric points over γ, say $\beta_1, \cdots, \beta_k$. Consider the k-fold product of Y over S, i.e. $Y^k = Y \times_S Y \cdots \times_S Y$.

We know that $\beta_1, \cdots, \beta_k$ define a geometric point α of Y^k. Recall that Y is Noetherian as well, so is Y^k. Then we can take the connected component of Y^k containing the image of α to be X. Then we only need to prove that $X \to S$ is a Galois covering. The morphism $X \to S$ is $X \hookrightarrow Y^k \to Y \to S$, and we know that $Y^k \to Y$, as the composition of several finite etale morphisms is finite etale. Since $X \hookrightarrow Y^k$ is a connected component, then $X \hookrightarrow Y^k$ is finite etale as well. Therefore $X \to S$ is finite etale.

Recall that to show $X \to S$ is a Galois covering, we only need to show that for any geometric point α', there is a S-morphism $u : X \to X$ such that $u \circ \alpha = \alpha'$. Notice that $p_i \circ j \circ \alpha = \beta_i$ where p_i is the canonical projection to the i-th component. Notice that $p_i \circ j$ are different morphisms from X to Y since we know that after we compose with α, we get different β_i, therefore we know that $p_i \circ j \circ \alpha'$ will all be different as there is only one S-morphism between two pointed morphisms in this case. Therefore we know that $p_i \circ j \circ \alpha'$ is just a permutation of $\beta_1, \cdots, \beta_k$. So there will be a permutation of $1, \cdots, k$ such that $p_i \circ j \circ \alpha = p_{\sigma(i)} \circ j \circ \alpha$ and let:

$$v : Y^k \to Y^k$$

be the morphism such that $p_i \circ v = p_{\sigma(i)}$. This implies that:

$$p_i v j \alpha = p_{\sigma(i)} j \alpha = p_i j \alpha'$$

Then we know that $vj\alpha = \alpha$. Note that v is an isomorphism of Y^k and the image of v of X is another connected component of Y^k. Since we know that that image contains the image of α', then it's just X. Therefore v induces an S-automorphism of X such that $u\alpha = \alpha'$, we are done. $\square$

Using what we prove above, we can finally define the **etale fundamental group**. Let (S, γ) be a pointed connected Noetherian scheme and let $\mathbf{Et}(S)$ be the category whose objects are etale covering spaces of S and morphisms are S-morphisms between these covering spaces. We have a functor F from $\mathbf{Et}(S)$ to the category of finite sets that maps each etale covering space X to the set $X(\gamma)$ of geometric points of X lying over γ. We will call this functor F the **fiber functor** for (S, γ).

Let I be the opposite category of the category of pointed connected Galois etale covering spaces of (S, γ). Then for each $i \in \mathrm{Obj}(I)$, we will denote the corresponding pointed Galois etale covering by (X_i, α_i). Given two objects i, j in I, there is a morphism $j \to i$ iff there is a S-morphism $f_{ij} : (X_i, \alpha_i) \to (X_j, \alpha_j)$. Such a morphism is unique if it exists by the results we proved before. The above dominating result tells us that the category I satisfies the conditions of a directed system. For each $X \in \mathrm{obj}\mathbf{Et}(S)$, consider the directed limit:

$$\varinjlim_{i \in \mathrm{ob} I} \mathrm{Hom}_S(X_i, X)$$

Since if $i \to j$ is a arrow, we have a morphism $X_j \to X_i$, thus a morphism $\mathrm{Hom}_S(X_i, S) \to \mathrm{Hom}_S(X_j, S)$. Then consider the map:

$$\varinjlim_{i \in \mathrm{ob} I} \mathrm{Hom}_S(X_i, X) \to F(X), \qquad f \mapsto f \circ \alpha_i, \quad f \in \mathrm{Hom}_S(X_i, X)$$

I will let you check this is indeed well defined. Then I claim that this is actually bijective. To see why, since S is connected Noetheiran, for any $\alpha \in F(X)$, we have a pointed etale covering (X, α), we have something that is connected etale covering (Y, α') of (X, α) dominating (X, α) and (S, γ). Moreover (Y, α') is Galois over (S, γ). This shows that this is surjective. To show this is injective, suppose that $f \in \mathrm{Hom}_S(X_i, X)$ and $g \in \mathrm{Hom}_S(X_j, X)$ is mapped to $f \circ \alpha_i = g \circ \alpha_j$. Then consider the following diagram:

$$\begin{tikzcd}
 & (X_k, \alpha_k) \arrow[dl] \arrow[dr] \arrow[dd, bend right] & \\
(X_i, \alpha_i) \arrow[r, "f"] \arrow[dr] & X \arrow[d] & (X_j, \alpha_j) \arrow[l, "g"'] \arrow[dl] \\
 & (S, \gamma) &
\end{tikzcd}$$

Then we know that after we consider the map from (X_k, α_k), the uniqueness result implies that $(X_k, \alpha_k) \to (X_i, \alpha_i) \circ f = (X_k, \alpha_k) \to (X_i, \alpha_i) \circ g$, therefore $f = g$. If you are familiar with the category theoretic terminology, this situation is called that F is **pro-represented** by the inverse system $(X_i)_{i \in \mathrm{obj} I}$ denoted by $\varprojlim_{i \in \mathrm{Obj} I} X_i$. Here we by $\varprojlim_{i \in \mathrm{Obj} I} X_i$, we don't need this inverse limit to actually exist, this is just a notation.

