Appendix 10 for Math 632: 2025 Winter Notes

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This part of the notes basically reviews cohomology of quasicoherent sheaves on schemes, which is surprisingly elegant and simple. We could have defined this far before right now, but as you will see, cohomology is most useful when companied with what we have developed so far.

1 Desired Properties of Cohomology

Before defining cohomology groups of quasicoherent sheaves on explicitly, we first describe the important properties they have which is in some sense more important than the formal definition.

Suppose that X is a quasicompact and separated A-scheme, we will then define for each quasicoherent sheaf $\mathcal{F}$ on X, an A-module denoted by $H^i(X, \mathcal{F})$ for each i. In particular, if $A = k$ a field, we get k-vector spaces and in this case we define $h^i(X, \mathcal{F}) = dim_k H^i(X, \mathcal{F})$. The following are the properties we will prove later:

1. Each H^i is a covariant functor from $Qcoh_X \to Mod_A$.
2. The functor H^0 is identified with the functor of global sections Γ in the sense that there is a natural isomorphism between the two functors.
3. If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence, then there is a long exact sequence in the form:
$$0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^0(X, \mathcal{G}) \to H^1(X, \mathcal{F}) \to \cdots$$
where $H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G})$ is induced by the functor H^0 and the connecting morphism $H^0(X, \mathcal{G}) \to H^1(X, \mathcal{F})$ need to be defined later.
4. If $\pi : X \to Y$ is any morphism of quasicoherent separated A-schemes, then notice that by cancellation theorem, π is also quasicompact and separated, then $\pi_*\mathcal{F}$ is also quasicoherent. Then there is a natural morphism $H^i(Y, \pi_*\mathcal{F}) \to H^i(X, \mathcal{F})$ extending $\Gamma(Y, \pi_*\mathcal{F}) \to \Gamma(X, \mathcal{F})$. On the other hand, if $\mathcal{G}$ is a quasicoherent sheaf on Y, we can define $\mathcal{F} = \pi^*\mathcal{G}$ and then via the natural adjunction morphism $\mathcal{G} \to \pi_*\pi^*\mathcal{G}$, we can form:
$$H^i(Y, \mathcal{G}) \to H^i(Y, \pi_*\pi^*\mathcal{G}) \to H^i(X, \pi^*\mathcal{F})$$
called the **pullback morphism**.
5. If $\pi : X \to Y$ is an affine morphism, then the natural morphism in the previous property is an isomorphism $H^i(Y, \pi_*\mathcal{F}) \simeq H^i(X, \mathcal{F})$ for all i. In particular, when π is a closed embedding and Y is $\mathbb{P}^n_A$, we can translate the computation of H^i on a projective scheme to the computation on $\mathbb{P}^n_A$.
6. If X can be covered by n affine opens, then $H^i(X, \mathcal{F}) = 0$ for all $i \geqslant n$ and all quasicoherent sheaf $\mathcal{F}$. In particular, if X is affine itself, then we can cover it with 1 affine open, thus H^1 vanishes for all quasicoherent sheaves, therefore $\Gamma(X, -)$ is exact, which is something we already saw.

   Remark 1.1. We have a vanishing result of cohomology that is of different flavor which states that if $\dim X = n$, then $H^i(X, \mathcal{F}) = 0$ for all $i > n$. We will prove this for projective k-schemes, and there will be a brief discussion on the general case later.

7. The functors H^i behaves well with respect to filtered colimits:

$$H^i(X, \text{colim}\mathcal{F}_j) = \text{colim}H^i(X, \mathcal{F}_j)$$

8. Finally, we will identify the cohomology of all $\mathcal{O}(m)$ on $\mathbb{P}^n_A$ in the following theorem:

   Theorem 1.2. (a): $H^0(\mathbb{P}^n_A, \mathcal{O}(m))$ is a free A-module of rank $\binom{n+m}{m}$ if $m \geqslant 0$.
   (b): The top degree cohomology $H^n(\mathbb{P}^n_A, \mathcal{O}(m))$ is a free A-module of rank $\binom{-m-1}{-m-m-1}$ if $m \leqslant -n-1$.
   (c): All the other $H^i(\mathbb{P}^n_A, \mathcal{O}(m)) = 0$.

Remark 1.3. The theorem in part 8 of the properties above is one of the most important theorems we will see in this chapter as it foreshadows a lot of facts we will prove later and it's also an example of other properties from 1 to 7. For example, we see that since $\mathbb{P}^n_A$ can be covered by n affine opens, H^i for $i \geqslant n$ vanishes. Also you see that all the cohomology groups are finitely generated A-modules, this holds more generally for all coherent sheaves on proper A-schemes, we will prove this later using **Grothendieck's Coherence Theorem**. Moreover, the fact that the top cohomology vanishes for $m > -n-1$ is an example of **Kodaira vanishing**.

Remark 1.4. If in particular we consider $\mathbb{P}^n_k$ where k is a field, we see that the top cohomology group is one dimensional for $m = -n-1$, which is the first example of a **dualizing sheaf**.

We will also prove that there is a perfect pairing:

$$H^i(\mathbb{P}^n, \mathcal{O}(m)) \times H^{n-i}(\mathbb{P}^n, \mathcal{O}(-m-n-1)) \to H^n(\mathbb{P}^n, \mathcal{O}(-n-1))$$

which is the first example of **Serre duality**.

Finally the alternating sum $\sum(-1)^i h^i(\mathbb{P}^n, \mathcal{O}(m))$ is a polynomial in m, which is the first example of the **Hilbert polynomial**.

Before we prove the properties 1 to 8 and the stuff we mentioned in the remarks, let's first try to use the properties to prove something useful, as a motivation:

Theorem 1.5. 1. For any coherent sheaf $\mathcal{F}$ on a projective A-scheme X where A is Noetherian, $H^i(X, \mathcal{F})$ is a coherent A-module (finitely generated since A is Noetherian.)

2. Further more, for $m \gg 0$, $H^i(X, \mathcal{F}(m)) = 0$ for all $i > 0$ even without the hypothesis that A is Noetherian.

Proof. Recall that any projective A-scheme is of the form $i : X \hookrightarrow \mathbb{P}^n_A$ where i is a closed embedding. Furthermore, the pushforward of a coherent sheaf by a closed embedding on locally Noetherian schemes is still coherent, therefore via the isomorphism $H^i(X, \mathcal{F}) \simeq H^i(\mathbb{P}^n_A, i_*\mathcal{F})$, we could just assume $X = \mathbb{P}^n_A$.

Now recall that since $\mathcal{O}(1)$ is ample, we have a surjective morphism:

$$\mathcal{O}(-m)^{\oplus i} \to \mathcal{F}$$

where the direct sum is finite and m is some positive integer. Let $\mathcal{G}$ be the kernel, which gives us a SES:

$$0 \to \mathcal{G} \to \mathcal{O}(-m)^{\oplus i} \to \mathcal{F} \to 0$$

This gives a LES ends with:

$$H^n(X, \mathcal{G}) \to H^n(X, \mathcal{O}(-m)^{\oplus i}) \to H^n(X, \mathcal{F}) \to 0$$

since we have $n+1$ affine opens to cover X and the vanishing theorem tells you that after degree n everything vanishes. But recall that H^n commutes with direct sums and $H^n(X, \mathcal{O}(-m))$ is finitely generated, thus $H^n(X, \mathcal{F})$ is finitely generated for all $\mathcal{F}$, thus in particular $H^i(X, \mathcal{G})$ is also finitely generated A-module, then consider the sequence in degree $n-1$, since A is Noetherian, any submoudle of finitely generated module is finitely generated, then break the long exact sequence into short exact sequence, similar arguments tell you that $H^{n-1}(X, \mathcal{F})$ is finitely generated, then keep going down to get the desired properties.

To prove the vanishing $H^i(X, \mathcal{F}(N))$ for large enough N and $i \geqslant 1$, we twist the short exact sequence:

$$0 \to \mathcal{G} \to \mathcal{O}(-m)^{\oplus i} \to \mathcal{F} \to 0$$

by $\mathcal{O}(N)$ for large enough enough N such that $H^i(X, \mathcal{O}(-m+N)) = 0$ for all $i \geqslant 1$. Then we can still use the LES to show $H^n(\mathcal{F}(-m+N)) = 0$ for all $\mathcal{F}$, thus $H^n(\mathcal{G}(-m+N)) = 0$ as well. Then plus the fact that $H^{n-1}(X, \mathcal{O}(-m+N)) = 0$, we conclude $H^{n-1}(X, \mathcal{F}(-m+N)) = 0$, then we keep going down, we are done. □

In particular, we have proved following, which we have came across to before:

Corollary 1.6. Any projective k-scheme has a finite dimensional space of functions. More generally if A is Noetherian and $\mathcal{F}$ is a coherent sheaf on a projective A-scheme then $H^0(X, \mathcal{F})$ is a coherent A-module.

Remark 1.7. Actually the above theorem is still true for proper k schemes, not just projective, which will be a consequence of Grothendieck's coherence theorem.

