Appendix 15 for Math 632: 2025 Winter Notes

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In this part of the notes, we spend some time discussing an important construction, the blow-up of a scheme along a closed subscheme cut out by a finite type ideal sheaf. The reason we discuss it after the differentials is because the interpretation of many classical examples is better understood using differentials.

We start by giving an informal motivating example, the blow-up of the affine plane at the origin. After this, we will define the blow-up using universal property and see discuss why we use this such a definition assume it exists. Finally, we show that it really exists by constructing it rather explicitly and then see that it's actually projective. We will also compute several examples as applications, which also let you see that in many cases, we can indeed compute the blow-up rather explicitly.

1 Motivating Example

Recall that we have previously discussed the following example: Consider the Cartesian product $\mathbb{A}^2_k \times_k \mathbb{P}^1_k$, recall that it's covered by 2 $\mathbb{A}^3_k$. Let's use x, y as the coordinates on $\mathbb{A}^2_k$ and u, v as the coordinates on $\mathbb{P}^1_k$. Then we can define a closed subscheme that is informally cut out by $xv = yu$. We want to mention it here again since this, as you will see later, is the first example we can compute by hand. Another reason is that we want to fill in some of the details we didn't fill in before.

To define it properly, we need to consider the invertible sheaf $p_1^*\mathcal{O}_{\mathbb{A}^2_k} \otimes p_2^*\mathcal{O}(1)$ and we define the closed subscheme to be the one cut out by the global section $p_1^*x \otimes p_2^*v - p_1^*y \otimes p_2^*u$. Here we mention again that the pullback section of a section x of a quasicoherent $\mathcal{O}_X$ module $\mathcal{F}$ is define to be be pullback morphism of $\mathcal{O}_X \to \mathcal{F}$ where $1 \mapsto x$, but here we explicitly requires the pullback functor to be defined by $f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Y$.

Defined this way, we can consider the following diagram:

$$\begin{tikzcd}
\mathbb{A}^1_k \times_k \mathbb{A}^2_k \arrow[r, hook] \arrow[d] & \mathbb{P}^1_k \times_k \mathbb{A}^2_k \arrow[r, "p_1"] \arrow[d, dashed, "p_2"] & \mathbb{A}^2_k \arrow[d] \\
\mathbb{A}^1_k \arrow[r, hook] & \mathbb{P}^1_k \arrow[r] & k
\end{tikzcd}$$

We know for example, $p_1^*x \otimes p_2^*v$ is defined to be the tensor product of the morphism p_1^*x and p_2^*v, then we know that since we have a canonical isomorphism between pullback of tensor and tensor of pullback, we know that after we pullback this tensor to $\mathbb{A}^1_k \times \mathbb{A}^2_k$, it's isomorphic to the morphism obtained by tensoring the pullback of x and v along $\mathbb{A}^1_k \times \mathbb{A}^2_k \to \mathbb{A}^2_k$ and $\mathbb{A}^1_k \times \mathbb{A}^2_k \to \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$, which I will let you check, is canonically isomorphic to the section $x\frac{v}{u}$, similarly, you can argue that the pullback of $p_1^*y \otimes p_2^*u$ is y, then since pullback is additive, we can conclude that the pullback of $p_1^*x \otimes p_2^*v - p_1^*y \otimes p_2^*u$ to $\mathbb{A}^1_k \times \mathbb{A}^2_k = Speck[\frac{v}{u}, x, y]$ is the section $x\frac{v}{u} - y$. Recall that sections of isomorphic line bundles define the same closed subscheme, therefore we know how to define the closed subscheme in $\mathbb{A}^1_k \times \mathbb{A}^2_k$.

But all being said, this is just trying to make the definition rigorous enough, what is important is still how to understand this closed subscheme of $\mathbb{P}^1_k \times \mathbb{A}^2_k$. As what we have discussed before, we can think of this closed subscheme as the subset of $\mathbb{A}^2_k \times \mathbb{P}^1_k$ defined by:

$$\{(p \in \mathbb{A}^2, [l] \in \mathbb{P}^2_k) : p \in l\}$$

To give a concrete sense of what happens here, let's consider the following diagram:

suppose that $[u : v]$ where $u, v \in k$ is a point in $\mathbb{P}^1_k$, to satisfies the condition $xv - yu = 0$, we know that this is actually defines a line in $\mathbb{A}^2_k$, but what is important here is that the line always contain the origin no matter what point $[u : v]$ you choose. What we are really doing here is that we are computing the fiber of the point $[u : v]$ in the blow-up.

For the notation, let's use $\mathrm{Bl}\mathbb{A}^2_k$ to denote the blow-up, then we can consider the map from the blow-up to $\mathbb{A}^2_k$, and for a point $(a, b) \in \mathbb{A}^2_k$, we can consider it's fiber. Recall that the morphism from $\mathrm{Bl}\mathbb{A}^2_2 \to \mathbb{A}^2_k$ can be described by two rings maps:

$$k[x, y] \to k[x, y, \frac{v}{u}]/(y - x\frac{v}{u}), \qquad k[x, y] \to k[x, y, \frac{u}{v}]/(x - y\frac{u}{v})$$

Then if we want to compute the fiber, we can do it locally by:

$$\begin{tikzcd}
? \arrow[r, hook] \arrow[d, hook] & ? \arrow[r, hook] \arrow[d, hook] & ? \arrow[r] \arrow[d, hook] & (x,y) \arrow[d, hook] \\
Speck[x, y, \frac{u}{v}, \frac{v}{u}]/(x - y\frac{u}{v}) \arrow[r, hook] & Speck[x, y, \frac{u}{v}]/(x - y\frac{u}{v}) \arrow[r, hook] & \mathrm{Bl}\mathbb{A}^2_k \arrow[r] & \mathbb{A}^2_k
\end{tikzcd}$$

From this diagram we know that if $y \neq 0$, the fiber is computed by $k[\frac{u}{v}, \frac{v}{u}] \to k$ where $\frac{u}{v} \mapsto \frac{x}{y}$, thus the fiber is a single point. Similarly, we can compute that if $x \neq 0$, the fiber is also a single point. On the other hand, if $x = y = 0$, then the fiber is computed by the following diagram:

$$\begin{tikzcd}
k[\frac{u}{v}] \arrow[dr] & & k[\frac{v}{u}] \arrow[dl] \\
& k[\frac{u}{v}, \frac{v}{u}] &
\end{tikzcd}$$

Therefore the fiber is $\mathbb{P}^1_k$!

So if you have ever seen a schematic illustration of the blow-up over $\mathbb{A}^2_k$ (which is everywhere on the internet) you can think of $\mathrm{Bl}\mathbb{A}^2_k \to \mathbb{A}^2_k$ as separating all lines on the plane that passes through the origin by make a replication of the origin $\mathbb{P}^1_k$ times! Indeed, we have even proved that the morphism $\mathrm{Bl}\mathbb{A}^2_k \to \mathbb{A}^2_k$ is an isomorphism away from the origin, only the origin gets replicated in the blow-up.

We can directly verify that this blow-up is smooth, as we know that locally, for example on $Speck[x, y, z]/(x - yz)$, we know that the Jacobian matrix is $(1, z, y)^T$, which has corank 2, similar results holds on another cover, therefore this is smooth of relative dimension 2 over k, thus in particular regular.

As we will see later, any blow of a surface is regular and will also be a surface and also regular, therefore we know that the inverse image of the origin is also a curve in $\mathrm{Bl}\mathbb{A}^2_k$, therefore is an effective Cartier divisor due to regularity (which implies locally principal and the element that cuts out the closed subscheme locally is not a zero divisor). This is called the **exceptional divisor** of the blow-up.

This construction can be useful as sometimes it can help us desingularize a curve $C \subset \mathbb{A}^2_k$ which is singular at the origin. Often what we do is to consider the inverse image of $C - \{(0, 0)\}$ and then take the closure, which is a curve in the blow-up, together with a birational morphism to C.

Definition 1.1. Suppose that X is a variety, then a **desingularization** or a **resolution of singularity** of X is a proper birational morphism $\widetilde{X} \to X$ from a regular variety $\widetilde{X}$.

For example, recall that the node $y^2 = x^3 + x^2$, which is singular at the origin, once you conduct the above procedure, what you get will be a regular curve in the blow-up and the node at the origin downstairs will be separated. This will be an example of the **proper transform** or **strict transform** of the curve.

We can also now define the blow-up of $\mathbb{A}^n_k$ at the origin, which is going to be a closed subscheme of $\mathbb{A}^n_k \times_k \mathbb{P}^{n-1}_k$, defined informally by $x_iX_j - x_jX_i$ where we use $x_1, \cdots, x_n$ as coordinates on $\mathbb{A}^n_k$ and $X_1, \cdots, X_n$ as coordinates on $\mathbb{P}^{n-1}_k$. You can make this a formal definition using what we have discussed before, and then check that this is smooth, but we will wait later for a more general definition rather than this ad hoc definition tailored for $\mathbb{A}^n_k$.

One obvious defect of the above definition is that it's not coordinate free. If we consider an isomorphism of $\mathbb{A}^2_k$ which preserves the closed point (x, y), then in this case, how do we define the blow-up with another different coordinate and are these two blow-ups isomorphic to each other. Later, our desired definition will be coordinate free which solves this problem. Even though when you first see the definition, it's not necessarily related to the above motivating example, but I promise you later you will understand the connection. Understanding the above motivating example will be useful to understand the general blow-up later.

2 Blowing Up, via Universal Property

As we mentioned before, we will start our discussion of the blow-up by it's universal property. The disadvantage of this definition is that in the beginning it doesn't seem to be related to our motivating example. But the advantage is that it lets you to prove a lot of things without any work.

Definition 2.1. Let $X \hookrightarrow Y$ be a closed subscheme corresponding to a finite type quasicoherent sheaf of ideals. (If Y is locally Noetherian, this is automatic) Then the **blow-up** of $X \hookrightarrow Y$ is a Cartesian diagram:

$$\begin{tikzcd}
E_X Y \arrow[r, hook] \arrow[d] & \mathrm{Bl}_X Y \arrow[d, "\beta"] \\
X \arrow[r, hook] & Y
\end{tikzcd}$$

such that $E_X Y$, the scheme theoretic preimage of X by β is an effective Cartier divisor of $\mathrm{Bl}_X Y$, such that for any other such Cartesian diagram:

$$\begin{tikzcd}
D \arrow[r, hook] \arrow[d] & W \arrow[d, "\beta"] \\
X \arrow[r, hook] & Y
\end{tikzcd}$$

where D is an effective Cartier divisor of W, it factors uniquely through the previous diagram, i.e. we have unique morphisms $D \to E_X Y$ and $W \to \mathrm{Bl}_X Y$ such that the diagram below commutes:

$$\begin{tikzcd}
D \arrow[r, hook] \arrow[drr] \arrow[ddr] & W \arrow[drr] \arrow[ddr] & & \\
 & & E_X Y \arrow[r, hook] \arrow[dl] & \mathrm{Bl}_X Y \arrow[dl] \\
 & X \arrow[r, hook] & Y &
\end{tikzcd}$$

We call $\mathrm{Bl}_X Y$ the **blow-up of Y along X** or the **blow-up of Y with center X**. We call $E_X Y$ the **exceptional divisor** of the blow-up. Obviously, here β, Bl stands for blow-up and E stands for exceptional.

Now consider the category whose objects are Cartesian diagrams in the above definition with morphisms defined to be:

$$\begin{tikzcd}
D \arrow[r, hook] \arrow[drr] \arrow[ddr] & W \arrow[drr] \arrow[ddr] & & \\
 & & D' \arrow[r, hook] \arrow[dl] & W' \arrow[dl] \\
 & X \arrow[r, hook] & Y &
\end{tikzcd}$$

Then we find that the blow-up is the final object in this category and indeed any two blow-up will be isomorphic via a unique isomorphism.

We will postpone the proof of the existence later and now let's focus on what can we say about the blow-up just based on this universal property definition. Consider a closed subscheme $Z \hookrightarrow X$, then consider the scheme theoretic preimage of Z, i.e. $\beta^{-1}Z$, it's called the **total transform** of Z. We can also consider the closed subset $\overline{\beta^{-1}(Z - X)}$, which is called the **proper transform** of Z (sometimes also called **strict transform** or **birational transform**, but we will use the first term). For now, it's just closed subset, but later we will give it a scheme structure.

Proposition 2.2. If X is the empty set, i.e. when the closed subscheme is cut out by the unit ideal, then $\mathrm{Bl}_X Y = Y$ and more generally, if X is an effective Cartier divisor, then the blow-up β is an isomorphism.

Proof. Recall that we do recogonize the empty set as an effective Cartier divisor, which is locally cut out by the unit element.

First, let's consider when X is the empty set, which is not that serious a discussion since when X is the empty set, no one really discuss the blow-up. Consider the following diagram:

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & Y \arrow[d, dashed, "\mathrm{Id}"] \\ \varnothing \arrow[r] & Y \end{tikzcd}$$

Then I will let you check that the universal property is just saying that if we have a morphism $\pi : W \to Y$, then we only have a unique morphism $\tau : W \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} W \arrow[rr, dashed, "\tau"] \arrow[rd, "\pi"'] & & Y \arrow[ld, "\mathrm{Id}_Y"] \\ & Y & \end{tikzcd}$$

and τ is just π.