Let $f_{ij} : (X_i, \alpha_i) \to (X_j, \alpha_j)$ be a morphism $j \to i$ in I. We have bijections for $v = i, j$:

$$\mathrm{Aut}(X_v/S) \to F(X_v), \qquad \sigma \mapsto \sigma(\alpha_v)$$

This is indeed a bijection. It's injective as we know that X_v is connected, therefore there is only at most one S morphism between (X_v, α_v) and $(X_v, \sigma\alpha_v)$, i.e. σ. To prove it's surjective, recall that the category I consists of pointed Galois etale covering spaces of (S, γ), therefore we know that the automorphism group $\mathrm{Aut}(X_v/S)$ acts transitively on all the the geometric points above γ. Under this bijection, we can identify the natural map:

$$F(X_i) \to F(X_j), \qquad \alpha \mapsto f_{ij} \circ \alpha$$

with:

$$\phi_{ij} : \mathrm{Aut}(X_i/S) \to \mathrm{Aut}(X_j/S)$$

Let $\sigma \in \mathrm{Aut}(X_i/S)$, consider what is $\phi_{ij}(\sigma)$, we know that σ is mapped to $\sigma(\alpha_i)$ in $F(X_i)$, then is mapped to $f_{ij}\sigma(\alpha_i)$, therefore we know that $\phi_{ij}(\sigma)(\alpha_j) = f_{ij}\sigma(\alpha_i)$, in particular:

$$\phi_{ij}(\sigma) f_{ij}(\alpha_i) = f_{ij}\sigma\alpha_i$$

Then we know that $\phi_{ij}(\sigma) f_{ij} = f_{ij}\sigma$, that is the following diagram is commutative:

$$\begin{tikzcd}
X_i \arrow[r, "\sigma"] \arrow[d, "f_{ij}"'] & X_i \arrow[d, "f_{ij}"] \\
X_j \arrow[r, "\phi_{ij}(\sigma)"'] & X_j
\end{tikzcd}$$

Then I claim that $(\mathrm{Aut}(X_i/S), \phi_{ij})_{i\in I}$ forms an inverse system. (Here there are again some set theoretic issues among whether objects of I forms a set, we will ignore these sort of questions.) To check this, we know that if we have $j \to i$ then there is a $\phi_{ij} : \mathrm{Aut}(X_i/S) \to \mathrm{Aut}(X_j/S)$. Then we need to check the composition is compatible, i.e. $\phi_{ij}(\sigma) = \phi_{jk}\phi_{ij}(\sigma)$. Again, this is because both S-automorphisms maps α_k to $f_{jk}f_{ij}\sigma(\alpha_i)$ and such automorphism is unique.

Definition 2.19. Let (S, γ) be a pointed connected Noetherian scheme, we will define the **fundamental group** of (S, γ) to be:

$$\pi_1(S, \gamma) = \varprojlim_{i\in I} \mathrm{Aut}(X_i/S)^\circ$$

where the transition functions between $\mathrm{Aut}(X_i/S)^\circ \to \mathrm{Aut}(X_j/S)^\circ$ is still defined by ϕ_{ij}.

By the previous remark, if we ignore the set theoretic issues, we are still think of the group as a subgroup of the product group $\prod_i \mathrm{Aut}(X_i/S)^\circ$.

One important property of this inverse system is that the transition maps are all **surjective**. This will be a very important property later to use when we discuss **pro-finite groups**:

Proposition 2.20. For all i, j, $\phi_{ij} : \mathrm{Aut}(X_i/S)^\circ \to \mathrm{Aut}(X_j/S)^\circ$ are all surjective.

Proof. It suffice to prove that the map $F(X_i) \to F(X_j)$ are all surjective. Let β be a geometric point of X_j lying above γ in S. Then since $X_i \to X_j$ is finite etale and X_i, X_j are connected, the morphism is surjective. We also know that the induced morphism on the residue field is finite separable. Let the image of β be x_j, we know that there is a x_i lying above x_j, therefore we can embed $k(x_i)$ into the separable closure we choose for γ, we know that this defines a geometric point of X_i lying above γ and it suffices for our purpose. $\square$

Notice that for each i, we know $\mathrm{Aut}(X_i/S)$ acts on $\mathrm{Hom}_S(X_i, X)$ on the right, we know that the opposite group $\mathrm{Aut}(X_i/S)^\circ$ acts on $\mathrm{Hom}_S(X_i, X)$ on the left.