One important consequence of the theorem above is:

Proposition 1.8. Suppose that $\pi : X \to Y$ is a projective morphism of locally Noetherian schemes, then the pushforward of a coherent sheaf on X is still coherent on Y.

Proof. This question is local on Y, therefore we can just assume $Y = SpecA$ where A is Noetherian thus X is a projective A-scheme, and $\mathcal{F}$ is coherent on X, we know that the pushforward is still quasicoherent, since projective morphism is proper, thus qcqs, then we only have to show that over $SpecA$, the global section of the pushforward is coherent, this is the consequence of the above theorem. □

Another characterization of finite morphisms using projectivity and finite fiber can also be given using cohomology. We first prove the following:

Proposition 1.9. Suppose that Y is locally Noetherian then $\pi : X \to Y$ is finite iff it's projective and affine.

Proof. We have already proved that finite always implies projectivity and it's of course affine.

For the other direction, we already know it's affine, thus we only need to show that suppose that $SpecA$ is an affine open of Y with inverse image $SpecB$, then B is a finite A-module. In other words, we need to prove that the pushforward of $\mathcal{O}_X$ is a coherent $\mathcal{O}_Y$-module, which is ensured by projectivity and Noetherian hypothesis. □

The following theorem is finally the statement that projectivity plus finite fibers is finite, and it has a few useful consequences:

Theorem 1.10. Suppose $\pi : X \to Y$ is a morphism with Y Noetherian, then π is projective and has a finite fibers iff it's finite. Or equivalently, it's projective and quasifinite iff it's finite.

Proof. First recall that quasifinite is finite fibers plus finite type, but projectivity implies proper, thus finite type.

We already know that finite morphisms is projective and have finite fibers without any Noetherian hypothesis on Y, thus we only have to assume $\pi : X \to Y$ is projective and of finite fibers, then show it's finite.

Recall that we can check finiteness affine locally on Y, thus suppose that q is a point in Y, we choose an affine open $SpecA$ of q and since π is projective, we know π can be decomposed to:

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "\pi"'] & \mathbb{P}^n_A \arrow[d, "\rho"] \\ & SpecA \end{tikzcd}$$

where i is a closed embedding and ρ is the structure morphism of $\mathbb{P}^n_A$. Now consider the following diagram:

$$\begin{tikzcd}
 & & \mathbb{P}^n_{k(q)} \arrow[d] \arrow[ddl] \\
\pi^{-1}q \arrow[r] \arrow[urr, dashed] \arrow[dr] & X \arrow[r, hook, "i"] \arrow[dr, "\pi"] & \mathbb{P}^n_A \arrow[d, "\rho"] \\
 & Speck(q) \arrow[r] & SpecA
\end{tikzcd}$$

Notice that since $\pi^{-1}q$ is a finite set, then the image in $\mathbb{P}^n_{k(q)}$ is a finite set, recall that we have proved that there is a hypersurface H_q in $\mathbb{P}^n_{k(q)}$ that misses the image. Say this hypersurface is defined by a homogeneous equation f with coefficients in $k(q)$. Recall that $\mathbb{P}^n_{k(q)} \to \mathbb{P}^n_A$ is induced by $A[x_0, \cdots, x_n] \to k(q)[x_0, \cdots, x_n]$. Since $A \to k(q)$ is just $A \to A_p \to A_p/pA_p$, we can assume that for each element in A_p/pA_p, say the class of $\frac{a}{b}$, where b is not in p, we know that the class of $\frac{b}{1}$ is not 0, then we can multiply a nonzero element such that it's the image of something in A, then consider the coefficients of f, we can multiply with an elements such that each of them comes from the image of something in A and since multiplying by a scalar doesn't change the hypersurface defined by f, we can assume we can lift f back to A which defines a hypersurface H, and we know that H lies in the image of H_q. I claim that H doesn't intersect with $\pi^{-1}q$ since if some point p in the image of π^{-1} in $\mathbb{P}^n_{k(q)}$ represents a homogeneous prime p in $k(q)[x_0, \cdots, x_n]$, then it's image q in $A[x_0, \cdot, x_n]$ is the inverse image of p. Then we know that f is not in p in $k(q)[x_0, \cdots, x_n]$, say it's lift in $A[x_0, \cdots, x_n]$ is F, we know that F is not in q since if so f would be in p. Then consider $H' = \pi(H \cap X)$, which is closed in $SpecA$ since projective morphism is closed. Then H' doesn't contain q since H doesn't contain $\pi^{-1}q$. Now let $U = SpecA - H'$, this will be an open subset of $SpecA$ that contains q. Finally, we know that $\pi^{-1}U \to U$ is also projective as projectivity is preserved by base change. But on the other hand, it's the composition:

$$\pi^{-1}U \hookrightarrow \mathbb{P}^n_U - H \to U$$

where the first morphism is a closed embedding and the second is affine, which comes from the following diagram:

$$\begin{tikzcd}
 & & \mathbb{P}^n_U \arrow[d, hook] \arrow[ddl] \\
\pi^{-1}U \arrow[r, hook] \arrow[urr, dashed] \arrow[dr] & X \arrow[r, hook, "i"] \arrow[dr, "\pi"] & \mathbb{P}^n_A \arrow[d, "\rho"] \\
 & U \arrow[r, hook] & SpecA
\end{tikzcd}$$

where you know $\pi^{-1}U \to U$ is the composition of $\pi^{-1}U \to \mathbb{P}^n_U \to U$. But recall that by definition of U H in $\mathbb{P}^n_U$ doesn't get mapped to U, also we know that the image of $\pi^{-1}U$ in $\mathbb{P}^n_U$ doesn't contain any point in H since if it does, then some point in H is mapped to U a contradiction. Therefore the above composition is:

$$\pi^{-1}U \hookrightarrow \mathbb{P}^n_U - H \to U$$

The first is a closed embedding since we know first that $\pi^{-1}U \to \mathbb{P}^n_U$ is a closed embedding. To see why this is true, notice that the morphism $\pi^{-1}U \to \mathbb{P}^n_U$ we described can also be interpreted as the morphism in the following diagram:

$$\begin{tikzcd}
\pi^{-1}U \arrow[r, hook] \arrow[d, hook] & \mathbb{P}^n_U \arrow[r] \arrow[d, hook] & U \arrow[d, hook] \\
X \arrow[r, hook] & \mathbb{P}^n_A \arrow[r] & SpecA
\end{tikzcd}$$

as it's the unique morphism that makes a certain diagram commute. Then notice that the image is in $\mathbb{P}^n_U - H$, which is open, then if $\pi : X \to Y$ is a closed embedding that factors through the open embedding $U \hookrightarrow Y$, then $X \to U$ is also a closed embedding.

The second morphism $\mathbb{P}^n - H \to U$ is affine since this could be thought as the restriction of $\mathbb{P}^n_A - H \to SpecA$ to U then since $\mathbb{P}^n_A - H \to SpecA$ is affine, it's restriction is still affine. Therefore, we know that $\pi^{-1}U \to U$ is projective and affine, with U locally Noetherian, thus it's finite, we are done. □

A similar argument can be used to show upper semicontinuity of fiber dimensions of projective morphisms:

Proposition 1.11. Suppose that $\pi : X \to Y$ is a projective morphism where Y is locally Noetherian then $\{q \in Y : \dim \pi^{-1}q > k\}$ is a closed subset of Y, informally speaking, fiber dimensions can jump up on closed subsets of Y.

In particular, you can interpret the case where $k = -1$ as the fact that projective morphisms are closed. We have seen a version of this between finite type k-schemes, but this approach here is much simpler.

Proof. The assertion in the proposition is equivalent to showing $\{q \in Y : \dim \pi^{-1}q \leqslant k\}$ is open. That is to say, suppose that $q \in Y$ is a point with $\dim \pi^{-1}q \leqslant k$, then there is an neighborhood of q such that every point in the neighborhood has fiber dimension at most k.

Again, we choose an affine open of q, and from the projectivity of π, we know it looks like:

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "\pi"'] & \mathbb{P}^n_A \arrow[d, "\rho"] \\ & SpecA \end{tikzcd}$$

since we know q has fiber dimension at most k, then we know that it's image in $\mathbb{P}^n_{k(q)}$ has dimension at most k since we know that $\pi^{-1}q \to \mathbb{P}^n_{k(q)}$ is a closed embedding. Then since the image is of dimension at most k, then are $r+1$ nonempty hypersurfaces missing the image, say they are $H_0, \cdots, H_r$, then we again lift them to $\mathbb{P}^n_A$, and we know that $X \cap H_0 \cap \cdots \cap H_r$ in $SpecA$ is closed, and we will take out this closed subset and the remaining stuff is called U. I claim that point in U has fiber dimension at most k.