What is interesting is when X is a nontrivial effective Cartier divisor. Consider the diagram:

$$\begin{tikzcd} D \arrow[r, hook] \arrow[rrd, dashed] \arrow[rdd, "f"'] & W \arrow[rrd, dashed] \arrow[rdd, "g"] & & \\ & & X \arrow[r, hook] \arrow[ld] & Y \arrow[ld] \\ & X \arrow[r, hook] & Y & \end{tikzcd}$$

We see that the two dotted morphisms are unique, which are f, g. We see that the essence of this proof is that in the category of Cartesian diagrams whose bottom side is $X \to Y$, we know that the below morphism is always the final object:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\mathrm{Id}"] & Y \arrow[d, "\mathrm{Id}"] \\ X \arrow[r] & Y \end{tikzcd}$$

The only thing that prevent us to use this as the final object is when X is not an effective Cartier divisor, this is not in the category we defined by definition. $\square$

The following result shows that blow-up can be computed locally:

Proposition 2.3. Suppose that U is an open subset of Y, then $\mathrm{Bl}_{X \cap U} U$ is isomorphic to $\beta^{-1} U$ canonically.

Proof. Consider the diagram:

$$\begin{tikzcd} & & \beta^{-1}U \cap E_X Y \arrow[lldd, hook] \arrow[rrdd, hook] \arrow[d, dashed] & & \\ & & X \cap U \arrow[lldd] \arrow[rrdd] & & \\ E_X Y \arrow[rr, hook] \arrow[d] & & \mathrm{Bl}_X Y \arrow[d, dashed, "\beta"] & & \beta^{-1}U \arrow[ll, hook'] \arrow[d] \\ X \arrow[rr, hook] & & Y & & U \arrow[ll, hook'] \end{tikzcd}$$

where the dotted morphism is induced by the universal property. I claim that the two squares on the left and right sides are Cartesian. I will let you check it, directly using the universal property. After this, notice that $\beta^{-1}U \cap E_X Y$ is an effective Cartier divisor of $\beta^{-1}U$. Therefore, to prove the assertion, it suffice to prove that that it's the final object. To prove this, consider the following diagram:

$$\begin{tikzcd}
& \beta^{-1}U \cap E_X Y \arrow[ddl] \arrow[d, dashed] \arrow[ddr] & D \arrow[dl] \arrow[ddr, hook] & \\
& X \cap U \arrow[ddl] \arrow[ddr] & & \\
E_X Y \arrow[r, hook] \arrow[d] & \mathrm{Bl}_X Y \arrow[d, dashed, "\beta"] & \beta^{-1}U \arrow[l, hook'] \arrow[d] & W \arrow[dl] \\
X \arrow[r, hook] & Y & U \arrow[l, hook'] &
\end{tikzcd}$$

where D is an effective Cartier divisor of W and $E, W, X \cap U, U$ is Cartesian. Then we know that we can consider the rectangle patched together by the square $D, W, X \cap U, U, X, Y$, then we know from the blow-up of Y along X is final that we have uniuqe morphisms from D to $E_X Y$ and from W to $\mathrm{Bl}_X Y$, which induces a morphism from W to $\beta^{-1}U$ since the square $\mathrm{Bl}_X Y, Y, \beta^{-1}U, U$ is Cartesian, then we know from the fact that $U \hookrightarrow Y$ is a monomorphism that the above information gives a unique morphism from D to $\beta^{-1}U \cap E_X Y$, thus this is indeed terminal. We are done.

In conclusion we know that if the blow-up of $X \to Y$ exists, then for any $U \subset Y$, we know that the blow-up of $X \cap U \hookrightarrow U$ exists which is given by:

$$\begin{tikzcd}
& \beta^{-1}U \cap E_X Y \arrow[r, hook] \arrow[dl, dashed] \arrow[dd, hook] & \beta^{-1}U \arrow[dl] \arrow[dd] \\
X \cap U \arrow[r, hook] \arrow[dd, hook] & U \arrow[dd, hook] & \\
& E_X Y \arrow[r, hook] \arrow[dl] & \mathrm{Bl}_X Y \arrow[dl, "\beta"] \\
X \arrow[r, hook] & Y &
\end{tikzcd}$$

$\square$

Corollary 2.4. Suppose that $X \hookrightarrow Y$ is a closed embedding defined by a finite type quasicoherent ideal sheaf, then I claim that the natural morphism:

$$\beta|_{\mathrm{Bl}_X Y - E_X Y} : \mathrm{Bl}_X Y - E_X Y \to Y - X$$

is an isomorphism.

Proof. Consider the diagram:

$$\begin{tikzcd}
& \varnothing \arrow[r] \arrow[dl] \arrow[dd] & \beta^{-1}(Y - X) \arrow[dl, "\simeq"'] \arrow[dd, hook] \\
\varnothing \arrow[r] \arrow[dd] & Y - X \arrow[dd, hook] & \\
& E_X Y \arrow[r, hook] \arrow[dl] & \mathrm{Bl}_Y X \arrow[dl, "\beta"] \\
X \arrow[r, hook] & Y &
\end{tikzcd}$$

We are done. $\square$

Proposition 2.5. If Y_α is an open cover of Y as α runs over some index set, then if the blow-up of Y_α along $X \cap Y_\alpha$ exists, then the blow-up of Y along X exists.

Proof. Recall that we have the following diagram for each Y_α:

$$\begin{tikzcd}
E_{X\cap Y_\alpha}Y_\alpha \arrow[r, hook, "\gamma_\alpha"'] \arrow[d, hook] & X \cap Y_\alpha \arrow[r, hook] \arrow[d, hook, "i_\alpha"] & X \arrow[d, hook] \\
\mathrm{Bl}_{X\cap Y_\alpha}Y_\alpha \arrow[r, hook, "\beta_\alpha"] & Y_\alpha \arrow[r, hook] & Y
\end{tikzcd}$$

If we consider Y_α and $Y_{\alpha'}$, we know from what we have seen above that the blow-up of $X \cap Y_\alpha \cap Y_{\alpha'} \to Y_\alpha \cap Y_{\alpha'}$ exists, and we know that they fit in the following diagram as well:

$$\begin{tikzcd}
E_{X\cap Y_\alpha}Y_\alpha \arrow[r, hook, "\gamma_\alpha"'] \arrow[d, hook] & X \cap Y_\alpha \arrow[d, hook, "i_\alpha"] & & X \cap Y_{\alpha'} \arrow[d, hook, "i_{\alpha'}"'] & E_{X\cap Y_{\alpha'}}Y_{\alpha'} \arrow[l, hook', "\gamma_{\alpha'}"'] \arrow[d, hook] \\
\mathrm{Bl}_{X\cap Y_\alpha}Y_\alpha \arrow[r, hook, "\beta_\alpha"] & Y_\alpha & & Y_{\alpha'} & \mathrm{Bl}_{X\cap Y_{\alpha'}}Y_{\alpha'} \arrow[l, hook', "\beta_{\alpha'}"] \\
& \beta_\alpha^{-1}(Y_\alpha \cap Y_{\alpha'}) \cap X \cap Y_\alpha \cap Y_{\alpha'} \arrow[r, hook] \arrow[d, hook] \arrow[uul, dashed] & X \cap Y_\alpha \cap Y_{\alpha'} \arrow[d, hook] \arrow[uul, hook] \arrow[uur] & \beta_{\alpha'}^{-1}(Y_\alpha \cap Y_{\alpha'}) \cap X \cap Y_\alpha \cap Y_{\alpha'} \arrow[l, hook'] \arrow[d, hook] \arrow[uur, dashed] & \\
& \beta_\alpha^{-1}(Y_\alpha \cap Y_{\alpha'}) \arrow[r, hook] \arrow[uul, hook] & Y_\alpha \cap Y_{\alpha'} \arrow[uul, hook] \arrow[uur] & \beta_{\alpha'}^{-1}(Y_\alpha \cap Y_{\alpha'}) \arrow[l, hook'] \arrow[uur] &
\end{tikzcd}$$

What is important to notice is that we know that since the two bottom squares are both the blow-up of $X \cap Y_\alpha \cap Y_{\alpha'} \hookrightarrow Y_\alpha \cap Y_{\alpha'}$, there is a canonical isomorphism between then, which is used to glue $E_{X\cap Y_\alpha}Y_\alpha$ and $E_{X\cap Y_{\alpha'}}Y_{\alpha'}$ and similarly to glue $\mathrm{Bl}_{X\cap Y_\alpha}Y_\alpha$ and $\mathrm{Bl}_{X\cap Y_{\alpha'}}Y_{\alpha'}$.

Now, once we check the cocycle condition holds, we can indeed glue these together and obtain:

$$\begin{tikzcd}
E_X Y \arrow[r, hook] \arrow[d] & \mathrm{Bl}_Y X \arrow[d] \\
X \arrow[r, hook] & Y
\end{tikzcd}$$

where the morphism between them are also glued together. To check this is Cartesian, recall that this can be check locally and to check it's final, it also suffices to check locally, therefore we are done. $\square$

Remark 2.6. Not only can we conclude that if the blow-up locally exists, then it exists in general, we can also conclude that if we already have a blow-up diagram, then locally the diagram remains a blow-up diagram, which means that we have the following diagram:

$$\begin{tikzcd}
& & E_Y X \arrow[r, hook] \arrow[d] & \mathrm{Bl}_Y X \arrow[d] \\
X \cap U \times_X E_Y X \arrow[r, hook] \arrow[d] \arrow[urr, hook] & U \times_Y \mathrm{Bl}_Y X \arrow[d] \arrow[urr, hook] & X \arrow[r, hook] & Y \\
X \cap U \arrow[r, hook] \arrow[urr, hook] & U \arrow[urr, hook] & &
\end{tikzcd}$$

where the square in the back is still blow-up if the square in front is blow-up. i will let you check this, and the key is to notice that $X \cap U \times_X E_Y X \hookrightarrow E_Y X$ and $U \times_Y \mathrm{Bl}_Y X \hookrightarrow \mathrm{Bl}_Y X$ are monomorphisms.

Conversely, if a priori, we have a Cartesian diagram:

$$\begin{tikzcd}
Z \arrow[r, hook] \arrow[d] & W \arrow[d] \\
X \arrow[r, hook] & Y
\end{tikzcd}$$

Such that there is an open cover of Y such that the pullback of the square is blow-up, then globally the diagram is pullback. The reason is the following diagram:

$$\begin{tikzcd}
& & Z \arrow[r, hook] \arrow[d] & W \arrow[d] \\
X\cap U\times_X Z \arrow[r, hook] \arrow[d] \arrow[urr, hook] & U\times_Y W \arrow[d] \arrow[urr, hook] & X \arrow[r, hook] & Y \\
X\cap U \arrow[r, hook] \arrow[urr, hook] & U \arrow[urr, hook] & A \arrow[r, hook] \arrow[u] & B \arrow[u] \\
A\times_X X\cap U \arrow[r, hook] \arrow[u] \arrow[urr, hook] \arrow[uu, bend left, "!\exists"] & U\times_Y B \arrow[u] \arrow[urr, hook] \arrow[uu, bend right, "!\exists"'] & &
\end{tikzcd}$$

The moral here is that if we have two Y schemes X, Z, if we want to prove that two Y-scheme morphisms between X, Z are the same, then we only need to find an open over of Y and show that the restriction of the two morphisms agree over that open.

Proposition 2.7. If in $X \hookrightarrow Y$, X is actually Y itself, then the blow-up is actually the empty scheme.

Proof. To see why, consider the identity morphism $Y \to Y$, in this case, suppose that $W \to Y$ is a morphism, then we know that the base change is actually W itself, therefore, only $\varnothing \to Y$ makes $\varnothing \to \varnothing$ an effective Cartier divisor. $\square$

Now we will see that blow-up actually preserves irreducibility and reducedness:

Proposition 2.8. If Y is a irreducible scheme and $X \hookrightarrow Y$ doesn't contain the generic point, then $\mathrm{Bl}_X Y$ is still irreducible. Moreover, if Y is reduced, then for any closed subscheme $X \hookrightarrow Y$, we have $\mathrm{Bl}_X Y$ is reduced.

In particular, if Y is integral and if X doesn't contain the generic point, then Bl_X is integral, this holds even when X is not reduced nor irreducible.

Proof. Recall that a topological space is irreducible iff any two nonempty open intersects each other. Let's assume that X is nonempty since it's it's empty, we know that the blow-up is just Y itself. Suppose that U_1, U_2 are two opens in $\mathrm{Bl}_X Y$, if they are both contained in $\mathrm{Bl}_X Y - E_X Y$, we know that in this case β gives us an isomorphism between $\mathrm{Bl}_X Y - E_X Y$ and $Y - X$. Since $Y - X$ is also irreducible, we know that the image of U_1, U_2 in $Y - X$ intersects, which implies U_1, U_2 intersects.

Now consider two arbitrary opens U_1, U_2, since $E_X Y$ is an effective Cartier divisor, I claim that U_1, U_2 can not be contained in $E_X Y$. Suppose that there is an open U contained in $E_X Y$. Suppose that we have a ring A and a nonzerodivisor $f \in A$, then consider $SpecA/(f)$. Suppose that $D(g) \subset V(f)$. This implies that $D(g) \cap D(f) = D(fg)$ is empty. However, we know that this implies that fg is in every prime, therefore fg is a nilpotent element, i.e. there is n such that $f^n g^n = 0$, but this implies that g is nilpotent thus $D(g)$ is empty, we are done.

Now, since U_1, U_2 are not contained in $E_X Y$, therefore to prove U_1, U_2 intersect, we can prove that $U_1 - E_X Y, U_2 - E_X Y$ intersects, they are open, not empty, therefore our discussion in the beginning shows that they intersect.

Finally, let's show that $\mathrm{Bl}_Y X$ is reduced, i.e. we need to show that the stalk is reduced for each point in $\mathrm{Bl}_Y X$. Again, we know that if a point is in $\mathrm{Bl}_X Y - E_X Y$, then it's isomorphic to the stalk downstairs, therefore we know that it's reduced. Therefore we only need to care about points in $E_X Y$. This reduces to the following algebraic fact, if A is a ring with a nonzero divisor f, if we assume that A_f is reduced, then A is reduced. Suppose that $x \in A$ is nilpotent, then we know that $\frac{x}{1}$ is nilpotent, since A_f is reduced, $\frac{x}{1}$ is 0, therefore $f^n x = 0$ in A, but since f is a nonzerodivisor, $f^n x = 0$ implies $x = 0$. $\square$

Notice that all the above discuss doesn't require any of the detailed description of the blow-up, all we know is the universal property and the exceptional divisor is an effective Cartier divisor. Now, let's try to show that the blow-up exists in a very special case.