Proposition 2.21. $\pi_1(S, \gamma)$ acts on $\varinjlim_i \mathrm{Hom}_S(X_i, X)$.

Proof. We are still thinking of the fundamental group as a subgroup of the product groups whose entries are compatible under the transition maps. Then to define the action of a tuple $(\sigma_i)_{i\in I}$ on something in the direct limit say represented by $g_i \in \mathrm{Hom}_S(X_i, X)$. We want to define it just by $g_i \circ \sigma_i$. We need to check this is well defined, i.e. for different representatives of g_i, we get the same result.

Let $j \to i$ be an arrow in I, representing a morphism $f_{ij} : X_i \to X_j$. We know that there is a morphism $\mathrm{Hom}_S(X_j, X) \to \mathrm{Hom}_S(X_i, X)$. Then we know that if this element in the direct limit is represented by $g_j : X_j \to X$, then we know that $g_i = g_j \circ f_{ij}$. Then we need to prove:

$$g_i \circ \sigma_i = g_j \circ \phi_{ij}(\sigma_i)$$

That is to say:

$$g_j \circ f_{ij} \circ \sigma_i = g_j \circ \phi_{ij}(\sigma_i) \circ f_{ij}$$

We know that $f_{ij} \circ \sigma_i = \phi_{ij}(\sigma_i) \circ f_{ij}$ by our definition, therefore we are done. $\square$

Since we have an isomorphism $\varinjlim_i \mathrm{Hom}_S(X_i, X) \to F(X)$, we know that the fundamental group $\pi_1(S, \gamma)$ also acts on $F(X)$. For some $\alpha \in F(X)$, let's assume that it comes from some $f(\alpha_i)$ for some (X_i, α_i) and $f \in \mathrm{Hom}_S(X_i, X)$. Then we know that the action of something (σ_i) in $\pi_1(S, \gamma)$ applied to f is $f \circ \sigma_i$, this is mapped to $f \circ \sigma_i(\alpha_i)$ while the original element is $f(\alpha_i)$.

Recall that each $F(X)$ is finite and $\varinjlim_i \mathrm{Hom}_S(X_i, X) \to F(X)$ is an isomorphism, then we know that there is an i, such that $\mathrm{Hom}_S(X_i, X) \to F(X)$ is a surjective morphism. Then we know that the action of $\pi_1(S, \gamma)$ on $F(X)$ factors through $\mathrm{Aut}(X_i/S)^\circ$. I claim that the projection $\pi_1(S, \gamma) \to \mathrm{Aut}(X_i/S)^\circ$ is surjective with $\mathrm{Aut}(X_i/S)^\circ$ finite. The projection is surjective since each $\mathrm{Aut}(X_i/S)^\circ \to \mathrm{Aut}(X_j/S)^\circ$ is surjective, therefore given any element in $\mathrm{Aut}(X_i/S)^\circ$, we can produce a compatible tuple. The reason that $\mathrm{Aut}(X_i/S)^\circ$ is finite is because the number of elements in this group is the same as the number of elements in the orbit of α_i, which is finite.

Proposition 2.22. If we give $\mathrm{Aut}(X_i/S)^\circ$ for each i $F(X)$ the discrete topology and $\pi_1(S,\gamma)$ the product then the subspace topology. The I claim that $\pi_1(S,\gamma)$ is a topological group and the topological group $\pi_1(S,\gamma)$ acts continuously on $F(X)$.

Proof. Recall that if G_i's are topological groups, then the product group with the product topology is a topological group. If G is a topological group, then any subgroup H with the subspace topology is a topological group. Since $\mathrm{Aut}(X_i/S)^\circ$ are given the finite topology, they are always topological groups, therefore $\pi_1(S,\gamma)$ is indeed a topological group.

Then consider the projection $\pi_1(S,\gamma) \to \mathrm{Aut}(X_i/S)^\circ$ where $\mathrm{Aut}(X_i/S)^\circ$ is the group such that $\mathrm{Hom}_S(X_i, X) \to F(X)$ is surjective. I will let you check this projection is continuous and surjective. Then to check that $\pi_1(S,\gamma)$ acts continuous on the topological space $F(X)$, we need to check that the map of topological spaces induced by the action is continuous:

$$\pi_1(S,\gamma) \times F(X) \to F(X)$$

Since $F(X)$ is discrete, I will let you check that this is equivalent to check that for each $\alpha \in F(X)$, the map:

$$\pi_1(S,\gamma) \to F(X)$$

is continuous. Recall that this map factors through:

$$\pi_1(S,\gamma) \to \mathrm{Aut}(X_i/S)^\circ \to F(X)$$

where $\mathrm{Aut}(X_i/S)^\circ \to F(X)$ comes from the continuous action map:

$$\mathrm{Aut}(X_i/S)^\circ \times F(X) \to F(X)$$

this is indeed continuous since $F(X)$ and $\mathrm{Aut}(X_i/S)^\circ$ are all discrete. ☐

Proposition 2.23. An etale covering space X of a pointed connected Noetherian scheme is connected iff the fundamental group $\pi_1(S,\gamma)$ act transitively on $F(X)$ where F is the fiber functor.