To see this, this time consider:

$$\pi^{-1}U \hookrightarrow \mathbb{P}^n_U - H_0 \cap \cdots \cap H_r \to U$$

with the first one being a closed embedding. Then say $p \in U$, then we can consider the closed embedding $\pi^{-1}p \hookrightarrow \mathbb{P}^n_{k(p)}$, I claim that the image misses the intersection of hypersurfaces $H_0, \cdots, H_r$ where not we think of the coefficients defining H_i as being in $k(p)$ by the canonical morphism $A \to A_p/pA_p$. This is because we have the following diagram:

$$\begin{tikzcd} \pi^{-1}p \arrow[r] \arrow[d, hook, "\text{closed}"] & \pi^{-1}U \arrow[r, hook] \arrow[d, hook, "\text{closed}"] & X \arrow[d, hook, "\text{closed}"] \\ \mathbb{P}^n_{k(p)} \arrow[r] \arrow[d] & \mathbb{P}^n_U \arrow[r, hook] \arrow[d] & \mathbb{P}^n_A \arrow[d] \\ Speck(p) \arrow[r] & U \arrow[r, hook] & SpecA \end{tikzcd}$$

and you notice that $\pi^{-1}p \to X \to \mathbb{P}^n_A$ missing $H_0 \cap \cdots \cap H_r$ thus $\pi^{-1}p \to \mathbb{P}^n_{k(p)}$ also misses $H_0 \cap \cdots \cap H_r$. This is again because if $f_0, \cdots, f_k$ lies in the point, then we know that $f_0, \cdots, f_k$ also lies in the preimage.

But this tells you that the image of $\pi^{-1}p$ in $\mathbb{P}^n_{k(p)}$ is at most k, otherwise, if it's dimension is $k+1$ or higher, then we know that since the codimension of $H_0 \cap \cdots \cap H_k$ is at most $k+1$, then we know that pick the irreducible component of $\pi^{-1}p$ whose dimension is $k+1$, we know it's codimension is $n-k-1$, then the codimension of $H_0 \cap \cdots \cap H_k$ plus the codimension of $\pi^{-1}p$ is at most $n-k-1+k+1 = n \leqslant n$, thus we know they should intersect, a contradiction. □

Remark 1.12. The key here is the fact that the codimension of intersection of $k+1$ hypersurfaces is at most $k+1$. We take out any irreducible component of this intersection, and all we need to prove is that for each such ireeducible component, the codimension is at most $k+1$, reduce this to prove that in each of the standard affine opens, it's of codimension at most $k+1$, which again reduces to the statement of Krull's theorem.

The final exercise of this chapter is on a different theme that we will come to later:

Proposition 1.13. Suppose that $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a SES of quasicoherent sheaves on X projective over A with $\mathcal{F}$ coherent, then after you twist by $\mathcal{O}(1)$ for enough times, the global sections functor becomes exact, i.e. for $n \gg 0$, we have a SES:

$$0 \to \Gamma(X, \mathcal{F}(n)) \to \Gamma(X, \mathcal{G}(n)) \to \Gamma(X, \mathcal{H}(n)) \to 0$$

Proof. Recall that for a scheme X projective over A, any coherent sheaf $\mathcal{F}$ enjoys the property that after you twist by $\mathcal{O}(1)$ for enough times $H^i(X, \mathcal{F}(n)) = 0$ for all $i \geqslant 1$, then the assertion follows from the LES. □

2 Definitions and Proofs of Key properties

In this section we give the formal definition cohomology of quasicoherent sheaves on a quasicompact and separated scheme X. We will also prove quite a chunk of the properties we listed in the previous section, but the last one that identifies the cohomology of $\mathcal{O}(m)$ on $\mathbb{P}^n_A$ should wait until the next section.

The general tool we will use is called Cech cohomology, which is usually defined using a limit over finer and finer covers of a space. But in algebraic geometry, this is much simpler and can be done using a single cover without worrying about the limits.

The reason we want quasicompact and separated is because we want the scheme to be covered by a finite collection of affine opens, the intersection of any two of which is still affine. This is true for example when X is quasiprojective over A. Now suppose that $\mathcal{F}$ is a qausicoherent sheaf on X and $\mathcal{U} = \{U_i\}_{i=1}^n$ is a finite collection of affine opens covering X. For $I \subset \{1, \cdots, n\}$, we define $U_I = \cap_{i \in I} U_i$, which is affine by separated assumption(A strong analogy for this definition is that in algebraic geometry or differential topology, people usually define cohomology using finite covers such that all the intersections are contractible). Now we consider the **Cech Complex**:

$$0 \longrightarrow \prod_{|I|=1} \mathcal{F}(U_i) \longrightarrow \prod_{|I|=2} \mathcal{F}(U_i) \longrightarrow \cdots \longrightarrow \prod_{|I|=n} \mathcal{F}(U_i) \longrightarrow \prod_{|I|=n+1} \mathcal{F}(U_i) \longrightarrow \cdots$$

where all the I are subsets of $\{1, \cdots, n\}$. To avoid confusion, we require the elements in I are ordered. The morphisms are defined to be $\mathcal{F}(U_I) \to \mathcal{F}(U_J)$ where $|J| = |I| + 1$ is 0 if I is not contained in J and if $J = I \cup \{j\}$ with j the k-th element in J, then it's defined to be $(-1)^{k-1}$ times the restriction map. It's obvious that this definition makes the sequence a complex: the composition is 0. This is easy to see since if you want to look at the morphism $\mathcal{F}(U_I) \to \mathcal{F}(U_K)$ then this is the sum of all possible $\mathcal{F}(U_I) \to \mathcal{F}(U_J) \to \mathcal{F}(U_K)$, and it's 0 if any of $\mathcal{F}(U_I) \to \mathcal{F}(U_J)$ or $\mathcal{F}(U_J) \to \mathcal{F}(U_K)$ is 0 and if none of them is zero then we know that $K = J \cup \{k\}$ and $J = I \cup \{j\}$, say j is the j-th element in J and k is the k-th element in K, then the morphism is just $(-1)^{j+k}$ times the restriction. However, if we define $J' = I \cap \{k\}$ and $K' = K$, we know that the restriction is $(-1)^{k+j-1}$ or $(-1)^{k+j+1}$ times the restriction, thus the sum is 0. This is because for example when $j < k$, then k is $k-1$-th element in J' and j is the j-th element in K. If $k < j$, it's reversed but and the sum is now $j + k + 1$, thus we are done, this suffice to show the composition is 0.

Definition 2.1. Define $H^i_{\mathcal{U}}(X, \mathcal{F})$ to be the i-th cohomology of the above complex, the index starts with 0, i.e. $\prod_{|I|=1} \mathcal{F}(U_i)$ is the 0-th element in this complex.

Note that if X is an A-scheme, then $H^i_{\mathcal{U}}(X, \mathcal{F})$ is an A-module for all i and we have almost successfully defined the Cech cohomology $H^i(X, \mathcal{F})$, except for the fact that a priori, our definition depends on the choice of the affine open cover $\mathcal{U}$.

Remark 2.2. By definition, it's already clear that $H^0_{\mathcal{U}}(X, \mathcal{F}) \simeq \Gamma(X, \mathcal{F})$ via a unique isomorphism since we can apply the sheaf axioms which states that the global section is the kernel of the first morphism in the Cech complex.

The following result is key for the construction of the connecting morphism we mentioned in the previous section:

Proposition 2.3. Suppose that:

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is a SES of sheaves of abelian groups on a topological space and $\mathcal{U}$ is a finite open cover such that on any intersection U_I, the map $\mathcal{F}(U_I) \to \mathcal{H}(U_I)$ is surjective, then we get a LES of cohomology for $H^i_{\mathcal{U}}$.

Proof. Recall that if we have a SES of complexes then cohomology gives you a LES in general, then the assumption we had that $\mathcal{F}(U_I) \to \mathcal{H}(U_I)$ is surjective is indeed the condition you need to have a SES of complexes, thus we are done. $\square$

Remark 2.4. Af this point, we can say that even though when we defined the Cech complex, it depends on the order we choose for the opens in $\mathcal{U}$, the cohomology and the corresponding LES doesn't as if you choose two different order of $\mathcal{U}$, you get isomorphic Cech complexes, and the naturality of connecting homomorphism and the funcotriality of cohomology tells you that we are good.

Remark 2.5. Recall that for a SES of quasicoherent sheaves on a quasicompact separated scheme X say $SpecA$ is any affine open, then we know that:

$$0 \to \Gamma(SpecA, \mathcal{F}) \to \Gamma(SpecA, \mathcal{G}) \to \Gamma(SpecA, \mathcal{H}) \to 0$$

is still exact (the key here is the last morphism is still surjective). Then we know that since any finite intersection of affine opens is still affine, the assumption in the previous proposition is always satisfied for arbitrary finite affine open covers, thus we get the connecting homomorphism naturally.

The following propositions tells you that the LES we obtained for $H^i_{\mathcal{U}}$ behaves well upon localization:

Proposition 2.6. Suppose that you have a SES of quasicoherent sheaves on an A-scheme X where $\pi : X \to SpecA$ is separated, then say $f \in A$ and we can consider it's pullback $\pi^* f$, we denote $X_f = \pi^{-1}(D(f))$, then say you have a finite affine cover $\mathcal{U}$ of X, whose restriction on X_f is called $\mathcal{U}$'. Then $\mathcal{U}'$ is still an affine cover of X_f, and the LES of cohomology associated to:

$$0 \to \mathcal{F}|_{X_f} \to \mathcal{G}|_{X_f} \to \mathcal{H}|_{X_f} \to 0$$

is identified with the LES of cohomology associated to:

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

localized at f.