Now suppose that $X \hookrightarrow Y$ is a locally principal closed embedding, i.e. there is an affine open cover of Y, say $SpecA$, such that $X \hookrightarrow Y$ is $SpecA/t \hookrightarrow SpecA$. Recall that from the previous proposition, to prove the blow-up exists, we only need to prove that locally it exists, therefore we can just assume that we are working with $SpecA/t \hookrightarrow SpecA$. A good example to think of here is $Speck[y] \hookrightarrow Speck[x,y]/(xy)$.

Consider the kernel:

$$I = \mathrm{Ker}(A \to A_t)$$

this consists of $a \in A$ such that it's annihilated by some power of t. Let $\phi : A \to A/I$ be the projection.

Proposition 2.9. The following diagram is a blow-up diagram:

$$\begin{tikzcd}
SpecA/(t,I) \arrow[r, hook] \arrow[d] & SpecA/I \arrow[d] \\
SpecA/t \arrow[r, hook] & SpecA
\end{tikzcd}$$

Proof. First, I claim that t in A/I is a nonzerodivisor. Suppose that $t \cdot x = 0$, which means that $t \cdot x \in I$, since I is the kernel of $A \to A_t$, we know that $t^{n+1} \cdot x = 0$ in A, which implies $x \in I$.

This diagram is obviously Cartesian, therefore we only need to show that it's final. Suppose that we have:

$$\begin{tikzcd}
& & W \arrow[r, hook] \arrow[dll, dashed] \arrow[ddl] & D \arrow[dll, dashed] \arrow[ddl] \\
SpecA/(t,I) \arrow[r, hook] \arrow[dr] & SpecA/I \arrow[dr] & & \\
& SpecA/t \arrow[r, hook] & SpecA &
\end{tikzcd}$$

We need to check the dotted morphisms exist and are unique. Recall that to define morphisms from, for example W to $SpecA/(t,I)$, we can define it locally on affines and then show that they agree on the intersections. If we can show that for $W \hookrightarrow D$, on every affine open of D, say $SpecB/f \to SpecB$, this morphisms exist and are unique, this proves that globally these two morphisms exist and are unique. The uniuqeness is clear, so we only need to focus on existence, i.e. these locally defined morphisms agree on intersections. But these is just the uniqueness on affines as the intersection can again be covered by distinguished affine opens.

Therefore we can assume that we work in the following situation:

$$\begin{tikzcd}
& & SpecB/t \arrow[r, hook] \arrow[dll, dashed] \arrow[ddl] & SpecB \arrow[dll, dashed] \arrow[ddl] \\
SpecA/(t,I) \arrow[r, hook] \arrow[dr] & SpecA/I \arrow[dr] & & \\
& SpecA/t \arrow[r, hook] & SpecA &
\end{tikzcd}$$

where we assume that $A \to B$ maps t to a nonzero divisor of B. To prove the existence and uniqueness of the two morphisms. Let's first consider the map $SpecB \to SpecA/I$, from the universal property of quotients, there is such a map and is unique iff the map $A \to B$ maps I to 0. Suppose $a \in I$, then to prove that $a = 0$ in B, recall that t is a nonzerodivisor in B, we only need to show that it's annihilated by t, but this is the definition of $a \in I$. Similarly, the universal property shows that $SpecB/(t) \to SpecA/(t,I)$ exists and is unique. Then you can check this indeed makes the diagram commute. $\square$

This description fits with many of the properties we showed before. For example if t is an effective Cartier divisor in $SpecA$, we know that $I = 0$, therefore the blow-up morphism is an isomorphism. You can also check other properties if you want.

Proposition 2.10. *$SpecA/I$ is the scheme theoretic closure of $D(t)$ in $SpecA$.*

Proof. Recall that the scheme theoretic image of $D(t)$ is just the scheme theoretic image of $D(t) \hookrightarrow SpecA$. Then recall that on affine opens, the scheme theoretic image is just cut out by functions that pullback to 0 on $SpecA_t$. $\square$

Remark 2.11. In our affine description above, we can think of what happens in $SpecA/I$ is that we are killing those stuff that makes t a zerodivisor in A. If in particular, t itself is already an effective Cartier divisor, we know that we have nothing to kill therefore the blow-up remains the same. Therefore we can think of what we are doing here is shaving $SpecA$ off those fuzzing stuff in $SpecA/t$. This is also suggested in $SpecA/I$ is the scheme theoretic image of $D(t)$, i.e. we shaved off those elements that is making $A \to A_t$ not injective, i.e. those making t a zerodivisor.

If you check this intuition on the example $Speck[y] \hookrightarrow Speck[x,y]/(xy)$. We know that the element we are modding out is x, and we know that those that is killed by powers of $x$are exactly (y), therefore we know that the blow-up is $Speck[x] \hookrightarrow Speck[x,y]$. I.e. we are having off $Speck[y]$ since this is exactly what makes x a zero divisor.

Then we discuss the famous **blow-up closure lemma**:

Suppose that we have a Cartesian diagram:

$$\begin{tikzcd}
W \arrow[r, hook] \arrow[d] & Z \arrow[d] \\
X \arrow[r, hook] & Y
\end{tikzcd}$$

Suppose that the bottom is a closed embedding which correspond to a finite type quasicoherent ideal sheaf. This implies that the closed embedding above also correspond to a finite type quasicoherent ideal sheaf since we know that locally this diagram is:

$$\begin{tikzcd}
SpecB/BI \arrow[r, hook] \arrow[d] & SpecB \arrow[d] \\
SpecA/I \arrow[r, hook] & SpecA
\end{tikzcd}$$

Since I is finitely generated over A, we know that BI is finitely generated over B, essentially by the same elements. If this is the first time you read about the blow-up closure lemma, maybe it's good to first consider the case when $Z \to Y$ is a closed embedding.

Let's consider the following diagram:

$$\begin{tikzcd}
W \times_X E_X Y \arrow[rr, dashed] \arrow[d] \arrow[ddr] & & Z \times_Y \mathrm{Bl}_X Y \arrow[d] \arrow[ddr] & \\
E_X Y \arrow[rr, hook] \arrow[ddr] & & \mathrm{Bl}_X Y \arrow[ddr] & \\
& W \arrow[rr, hook] \arrow[d] & & Z \arrow[d] \\
& X \arrow[rr, hook] & & Y
\end{tikzcd}$$

where the dotted morphism is induced by the universal property. I claim that the diagram consisting of $W \times_Z E_X Y, Z \times_Y \mathrm{Bl}_X Y, \mathrm{Bl}_X Y$ is Cartesian, and the reason is just we know the outer rectangle is Cartesian as the stack of two Cartesian square is Cartesian, and we know that the square consisting of $Z \times_Y \mathrm{Bl}_X Y, \mathrm{Bl}_X Y, Z, Y$ is Cartesian.

One useful observation we make here is that since $X \hookrightarrow Y$ is a closed embedding, which is a monomorphism, therefore we know that $W \times_X E_X Y$ is actually satisfies the universal property of $W \times_Y E_X Y$. We also observe that the following two squares stack together to form a new Cartesian square:

$$\begin{tikzcd}
E_X Y \times_X W \arrow[r] \arrow[d] & W \arrow[r, hook] \arrow[d] & Z \arrow[d] \\
E_X Y \arrow[r] & X \arrow[r, hook] & Y
\end{tikzcd}$$

Therefore we have the canonical isomorphisms:

$$E_X Y \times_Y W \simeq E_X Y \times_X W \simeq E_X Y \times_Y Z$$

On the other hand, if you stack up the following two Cartesian squares:

$$\begin{tikzcd} W \times_X E_X Y \arrow[r] \arrow[d] & E_X Y \arrow[r, hook] \arrow[d] & \mathrm{Bl}_X Y \arrow[d] \\ W \arrow[r] & X \arrow[r, hook] & Y \end{tikzcd}$$

We have another useful and canonical isomorphism:

$$W \times_X E_X Y \simeq W \times_Y \mathrm{Bl}_X Y$$

Finally, we observe that the in the Cartesian square:

$$\begin{tikzcd} W \times_X E_X Y \arrow[r, hook] \arrow[d] & Z \times_Y \mathrm{Bl}_X Y \arrow[d] \\ E_Y X \arrow[r, hook, "\text{eff.Car}"'] & \mathrm{Bl}_X Y \end{tikzcd}$$

Since the bottom closed embedding is locally principal, the top closed embedding is locally principal, however there is no guarantee that it's going to be an effective Cartier divisor since we know that a nonzerodivisor may not be mapped to another nonzerodivisor.

Now consider the open subset of $Z \times_Y \mathrm{Bl}_X Y$ which is $Z \times_Y \mathrm{Bl}_X Y - W \times_X E_X Y$, since we have the canonical isomorphism $W \times_X E_X Y \simeq W \times_Y \mathrm{Bl}_X Y$, you can also think of this as $Z \times_Y \mathrm{Bl}_X Y - W \times_Y \mathrm{Bl}_X Y$. Then we define $\overline{Z}$ to be the scheme theoretic closure of $Z \times_Y \mathrm{Bl}_X Y - W \times_X E_X$ or $Z \times_Y \mathrm{Bl}_X Y - W \times_Y \mathrm{Bl}_X Y$. Recall that if $U \hookrightarrow Y$ is an open embedding, it's not in general quasicompact. However, if we have a locally principal closed embedding $X \hookrightarrow Y$, then we consider the open embedding $Y - X \hookrightarrow Y$, it's quasicompact(do you see why). Therefore in this case we know that the scheme-theoretic closure of $Z \times_Y \mathrm{Bl}_X Y - W \times_X E_X Y$ can be computed affine locally, so it's not a mystery.

Suppose that $Z \to Y$ is actually a closed embedding, then I will let you check that the above defined $\overline{Z}$ is the **proper transform** of Z we defined earlier. For this reason, even if $Z \to Y$ is not a closed embedding, we will also call $\overline{Z}$ the **proper transform** of Z. similarly, since if $Z \hookrightarrow Y$ is a closed embedding, $Z \times_Y \mathrm{Bl}_X Y$ is the **total transform** of Z, then even if $Z \to Y$ is not closed, we will still call $Z \times_Y \mathrm{Bl}_X Y$ the **proper transform** of Z.

Also consider the following diagram:

$$\begin{tikzcd} E_{\overline{Z}} \arrow[r] \arrow[d, hook] & \overline{Z} \arrow[d, hook] \\ W \times_X E_X Y \arrow[r, hook] \arrow[d] & Z \times_Y \mathrm{Bl}_X Y \arrow[d] \\ E_Y X \arrow[r, hook, "\text{eff.Car}"'] & \mathrm{Bl}_X Y \end{tikzcd}$$

where we define $E_{\overline{Z}}$ to be the pullback of $\overline{Z}$. Notice that it's still a locally closed embedding, and it's even an effective Cartier divisor. This is a purely general reasoning, consider the following diagram:

$$\begin{tikzcd} X \times_Y \overline{Y - X} \arrow[r, hook] \arrow[d, hook] & \overline{Y - X} \arrow[d, hook] \\ X \arrow[r, hook, "\text{locally prin.}"] & Y \end{tikzcd}$$

We know that if we assume that $X \hookrightarrow Y$ is locally $SpecA/f \hookrightarrow SpecA$, we know that $\overline{Y-X} \hookrightarrow Y$ can be locally identified by $SpecA/I \hookrightarrow SpecA$ where I is the ideal consisting of elements of A that are annihilated by some powers of f. Then $X \times_Y \overline{Y-X}$ is locally $SpecA/(f,I)$, but we know that f is a nonzerodivisor in A/I, therefore this is indeed an effective Cartier divisor.

Finally, I claim that the following square is a blow-up square:

$$\begin{tikzcd}
E_{\overline{Z}} \arrow[r, hook] \arrow[d, hook] & \overline{Z} \arrow[d, hook] \\
W \times_X E_X Y \arrow[r, hook] & Z \times_Y \mathrm{Bl}_X Y
\end{tikzcd}$$

This is because similar to how we proved blow-up can be computed locally, if we want to check this diagram is Cartesian, we can also do it locally on an affine open cover of $Z \times_Y \mathrm{Bl}_X Y$, but recall that $W \times_X E_X Y \hookrightarrow Z \times_Y \mathrm{Bl}_X Y$ is locally principal, therefore we know that the above square locally is:

$$\begin{tikzcd}
SpecA/(t,I) \arrow[r, hook] \arrow[d, hook] & SpecA/I \arrow[d, hook] \\
SpecA/t \arrow[r, hook] & SpecA
\end{tikzcd}$$

By our above understanding of blow-up of locally principal embeddings on the affine case, this is indeed a blow-up diagram, therefore we are done.

After all these discussion, we can finally state the blow-up closure lemma:

Lemma 2.12. Using all the above notation, if the vlow-up of $X \hookrightarrow Y$ exists, then the blow-up of $W \hookrightarrow Z$ also exists which is given by:

$$\begin{tikzcd}
E_{\overline{Z}} \arrow[r, hook] \arrow[d, hook] & \overline{Z} \arrow[d, hook] \\
W \times_X E_X Y \arrow[r, hook] \arrow[d] & Z \times_Y \mathrm{Bl}_X Y \arrow[d] \\
W \arrow[r, hook] & Z
\end{tikzcd}$$

Before we prove this lemma, let's say why this can be quite useful.

First, if $Z \hookrightarrow Y$ is a closed embedding, then the proper transform of Z is computed by the blow-up of Z along it's scheme theoretic intersection with $W = X \cap Z$.