Proof. Let $C_1, \cdots, C_k$ be all connected components of X. Then we know that $F(X) = \coprod_{v=1}^k F(C_v)$ and I claim that each $F(C_v)$ is stable under the action of $\pi_1(S,\gamma)$. To prove this, recall that if α is a geometric point of X lying above γ with image in C_i, then we know that it's the image of some morphism f in $\mathrm{Hom}_S(X_j, X)$, i.e. $\alpha = f \circ \alpha_j$. The action of any element in the fundamental group, say with the element σ_j in the X_j-th slot is $f \circ \sigma_j(\alpha_j)$. Recall that the morphism f is again finite an etale, therefore we know that the image of f is one of the connected components of X, and thus C_i. That is we have:

$$\begin{tikzcd}
\alpha_j \arrow[r] & X_j \arrow[r, "\sigma_j"] \arrow[drr] & X_j \arrow[r, "f"] \arrow[dr] & C_i \arrow[r, hook] \arrow[d] & X \arrow[dl] \\
 & & & S &
\end{tikzcd}$$

Therefore you see that after the action, this geometric point still lies in C_i. This proves that if X is not connected, then the $\pi_1(S,\gamma)$ action is not transitive.

Conversely, let's assume that X is connected and prove $\pi_1(S,\gamma)$ indeed acts transitively. Consider the following diagram:

$$\begin{tikzcd}
\alpha_i' \arrow[r] \arrow[d] & \alpha' \arrow[d] & \\
X_i \arrow[r, "f"] & X \arrow[r] & S \\
\alpha_i \arrow[r] \arrow[u] & \alpha \arrow[u] &
\end{tikzcd}$$

where $(X_i, \alpha) \to (S, \gamma)$ is a pointed Galios etale covering dominating (X, α). Since X is connected, we know that $X_i \to X$ is surjective, then we know that there is a geometric point over α'. Since $X_i \to S$ is Galois, we know that $\mathrm{Aut}(X_i/S)$ acts transitively on the geometric points of X_i lying above γ, therefore there is a σ_i such that $\sigma_i(\alpha_i) = \alpha_i'$, then we know that after we post-compose $X_i \to X$ we get α'. This is exactly the action of σ_i on the element α. ☐

Remark 2.24. Recall that we have proved a similar result before: if X is a connected etale covering of (S, γ), then we know that the group $\mathrm{Aut}(X_i/S)^\circ$ acts transitively on $X(\gamma) = F(X)$ iff X is a Galios covering of S.

In the above result, by passing somehow to the inverse limit of the groups $\mathrm{Aut}(X_i/S)^\circ$, i.e. the fundamental group $\pi_1(S, \gamma)$, we get the result that now we don't X to be Galois over S, it only need to be a connected etale covering and then we have $\pi_1(S, \gamma)$ acts transitively on $F(X)$. This result is going to look very similar to the classic result of covering spaces of topological spaces.

Theorem 2.25. Let (S, γ) be a pointed connected Noetherian scheme, then the fiber functor $F : X \mapsto X(\gamma)$ defines an equivalence of categories from the $\mathbf{Et}(S)$ to the category of finite sets which $\pi_1(S, \gamma)$ acts continuously on the left.

Proof. Recall that a functor is an equivalence of categories if it's fully-faithful and essentially surjective. Let's deal with the fully-faithful part first.

Suppose that X, Y are etale covering spaces of S we want to prove the following map is bijective:

$$\mathrm{Hom}_S(X, Y) \to \mathrm{Hom}_{\pi_1(S,\gamma)}(F(X), F(Y))$$

Clearly, a morphism $f : X \to Y$ gives us a morphism $\alpha \mapsto f(\alpha)$ in $\mathrm{Hom}_{\pi_1(S,\gamma)}(F(X), F(Y))$. The morphisms in $\mathrm{Hom}_{\pi_1(S,\gamma)}(F(X), F(Y))$ should be compatible with the group action, so let's check that the above morphims $\alpha \mapsto f(\alpha)$ is compatible with the group action. Suppose that α is the image of $f \in \mathrm{Hom}_S(X_i, X)$, then we have the sequence:

$$(X_i, \alpha_i) \xrightarrow{g} X \xrightarrow{f} Y$$

we know that suppose that $\sigma_i \in \mathrm{Hom}_S(X_i/S)$ then the action is $g\sigma_i(\alpha_i)$, then it's mapped to $f(g\sigma_i(\alpha_i))$. Then the action of σ_i in $f(\alpha)$ is $f \circ g \circ \sigma(\alpha_i)$, therefore indeed compatible with the group action.