Proof. Let's first show that there are natural isomorphisms such that the following diagram commutes:

$$\begin{tikzcd}
(H^i_{\mathcal{U}}(X, \mathcal{F}))_f \arrow[r] \arrow[d, "\simeq"'] & (H^i_{\mathcal{U}}(X, \mathcal{G}))_f \arrow[d, "\simeq"] \\
H^i_{\mathcal{U}'}(X_f, \mathcal{F}|_{X_f}) \arrow[r] & H^i_{\mathcal{U}'}(X_f, \mathcal{G}|_{X_f})
\end{tikzcd}$$

this is basically a restatement of quotient commutes with localization or in other words, localization is an exact functor.

We first notice that there are natural isomorphisms:

$$\begin{tikzcd}
(\mathcal{F}(U_I))_f \arrow[r] \arrow[d, dashed, "\simeq"] & (\mathcal{G}(U_I))_f \arrow[d, dashed, "\simeq"] \\
\mathcal{F}(U'_I) \arrow[r] & \mathcal{G}(U'_I)
\end{tikzcd}$$

where U'_I is the restriction of U_I to X_f. This is because $\mathcal{F}$ is a quasicoherent sheaf and localization of $\mathcal{F}(U_I)$ at f is the same as localization of $\mathcal{F}(U_I)$ at $\pi^* f$ restricted to U_I, the the isomorphism is just the fact that for quasicoherent sheaves, restriction to a distinguished open is just localization.

Therefore, suppose the Cech complex of $\mathcal{F}$ associated to the affine open cover $\mathcal{U}$ is denoted by $C(\mathcal{F},\mathcal{U})$ we have a isomorphism of SES of complexes:

$$\begin{tikzcd} 0 \arrow[r] & C(\mathcal{F},\mathcal{U})_f \arrow[r] \arrow[d, dashed, "\simeq"] & C(\mathcal{G},\mathcal{U})_f \arrow[r] \arrow[d, dashed, "\simeq"] & C(\mathcal{H},\mathcal{U})_f \arrow[r] \arrow[d, dashed, "\simeq"] & 0 \\ 0 \arrow[r] & C(\mathcal{F}|_{X_f},\mathcal{U}') \arrow[r] & C(\mathcal{G}|_{X_f},\mathcal{U}') \arrow[r] & C(\mathcal{H}|_{X_f},\mathcal{U}') \arrow[r] & 0 \end{tikzcd}$$

Then the functorializty of cohomology and naturality of the connecting homomorphism shows that the induced LES using $\mathcal{U}'$ is isomorphic to the LES of the localization of original complex. Therefore, the assertion in the proposition translates to the original LES localized at f is isomorphic to the LES of the localized complex. We only have to worry about the connecting morphism since with the same degree, the isomorphism just comes from the fact that localization commutes with quotient, i.e. there is a a natural isomorphism between the functor that localize first and then quotient and the functor that quotient then take the localization.

As for the connecting morphism, we only have to show that the following diagram commute:

$$\begin{tikzcd} (H^i_{\mathcal{U}}(X,\mathcal{H}))_f \arrow[r, "\delta_f"] \arrow[d, dashed, "\simeq"] & (H^{i+1}_{\mathcal{U}}(X,\mathcal{F}))_f \arrow[d, dashed, "\simeq"] \\ H^i(C(X,\mathcal{H})_f) \arrow[r] & H^{i+1}(C(X,\mathcal{F})_f) \end{tikzcd}$$

this requires us to have a clear understanding of how this connecting morphism is defined in our case. Notice that the origianl δ defined is via:

$$\delta([\gamma]) = [d\sigma]$$

where σ is any lift of γ in $C^i(\mathcal{G},\mathcal{U})$ which turns out to also be an element of $C^i(\mathcal{F},\mathcal{U})$. Then we know that after we localize:

$$\delta_f(\frac{[\gamma]}{f^n}) = \frac{[d\sigma]}{f^n}$$

and $[d\sigma]/f^n$ is mapped via the isomorphism to $[\frac{d\sigma}{f^n}]$. On the other hand, you know that $\frac{[\gamma]}{f^n}$ is mapped via the isomorphism to $[\frac{\gamma}{f^n}]$, then notice that a lift of $\frac{\gamma}{f^n}$ will be $\frac{\sigma}{f^n}$, then we only need to prove that $[d\frac{\sigma}{f^n}] = [\frac{d\sigma}{f^n}]$ where the first d is the differential in the localized complex, thus it's equal to $\frac{d\sigma}{f^n}$, we are done. ☐

We are finally ready to prove the following theorem:

Theorem 2.7. Again, let X be a quasicompact and separated scheme over A for some ring A, then $H^i_{\mathcal{U}}(X,\mathcal{F})$ is independent of the finite affine cover $\mathcal{U}$. More precisely, it will be a corollary of the following statement. Say $\mathcal{V} = \{V_i\}$ and $\mathcal{U} = \{U_i\}$ are two affine covers such that for each $U_i \in \mathcal{U}$, it's also in $\mathcal{V}$, i.e. $\mathcal{U} \subset \mathcal{V}$, we order $\mathcal{U}$ using the order of $\mathcal{V}$ (as we said before, the order doesn't matter), then the natural maps $H^i_{\mathcal{V}}(X,\mathcal{F}) \to H^i_{\mathcal{U}}(X,\mathcal{F})$ induced by the natural morphism of Cech complexes are isomorphism.

We will define the Cech cohomology group $H^i(X,\mathcal{F})$ to be $H^i_{\mathcal{U}}(X,\mathcal{F})$ where $\mathcal{U}$ is any finite affine cover.

Remark 2.8. Before the proof, you should realize that the natural morphism between two complexes associated to $\mathcal{U}$ and $\mathcal{V}$ are induced by projection. And the statement in the theorem can be simplified to just say the morphism of complexes is a quasi-isomorphism.

Proof. Notice that you only have to prove this for the case where $|\mathcal{V}| = |\mathcal{U}| + 1$ since other cases can be done inductively.

We will show that if $\mathcal{U} = \{U_i\}_{1 \leqslant i \leqslant n}$ is an affine open cover of X with U_0 is any other affine open, the natural map:

$$H^i_{\{U_i\}_{0 \leqslant i \leqslant n}} \to H^i_{\{U_i\}_{1 \leqslant i \leqslant n}}$$

is an isomorphism. The key is the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & \prod_{|I|=i-1,0\in I}\mathcal{F}(U_I) \arrow[r] \arrow[d] & \prod_{|I|=i,0\in I}\mathcal{F}(U_I) \arrow[r] \arrow[d] & \prod_{|I|=i+1,0\in I}\mathcal{F}(U_I) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \prod_{|I|=i-1}\mathcal{F}(U_I) \arrow[r] \arrow[d] & \prod_{|I|=i}\mathcal{F}(U_I) \arrow[r] \arrow[d] & \prod_{|I|=i+1}\mathcal{F}(U_I) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \prod_{|I|=i-1,0\notin I}\mathcal{F}(U_I) \arrow[r] \arrow[d] & \prod_{|I|=i,0\notin I}\mathcal{F}(U_I) \arrow[r] \arrow[d] & \prod_{|I|=i+1,0\notin I}\mathcal{F}(U_I) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

This is a SES of complexes with the lower two complexes being the Cech complex associated to $\{U_i\}_{0\leqslant i\leqslant n}$ and $\{U_i\}_{1\leqslant i\leqslant n}$(You need to think through what are the morphisms in the diagram). Then we know that the cohomology induces a LES, then if the top complex is exact, i.e. the cohomology vanishes, we are done.

But if you compute the cohomology of the top row, the i-th cohomology is just the cohomology $H^{i-1}_{\{U_j\cap U_0\}_{j>0}}(U_0,\mathcal{F}|_{U_0})$ except at H^0 since the start of this complex is $0\to\mathcal{F}(U_0)\to\prod_{j=1}^n\mathcal{F}(U_i\cap U_0)$ and by sheaf axioms, the cohomology is 0. Therefore, we only need to show the higher cohomology H^i with $i\geqslant 1$ is 0 for an affine A-scheme, which is the following theorem: □

Proposition 2.9. The higher Cech cohomology $H^i_{\mathcal{U}}(X,\mathcal{F})$ with $i\geqslant 1$ of any quasicoherent sheaf $\mathcal{F}$ on an affine A-scheme X vanishes for any finite affine cover $\mathcal{U}$.

Proof. This will be another partition of unity argument.