Then the lemma actually lets you compute some blow-ups by hand, as you will see in the examples we will do later. For example, if we have a plane curve $C \hookrightarrow \mathbb{A}^2_k$, which is singular at $p \in C$, and we would like to blow-up C at p in order to see if this can resolve the singularity or not. To compute the blow up of C along p, I claim that it suffice to consider the scheme theoretic closure of $C-p$ in the blow-up of the plane at p, which we have essentially described how to compute without checking it satisfies the universal property. (Do you see why this is true? Hint: Blow-up is an isomorphism away from the point p.)

Now, suppose that Z is finite type over k(now Z is not a closed subscheme of any X). If we have some nasty closed subscheme $W \hookrightarrow Z$ and we would like to compute the blow-up. Since blow-up can be computed locally, let's assume that Z is affine and W is computed by $f_1, \cdots, f_r$ in $\Gamma(Z, \mathcal{O}_Z)$. Since we assume that Z is finite type over k, let's assume that we can complete $f_1, \cdots, f_r$ to a generating set $f_1, \cdots, f_n$ of $\Gamma(Z, \mathcal{O}_Z)$ over k. Then we have a closed embedding $Z \hookrightarrow \mathbb{A}^n_k$ such that W is the scheme theoretic intersection of Z with $\mathbb{A}^{n-r}$, i.e. we have:

$$\begin{tikzcd}
W \arrow[r, hook] \arrow[d, hook] & Z \arrow[d, hook] \\
\mathbb{A}^{n-r} \arrow[r, hook] & \mathbb{A}^n_k
\end{tikzcd}$$

Then the blow-up closure lemma can be helpful for the computation of the blow-up.

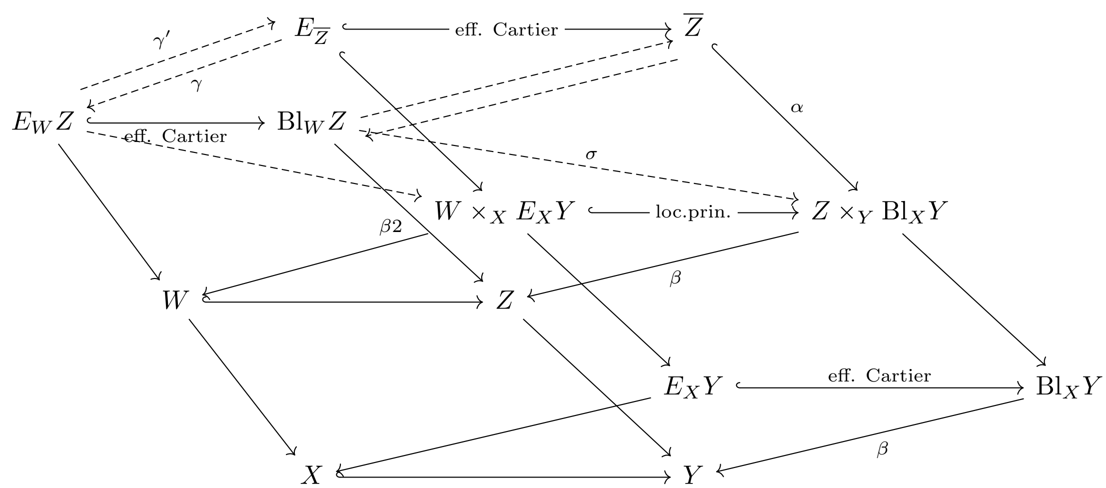
Finally, this can be used to reduce the existence of the general blow-up to a special case, even though we won't use it as our construction. To see how it works, I claim that if the blow-up $\mathrm{Bl}_{(x-1,\cdots,x_n)}Spec\mathbb{Z}[x_1,\cdots,x_n]$ exists, then I claim that general blow-ups always exist (again assume that the closed subscheme is cut out by a finite type ideal sheaf). To see why, recall that blow-up can be computed locally, then we can assume that $Y = SpecB$ and $X = SpecB/(f_1,\cdots,f_n)$. Then consider the following diagram:

$$\begin{tikzcd}
X = SpecB/(f_1,\cdots,f_n) \arrow[r, hook] \arrow[d, hook] & Y = SpecB \arrow[d, hook] \\
(x_1,\cdots,x_n) \arrow[r, hook] & Spec\mathbb{Z}[x_1,\cdots,x_n]
\end{tikzcd}$$

Then we know that the blow-up closure lemma tells you that once the blow-up of $(x_1,\cdots,x_n) \hookrightarrow Spec\mathbb{Z}[x_1,\cdots,x_n]$ exists, we know how to compute the blow-up of $X \hookrightarrow Y$. (We can also replace $\mathbb{Z}$ to k if we prefer to work over a field k.)

Finally, let's try to prove this lemma, which is again fairly categorical:

Proof. As we said, the proof is quite categorical, and the way we do it is to construct morphisms in both directions and then show they are inverses of each other. The following diagram is helpful:



Notice that the bottom cube diagram is just what we saw before and every dotted arrow is induced by some sort of universal property, don't forget that $E_{\overline{Z}} \hookrightarrow \overline{Z}$ is also a blow-up.

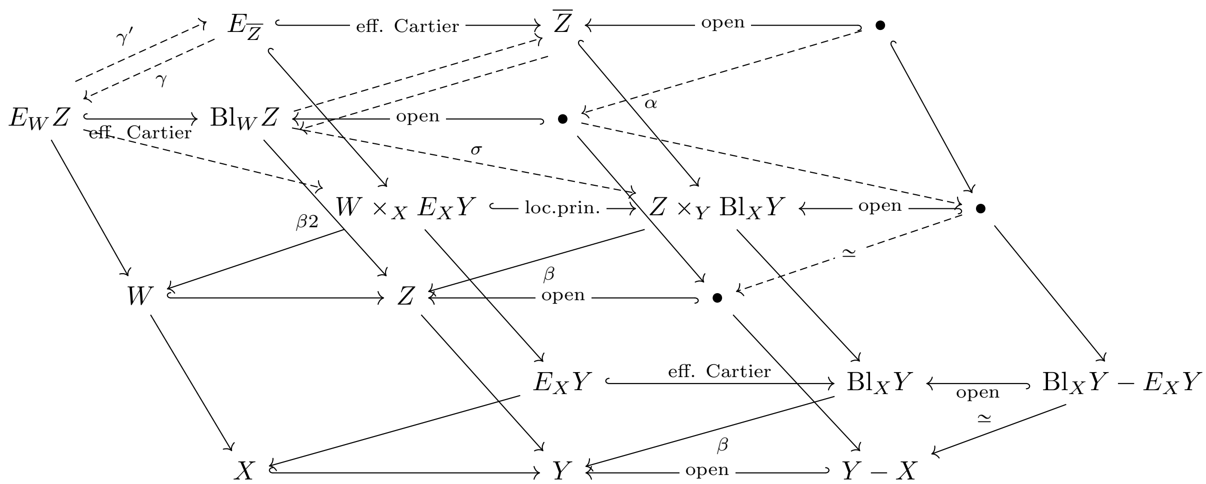
I claim that this is enough to show our desired isomorphism. To see why, for example we consider the diagram:

$$\begin{tikzcd}
E_W Z \arrow[rr, hook] \arrow[dr] & & \mathrm{Bl}_W Z \arrow[drrr] & & E_W Z \arrow[rr, hook] \arrow[dlll] \arrow[llll, bend right] & & \mathrm{Bl}_W Z \arrow[dl] \arrow[llll, bend right] \\
& W \arrow[rrrr, hook] & & & & Z &
\end{tikzcd}$$

If this commutes, then the two curved arrows must be identity due to the universal property of the final object.

I will let you check that this is enough to check that $\gamma \circ \gamma' = \mathrm{Id}$. This shows that γ' is monomorphic and γ is epimorphic. Conversely, to check $\gamma' \circ \gamma = \mathrm{Id}$, we need to use a previous result on the unique extension of morphisms away from an effective Cartier divisor. The statement goes: suppose that X is a Z-scheme and Y is a separated Z-scheme. Let D be an effective Cartier divisor on X, then any Z-morphism $X - D \to Y$ can be extended in at most one way to a Z-morphism $X \to Y$. Knowing this, I will let you show that the main obstacle for us to show the converse using the same argument lies in proving the following statement:

suppose that a morphism $f : \overline{Z} \to \mathrm{Bl}_W Z$ satisfies that after we compose with $\mathrm{Bl}_W Z \to Z \times_Y \mathrm{Bl}_X Y \to Z$ it's equal to $\overline{Z} \to Z \times_Y \mathrm{Bl}_X Y \to Z$. Then we can conlcude that the two ways from $\overline{Z}$ to $Z \times_Y \mathrm{Bl}_X Y$ is actually the same morphism. You can see that one key obstruction to this is that β is not a monomorphism. However, recall that away from $E_X Y$, we know that $\mathrm{Bl}_X Y$, the morphism β is an isomorphism, which means that away from $W \times_Z E_X Y$, β is also an isomorphism and in particular a monomorphism. To utilize this, consider the following diagram:



where I have on purpose omitted some notions and used $\cdots$ as it's not important to know what are they concretely. I will let you see that the isomorphism on the second floor is the restriction of β, and I will let you see that we can check that the compositions agree by using this isomorphism. Finally, what we have proved is that they agree when we delete the effective Cartier divisor $E_{\overline{Z}}$ in $\overline{Z}$, we need to show that they agree globally, which uses the fact that Z is separated (it's a closed subscheme), we are done. $\square$

3 The Blow-up Exists and Is Projective

In this section, let's show that in general blow-up always exists and in particular, the blow-up morphism $\mathrm{Bl}_X Y \to Y$ is going to be projective, thus quasicompact, proper, finitye type and separated. If $Y \to Z$ is a morphism which is quasicompact, proper, finitye type and separated, then so is the composition $\mathrm{Bl}_X Y \to Z$. If $Y \to Z$ is projective and Z is quasicompact, then $\mathrm{Bl}_X Y \to Z$ is also projective. Before you read what follows, I recomend to take a look at the etale cohomology series on the discussion of relative proj if you are not familiar with it. Without further discussion, let's state the existence theorem:

Theorem 3.1. Relative Proj description of the blow-up Suppose that $X \hookrightarrow Y$ is a closed embedding corresponding to a finite type quasicoherent ideal sheaf $\mathcal{I}$, then consider the following graded $\mathcal{O}_Y$-algebra:

$$\mathcal{O}_Y \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \mathcal{I}^3 \oplus \cdots$$

Notice that this is indeed a graded $\mathcal{O}_Y$-algebra with the grading induced from the multiplication $\mathcal{I} \hookrightarrow \mathcal{I} \hookrightarrow \mathcal{O}$. Notice that this is clearly generated in degree 1 as an $\mathcal{O}_Y$-algebra, and furthermore, since $\mathcal{I}$ is finite type, we can assume that it's finitely generated by degree 1, which has really good properties when we take the proj. Then I claim that:

$$\mathcal{P}roj_Y(\mathcal{O}_Y \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \mathcal{I}^3 \oplus \cdots) \to Y$$

satisfies the universal property of the diagram.

We will later prove it after we see what it tells us. But obviously, we need to first show that the preimage of $X \hookrightarrow Y$ is an effective Cartier divisor. Clearly, we can work affine locally on Y where we assume that $X \hookrightarrow Y$ is $SpecB/I \to SpecB$, and we know that the relative proj locally just looks like:

$$B_X Y = Proj_B(B \oplus I \oplus I^2 \oplus I^3 \oplus \cdots) \to SpecB$$

Here we use B_XY to denote the proj since we know that if globally it's a blow-up diagram, then locally it's also a blow-up diagram. Conversely, if for each affine of Y is satisfies the universal property, then globally it also satisfies the universal property. Recall that $B(I)_\bullet$ is called the Rees algebra of the ideal I of B.