First, we can assume that X, Y are connected since we can always decompose X, Y to $\coprod_i X_i$ and Y to $\coprod_j Y_j$ and the morphism $X \to Y$ always maps a connected component of X surjectively to a connected component of Y. Similarly, we know that under this connected component decomposition, we can decompose $F(X), F(Y)$ to $\coprod_i F(X_i), \coprod_j F(Y_j)$ such that each component in the union is stable under the $\pi_1(S, \gamma)$ action. Again, since these are with the discrete topology, we know that $\pi_1(S, \gamma)$ acts continuous on these components.

Therefore we have a diagram:

$$\begin{tikzcd}
\mathrm{Hom}_S(X, Y) \arrow[r] \arrow[d] & \mathrm{Hom}_{\pi_1(S,\gamma)}(F(X), F(Y)) \arrow[d] \\
\coprod_{i,j} \mathrm{Hom}_S(X_i, X_j) \arrow[r] & \coprod_{i,j} \mathrm{Hom}_{\pi_1(S,\gamma)}(F(X_i), F(Y_j))
\end{tikzcd}$$

This proves the assertion we can assume X, Y are connected. Moreover, when they are connected, the uniqueness result of morphisms between pointed etale coverings suggests that the map is injective.

To prove the map is surjective, fix $\alpha \in F(X)$ and $\beta \in F(Y)$, consider a pointed Galois etale covering (X_i, α_i) of (S, γ) dominating both (X, α) and (Y, β).

$$\begin{tikzcd}
 & (X, \alpha) \arrow[dr] & \\
(X_i, \alpha_i) \arrow[ur, "p"] \arrow[dr, "q"'] & & (S, \gamma) \\
 & (Y, \beta) \arrow[ur] &
\end{tikzcd}$$

Then since Y is connected, $q : X_i \to Y$ is surjective, then consider any morphism $\phi \in \mathrm{Hom}_{\pi_1(S,\gamma)}(F(X_1), F(Y))$, I claim that there is a morphism in $\mathrm{Hom}_S(X_i, Y) \to \mathrm{Hom}_{\pi_1(S,\gamma)}(F(X_i), F(Y))$ that is mapped to ϕ_i. First, consider $\phi(\alpha_i)$, since q is surjective, finite etale, there is a α_i' in $F(X_i)$ such that $q(\alpha_i') = \phi(\alpha_i)$. Recall that since $(X_i, \alpha_i) \to (S, \gamma)$ is Galios, there is a $\sigma \in \mathrm{Aut}(X_i/S)$ such that $\sigma(\alpha_i) = \alpha_i'$, then we know that $q\sigma(\alpha_i) = q(\alpha_i') = \phi(\alpha_i)$. Again, since $\pi_1(S,\gamma)$ acts transitively on $F(X_i)$, we know that the agreement of the induced map of $q \circ \sigma$ and ϕ on α_i implies that they agree globally. Therefore we have a surjective morphism:

$$\mathrm{Hom}_S(X_i, Y) \to \mathrm{Hom}_{\pi_1(S,\gamma)}(F(X_i), F(Y))$$

Now let $X_i(\alpha)$ be the set of geometric points of X_i lying above the geometric point α in X. Given two elements α_1, α_2 in $X_i(\alpha)$, we know that they are also in $X_i(\gamma)$, then since X_i is Galios over (S, γ), there is a σ in $\mathrm{Aut}(X_i/S)$ such that $\sigma\alpha_1 = \alpha_2$. This implies that σ is actually in $\mathrm{Aut}(X_i/X)$ as we have the following diagram:

$$\begin{tikzcd}
(X_i,\alpha_2) \arrow[r, "p"'] & (X,\alpha) \arrow[dr] & \\
& & (S,\gamma) \\
(X_i,\alpha_1) \arrow[uu, "\sigma"] \arrow[r, "p"] & (X,\alpha) \arrow[ur] &
\end{tikzcd}$$

and we know that $p \circ \sigma$ and p are both pointed morphisms over (S, γ) mapping (X_i, α_1) to (X, α), with X_i, X connected, then we know that the unique result implies $p \circ \sigma = p$.

Thenfore we know that $H = \mathrm{Aut}(X_i/X)$ acts transitively on $X_i(\alpha)$, therefore we know that $p : X_i \to X$ is isomorphic to the canonical morphism $X_i \to X_i/H$. In particular p is a Galios covering.