First notice that appending $\mathcal{F}(X)$ to the beginning of this sequence:

$$0\longrightarrow\mathcal{F}(X)\longrightarrow\prod_{i=1}^n\mathcal{F}(U_i)\longrightarrow\cdots$$

doesn't change the cohomology H^i of the original Cech complex with $i\geqslant 1$. Then if we can show this complex after we appending $\mathcal{F}(X)$ is exact, we are done.

To show it's exact, we use the following trick: we will first consider the case when one of the affine opens in $\mathcal{U}$ is the whole space, since the order doesn't matter, let's just assume it's U_0, then we consider the following SES of complexes:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \prod_{|I|=1,0\in I}\mathcal{F}(U_i) \arrow[r] \arrow[d] & \prod_{|I|=2,0\in I}\mathcal{F}(U_i) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathcal{F}(X) \arrow[r] \arrow[d] & \prod_{|I|=1}\mathcal{F}(U_i) \arrow[r] \arrow[d] & \prod_{|I|=2}\mathcal{F}(U_i) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathcal{F}(X) \arrow[r] & \prod_{|I|=1,0\notin I}\mathcal{F}(U_i) \arrow[r] \arrow[d] & \prod_{|I|=2,0\notin I}\mathcal{F}(U_i) \arrow[r] \arrow[d] & \cdots \\
 & & 0 & 0 &
\end{tikzcd}$$

You should notice that the middle row is the complex that we want to be exact. Moreover, the top row is exactly the bottom row, just shifted towards right by 1 and the differential differs from the bottom row by alternating powers of -1. The trick here is to show that the connecting morphism in the LES is an isomorphism, this proves that the cohomology of the middle row is 0 for each degree. This again requires the understanding of the definition of the connecting morphim. Say x is in the i-th place of the bottom row that is mapped to 0 using d of the bottom row, we need to find a lift of that x in the middle row and apply the differential there. The lift is easy to find, we just keep what we have for those I without 0 and for those I with 0, we can use any element we want, for consistency, we just choose 0. Then consider it's image under the differential, we need to find which element it correspond to in the top row, and our claim is that you only need to adjust signs on different components, I will let you prove this claim, but this tells you that the connecting homomorphism is indeed an isomorphism.

For the general case say $X = SpecR$, we wish to show the appended complex is exact as an A-module, but recall that via $R \to A$, this is also a complex of R-module so we can also show this is exact as a complex of R-modules. Recall that to show a sequence of R-modules is exact, it suffice to find a finite cover of $SpecR$ by $D(f_i)$ and we localize at each f_i and show each of the localized sequence is exact (The translation of this argument is just exactness of sequence of sheaves can be checked locally and we are just applying this to quasicoherent sheaves on affine schemes).

Now suppose $(f_1, \cdots, f_r) = (1)$ in R and we choose such $D(f_i)$ such that each $D(f_i)$ is contained in some U_j, then we localized the sequence at f_i, recall that localization plays well with the Cech cohomology, and after the localization we know the cohomology at degree i for $i \geqslant 2$ is just the check cohomology of $\mathcal{F}|_{D(f_i)}$ using the open cover $D(f_i) \cap U_i$, but this is already dealt with in the previous case since $D(f_i) \cap U_j = D(f_i)$. □

Remark 2.10. After we prove the preceding theorem, we have completed the definition of Cech cohomology for a quasicoherent sheaf on a quasicompact and separated scheme, together with the proof of the properties 1, 2 and 3. And property 4 is also easy, since if you have an affine open cover with n affine opens in total, degree n in the Cech complex is 0, thus of course the cohomology is 0.

Now let's prove property 5:

Proposition 2.11. Suppose that $\pi : X \to Y$ is affine and Y is a quasicoherent and separated A-scheme (X is also quasicompact and speared since π is an affine morphism), then if $\mathcal{F}$ is a quasicoherent sheaf on X, there is a natural isomorphism:

$$H^i(Y, \pi_* \mathcal{F}) \simeq H^i(X, \mathcal{F})$$

Proof. Choose a finite affine open cover of Y, say it's $\mathcal{U}$, then since π is affine, we can consider the inverse images of the opens in $\mathcal{U}$, which is again a finite affine open cover of X, we will call it $\mathcal{U}'$.

Now we have an isomorphism of Cech complexes

$$C^\bullet(\mathcal{F}, \mathcal{U}') \simeq C^\bullet(\pi_* \mathcal{F}, \mathcal{U})$$

which is just given by identity by the definition of $\pi_* \mathcal{F}$. This will induce isomorphisms of cohomology, we are done. □

Property 6 is also not hard:

Proposition 2.12. Suppose that $\pi : X \to Y$ is a quasicompact and separated morphism, say $\mathcal{F}$ is a quasicoherent sheaf on X with Y a quasicompact and separated scheme over $SpecA$. The hypothesis ensures $\pi_* \mathcal{F}$ is again quasicoherent on Y, then there is a natural morphism $H^i(Y, \pi_* \mathcal{F}) \to H^i(X, \mathcal{F})$ extending $\Gamma(Y, \pi_* \mathcal{F}) \to \Gamma(X, \mathcal{F})$.

One remark is that when π is not affine, even though the morphism in degree 0 is an isomorphism, in higher degrees, it need not be an isomorphism. For example, let $X = \mathbb{P}^1_k$ and $Y = Speck$ with $\mathcal{F} = \mathcal{O}(2)$, then after the study of next section, you will be able to use this to give a counterexample.

Proof. Again, choose a finite affine open cover of Y say $\mathcal{U}$, we again take the inverse image. However, since π is not affine, the inverse image is not necessarily affine. But this doesn't matter, we can again choose a finite affine open cover of each of the inverse image and ontain another finite affine open cover of X, say $\mathcal{U}'$.

This again gives you a morphism of Cech complex:

$$\begin{tikzcd} 0 \arrow[r] & \prod_{|J|=1}\mathcal{F}(U'_J) \arrow[r] & \prod_{|J|=2}\mathcal{F}(U'_J) \arrow[r] & \cdots \\ 0 \arrow[r] & \prod_{|I|=1}\mathcal{F}(\pi^{-1}U_I) \arrow[r] \arrow[u, "res"] & \prod_{|I|=2}\mathcal{F}(\pi^{-1}U_I) \arrow[r] \arrow[u, "res"] & \cdots \end{tikzcd}$$

where the morphism is defined to be $\mathcal{F}(\pi^{-1}U_I) \to \mathcal{F}(U'_J)$ is the restriction if J consists of exactly one element in the inverse image of U_i for each i and is defined to be 0 otherwise. And for convenience we will order the elements in $\mathcal{U}'$ lexicographically i.e. it has two label ij, with i indicating it's in the inverse image of $\pi^{-1}U_i$ and j indicating it's order within other U'_{ij} which is in the inverse image of U_i.

Then we need to prove the following is commutative:

$$\begin{tikzcd} \prod_{|J|=k}\mathcal{F}(U'_J) \arrow[r] & \prod_{|J|=k+1}\mathcal{F}(U'_J) \\ \prod_{|I|=k}\mathcal{F}(\pi^{-1}U_I) \arrow[r] \arrow[u, "res"] & \prod_{|I|=k+1}\mathcal{F}(\pi^{-1}U_I) \arrow[u, "res"] \end{tikzcd}$$

this is easy since if you want see what is the morphism from $\mathcal{F}(\pi^{-1}U_I)$ to $\mathcal{F}(U'_J)$ with $|I|+1=|J|$, then it's not 0 iff the $k+1$ elements in J has k elements U'_{ij} with $i \in I$ and the last element U_{st} should satisfy the property that $s \notin I$. Then I will let you prove that the lexicographic ordering compensates the sign change and it makes the diagram commute.

Finally we need to show that in degree 1 this induces the identity $\mathcal{F}(X) \to \mathcal{F}(X)$, which is simple since at degree 0, the induced morphism of the cohomology is the unique morphism that makes the following diagram commute:

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r] & \prod_{|J|=1}\mathcal{F}(U'_J) \\ \mathcal{F}(X) \arrow[r] \arrow[u, dashed] & \prod_{|I|=1}\mathcal{F}(U_I) \arrow[u, "res"'] \end{tikzcd}$$

and you can see the identity morphism satisfies the property. ☐

Finally we show cohomology commutes with filtered colimits:

Proposition 2.13. Suppose that X is quasicompact and spearated, and $\mathcal{F}_i$'s are quasicoherent sheaves on X, then we have natural isomorphism:

$$colim_i H^i(X, \mathcal{F}_i) \simeq H^i(X, colim_i \mathcal{F}_i)$$

Proof. This is basically the fact that filtered colimit is an exact functor. Suppose that $\mathcal{I}$ is a filtered index category and for each $i \in \mathcal{I}$ we have a SES:

$$0 \to A_i \to B_i \to C_i \to 0$$

of R-modules then the induced sequence:

$$0 \to colim_i A_i \to colim_i B_i \to colim_i C_i \to 0$$

is also a sequence of exact R-module. Therefore I claim if you consider the complex induced by the colimit:

$$colim_i C^\bullet(X, \mathcal{F}_i)$$

The cohomology of this colimt complex is the colimit of the cohomology. To see why, notice that since colimit is exact, it preserves kernels and images thus the unique morphim from the image to the kernel is also induced by the colimit universal property, therefore we have an isomorphism:

$$colimt_i H^j(X, \mathcal{F}_i) \simeq H^j(colim_i C^\bullet(X, \mathcal{F}_i))$$