Now we need to compute the inverse image of $SpecB/I$ in $Proj(B(I)_\bullet)$, recall that we know that it should be in the following diagram:

$$\begin{tikzcd}
Proj(B(I)_\bullet \otimes_B B/I) \arrow[r] \arrow[d] & Proj(B \oplus I \oplus I^2 \oplus \cdots) \arrow[d] \\
SpecB/I \arrow[r, hook] & SpecB
\end{tikzcd}$$

We know that if the upper left corner is the graded ring $B(I)_\bullet \otimes_B B/I$, then we know that the morphism is Cartesian. Notice that the new graded ring is just:

$$B(I)_\bullet \otimes_B B/I = B(I)_\bullet/IB(I)_\bullet = B/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$$

Therefore we can identify the above diagram with:

$$\begin{tikzcd}
Proj(B(I)_\bullet/IB(I)_\bullet) \arrow[r, hook] \arrow[d] & Proj(B(I)_\bullet) \arrow[d] \\
SpecB/I \arrow[r, hook] & SpecB
\end{tikzcd}$$

We know that the exact sequence of the closed embedding on the top corresponds to apply the $\tilde{-}$ construction to:

$$0 \to IB(I)_\bullet \hookrightarrow B(I)_\bullet \to B(I)_\bullet/IB(I)_\bullet \to 0$$

Therefore we know that the ideal sheaf corresponds to $\widetilde{IB(I)_\bullet}$. I claim that this is just $\mathcal{O}(1)$ of $Proj(B(I)_\bullet)$ which is defined to be $\widetilde{B(I)_\bullet(1)}$. We know that in order to compute $\widetilde{B(I)_\bullet(1)}$, the negative degree part doesn't matter, therefore we know that since starting from degree 0, i.e. when $d \geqslant 0$, $B(I)_\bullet(1)_d = (IB(I)_\bullet)_d$, $\mathcal{O}(1)$ is canonically isomorphic to $\widetilde{B(I)_\bullet(1)}$. Recall that a closed embedding is an effective Cartier divisor iff the ideal sheaf is invertible, we are done. Note that we have the graded ideal sheaf:

$$\mathcal{I} \oplus \mathcal{I}^2 \oplus \mathcal{I}^3 \oplus \cdots \hookrightarrow \mathcal{O}_Y \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \cdots$$

where on each affine open $SpecB$ of Y correspond to $IB(I)_\bullet \hookrightarrow B(I)_\bullet$, therefore we know that the closed subscheme defined by $\mathcal{I} \oplus \mathcal{I}^2 \oplus \mathcal{I}^3 \oplus \cdots$ is the preimage of X. Finally, recall that we have a graded $\mathcal{O}_X$-algebra:

$$\mathcal{O}_XY/\mathcal{I} \oplus \mathcal{I}/\mathcal{I}^2 \oplus \mathcal{I}^2/\mathcal{I}^3 \oplus \cdots$$

where the $\mathcal{O}_X$-algebra structure comes from the surjective morphism $\mathcal{O}_Y \to i_*\mathcal{O}_X$. For notational simplicity, we will use $\mathcal{O}_Y(\mathcal{I})$ to denote the graded ring on Y, then we know that the above graded ring is $\mathcal{O}_Y(\mathcal{I})/\mathcal{I}\mathcal{O}_Y(\mathcal{I})$ Then we can consider the relative proj structure:

$$\mathcal{P}roj_X(\mathcal{O}_Y(\mathcal{I})/\mathcal{I}\mathcal{O}_Y(\mathcal{I})) \to X$$

Then you can check that the functoriality of relative gluing we discussed in the etale cohomology gives you a global morphism (though I don't recommen you check it here since everything is pretty canonical so you should be able to convince yourself that the functoriality holds):

$$\begin{tikzcd}
\mathcal{P}roj_X(\mathcal{O}_Y(\mathcal{I})_\bullet/\mathcal{I}\mathcal{O}_Y(\mathcal{I})_\bullet) \arrow[r, hook] \arrow[d] & \mathcal{P}roj_Y(\mathcal{O}_Y(\mathcal{I})_\bullet) \arrow[d] \\
X \arrow[r, hook] & Y
\end{tikzcd}$$

Since this diagram is locally Cartesian, it's globally Cartesian, and we also know that the morphism on the top is an effective Cartier divisor. Therefore, after we check that the universal property, this will be the blow-up diagram.

The reason we discussed this is because later when we have a regular embedding $X \hookrightarrow Y$, you will see that $\mathcal{O}_Y(\mathcal{I})_\bullet/\mathcal{I}\mathcal{O}_Y(\mathcal{I})_\bullet$ is a vector bundle, therefore we will have a projectivization of a vector bundle. But for now, let's first prove that the above diagram indeed satisfies the universal property:

Proof. Let's assume that I is generated by $(x_1, \cdots, x_n)$. Then the first observation we make is that this morphism $Proj(B(I)_\bullet) \to SpecB$ is an isomorphism away from $SpecB/I$. To prove it, we need to consider $D(x_1) \subset SpecB$ and let's consider the following diagram:

$$\begin{tikzcd} Proj(B(I)_\bullet) \arrow[d] & Proj(B(I)_\bullet \otimes_B B_{x_1}) \arrow[l, hook'] \arrow[d] \\ SpecB & SpecB_{x_1} \arrow[l, hook'] \end{tikzcd}$$

then we need to prove that the vertical morphism on the right is an isomorphism. Let's think about what is this graded ring $B(I)_\bullet \otimes_B B_{x_1}$. I claim that it's isomorphic to $B_{x_1}[t]$. To see why, we know that:

$$B(I)_\bullet \otimes_B B_{x_1} = B_{x_1} \oplus I_{x_1} \oplus I_{x_1}^2 \oplus \cdots$$

However, since $x_1 \in I$, we know that $I_{x_1} = B_{x_1}$, therefore this localization just gives a infinite sum of B_{x_1} indexed by $n \geqslant 0$:

$$B_{x_1} \oplus B_{x_1} \oplus \cdots$$

This is isomorphic to $B_{x_1}[t]$ where we define:

$$B_{x_1}[t] \to B_{x_1} \oplus B_{x_1} \oplus \cdots, \qquad t \mapsto 1_1$$

where the subscript of 1 means that it's in degree 1. The inverse is defined to be $a \mapsto at^d$ if $ai \in B_{x_1}$ is in degree d. Therefore we know that the morphism $Porj(B(I)_\bullet \otimes_B B_{x_1}) \to SpecB_{x_1}$ is just the usual morphism we see $ProjB_{x_1}[t] \to SpecB_{x_1}$, which is an isomorphism. We are done.

As we said before, it suffice to consider the case where we have reduced Y to an affine open $SpecB$ and π is $SpecB/I \to SpecB$. Now to check that the following diagram satisfies the universal property:

$$\begin{tikzcd} Proj(B(I)_\bullet/IB(I)_\bullet) \arrow[r, hook] \arrow[d] & Proj(B(I)_\bullet) \arrow[d] \\ SpecB/I \arrow[r, hook] & SpecB \end{tikzcd}$$

We need to assume that we have another Cartesian diagram over $SpecB/I \hookrightarrow SpecB$, where the top morphism is an effective Cartier divisor, then we know that to check the universal property, we can assume that the morphism on the top is affine as morphisms are uniquely determined by how they behave locally: Therefore we can assume we work with the following diagram:

$$\begin{tikzcd} SpecR/(t) \arrow[r, hook] \arrow[d] & SpecR \arrow[d] \\ SpecB/I \arrow[r, hook] & SpecB \end{tikzcd}$$

by our understanding of the pullback on affines, we can assume that if the morphism $SpecR \to SpecB$ is induced by $B \to R$, then $(t) = (\phi(x_1), \cdots, \phi(x_n))$. Since t is a nonzerodivisor, we know that there is a unique element a_n in R such that $a_n t = \phi(x_n)$, we will call $a_n = \phi(x_n)/t$. And if you think about what is $(\phi(x_1)/t, \cdots, \phi(x_n)/t)$, we know that since there are some $b_1, \cdots, b_n$ such that:

$$b_1\phi(x_1) + \cdots + b_n\phi(x_n) = t$$

Then we know that $b_1\phi(x_1)/t + \cdots + b_n\phi(x_n)/t = 1$, therefore $(\phi(x_1)/t, \cdots, \phi(x_n)/t)$ is $(1) = R$. Then, consider the following diagram:

$$\begin{tikzcd}
 & & & SpecR/(t) \arrow[r, hook] \arrow[dddll] & SpecR \arrow[dll, dashed, "!\exists"'] \arrow[dddll, "\pi"] \\
 & & & & \\
Proj(B(I)_\bullet/IB(I)_\bullet) \arrow[r, hook] \arrow[dr] & Proj(B(I)_\bullet) \arrow[drr] & & & \\
 & SpecB/I \arrow[rr, hook] & & SpecB &
\end{tikzcd}$$

We know that if there is a unique morphism from $SpecR$ to $Proj(B(I)_\bullet)$ lifting π, then there is a unique morphism from $SpecR/t$ to $Proj(B(I)_\bullet/IB(I)_\bullet)$ due to the universal property of Cartesian diagrams and the fact that $Proj(B(I)_\bullet/IB(I)_\bullet) \hookrightarrow Proj(B(I)_\bullet)$ is a monomorphism. Therefore out job is just to figure out this unique morphism from $SpecR$ to $Proj(B(I)_\bullet)$ that lifts π.

To see this morphism exists and is unique, recall $Proj(B(I)_\bullet) \to SpecB$ is projective thus separated, therefore we know that if we can describe a morphism $SpecR \to Proj(B(I)_\bullet)$ that extends the morphism $SpecR_t \to SpecB$, we want to conclude that the morphism must be unique. But here we face a similar question we saw when we prove the blow-up closure lemma, that is $Proj(B(I)_\bullet) \to SpecB$ is not a monomorphism, therefore after we compose it we can not assume that the two original morphisms agree.

However, we have a way to fix it just like what we did in the proof of the blow-up closure lemma, which is to consider:

$$\begin{tikzcd}
 & & SpecR/(t) \arrow[r, hook] \arrow[dl] & SpecR \arrow[dl, dashed, "!\exists"'] \arrow[d, "\pi"] & SpecR_t \arrow[l, hook'] \arrow[dl, dashed] \arrow[d] \\
Proj(B(I)_\bullet/IB(I)_\bullet) \arrow[r, hook] \arrow[d] & Proj(B(I)_\bullet) \arrow[dr] & & U \times_B Proj(B(I)_\bullet) \arrow[ll, hook'] \arrow[r, "\simeq"] & U \\
SpecB/I \arrow[rr, hook] & & SpecB \arrow[rr, hookleftarrow] & & U
\end{tikzcd}$$

Then we know that if we have a morphism $SpecR \to Proj(B(I)_\bullet)$ that lifts $SpecR \to SpecB$, we know that we automatically get a morphism $SpecR_t \to U \times_B Proj(B(I)_\bullet) \to U$, however we know that since $U \times_B Proj(B(I)_\bullet) \to U$ is an isomorphism, we know that if there is such a morphism, it is unique. This together with the unique extension from away an effective Cartier divisor means that if there is a morphism $SpecR \to ProjB(I)_\bullet$, it's unique, thus the question is only existence.

For the existence, let's first consider ring map:

$$\phi' : B(I)_\bullet \to R$$

sending elements X in degree d to $\phi(X) = \phi(X)/t^d$. This makes sense since if X is in degree 0, then the map is just ϕ, then for X in higher degree, it's first in I^d, therefore we know that $\phi(X)$ is in $(\phi(x_1), \cdots, \phi(x_n))^d = (t)^d$, therefore is something times t^d, again t^d is a nonzero divisor, therefore it makes sense to talk about $\phi(X)/t^d$. This is obviously a ring morphism, thus it gives us a morphism $SpecR \to SpecB(I)_\bullet$. For notational reasons, let's use X_i be the degree 1 element x_i, here we use the capital case to indicate that it's in degree 1 not 0. Then we know that the inverse image of $D(X_i)$ is $D(\phi(x_i)/t)$. Since we have already know that $\phi(x_i)/t$'s generate R, therefore we know that the image of the morphism $SpecR \to SpecB(I)_\bullet$ is contained in $SpecB(I)_\bullet - V(X_1, \cdots, X_n)$, therefore we have the diagram:

$$\begin{tikzcd}
SpecB(I)_\bullet & & SpecB(I)_\bullet - V(X_1, \cdots, X_n) \arrow[ll, hook'] \arrow[dr] & \\
 & SpecR \arrow[ul] \arrow[ur] \arrow[rr] & & SpecB
\end{tikzcd}$$

which tells us that every morphism essentially is a morphism of $SpecB$-schemes.

On the other hand, we know that we have a $SpecB$-scheme morphism $SpecB(I)_\bullet - V(X_1, \cdots, X_n) \to ProjB(I)_\bullet$, which is defined to be the n sections $X_1, \cdots, X_n$ in $B(I)_\bullet$, which is the global section of the structure sheaf, which is really like how morphism to the projective n-space behave. Here we define the morphism first locally, for example:

$$Spec(B(I)_\bullet)_{X_1} \to D_+(X_1) = Spec((B_\bullet)_{X_1})_0$$

which is defined by the inclusion. Then we need to check it's compatible on the intersection, which is the following diagram for a general graded ring S which is finitely generated in degree 1:

$$\begin{tikzcd}
S_{x_1} \arrow[d] & (S_{x_1})_0 \arrow[l, hook'] \arrow[d] \\
S_{x_1x_2} & (S_{x_1x_2})_0 \arrow[l, hook'] \\
S_{x_2} \arrow[u] & (S_{x_2})_0 \arrow[l, hook'] \arrow[u]
\end{tikzcd}$$

which is basically what we need to check in order to define the proj, therefore we are done. But this essentially gives us the following extended diagram:

$$\begin{tikzcd}
SpecB(I)_\bullet & & SpecB(I)_\bullet - V(X_1, \cdots, X_n) \arrow[ll, hook'] \arrow[r] \arrow[drr] & & Proj(B(I)_\bullet) \arrow[d] \\
 & SpecR \arrow[ul] \arrow[ur] \arrow[rrr, "\pi"] & & & SpecB
\end{tikzcd}$$

so we just define our desired morphism to be the composition $SpecB \to SpecB(I)_\bullet - V(X_1, \cdots, X_n) \to Proj(B(I)_\bullet)$, we are done. $\square$

After we are done with the proof, we can immediately check that the motivating example we did in the first section is indeed a blow-up:

Example 3.2. Notice that the closed embedding of the origin into the affine plane is $I = (x, y) \subset k[x, y] = B$, therefore I claim the Rees-algebra is:

$$B[X, Y] = k[x, y][X, Y]/(xY - yX)$$

where X, Y are degree 1 and x, y are degree 0. To see why this is true, consider the morphism:

$$k[x, y][X, Y] \to B(I)_\bullet$$

where $X \mapsto x$ and $Y \mapsto y$ but both x, y are considered to be degree 1 elements, not degree 0 elements. It's clear that $xY - yX \mapsto 0$. Therefore we have an induced morphism:

$$k[x, y][X, Y]/(xY - yX) \to B(I)_\bullet$$

clearly this is still a morphism of graded rings, and clearly it's surjective. We want to show it's an isomorphism, I claim that first $k[x, y][X, Y]/(xY - yX)$ is a domain, since $xY - yX$ is linear in X, Y, therefore irreducible and in a UFD it implies it's prime. Then we know that we can compute it's dimension by looking at the transcendence degree of the fraction field over k. Notice that the stuff we are quotienting out becomes $\frac{xY}{X} - y = 0$, therefore in the fraction field, we can just think of it as a field generated over k by three algebraically independent variables, more precisely, we have an isomorphism:

$$Frac(k[x, y][X, Y]/(xY - yX)) \to k(x, X, Y), \quad k(x, y, z) \mapsto Frac(k[x, y][X, Y]/(xY - yX))$$

therefore we know that the dimension of the domain is 3. Similarly, what is the fraction field of $B(I)_\bullet$, to answer this question, we first notice that we can think of $B(I)_\bullet$ as $k[x, y][xt, yt] \subset k[x, y][t]$, where we use powers of t to show the degree, therefore we know that the faction field is $k(x, y, t)$ (the isomorphism here is easy to make), therefore is still of dimension 3. Therefore this surjective morphism is an isomorphism.