Now let $\phi \in \mathrm{Hom}_{\pi_1(S,\gamma)}(F(X), F(Y))$, consider the geometric point $\phi(\alpha)$, by the above surjectivity discussion, there is an S-morphism $f' : X_i \to Y$ such that $f'(\alpha_i) = \phi(\alpha)$. Then let any $h \in \mathrm{Aut}(X_i/X) \subset \mathrm{Aut}(X_i/S)$, I claim we have:

$$f' \circ h(\alpha_i) = h \cdot f'(\alpha_i) = h(\phi(\alpha)) = \phi(h(\alpha)) = \phi(\alpha) = f'(\alpha_i)$$

Here we abbreviated some notation. The first equality holds since the h action on $f'(\alpha_i)$ is just $f' \circ h(\alpha)$. The third equality $h(\phi(\alpha)) = \phi(h(\alpha))$ is because we know that ϕ is assumed to commute with the $\pi_1(S,\gamma)$ action and h can be viewed as an element in $\pi_1(S,\gamma)$. The the action of h on α is computed by $p \circ h(\alpha_i)$, however since h is in $\mathrm{Aut}(X_i/X)$, we know that $p \circ h = p$, therefore the action is trivial. This is why $\phi(h(\alpha)) = \phi(\alpha)$.

Then again the uniqueness result tells us $f' \circ h = f'$. Then consider the diagram:

$$\begin{tikzcd}
X_i \arrow[dr] \arrow[dd, "h"] & & \\
& X \arrow[dd, dashed, "f"] \arrow[dr] & \\
X_i \arrow[ur, "p"] \arrow[dr, shift left, "f'"] \arrow[dr, shift right, "q"'] & & S \\
& Y \arrow[ur] &
\end{tikzcd}$$

I claim that there is a S-morphism $f : (X, \alpha) \to (Y, \beta)$ such that $f \circ p = f'$. This is because we know that f' is invariant under the action of H, and X is the quotient of X_i by H, therefore the universal property tells us there is such a f.

Then $f(\alpha) = fp(\alpha_i) = f'(\alpha_i) = \phi(\alpha)$, and since $\pi_1(S,\gamma)$ acts transitively on $F(X)$, we conclude $f = \phi$. This shows the functor is indeed fully-faithful.

Then we need to show that it' essentially surjective. Let A be a finite discrete set on which $\pi_1(S,\gamma)$ acts continuously. Again, since A is discrete, we can consider it's orbits and assume that $\pi_1(S,\gamma)$ acts transitively on A. Then I claim that there is a pointed Galois covering (X_i, α_i) of (S, γ) such that the action of $\pi_1(S,\gamma)$ factors through $\mathrm{Aut}(X_i/S)^\circ$. To see the reason, consider the following result about topological groups.

Suppose that (G_i, ϕ_{ij}) is an inverse system of discrete topological groups such that transition maps are all continuous. Moreover we assume that for any finitely many $G_{i_1}, \cdots, G_{i_n}$, there is always a G_k that maps to all G_{i_k}. Then if the inverse limit $\varprojlim_j G_i$ continuously act on a finite discrete set A and the projections p_i to all G_i are all surjective, then if factors through some projection p_i to G_i. First, notice that a continuous action on A is just a continuous morphism $\rho : \varprojlim_j G_i \to \mathrm{Sym}A$ where the symmetric group of A is equipped with the discrete topology. I will let you check this. Then we say that it factors though G_i is the same as say that we have a continuous factorization:

$$\begin{tikzcd}
\varprojlim_i G_i \arrow[rr, "\rho"] \arrow[dr, "p_i"'] & & \mathrm{Sym}A \\
& G_i \arrow[ur] &
\end{tikzcd}$$

since G_i and $\mathrm{Sym}A$ are all discrete, we know that any $G_i \to \mathrm{Sym}A$ is continuous, thus we only need to find such a factorization.

To find such a factorization, consider the kernel of ρ, which is an open neighborhood of the identity in the inverse limit. Recall that the basis open neighborhoods of the identity of the identity is given by $\varprojlim G_i \cap \prod_i U_i$ where U_i are all open subsets containing the identity in each of the G_i and only finitely many U_i are not 0. I want to conclude that this open contains some the kernel of p_i for some i. Notice that suppose that this open neighborhood of the identity contains some basic open $\varprojlim G_i \cap \prod_i U_i$, then we know that in these U_i we only have finitely many ones that are not the whole G_i. We know that in our case these groups G_i are $\mathrm{Aut}(X_i/S)$, then we know that given finitely many such groups there are always some $\mathrm{Aut}(X_k/S)$ that maps to all of them. Then I claim that this implies that this implies that the kernel contains G_k. We also know that p_k to G_k is surjective and the kernel of p_k is contained in the kernel of ρ. We are done.

Then, we can assume that the action of $\pi_1(S, \gamma)$ factors through $\mathrm{Aut}(X_i/S)^\circ$. Then we know that since $\mathrm{Aut}(X_i/S)^\circ$ acts transitively on A, then let H be the stabilizer subgroup of say $a_1 \in A$, then there is an isomorphism of $\mathrm{Aut}(X_i/S)^\circ$-sets $\mathrm{Aut}(X_i/S)^\circ/H \to A$. Intuitively, the left cosets of $\mathrm{Aut}(X_i/S)^\circ/H$ are those consisting elements of $\mathrm{Aut}(X_i/S)^\circ$ which maps a_1 to a_i. I will let you check that this is indeed an isomorphism of $\mathrm{Aut}(X_i/S)^\circ$-sets, for example you can check it by choosing representatives in each coset.