Thus the problem now becomes to prove the cohomology of the colimit complex is the cohomology of $colim_i \mathcal{F}_i$. To show this, I claim that for quasicoherent sheaves $\mathcal{F}_i$ on affine open subsets the colimit doesn't need to be sheafified, which comes to the fact that colimits commutes with localizations, just like how we proved tensor product on affine open doesn't need to be sheafified. The following is the proof where we show localization commutes with colimits (even though we can just say localization is a left adjoint, thus preserves colimits, we can also give a concrete proof). First via the universal property of localization, there is a natural morphism going out of the localization of the colimit and similarly construct it's inverse and explicitly you will find the isomorphism maps $\overline{\frac{x}{f^n}} \to \frac{\overline{x}}{\overline{f^n}}$. ☐

We have now proved all the properties of the previous section, except for the last one where we identify the cohomology of all the line bundles on $\mathbb{P}^n_A$ which we will do in the next section. Before we do it, let's see some useful facts about cohomology of k-schemes. The first result is about cohomology and base change:

Proposition 2.14. Suppose X is a quasicompact and separated k-scheme, and $\mathcal{F}$ is a quasicoherent sheaf on X, then there is an isomorphism:

$$H^i(X,\mathcal{F}) \otimes_k K \simeq H^i(X \times_{Speck} SpecK, \mathcal{F} \otimes_k K)$$

where $\mathcal{F} \otimes_k K$ means the pullback of $\mathcal{F}$ to $X \times_k K$.

Proof. The first important point you should notice is that since everything here is is a k-vector space, tensors of k-vector spaces is an exact functor, thus it preserves kernels and cokernels, therefore, if we tensor the Cech complex $C^\bullet(\mathcal{F},\mathcal{U})$ with K over k, we get $C^\bullet(\mathcal{F},\mathcal{U}) \otimes_k K$ who cohomology is isomorphic to $H^i(X,\mathcal{F}) \otimes_k K$, therefore it suffice to show we have an isomorphism of complexes between $C^\bullet(\mathcal{F},\mathcal{U}) \otimes_k K$ and the Cech complex of $\mathcal{F} \otimes_k K$ with some affine open cover. Recall that if $\mathcal{U}$ is an affine open cover of X, then $\mathcal{U} \times_k K$ is an affine open cover of $X \times_k SpecK$, then consider the following diagram:

$$\begin{tikzcd}
V \times_k K \arrow[r, hook] \arrow[d] & U \times_k K \arrow[r, hook] \arrow[d] & X \times_k K \arrow[r] \arrow[d] & SpecK \arrow[d] \\
V \arrow[r, hook] & U \arrow[r, hook] & X \arrow[r] & Speck
\end{tikzcd}$$

then you know that the sections of $\mathcal{F} \otimes_k K$ on $U \times_k K$ is just the section of $\mathcal{F}$ on U tensor with K and the restriction map to $V \times_k K$ where V is a smaller affine is given by the restriction tensor with K, then this tells you that the Cech complex of $\mathcal{F} \otimes_k K$ with respect to $\mathcal{U} \otimes_k K$ is $C^\bullet(\mathcal{F},\mathcal{U}) \otimes_k K$, we are done. ☐

Proposition 2.15. Suppose that X is a separated and quasicompact k-scheme with $\mathcal{L}$ an invertible sheaf on X, if K/k is a field extension, then $\mathcal{L}$ is base point free iff it's pullback to $X \times_k K$ is base point free.

Proof. We already know that if $\mathcal{L}$ is base point free on X, then it's pullback is always base point free since pullback is right exact.

Conversely, suppose that $\mathcal{L} \otimes_k K$ is base point free, we want to show that $\mathcal{L}$ is base point free, then we choose a basis $\{e_i\}$ of the basis of global sections of $\mathcal{L}$ on X, then we know by the isomorphism of the previous section, $\{e_i \otimes 1\}$ is a basis of the global section of $\mathcal{L} \otimes_k K$, then if you investigate the isomorphim, you will know that if we choose $U_i = SpecA_i$ as an affine open cover of X, then the restriction of $e_i \otimes 1$ restricted to $U_i \otimes_k K$ is just $e_i \otimes_k K$, then since e_i generates $\Gamma(U_i,\mathcal{L})$, then we know $\Gamma(U_i \times_k K, \mathcal{L} \otimes_k K)$ is generated by $e_i \otimes 1$, we are done. ☐

The last result in this section is the following vanishing theorem related to the dimension of X:

Theorem 2.16. Suppose that X is a projective k-scheme and $\mathcal{F}$ is a quasicoherent sheaf on X, then $H^i(X,\mathcal{F}) = 0$ for $i > dimX$.

Proof. Consider the closed embedding $X \hookrightarrow \mathbb{P}^N_k$ of X, which has dimension n. Then there are $n+1$ hypersurfaces in $\mathbb{P}^N_k$ missing X. Recall that the complement of each of the hypersurface, denoted by U_i, is affine, thus $U_i \cap X$ is affine since closed embedding is affine, thus X is covered by $n+1$ affine, thus we are done by the affine cover vanishing theorem. ☐

Remark 2.17. Dimensional cohomology vanishing is even true in much more generality. To state it, we need to define cohomology using the language of derived functors, which we will do later. For example, this is true for k-varieties of dimension n, even though it can't be covered by $n+1$ affines.

3 Cohomology of Line Bundles on Projective Spaces

We finally prove the last promised property of cohomology. But first we will discuss a seeming non related problem, the cohomology of the affine space minus the origin:

Proposition 3.1. For any ring A, let $X = \mathbb{A}_A^{n+1} - V(x_0, \cdots, x_n)$ for $n \geqslant 1$. Here the coordinates on $\mathbb{A}_A^{n+1}$ are $x_0, \cdots, x_n$. Then $H^i(X, \mathcal{O}_X) = 0$ if $i \neq 0$ or n .

Proof. We will see that $H^0(X, \mathcal{O}_X)$ is $A[x_0, \cdots, x_n]$ and $H^n(X, \mathcal{O}_X)$ is the "Laurent Polynomials" of the form $(x_0 x_1 \cdots x_n)^{-1} A[x_0^{-1}, \cdots, x_n^{-1}]$.

We will discuss the case where $n = 2$ since the complications present themselves in this situation and for larger n situations are similar and will be done later. We will choose the natural affine open cover $U_i = D(x_i)$ and use them to compute the Cech cohomology. We first write out the Cech complex:

$$0 \longrightarrow A[x_0, x_1, x_2, x_0^{-1}] \times A[x_0, x_1, x_2, x_1^{-1}] \times A[x_0, x_1, x_2, x_2^{-1}] \longrightarrow$$

$$A[x_0, x_1, x_2, x_0^{-1}, x_1^{-1}] \times A[x_0, x_1, x_2, x_0^{-1}, x_2^{-1}] \times A[x_0, x_1, x_2, x_1^{-1}, x_2^{-1}] \longrightarrow A[x_0, x_1, x_2, x_0^{-1}, x_1^{-1}, x_2^{-1}]$$

The claim is that if we append the global section to the left, then the result complex is exact except at the last term, with cokernel $x_0^{-1} x_1^{-1} x_2^{-1} A[x_0^{-1}, x_1^{-1}, x_2^{-1}]$, which proves the result for $n = 2$.

Let's first prove exactness. The key observation is that the maps in the appended complex respects the multidegree of monomials. For example, you have a monomial $x_0^{a_0} x_1^{a_1} x_2^{a_2}$ no matter what mapped you are looking at, the images are still $x_0^{a_0} x_1^{a_1} x_2^{a_2}$ possibly with a sign change. In other words, if we decompose each stuff in the complex by multidegree of x_0, x_1, x_2, the morphisms respects this direct sum decomposition, in other words, to show that it's exact, we only have to look at a single multidegree and show that it's exact on each multidegree. This can be again divided to three cases:

1. $a_0, a_1, a_2 < 0$, i.e. all the degrees are negative. Then you find that there are no such stuff in the first three terms since we are at most localizing at 2 variables, we can use the following sequence to record what we got:
$$0_{H^0} \longrightarrow 0_0 \times 0_1 \times 0_2 \longrightarrow 0_{01} \times 0_{02} \times 0_{12} \longrightarrow A_{012}$$
Here the 0's indicate there are non trivial element satisfying the condition and A represents the coefficients of a certain multidegree (a_0, a_1, a_2) satsifying the condition $a_0, a_1, a_2 < 0$ and the subscripts here are used to record which variable we are inverting.

2. The second case is 2 of the multidegrees are negative, say a_0, a_1 are negative but a_2 are nonnegative, then similar to the first case we can use the following sequence to record what happens:
$$0_{H^0} \longrightarrow 0_0 \times 0_1 \times 0_2 \longrightarrow A_{01} \times 0_{02} \times 0_{12} \longrightarrow A_{012}$$
then you see the last morphism is injective with image A_{012}, thus this sequence is exact.