We can also compute that $B(I)_\bullet/IB(I)_\bullet$ is $k \oplus \langle x, y\rangle \oplus \langle x^2, xy, y^2\rangle \oplus \cdots$, i.e. $k[x, y]$, and we have a map of graded rings:

$$k[x, y][X, Y]/(xY - yX) \to k[x, y] \qquad X \mapsto x, Y \mapsto y$$

Therefore, we know that the blow-up diagram is:

$$\begin{tikzcd} Projk[x,y] \arrow[r, hook] \arrow[d] & Projk[x,y][X,Y]/(xY-yX) \arrow[d] \\ (0,0) \arrow[r, hook] & Speck[x,y] \end{tikzcd}$$

I will then let you see that this description is the same as in the motivating example.

Remark 3.3. Recall that if $D \hookrightarrow X$ is an effective Cartier divisor, we have the normal bundle to D, defined by $\mathcal{O}_X(D)|_D$, i.e. the invertible sheaf corresponding to D on the total space restricted to D.

In this case of blow-up a point on the plane, we know that the effective Cartier divisor in $Projk[x, y][X, Y]/(xY-yX)$ corresponds to $\mathcal{O}(1)$, therefore we know that the invertible sheaf is $\mathcal{O}(-1)$ on $Projk[x, y][X, Y]/(xY - yX)$, and after you restrict in on $Projk[x, y]$, you still get $\mathcal{O}(-1)$ as this is typically true for any $ProjS/I \to ProjS$ where S is a graded ring, finitely generated in degree 1 by degree 0 elements and I is a graded ideal of S, we have seen this before, where you can check the isomorphism using transition functions.

Later, if we blow-up a regular subvariety inside a regular k-variety, we will see that the effective Cartier divisor will be a projective bundle over X.

Our example above can be generalized to:

Proposition 3.4. If $i : X \hookrightarrow \mathbb{P}^n_k$ is a projective scheme, then we know that this closed embedding $i : X \hookrightarrow \mathbb{P}^n_k$ is isomorphic to some $Projk[x_0, \cdots, x_n]/I \hookrightarrow Projk[x_0, \cdots, x_n]$, therefore let's assume that X is just $Projk[x_0, \cdots, x_n]/I$ where I is a graded ideal not containing $(x_1, \cdots, x_n)$, we can assume that I is not the whole ring. Then consider the **affine cone** over X, which is $Speck[x_0, \cdots, x_n]/I$, I claim that if we blow it up at the origin, the exceptional divisor we get is isomorphic to X. Moreover, the normal bundle is isomorphic to $\mathcal{O}_X(-1) = i^*\mathcal{O}(-1)$.

Proof. For notational simplicity, let's just write $B = k[x_0, \cdots, x_n]/I$ and $J = (x_0, \cdots, x_n)/I$, then we we know that the blow-up diagram is:

$$\begin{tikzcd} ProjB(J)_\bullet/JB(J)_\bullet \arrow[r, hook] \arrow[d] & ProjB(J)_\bullet \arrow[d] \\ SpecB/J \arrow[r, hook] & SpecB \end{tikzcd}$$

We only need to figure out what is $B(J)_\bullet/JB(J)_\bullet$. If you consider what this quotient does here, first we know that:

$$B(J)_\bullet = B \oplus J \oplus J^2 \oplus J^3 \oplus \cdots$$

Then after we quotient out $JB(J)_\bullet$, we know that what it does on each degree, for example, what it does on B/I is $(B/I)/(J/I) = B/J + I = B/J = k$. As another example, we know that on degree 2 we have $(J/I)/(J^2/I) = J/(I + J^2)$, which is all the linear combinations of x, y, subject to the relations in I, we keep going and find that the ring is just $k[x_0, \cdots, x_n]/I$, therefore after we take the proj, it's isomorphic to X.

We also want to figure what is the normal bundle of this effective Cartier divisor $ProjB(J)_\bullet/JB(J)_\bullet \hookrightarrow ProjB(J)_\bullet$. Let's consider the following diagram:

$$\begin{tikzcd} ProjB(J)_\bullet/JB(J)_\bullet \arrow[r, hook] \arrow[d, "\simeq"] & ProjB(J)_\bullet \arrow[d] \\ X = ProjB \arrow[r, hook] & Projk[x_0, \cdots, x_n] \end{tikzcd}$$

This is how we identify the exceptional divisor with X, i.e. the vertical left isomorphism. I will let you check that if the right vertical morphism is defined by the graded k-algebra morphism defined by x_i maps to x_i where the second x_i is in degree 1 of $B(J)_\bullet$ makes the diagram commute. Obviously, $B(J)_\bullet$ is also generated by $x_0, \cdots, x_n$, in degree 1 over degree 0, therefore if we pullback $\mathcal{O}(1)$ on $Projk[x_0, \cdots, x_n]$ via the morphism $ProjB(J)_\bullet \to Projk[x_0, \cdots, x_n]$, it's exactly $O(1)$ on $ProjB(J)_\bullet$. Therefore we are done, via the commutativity of pulling back line bundles among composition of morphisms. □

Remark 3.5. Consider the closed embedding:

$$X = \mathbb{P}^1_k = Projk[x_0, x_1] \to Projk[x_0, \cdots, x_n] = \mathbb{P}^n_k$$

Then the affine cone of X is just the affine plane, then we know that the blow-up at the origin of the affine cone is X, whose normal bundle is the pullback of $\mathcal{O}(-1)$ on $\mathbb{P}^n$ to $\mathbb{P}^1$. This complete the proof of our earlier claim that the normal bundle of the exceptional divisor of the blow-up of the plane at the origin is $\mathcal{O}(-1)$ on $\mathbb{P}^1$.

Definition 3.6. Suppose that $X \hookrightarrow Y$ is a closed subscheme cut out by a finite type quasicoherent ideal sheaf $\mathcal{I}$, then we define the **normal cone** of X in Y to be:

$$N_X Y = \mathcal{S}pec_X(\mathcal{O}_Y/\mathcal{I} \oplus \mathcal{I}/\mathcal{I}^2 \oplus \mathcal{I}^2/\mathcal{I}^3 \oplus \cdots)$$

Recall that the exceptional divisor is defined using the same graded $\mathcal{O}_X$-algebra, but it's taking the relative proj, here we take the relative spec.

In particular, if $X = p \in Y$ is just a closed point, we will call the normal cone the **tangent cone** to Y at p. Since the exceptional divisor is the relative proj of the same graded $\mathcal{O}_X$-algebra, we will call the exceptional divisor the **projectivized tangent cone**.

To give some intuition of the tangent cone, let's consider the following example:

Example 3.7. Let's consider the nodal cubic $Speck[x,y]/(y^2 - x^2 - x^3)$, now to avoid distraction, let's assume k is not of characteristic 2. Then let's compute the tangent cone at the closed point (x, y). Since the scheme we are considering here is affine, by definition, we only need to consider the usual spec of the following algebra:

$$k[x,y]/(y^2 - x^2 - x^3)/(x,y) \oplus (x,y)/(y^2 - x^2 - x^3)/(x,y)^2 \oplus (x^2, xy, y^2)/(y^2 - x^2 - x^3)/(x,y)^3 \cdots$$

This is not difficult to see that this is isomorphic to the graded $k[x,y]/(y^2 - x^2 - x^3)$ algebra:

$$k[x,y]/(y^2 - x^2)$$

and the algebra structure is given by:

$$k[x,y]/(y^2 - x^2 - x^3) \to k[x,y]/(y^2 - x^2), \qquad x, y \mapsto 0$$

If you remember that the nodal cubic assumes an X-shape near the origin, i.e. the closed point (x, y) and the tangent cone we compute is globally an X-shape, you get the intuition that the tangent cone at the origin is just a close-up of the local situation at the origin.

Proposition 3.8. Suppose that $S_\bullet$ is a finitely generated graded k algebra generated by degree 1 over degree 0. We have seen that $S_\bullet$ is just $k[x_0, \cdots, x_n]/I$ where I is an homogeneous ideal contained in $(x_0, \cdots, x_n)$ and we have an isomorphism between $ProjS_\bullet$ and the exceptional divisor of the blow-up at the origin of $SpecS_\bullet$. Here we will show that the tangent cone to $SpecS_\bullet$ at the origin is isomorphic to $SpecS_\bullet$ itself.

Proof. This proposition should be contrasted with the above example as we see that in the nodal cubic case, the ideal is generated by an element that is not homogeneous, thus the ring is not a graded ring.

Let's say I is generated by $f_1, \cdots, f_n$ all homogeneous elements with degree $d_1, \cdots, d_n$. Then our job, if we denote $J = (x_1, \cdots, x_n)/I$, is just to compute the ring:

$$S_\bullet/J \oplus J/J^2 \oplus \cdots$$

Say $f_1, \cdots, f_n$ is in increasing order, then before we hit J^{f_1}/J^{f_1+1}, we know that quotient by I doesn't play any role, and beyond that if we keep going, what happens is that what we quotient in I gets kept, thus in the end we indeed get $S_\bullet$ in the end. □

Now we look at a very special class of blow-up, which is **blow-up of regular embedding**, i.e. the case where $X \hookrightarrow Y$ is a regular closed embedding.

We begin with an important algebra result:

Theorem 3.9. Let A be any ring, and if I is a ideal generated by a regular sequence $a_1, \cdots, a_d$, then we have an isomorphism:

$$Sym^n_{A/I}(I/I^2) \to I^n/I^{n+1}$$

Corollary 3.10. If $i : X \hookrightarrow Y$ is a regular embedding with ideal sheaf $\mathcal{I} \subset \mathcal{O}_Y$, then the natural map $Sym^n_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2) \to \mathcal{I}^n/\mathcal{I}^{n+1}$ is an isomorphism.

Moreover, since we know that the conormal sheaf $\mathcal{I}/\mathcal{I}$ is locally free on X, it's symmetric power is locally free thus $\mathcal{I}^n/\mathcal{I}^{n+1}$ is also locally free.

Before we see how to prove the theorem, which is quite technical, let's see how it can be useful:

Proposition 3.11. Suppose that $X \hookrightarrow Y$ is a regular closed embedding with ideal sheaf $\mathcal{I}$, the total space of the normal sheaf, i.e. the normal bundle is the normal cone $N_X Y$, and the exceptional divisor is a projective bundle over X.

Proof. Recall that if X is a scheme and $\mathcal{F}$ is a locally free sheaf of finite rank, the total space of $\mathcal{F}$ is defined to be:

$$\mathcal{S}pec(\mathrm{Sym}^\bullet_{\mathcal{O}_X}\mathcal{F}^\vee)$$

Recall that the conormal sheaf is $\mathcal{I}/\mathcal{I}^2$, which is locally free on X when $X \hookrightarrow Y$ is regular, then we know that the normal bundle is just:

$$\mathcal{S}pec(\mathrm{Sym}^\bullet_{\mathcal{O}_X}\mathcal{I}/\mathcal{I}^2)$$

Then recall that the definition of the normal cone of $X \hookrightarrow Y$ is:

$$\mathcal{S}pec(\oplus_{n\geqslant 0}\mathcal{I}^n/\mathcal{I}^{n+1})$$

Clearly we have an isomorphism between the two graded algebras coming from the above corollary.

Clearly, if we take the relative proj, which gives us the exceptional divisor $E_X Y$, by definition, it's a projective bundle over X. $\square$

Proposition 3.12. With the notation in the previous proposition, we know that the normal bundle to $E_X Y$ in $\mathrm{Bl}_X Y$ is $\mathcal{O}(-1)$, where we consider the exceptional divisor as the projective bundle over X.

Proof. Recall that by definition the normal bundle of an effective Cartier divisor $D \hookrightarrow Y$ is $\mathcal{O}(D)|_D$. Recall that in $\mathrm{Bl}_X Y$, the divisor corresponding to $E_X Y$ is always $\mathcal{O}(-1)$. Then the assertion holds from the general fact that the pullback of $\mathcal{O}(-1)$ from $\mathcal{P}roj_Y(\mathcal{O}_Y(\mathcal{I})_\bullet)$ to $\mathcal{P}roj_X(\mathcal{O}_Y(\mathcal{I})_\bullet/(\mathcal{I}\mathcal{O}_Y(\mathcal{I})_\bullet))$ is still $\mathcal{O}(-1)$. $\square$

Proposition 3.13. If X is a reduced closed point of Y then first X is a regular point of Y, then in particular the total space of the tangent space at p is the tangent cone to Y at p.

Proof. Actually, what we need to prove is just that p is indeed a regular point, since all the other assertions are just the special case of a regular embedding where we have a regular point. (Recall that in this case $\mathcal{I}/\mathcal{I}^2$ is isomorphic only supported at the single point p and the stalk at that point is just the cotangent space.)

To show a reduced closed point is a regular point, we only need to recall that any dimension 0 reduced ring is a field. Therefore, if A is a ring and m is a maximal ideal such that A_m is reduced, then A_m is necessarily a field, therefore m_m is 0, so is m/m^2, therefore indeed regular. $\square$

One quite dramatic result of regular embedding is the following:

Proposition 3.14. If $X \hookrightarrow Y$ is a closed embedding of smooth varieties over k, then $\mathrm{Bl}_X Y$ is also smooth.