Then consider $X = X_i/H$, we ask what is $F(X)$. First recall that $F(X_i)$ has the same number of elements as $\mathrm{Aut}(X_i/S)$. Now I claim that after we quotient by H to get $X = X_i/H$, $F(X)$ is isomorphic to $\mathrm{Aut}(X_i/S)^\circ/H$. To see why, for each point α in $F(X)$, we have something β in $F(X_i)$ that is mapped to α. Therefore we have a surjective map:

$$F(X_i) \to F(X)$$

Induced by $X \to X/H$ in the following diagram:

$$\begin{tikzcd}
X_i \arrow[rr, "q"] \arrow[dr] & & X = X_i/H \arrow[dl] \\
& S &
\end{tikzcd}$$

Recall that for each $x \in X$, let $G = \mathrm{Aut}(X_i/S)^\bullet$, we know that the inertia group of i is trivial. Then we know that $\mathrm{Aut}(X_i/X)^\circ = H$. Moreover, let's choose a α_i in $F(X)$, then the morphism:

$$G \to F(X_i), \qquad \sigma \mapsto \sigma \circ \alpha_i$$

is a bijection this identifies G and $F(X_i)$ as a G-set. Therefore we are actually using (X_i, α_i) Then consider the following surjection:

$$F(X_i) \to F(X)$$

defined by composing with q. Then we know that since H acts on $F(X_i)$ and we know that geometric points connected by elements in H maps to the same geometric point in $F(X)$, therefore we know that this morphism factors through:

$$F(X_i)/H \to F(X)$$

This is again a surjection and let's prove that it's injective. Suppose that α_1, α_2 in X_i is mapped to the same geometric point in X, since $X \to X_i/H$ is a Galois covering, there is something in H that takes α_1 to α_2. Therefore this is is also injective. Therefore we get an isomorphism $G/H \to F(X)$.

Recall that $\pi_1(S,\gamma)$ acts on X by acting on $\mathrm{Hom}_S(X_i, X) \to F(X), \quad f \mapsto f(\alpha)$. We know that this is surjective here since we know that there are always $\sigma \in \mathrm{Aut}(X_i/S)$ such that for any element $\beta \in F(X)$ we have $q \circ \sigma$ is mapped to β.

Recall that A has been identified as a G-set as G/H and then $F(X_i)/H$, then if we can show that $F(X_i)/H \to F(X)$ is a morphism that is compatible with the $\pi_1(S,\gamma)$ action, we are done. Let's think about what is the G-action of $F(X_i)$. Recall that it's action is inherited from G and we know that the left action of G on itself is that $g_1 \cdot g_2 = g_2 \circ g_1$, then we know that the left action of G on $F(X_i)$ is if $\alpha_j = g_j \circ \alpha_i$, then $\sigma \cdot (\alpha_j) = g_j \circ \sigma \circ \alpha_i$.

Then we need to show that say the class of $\alpha_j = g_j \circ \alpha_i$ is mapped to the class of β in $F(X)$. Let σ first act on α_i, we know that it goes to the class of $g_j \circ \sigma \circ \alpha_i$, then mapped to $p \circ g_j \circ \sigma \circ \alpha_i$. Then if we let σ act on $p(g_j \circ \alpha_i)$, we know that we get $p \circ g_j \circ \sigma \circ \alpha_i$, therefore indeed compatible, we are done. □

Remark 2.26. One mistake one could make is that they will think this action is a little bit weird to look at in the beginning since this seems is saying that for $\beta \in F(X)$, we are picking a lift α and define the action to be $q \circ g \circ \alpha$, but when you check that this is a well define $\mathrm{Aut}(X_i/S)$-morphism, you went to trouble as we know that if α' is another lift, we only know that $\alpha' = h \circ \alpha$, this doesn't implies $q \circ g \circ \alpha = q \circ g \circ \alpha' = q \circ gh \circ \alpha$. You will notice that this is true when H is normal while we have no guarantee on the normality of H.

The reason this looks weird is that even though this may looks like a natural way to define the action, this is not how this action is defined at all! Following the correct definition of the action, you won't find this weird.

Proposition 2.27. Let S be a Noetherian scheme and let γ, γ' be two geometric points in S, if S is connected then we have an isomorphism:

$$\pi_1(S,\gamma') \to \pi_1(S,\gamma)$$

and any such isomorphisms differ by an inner automorphism of $\pi_1(S,\gamma)$, i.e. the automorphism you get by conjugating one element in $\pi_1(S,\gamma)$.