3. Only one of the multidegree is negative, say a_0 is negative, then the sequence is:
$$0_{H^0} \longrightarrow A_0 \times 0_1 \times 0_2 \longrightarrow A_{01} \times A_{02} \times 0_{12} \longrightarrow A_{012}$$
then the kernel of the second map is obvious 0, thus it's exact there. For the third morphism, the kernel is the subset of $A \times A$ where elements are of the form (x, x), which is the image of the second morphism, thus it's clearly exact, and similarly you can show it's also exact at the last term.

4. The last case is all of the multidegrees are nonnegative, then the sequence is:
$$A_{H^0} \longrightarrow A_0 \times A_1 \times A_2 \longrightarrow A_{01} \times A_{02} \times A_{12} \longrightarrow A_{012}$$

To show this is exact, we use a similar method as what we did before. We consider a SES of complex:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_{02} \times A_{12} \arrow[r] \arrow[d] & A_{012} \arrow[d] \\
0 \arrow[r] & A_{H^0} \arrow[r] \arrow[d] & A_0 \times A_1 \times A_2 \arrow[r] \arrow[d] & A_{01} \times A_{02} \times A_{12} \arrow[r] \arrow[d] & A_{012} \arrow[d] \\
0 \arrow[r] & A_{H_0} \arrow[r] & A_0 \times A_1 \arrow[r] & A_{01} \arrow[r] & 0
\end{tikzcd}$$

where in the last row we don't want 2 to appear in the intersection and for the top row, we require 2 must appear in the intersection. We want to to the row in the middle is exact, and it suffice to show the top row and the bottom row are exact then use the LES. But the top row and bottom row can be easily shown to be exact, we are done.

This finishes the proof since we see that the sequence is exact except for the last term and for the last term, it has a nontrivial cohomology if the polynomials consists of monomials of negative multidegree, thus the cohomology of the last term, i.e. the top degree cohomology is $x_0^{-1}x_1^{-1}x_2^{-1}A[x_0^{-1}, x_1^{-1}, x_2^{-1}]$.

Now we prove this for general $n \geqslant 1$. Even though we divided into $0, 1, 2, 3$-negative situations in the proof above, actually, in the following general argument, you will see that except for the case of 3-negative situation, we can use the same method for all the remaining cases, i.e. construct a homotopy between the identity and 0, thus it's exact. To see how this is done, let's say you have a particular multidegree $A = (a_0, \cdots, a_n)$ where we will denote $S = \{i : a_i < 0\}$, i.e. all the subscript of variables whose degree is negative, we denote the complement of S by T, i.e. the location where the degree is not negative. Then we can collect the pieces of the complex with this multidegree, denoted by $C^\bullet(A)$. Then we will define a homotopy:

$$C^\bullet(A)^p \to C^\bullet(A)^{p-1}$$

To see how it's defined, notice that $C^\bullet(A)$ will be 0 on degrees less than $|S| - 1$ since on degrees smaller than this number, there won't be any polynomial with the desired multidegree. So let's just assume $p \geqslant |S|$, otherwise, we will just define it by 0. Then we can just assume that $C^\bullet(A)^p$ is $C^\bullet(A)^{|S|-1+k}$ with $k \geqslant 0$. For $C^\bullet(A)^{|S|-1+k}$ is $\oplus_{J \subset T, |J|=k} A \cdot e_{S \cup J}$. Let's choose $t \in T$ to be the largest element in T, and there is such an element if S is not the whole set, thus this argument works for all the multidegree except for the multidegree where each one of them is negative. Then we will define $C^\bullet(A)^p \to C^\bullet(A)^{p-1}$ by $e_{S\cup J}$ is mapped to $(-1)^{\epsilon(J,t)} e_{S \cup J - \{t\}}$ if $t \in S \cup J$ where $\epsilon(J, t)$ stands for the number of elements in J that is less than or equal to t. Well, I claim this is a homotopy between the identity and zero. To see why this is true, we only have to check this for $e_{S \cap J}$, notice that if you first apply the homotopy and then the differential, it's nonzero only when t in in J otherwise the homotopy mapps it to 0, thus what we got will be $d((-1)^{\epsilon(t,J)} e_{S\cup J - \{t\}}) = (-1)^{\epsilon(t,J)}((-1)^{pos(t,I)-1} e_{S \cap J} + (-1)^{pos(s,I)-1} e_{S \cup J - \{t\} + \{s\}})$ for some $s \neq t$, or it's juts 0 if $t \notin J$. Then if we first apply the differential, if $t \in J$, then after the differential you got the summation of $(-1)^{pos(s,I)-1} e_{S \cup J \cup \{s\}}$ for $s \neq t$. Then if you apply the homotopy, you just got the summation $(-1)^{\epsilon(t,J)}(-1)^{pos(s,I)-1} e_{S\cup J \cup \{s\}}$ for all $s \neq t$. Or if $t \notin J$, you just got $h((-1)^{pos(t,I)-1} e_{S \cup J \cup \{t\}})$ which is $(-1)^{pos(t,I)-1}(-1)^{\epsilon(t,J)} e_{S\cup J}$. In conclusion, you just need to make sure $(-1)^{\epsilon(t,J)+pos(t,I)-1} = 1$ for all possible J, but we know that we define the homotopy for different $J \subset T$, thus we can modify $\epsilon(t, J)$ by possibly adding 1 to make it an even number for different J, thus we are done.

But notice that for the case where we require all the multidegree to be negative, the only possibly cohomology is the last term with cohomology A, thus in total, we showed that the only nonvanishing cohomology is the first and last term, with the last term being $x_0^{-1} \cdots x_n^{-1} A[x_0^{-1}, \cdots, x_n^{-1}]$. ☐

Remark 3.2. If you have ever seen the notion of reduced homology, the above proof is exactly the same to that of proving the reduced cohomology of the boundard of a simplex S is 0 unless S is the empty set.

We are now ready to identify the cohomology of $\mathcal{O}(m)$:

Proof. Again, we cover $\mathbb{P}^n_A$ using the standard affine open cover $D(x_0), \cdots, D(x_n)$. We now come to the central trick: the Cech complex for $\mathcal{O}(m)$ is the degree m part of the Cech complex for $\mathcal{O}_X$ where X is the affine $n+1$ space minus the origin. To be more precise, there is an isomorphism of A-modules: $\Gamma(U_I, \mathcal{O}(m))$ with the homogeneous degree m Laurent monomials in $x_0, \cdots, x_n$ with coefficients in A such that each x_i for $i \notin I$ appears with a nonnegative degree.

We will use $(R_I)_m$ to denote the A-module span of monomials in $\{x_0^{a_0} \cdots x_n^{a_n} \in A[x_0, \cdots, x_n, x_i^{-1} : i \in I] : \sum_i a_i = m\}$ and this is homogeneous degree m Laurent monomials in $x_0, \cdots, x_n$ with coefficients in A such that each x_i for $i \notin I$ appears with a nonnegative degree, and notice that m could be a nonpositive number. To see what is this isomorphism, recall how we define $\mathcal{O}(m)$, on $D_+(X_I)$, it's just defined to be the zero degree component of $(S(m))_{X_I}$ where X_I stands for localize at the product of x_i where $i \in I$ and for $U_J \subset U_I$, i.e. $I \subset J$, the restriction is just the usual restriction induced by $(S(m))_{X_I} \to (S(m))_{X_J}$, then notice that we have the following diagram:

$$\begin{tikzcd}
((S(m))_{X_I})_0 \arrow[r] \arrow[d, "\simeq"] & ((S(m))_{X_J})_0 \arrow[d, "\simeq"'] \\
(S_{X_I})_m \arrow[r] & (S_{X_J})_m
\end{tikzcd}$$

I will let you think how you get the isomorphism. What's important is that the bottom stuff is just $(R_I)_m$ and $(R_J)_m$ and the restriction is just mapping a Laurent polynomial to the same Laurent polynomial, thus we are done.

This suffice to prove the theorem since using the above description, you find that the Cech complex of $\mathcal{O}(m)$ using the standard affine open cover is part of the Cech complex of the affine space minus the origin. To be precise, $(R_I)_m$ is just

$$\bigoplus_{\sum a_i = m, a_j \geqslant 0, j \notin I} A \cdot x_0^{a_0} \cdots x_n^{a_n}$$

this is just the direct sum of a bunch of multidegree components of the Cech complex we saw before! And you see that the restricition of the Laurent series respects this direct sum decomposition.