Proof. We have to use two results we still have to complete in later chapters: first, we need to use the fact that over any field k smooth varieties are regular, then we need to use the result that this result can be checked when $k = \overline{k}$, i.e. algebraically closed. The second result will depend on the fact that the blow-up construction commutes with base change to the algebraic closure, which we will prove later in a generalized result.

Now, we know that away from $E_X Y$, $\mathrm{Bl}_X Y$ is isomorphic to Y, therefore smooth. Therefore we only need to show that the points on $E_X Y$ is smooth. Since $X \hookrightarrow Y$ is a closed embedding of smooth varieties over an algebraically closed field, then X, Y are both regular, then we recall the result that regular closed subschemes in a regular scheme is regularly embedded. Therefore we know that $X \hookrightarrow Y$ is a regular closed embedding, therefore we know that $E_X Y$ over X is a projective bundle over X, thus $E_X Y$ over k is also smooth. To check this, recall that if X is any scheme and $\mathcal{F}$ is a locally free sheaf over X of finite rank n, then:

$$\mathbb{P}\mathcal{F} \to X$$

is a smooth morphism. To see why, we know that to check smoothness, we only need to do it locally on the target, and we can choose affine opens of X such that the bundle is trivial, then over this affine open, say $SpecA$ the morphism is just $\mathbb{P}^n_A \to SpecA$, which is clearly smooth. Then we use the composition of smooth morphisms is smooth.

Therefore we know that $E_X Y$ is again regular since again we work over algebraically closed fields. Then we need to use the regular slicing criterion for Noetherian schemes for example a k-variety, which means that if $X \hookrightarrow Y$ is an effective Cartier divisor where X is regular, then for points in Y that is also in X, it's regular in Y. This is because if (A, m, k) is a local ring of finite dimension, say n, with A/f regular local and f a nonzero divisor in m, we know that the dimension of A/f is $n-1$ and we know that $(m/f)/(m/f)^2$ is just $m/(m^2 + (f))$. Then we know since f is a nonzero-divisor and f is in m, from A/f is again regular, we know f is not in m^2. Then we know that the dimension of m/m^2 over k is n and we know that the dimnension of A is also n, thus regular local.

Therefore again, since we are over a algebraic closed field, regular implies smooth. □

Now let's prove the theorem: We begin with a rather simple observation. Consider the following morphism of graded A/I-algebra:

$$(A/I)[X_1, \cdots, X_d] \to \oplus_{n=0}^{\infty} I^n/I^{n+1}$$

where X_i is mapped to a_i in I/I^2. Clearly this is surjective, and if this is an isomorphism, consider the following diagram:

$$\begin{tikzcd}
\mathrm{Sym}^{\bullet}_{A/I}(I/I^2) \arrow[rr, two heads] \arrow[dr, "\simeq"'] & & \oplus_{n\geqslant 0}^{\infty} I^n/I^{n+1} \\
& (A/I)[X_1, \cdots, X_d] \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

I claim that the natural morphism $\mathrm{Sym}^{\bullet}_{A/I}(I/I^2) \to (A/I)[X_1, \cdots, X_d]$ is an isomorphism as well. To see why, recall that since I is generated by a regular sequence $a_1, \cdots, a_d$, we know that $(A/I)^{\oplus d} \to I/I^2$ defined by $e_i \mapsto a_i$ is an isomorphism, therefore we have the diagram:

$$\begin{tikzcd}
& \mathrm{Sym}^{\bullet}_{A/I}(I/I^2) \arrow[dr, "\simeq"] & \\
\mathrm{Sym}^{\bullet}_{A/I}((A/I)^{\oplus d}) \arrow[ur, "\simeq"'] \arrow[rr, "\simeq"] & & (A/I)[X_1, \cdots, X_d]
\end{tikzcd}$$

Therefore, we only need to prove the surjective morphism is indeed an isomorphism. Recall that if A is a Noetherian ring (here for practical reasons we impose the Noetherian condition), we have the notion of the so called total fraction ring $K(A)$, which is defined to be the the product of A_p where p is an associated point of A, or understood as the ring of rational functions on $SpecA$. As A is Noetherian, it only has finitely many associated points, therefore we can consider the following diagram:

$$\begin{tikzcd} A \arrow[rr, hook] \arrow[dr, hook] & & K(A) \\ & A[a_2/a_1, \cdots, a_d/a_1] \arrow[ur, hook] & \end{tikzcd}$$

To be clear about what does this diagram mean, recall that a_1 is not a zerodivisor, then I claim that $D(a_1)$ contains all the associated points, as for Noetherian rings, the union of all associated points are all the zero-divisors, so $a_2/a_1, \cdots, a_d/a_1$ make sense as regular functions living on $D(a_1)$. Then the notation $A[a_2/a_1, \cdots, a_d/a_1]$ is used to denote the subring of $K(A)$ generated as an A-algebra by $a_2/a_1, \cdots, a_d/a_1$ in $K(A)$.

Now we can consider the A-algebra morphism:

$$\beta : A[T_2, \cdots, T_d] \to A[a_2/a_1, \cdots, a_d/a_1]$$

where $T_i \mapsto a_i/a_1$. It's very easy to see that $a_1T_i - a_i$ is always in the kernel. But moreover we can show that the kernel is indeed only generated by $L_i = a_1T_i - a_i$:

Lemma 3.15. The kernel of β is $(L_2, \cdots, L_d)$.

Proof. What happens when $d = 1$? We will just consider the morphism $A \to A$, which is clearly an isomorphism. Then what happens when $d = 2$? In this case, β is just:

$$A[T_2] \to A[a_2/a_1]$$

Say $F[T_2]$ is in the kernel, then we know $F(a_2/a_1)$ is 0. Then consider the new polynomial $a_1^{\deg F}F[T_2]$, whose leading term has a a_1 factor, therefore we can perform the division algorithm and get:

$$a_1^{\deg F}F[T_2] = G[T_2](a_1T - a_2) + R$$

Since we have multiplied enough powers of a_1 to F, we can get that R is in fact in A, and then we plug $T_2 = a_2/a_1$, we can conclude that R is in fact 0, thus $G[T_2](a_1T - a_2) = 0 \bmod a_1^{\deg F}$, or in other words in $A/(a_1^{\deg F})$. Now since a_2 is not a zerodivisor in $A/(a_1^{\deg F})$, we consider the coefficients of $G[T_2]$, we know that if it's not divisible by $a_1^{\deg F}$, then after you multiply by a_2, it's still not, and since the degree of $a_1T_2G[T_2]$ is one larger, $G[T_2](a_1T - a_2)$ can't be 0. Therefore all the coefficients should be divided by $a_1^{\deg F}$. Therefore $F[T_2]$ is divisible by $(a_1T - a_2)$, we are done.

For the general situation where $d > 2$, let's assume the result for all smaller d. Let $A' = A[a_2/a_1]$, then we know that $a_1, a_3, a_4, \cdots, a_d$ is a regular sequence in A' to see why, recall that $A' = A[T_2]/(a_1T_2 - a_2)$, then we know that if $a_1F[T_2] = 0$ in A', we know that $a_1^{\deg F}F[T_2]$ is divisible by $a_1T_2 - a_2$. But using the same argument as in the case when $d = 2$ we can conclude that $F[T_2]$ itself is also divisible by $a_1T_2 - a_2$, thus we are done. Then consider $A'/(a_1)$, which is $A[T_2]/(a_1, a_2)$, then since a_3 is a nonzerodivisor in $A/(a_1, a_2)$, we know that it's also a nonzerodivisor in $A[T_2]/(a_1, a_2)$. Similarly we get $a_1, a_3, \cdots, a_d$ is a regular sequence. Then consider the composition:

$$A[T_2, \cdots, T_d] \to A'[T_3, \cdots, T_d] \to A'[a_3/a_1, \cdots, a_d/a_1] = A[a_2/a_1, \cdots, a_d/a_1]$$

This is the morphism β we care about. Let's first consider the first morphism, this time consider $B = A[T_3, \cdots, T_d]$, then we know that a_1, a_2 are still a regular sequence in B. Then I will let you to show that using similar methods as in $d = 2$ that this morphism can be written as:

$$B[T_2] \to B[a_2/a_1]$$

where $B[a_2/a_1]$ is considered as a subring of the total fraction ring of B, then we know that the kernel is (L_2). By our inductive hypothesis, the kernel of the second morphism is $(L_3, \cdots, L_d)$. Recall that if we consider $A \to A/J$ then the inverse image of I/J in A is $I+J$. Then we know that since $A'[T_3, \cdots, T_d]$ is $A[T_2, \cdots, T_d]/L_2$, and $(L_3, \cdots, L_d)$ is generated again by $L_3, \cdots, L_d$, then we know that the kernel is:

$$(L_2) + (L_3 + \cdots, L_d) = (L_2, \cdots, L_d)$$

□

This result helps to prove the theorem:

Proof. Recall that our goal is to prove that the following A/I algebra morphism is an isomorphism:

$$\alpha : (A/I)[X_1, \cdots, X_d] \to \oplus_{n \geqslant 0}^{\infty} I^n/I^{n+1}$$

Consider the following diagram:

$$\begin{tikzcd} & A[X_1, \cdots, X_d] \arrow[dl, two heads] \arrow[dr, two heads, "\alpha'"] & \\ (A/I)[X_1, \cdots, X_d] \arrow[rr, two heads, "\alpha"] & & \oplus_{n \geqslant 0}^{\infty} I^n/I^{n+1} \end{tikzcd}$$

If we can show that the morphism α' has kernel $(a_1, \cdots, a_r)$, we are done since we know that the zero ideal in $(A/I)[X_1, \cdots, X_d]$ has preimage in $A[X_1, \cdots, X_d]$ $(a_1, \cdots, a_d)$ and you can check that for any nonzero element in $(A/I)[X_1, \cdots, X_d]$ by definition, couldn't be in $(a_1, \cdots, a_r)$.

Now suppose that we have some F in $A[X_1, \cdots, X_d]$ that is mapped to 0, we need to show it's in $(a_1, \cdots, a_d)$. We can assume that F is homogeneous, say of degree n, then we know that $x = F(a_1, \cdots, a_d)$ is in I^{n+1} by assumption. Then we know that x can be written as $F'(a_1, \cdots, a_d)$ where the coefficients of F' is in I. Then consider $G = F - F'$, we know that $G(a_1, \cdots, a_d) = 0$. If we can show that G is in $(a_1, \cdots, a_n)$, then since F' is in $(a_1, \cdots, a_n)$, we can conclude F is in $(a_1, \cdots, a_d)$.

Now from our assumption that $F(a_1, \cdots, a_d) = 0$, I claim that $F(1, a_2/a_1, \cdots, a_d/a_1) = 0$ in $A[a_2/a_1, \cdots, a_d/a_1]$ as well. This is because that in we can do everything in the total fraction ring of A, where a_1 is invertible. But the previous lemma tells us:

$$F(1, T_2, \cdots, T_d) \in (a_1T_2 - a_2, \cdots, a_1T_d - a_d)$$

Notice that the coefficients of $F(1, T_2, \cdots, T_d)$ are the coefficients of F, in $(a_1, \cdots, a_n)$, we are done. □

4 Examples and Calculations

In this section, we will work through a few examples, partially because we will keep use blow-up from time to time later, but more important to see how blow-up can be useful.

Throughout this section, let's assume k is of characteristic 0 to avoid distraction.

Example 4.1. Let's again first look back on our old example of the blow-up of the plane $\mathbb{A}^2$ at the origin. The explicit description of this is going to help with other examples later.

We have given a detailed explanation of the blow-up and the exceptional divisor. What we want to say here is that since the blow-up is:

$$Projk[x, y, X, Y]/(xY - yX)$$

It's clearly a closed subscheme of $Projk[x, y, X, Y]$, and if you do the local computation and gluing, this is isomorphic to $\mathbb{A}^2 \times \mathbb{P}^1$. Therefore the blow-up is naturally a closed subsheme of $\mathbb{A}^2 \times \mathbb{P}^1$, defined by $(xY - yX)$ as defined before.

Let's also try to describe the exceptional divisor locally in charts. We know that the blow-up can be described in two charts, first $Speck[x, y, X/Y]/(xY/X - y)$ and second $Speck[x, y, Y/X]/(x - yX/Y)$. We know that the homogeneous ideal in $k[x, y, X, Y]/(xY - yX)$ that cut out by (x, y), therefore we know that in both charts, the closed subscheme is again cut out by (x, y). Notice that:

$$k[x, y, X/Y]/(xY/X - y) \simeq k[x, X/Y], \qquad k[x, y, Y/X]/(x - yX/Y) \simeq k[y, Y/X]$$

Then x, y in this new coordinates are $x, X/Yx$ and $y, yX/Y$, therefore the ideal is principal and generated by a nonzero divisor.

As a first example that is not blow-up up something affine, we consider:

Example 4.2. Let p be a k-valued point of $\mathbb{P}^2_k$, we will show that there is an isomorphism between $\mathrm{Bl}_p\mathbb{P}^2$ and Hirzebruch surface $\mathbb{F}_1 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(1))$.

Let's use the projective coordinate x, y, z on $\mathbb{P}^2$. Let's first prove that blow-up distinct k-valued points gives isomorphic surfaces, therefore we don't have to work with an arbitrary point but only need to work with one specific k-valued point. To show this, notice that we have the following diagram:

$$\begin{tikzcd} p_1 \arrow[r, hook] \arrow[d, "\mathrm{Id}"] & \mathbb{P}^2 \arrow[d, "\simeq"] \\ p_2 \arrow[r, hook] & \mathbb{P}^2 \end{tikzcd}$$

The right vertical morphism is given by a element in the projective linear group, as it acts transitively on all k-valued points on $\mathbb{P}^2$. Then I will let you show by the universal property of blow-up that the blow-up and the exceptional divisors are all isomorphic.