Remark 2.28. This is again very like what we learn in the classical theory of covering spaces of topological spaces, where we get if a topological space is connected, then the fundamental group does not depend on the base point.

Proof. Let I be the opposite category of the category of pointed Galois covering of (S,γ) and let J be the opposite category of the category of pointed Galois covering of (S,γ'). Then since $f_{ij} : (X_i,\alpha_i) \to (X_j,\alpha_j)$ is surjective, we may find $\alpha_i \in X_i(\gamma')$ for each i, such that for each pair i,j we have $f_{ij}(\alpha_i') = \alpha_j'$. (Here again there are some set-theoretic issues involved to prove the existence, we don't want to deal with that). We want to find such α_i''s such that we have a inverse system $\{(X_i,\alpha_i'), f_{ij}\}$ that is again indexed by I.

Then Let (X,α') be any pointed etale covering space of (S,γ'), we know that if we consider the connected component X' of X containing the image of α', we know that the morphism to S is surjective, therefore there will be a geometric point α of X lying above γ whose image is in X'. Now consider (X,γ), we know that there is some $p : (X_i,\alpha_i) \to (X,\gamma) \to (S,\gamma)$ such that $X_i \to S$ is Galois. Notice that the image of X_i is onto X', therefore there will be a α_i'' over α'. Again since X_i is Galois over S, there is a $g \in \mathrm{Aut}(X_i/S)$ such that $g \circ \alpha_i' = \alpha_i''$. Then pg is a morphism from (X_i,α_i') to (X,α'). Then in particular, for any element in J, it's dominated by some (X_i,α_i'), therefore the family $\{(X_i,\alpha_i')\}$ is cofinal in the category J.

Then we have the isomorphism:

$$\pi_1(S,\gamma') = \varprojlim_{j} \mathrm{Aut}(X_j/S)^\circ \simeq \varprojlim_{\{X_i,\alpha_i'\}} \mathrm{Aut}(X_i/S) = \pi_1(S,\gamma)$$

where the isomorphism holds since the system $\{X_i,\alpha_i'\}$ is cofinal in the category J.

Then suppose that we have another family (X_i,α_i'') such that $f_{ij}(\alpha_i') = \alpha_j''$, we get another isomorphism:

$$\pi_1(S,\gamma') \to \pi_1(S,\gamma)$$

Recall that α_i', α_i'' are both above γ', then we know that there will be $g_i \in \mathrm{Aut}(X_i/S)$ such that $g_i \circ \alpha_i' = \alpha_i''$. First we know that we have:

$$\phi_{ij}(g_i) \circ \alpha_j' = \phi_{ij}(g_i) \circ f_{ij} \circ \alpha_i' = f_{ij} \circ g_i \circ \alpha_i' = f_{ij} \circ \alpha_i'' = \alpha_j''$$

Then notice that g_j and $\phi_{ij} g_i$ both take α'_j to α''_j over γ', therefore by uniqueness, we know that are equal. Therefore we know that (g_i) is an element in $\pi_1(S, \gamma)$.

Then let σ be an element in $\pi_1(S, \gamma')$, suppose that it's mapped to (σ_i) using the isomorphism defined by (α'_i). Then we know that if f is in $\mathrm{Aut}(X_j/S)$ it's mapped to σ_i in $\mathrm{Aut}(X_i/S)$, then we have the following diagram:

$$\begin{tikzcd} X_i \arrow[r, "\sigma_i"] \arrow[d] & X_i \arrow[d] \\ X_j \arrow[r, "f"'] & X_j \end{tikzcd} \qquad \begin{tikzcd} X_i \arrow[r] \arrow[d, "g_i"'] & X_i \arrow[d, "g_i"'] \\ X_i \arrow[r, "\sigma_i"'] \arrow[d] & X_i \arrow[d] \\ X_j \arrow[r, "f"'] & X_j \end{tikzcd}$$

where the second diagram exists because we need to move α'_i to α''_i, this is where the conjugation comes from, and we know the image is $(g_i^{-1} \sigma_i \circ g_i)$, therefore is indeed an inner isomorphism. $\square$

Proposition 2.29. Let K be a field and Ω a separably closed field containing K, $\gamma : Spec\Omega \to SpecK$ the corresponding geometric point and K_s the separable closure of K in Ω. Then we have a canonical isomorphism:

$$\pi_1(SpecK, \gamma) = \mathrm{Gal}(K_s/K)$$

Proof. Let $\{K_i\}$ be the family of finite Galois extensions of K contained in Ω, and let $\alpha_i : Spec\Omega \to SpecK_i$ be the corresponding geometric points.

Consider pointed connected Galois etale coverings of $(SpecK, \Omega)$, which are just finite separable extensions L of K in Ω. Notice that the automorphisms are just $\mathrm{Gal}(L/K)$ and indeed K is the fixed subring of this field, therefore is indeed Galois. **** completed later* $\square$

References