Let's compute the Hirzebruch surface $\mathbb{F}_1$. By definition, it's:

$$\mathbb{P}(\mathrm{Sym}^\bullet(\mathcal{O} \oplus \mathcal{O}(1))^\vee)$$

Recall that the dual of $\mathcal{O} \oplus \mathcal{O}(1)$ is:

$$\mathrm{Hom}(\mathcal{O} \oplus \mathcal{O}(1), \mathcal{O}) = \mathcal{O} \oplus \mathcal{H}om(\mathcal{O}(1), \mathcal{O})$$

Recall that we have an isomorphism between:

$$\mathcal{H}om(\mathcal{O}(1), \mathcal{O}) \simeq \mathcal{H}om(\mathcal{O}, \mathcal{O}(-1))$$

defined on each open U by if we have a morphism: $\mathcal{O}(1)|_U \to \mathcal{O}|_U$, we map it to the morphism after we tensor with $\mathcal{O}(-1)$. Obviously this is an isomorphism on each U, it's also compatible with the restriction since tensor does. Therefore we know the dual of $\mathcal{O} \oplus \mathcal{O}(1)$ is just $\mathcal{O} \oplus \mathcal{O}(-1)$.

We know that $\mathbb{P}^1$ is covered by two $\mathbb{A}^1$, i.e. $D_+(x), D_+(y)$, then we know that the Hirzebruch surface can be described by the following diagram:

$$\begin{tikzcd} Proj\mathrm{Sym}^\bullet(k[\frac{x}{y}] \oplus k[\frac{x}{y}]) \arrow[d] & Proj\mathrm{Sym}^\bullet(k[\frac{x}{y}, \frac{y}{x}] \oplus k[\frac{x}{y}, \frac{y}{x}]) \arrow[l, hook'] \arrow[r, hook] \arrow[d] & Proj\mathrm{Sym}^\bullet(k[\frac{y}{x}] \oplus k[\frac{y}{x}]) \arrow[d] \\ Spec k[\frac{x}{y}] & Spec k[\frac{x}{y}, \frac{y}{x}] \arrow[l, hook'] \arrow[r, hook] & Spec k[\frac{y}{x}] \end{tikzcd}$$

What matters is how the top row is glued together. First recall that $\mathcal{O}(-1)$ on $\mathbb{P}^1$ is defined by two trivial line bundle on $Spec k[\frac{x}{y}]$ and $Spec k[\frac{y}{x}]$, with transition function on the overlap defined by $\frac{x}{y} \mapsto \frac{x}{y}^2$. Then we know the above diagram can be written as:

$$\begin{tikzcd} Proj k[\frac{x}{y}, u, v] \arrow[d] & Proj k[\frac{x}{y}, \frac{y}{x}, u, v] \arrow[l, hook'] \arrow[r, hook] \arrow[d] & Proj k[\frac{y}{x}, u, v] \arrow[d] \\ Spec k[\frac{x}{y}] & Spec k[\frac{x}{y}, \frac{y}{x}] \arrow[l, hook'] \arrow[r, hook] & Spec k[\frac{y}{x}] \end{tikzcd}$$

where $\frac{x}{y}, \frac{y}{x}$ on the top row is understand to be degree 0 and u, v are understood to be degree 1. The morphism $Proj k[\frac{x}{y}, \frac{y}{x}, u, v] \to Proj k[\frac{x}{y}, u, v]$ is defined by $\frac{x}{y} \mapsto \frac{x}{y}, u \mapsto u, v \mapsto \frac{x}{y}v$. The morphism $Proj k[\frac{x}{y}, \frac{y}{x}, u, v] \to Proj k[\frac{y}{x}, u, v]$ is defined by $\frac{y}{x} \mapsto \frac{y}{x}, u \mapsto u, v \mapsto v$. Therefore, we can use four $\mathbb{A}^2$ to describe the Hirzebruch surface, illustrated as follows:

$$\begin{tikzcd}
Speck[\frac{x}{y}, \frac{u}{v}] & Speck[\frac{x}{y}, \frac{y}{x}, \frac{u}{v}] \arrow[l] \arrow[r] & Speck[\frac{y}{x}, \frac{u}{v}] \\
Speck[\frac{x}{y}, \frac{u}{v}, \frac{v}{u}] \arrow[u] \arrow[d] & & Speck[\frac{y}{x}, \frac{u}{v}, \frac{v}{u}] \arrow[u] \arrow[d] \\
Speck[\frac{x}{y}, \frac{v}{u}] & Speck[\frac{x}{y}, \frac{y}{x}, \frac{v}{u}] \arrow[l] \arrow[r] & Speck[\frac{y}{x}, \frac{v}{u}]
\end{tikzcd}$$

the gluing morphisms are indicated above.

To compute the blow-up, for simplicity, let's use the point $[1:0:0]$, i.e. the point (y,z). We will describe the blow-up using four charts of $\mathbb{A}^2$, we will use x, y, z as the projective coordinate on $\mathbb{P}^2$. First notice that the point is in $D_+(x)$ while not in $D_+(y)$ and $D_+(z)$, therefore we know that locally over $D_+(y), D_+(z)$, the blow-up is trivial, i.e. the blow-up is the space downstairs. Over $D_+(x)$, the blow-up is again the blow-up of the plane at the origin, therefore, could be described by 2 $\mathbb{A}^2$, as what we have saw before. Recall that the blow-up of the origin is covered by:

$$Speck[x, y, \frac{X}{Y}]/(\frac{X}{Y}y - x) = Speck[t, \frac{X}{Y}], Speck[x, y, \frac{Y}{X}]/(x\frac{Y}{X} - y) = Speck[t, \frac{Y}{X}]$$

The gluing is given by:

$$\begin{tikzcd}
Speck[t, \frac{X}{Y}] \arrow[dr, "t \mapsto t\frac{Y}{X}"'] & & Speck[t, \frac{Y}{x}] \arrow[dl, "t \mapsto t"] \\
& Speck[t, \frac{X}{Y}, \frac{Y}{X}] &
\end{tikzcd}$$

Then globally, the blow-up is described by:

$$\begin{tikzcd}
? \arrow[r] \arrow[d] & Projk[\frac{x}{y}, \frac{z}{y}, T] = Speck[\frac{x}{y}, \frac{z}{y}] \\
Proj[\frac{y}{x}, \frac{z}{x}, \frac{Y}{X}, \frac{Z}{X}]/(\frac{y}{x}\frac{Z}{X} - \frac{z}{x}\frac{Y}{X}) & Projk[\frac{x}{y}, \frac{z}{y}, \frac{y}{z}, T] = Speck[\frac{x}{y}, \frac{z}{y}, \frac{y}{z}] \arrow[u] \arrow[d] \\
? \arrow[r] \arrow[u] & Projk[\frac{x}{z}, \frac{y}{z}, T] = Speck[\frac{x}{z}, \frac{y}{z}]
\end{tikzcd}$$

We need to figure out what is the mysterious intersection in the two questions marks. The representation of the blow-up of the plane at the origin as $Projk[\frac{y}{x}, \frac{z}{x}, \frac{Y}{X}, \frac{Z}{X}]/(\frac{y}{x}\frac{Z}{X} - \frac{z}{x}\frac{Y}{X})$ makes what happens here not that clear. But if we consider the following diagram, it will be come much clear:

$$\begin{tikzcd}
Projk[\frac{y}{x}, \frac{z}{x}, \frac{x}{y}] \oplus k[\frac{y}{x}, \frac{z}{x}, \frac{x}{y}] \oplus k[\frac{y}{x}, \frac{z}{x}, \frac{x}{y}] \oplus \cdots \arrow[r, hook] \arrow[d] & Projk[\frac{y}{x}, \frac{z}{x}] \oplus (\frac{y}{x}, \frac{z}{x}) \oplus (\frac{y}{x}, \frac{z}{x})^2 \oplus \cdots \arrow[d] \\
Speck[\frac{y}{x}, \frac{z}{x}, \frac{x}{y}] \arrow[r, hook] & Speck[\frac{y}{x}, \frac{z}{x}]
\end{tikzcd}$$

This is the definition of the relative proj on the intersections, and we know that we have the isomorphism:

$$\begin{tikzcd}
Projk[\frac{y}{x}, \frac{z}{x}, \frac{x}{y}, T] \arrow[r, hook] \arrow[d, "\simeq"'] & Projk[\frac{y}{x}, \frac{z}{x}, \frac{Y}{X}, \frac{Z}{X}]/(\frac{y}{x}\frac{Z}{X} - \frac{z}{x}\frac{Y}{X}) \arrow[d, "\simeq"] \\
Projk[\frac{y}{x}, \frac{z}{x}, \frac{x}{y}] \oplus k[\frac{y}{x}, \frac{z}{x}, \frac{x}{y}] \oplus k[\frac{y}{x}, \frac{z}{x}, \frac{x}{y}] \oplus \cdots \arrow[r, hook] & Projk[\frac{y}{x}, \frac{z}{x}] \oplus (\frac{y}{x}, \frac{z}{x}) \oplus (\frac{y}{x}, \frac{z}{x})^2 \oplus \cdots
\end{tikzcd}$$

We know that the morphism on the top is defined by $\frac{Y}{X} \mapsto \frac{y}{x}T$, $\frac{Z}{X} \mapsto \frac{z}{x}T$, therefore we know that since $\frac{x}{y}$ is invertible, the image of the open immersion of $D_+(\frac{Y}{X})$. This shows that we again can describe the blow-up in four charts, and I will let you check it agrees with the above description of $\mathbb{F}_1$.

So far we have only computed some blow-ups and exceptional divisors. Recall that another set of very important notion in computing blow-up is the **total** an **proper transform**, which is described in the Blow-up Closure lemma. In this part, we will see how we can use these ideas to resolve singularities:

Example 4.3. Recall that the nodal cubic in the plane $\mathbb{A}^2$ is $Speck[x,y]/(y^2-x^3-x^2)$, a closed subscheme of the plane. Consider the blow-up of the origin of $\mathbb{A}^2$ and we will compute the total proper transform of the nodal cubic and see how it resolves the singularity at the origin. It's indicated in the diagram below, using the notation we used in the blow-up closure lemma:

$$\begin{tikzcd} W = (x,y) \arrow[r, hook] \arrow[d] & Speck[x,y]/(y^2-x^3-x^2) = Z \arrow[d, hook] \\ X = (x,y) \arrow[r, hook] & Speck[x,y] = Y \end{tikzcd}$$

Recall that the proper transform of Z, by the blow-up closure lemma, is the blow-up of Z along (x,y). Let's compute it here. Let's write $A = k[x,y]/(y^2-x^3-x^2)$, then I claim that the blow-up is the proj of:

$$A[X,Y]/(xY-yX)$$

To see why, we first see that we have a clear surjective morphism:

$$A[X,Y] \to A \oplus m \oplus m^2 \oplus \cdots$$

Clearly, $xY-yX$ is in the kernel and we need to prove that the kernel is no larger than this. What we notice that is A is a domain, therefore we can talk about it fraction field K, and we have:

$$\begin{tikzcd} A[X,Y] \arrow[r, two heads] \arrow[d, hook] & A \oplus m \oplus m^2 \arrow[r, hook] & A[T] \arrow[d, hook] \\ K[X,Y] \arrow[rr, "X\mapsto xT, Y\mapsto yT"] & & K[T] \end{tikzcd}$$

Notice that the bottom morphism is clearly surjective as x, y are not 0 in K, then we have a surjective morphism:

$$K[X,Y]/(yX-xY) \to K[T]$$

Clearly these are domains, then if we can prove they are of the same dimension, we are done. Clearly, $K[X,Y]/(yX-xY) \simeq K[Z]$, dimension 1, we are done.

We know that the preimage of $(yX-xY)$ in $A[X,Y]$ is $(yX-xY)$, the kernel of the first map in the top horizontal row is $(yX-xY)$. Therefore we can again use two charts to describe the proper transform, i.e. $D_+(X)$ and $D_+(Y)$.

In the first chart, i.e. $SpecA[\frac{Y}{X}]/(x\frac{Y}{X}-y)$, we know it's actually:

$$Speck[x,y,\frac{Y}{X}]/(y^2-x^2-x^3, x\frac{Y}{X}-y) = Speck[x,\frac{Y}{X}]/(x^2\frac{Y}{X}^2-x^2-x^3)$$

Notice that can be considered as the scheme theoretic union of the exceptional divisor defined by x^2 and the curve defined by $\frac{Y}{X}^2-1-x$. Using the Jacobian partial derivative criterion, since the partial with respect to x is always 1, the corank is always 1, therefore smooth thus regular. When $x=0$, we have $\frac{Y}{X}^2-1=0$, i.e. the exceptional divisor meets the curve at $\frac{Y}{X} = \pm 1$. What about the other patch, we have

$$Speck[x,y,\frac{X}{Y}]/(x-y\frac{X}{Y}, y^2-x^2-x^3) = Speck[y,\frac{X}{Y}]/(y^2-y^2\frac{X}{Y}^2-y^3\frac{X}{Y}^3)$$

Again, this factors as the exceptional Cartier divisor defined by y^2 and the curve $1 - \frac{X}{Y}^2 - y\frac{X}{Y}^3$, again, we can use the Jacobian criterion to show smoothness.

What about the total transform of the nodal cubic curve. It's defined as the fibered product $\mathrm{Bl}_X Y \times_Y Z$, that is:

$$\begin{tikzcd}
T_Z \arrow[r] \arrow[d, hook] & Speck[x,y]/(y^2 - x^2 - x^3) \arrow[d, hook] \\
\mathrm{Bl}_X Y \arrow[r] & Speck[x,y]
\end{tikzcd}$$

We know that we can define $\mathrm{Bl}_X Y$ using two charts, thus we can define T_Z using two charts, we omit the computation.

References