Now we know that $H^0(\mathbb{P}^n_A, \mathcal{O}(m))$ is just the homogeneous degree m polynomials with coefficients in A. For $1 \leqslant i \leqslant n-1$, since the complex is exact there, we know we only have trivial cohomology. For the top degree, we know only when the multidegree consist of only negative index can we have nontrivial cohomology, thus the cohomology is identified with homogeneous degree m Laurent polynomials in $x_0^{-1} \cdots x_n^{-1} A[x_0^{-1}, \cdots, x_n^{-1}]$, since you see that the largest degree homogeneous Laurent polynomial is of degree $-n-1$, the top degree cohomology vanishes unless $m \leqslant -n-1$, and the rank is $\binom{-m-1}{-n-m-1}$ □

Remark 3.3. About the rank of the free A-modules appearing in the above proof, we can understand it from the classic combinatorics problem called stars and bars question.

There are two versions of that problem, but it's basically just trying to see if we use $k-1$ bars to separate n indistinguishable points into k blocks, how many different ways you have. One version of that is in each block, you need to have at least one star and another version is that you allow empty blocks. The answer for the second version is $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$ since it's just you have $n+k-1$ slots and you are picking $k-1$ slots for the bars. The answer for the first version is $\binom{n-1}{k-1}$ and the reason for this is that since each block need to have at least 1 star, thus the only possible slots you can use is the slots between the stars. This explains the binomial formula appearing in the theorem.

Remark 3.4. There is a better way to say that $\Gamma(\mathbb{P}^n_A, \mathcal{O}(m))$ is the space of degree m homogeneous polynominals in $n+1$ variables.

To see this, notice that we have a canonical morphism:

$$\pi : X = \mathbb{A}^{n+1}_A - \{0\} \to \mathbb{P}^n_A$$

which is an affine morphism by definition. Then we have a natural inclusion of complexes obtained by using $D_+(x_i)$ to cover $\mathbb{P}^n_A$ and use $D(x_i)$ to cover $\mathbb{A}^{n+1} - \{0\}$ which gives a natural morphism:

$$C^\bullet(\mathcal{O}(m), D_+(x_i)) \hookrightarrow C^\bullet(\mathcal{O}, D(x_i))$$

using the description $\Gamma(\mathcal{O}(m), U_I)$ and restrictions using homogeneous degree m Laurent polynomials, then this naturally gives you a natural inclusion of $\Gamma(\mathcal{O}(m), \mathbb{P}^n_A) \hookrightarrow \Gamma(\mathcal{O}, X)$ that identifies the global section with degree m homogeneous polynomials.

Moreover, you can prove that the induced map on degree n is also an inclusion. You can do this by constructing a SES of complexes using the old method, then the induced LES tells you this and this identifies the top degree cohomology $\mathcal{O}(m)$ as homogeneous degree m Laurent polynomials in $(x_0 \cdots x_n)^{-1} A[x_0^{-1}, \cdots, x_n^{\bullet}]$.

4 Riemann-Roch, Arithmetic Genus

After the basic properties of cohomology, we will see a lot classical constructions appear very cheaply and quickly.

In this section, we work with a field k and $\mathcal{F}$ is a coherent sheaf on a projective k-scheme X, recall that we already saw $h^i(X, \mathcal{F})$ is finite and it vanishes for high enough i, for example when i is larger than the dimension of X. Notice that a variety over a field k is always finite dimensional and a projective k-scheme is also finite dimensional, it has a cover each of which is finite dimensional.

Definition 4.1. The Euler characteristic of $\mathcal{F}$ is:

$$\chi(X, \mathcal{F}) = \sum_{i=0}^{\dim X} (-1)^i h^i(X, \mathcal{F})$$

We will see here and later that Euler characteristic behaves better than individual cohomology groups. For example, we know that for $m \geqslant 0$:

$$h^0(\mathbb{P}^n_A, \mathcal{O}(m)) = \binom{n+m}{n} = \frac{(m+1)(m+2)\cdots(m+n)}{n!}$$

This is good since this is a polynomial in m with rational coefficients, for example, the coefficients of m^n is $\frac{1}{n!}$. But this is not true for all m, for example, even it's true for $m \geqslant -n-1$, but for $m \leqslant -n-1$, it turns out that for $m \leqslant -n-1$, the multiplication is either positive or negative, never 0. On the other hand, if you look at the Euler characteristic of $\mathcal{O}(m)$ on $\mathbb{P}^n_k$, the top degree cohomology when $m \leqslant -n-1$ is of dimension $\binom{-m-1}{-n-m-1}$. Then if you think about the alternating sum, no matter what m you have, the result is:

$$\chi(\mathbb{P}^n_k, \mathcal{O}(m)) = h^0(\mathbb{P}^n_k, \mathcal{O}(m)) + (-1)^n h^n(\mathbb{P}^n_k, \mathcal{O}(m))$$

when $m > -n-1$ it's just $\frac{(m+1)(m+2)\cdots(m+n)}{n!}$, but when $m \leqslant -n-1$,, even though the first one is 0, the last one is $\binom{-m-1}{-n-m-1} = \binom{-m-1}{n} = \frac{(-1)^n(m+n)(m+n-1)\cdots(m+1)}{n!}$, thus the sum is always $\frac{(m+1)(m+2)\cdots(m+n)}{n!}$.

So one lesson is that if one cohomology group behaves behaves well in a certain range and suddenly messes up, it's likely that it's because the Euler characteristic behaves well always and maybe other cohomology groups vanishes in that certain range.

In fact we will see later that sometimes it's hard to calculate the cohomology groups, even h^0, but the Euler characteristic is easier, so one important way of controlling cohomology is to first compute the Euler characteristic and show all the other cohomology vanishes. This is also the reason why we have a ubiquitous amount of vanishing theorem of cohomology.

The following proposition show that Euler characteristic behaves well in the sense that it's additive in exact sequences:

Proposition 4.2. If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a SES of coherent sheaves on a projective k-scheme X, then $\chi(X, \mathcal{G}) = \chi(X, \mathcal{F}) + \chi(X, \mathcal{H})$. More generally, if:

$$0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_n \to 0$$

is an exact sequence of coherent sheaves, then:

$$\sum_{i=1}^{n} (-1)^i \chi(X, \mathcal{F}_i) = 0$$

Proof. Let's first consider the SES case since another assertion follows from this.

Recall that for a SES of quasicoherent sheaves, we have a induced LES of cohomology, recall that after the dimension of X, all the cohomology vanishes we just have a finite exact sequence. Then we will break this exact sequence into SES like the following diagram:

$$\begin{tikzcd}
0 & H^0(X,\mathcal{F}) \arrow[r] & H^0(X,\mathcal{G}) \arrow[r] \arrow[dr] & H^0(X,\mathcal{H}) \arrow[r] \arrow[dr] & H^1(X,\mathcal{F}) & \cdots \\
 & & & Ker \arrow[u, hook] & Ker \arrow[u, hook] &
\end{tikzcd}$$

you got $h^0(X,\mathcal{G}) = h^0(X,\mathcal{F}) + \dim Ker$, but you find $h^0(X,\mathcal{H})$ is just summation of the dimension of two kernel, thus we know that $h^0(X,\mathcal{G}) = h^0(X,\mathcal{F}) + h^0(X,\mathcal{H}) - \dim Ker - 0$ where 0 can be though of the dimension of the kernel of $H^0(X,\mathcal{F}) \to H^0(X,\mathcal{G})$. Then we consider $h^1(X,\mathcal{G})$, similar arguments apply, we know that $h^1(X,\mathcal{G}) = h^1(X,\mathcal{F}) + h^1(X,\mathcal{H}) - \dim Ker - \dim Ker$, this holds all the way up until $h^0(X,\mathcal{G}) = h^n(X,\mathcal{F}) + h^n(X,\mathcal{H}) - \dim Ker - \dim Ker$ while you know one of the kernel is 0, thus you sum all this up, with appropriate sign changes, we are done.

Then for the finite exact sequence of coherent sheaves, we still break them into SES of coherent sheaves, and for each SES, since Euler charateristics is just like dimensions of vector spaces which is additive in SES, thus it's alternating sum is 0. ☐

Now we want to talk about Riemann-Roch theorem for line bundles on projective curves:

Proposition 4.3. Suppose that C is an integral 1 variety over k which is also projective over k. Suppose that $D = \sum_{p\in C^{reg}} a_p[p]$ is a divisor on C, here C^{reg} is the set of closed regular points of C, then we define the degree of D by:

$$\deg D = \sum a_p \deg p$$

where the degree of a point is just the degree of the field extension, finite here since for a closed point, it's residue field is always a finite extension of k. Since each of the points p are regular, then we know that D is also an effective divisor (see the arguments below if you are not sure what happened), thus $\mathcal{O}(D)$ is invertible, then:

$$\chi(X,\mathcal{O}_C(D)) = \deg D + \chi(C,\mathcal{O}_C)$$

Proof. To do this, first notice that if p is a regular closed point, then the canonical morphism:

$$Speck(p) \to C$$

is a closed embedding since you only have to show this locally, say $Speck(p) \to SpecA$ where A is locally finite type over k, and p is a maximal ideal of A, then the morphism $A \to k(p)$ is surjective $A \to A_m/mA_m \simeq A/m$, thus it's a closed embedding, then we can consider the SES of the closed embedding:

$$0 \to \mathcal{I}_p \to \mathcal{O}_C \to \mathcal{O}|_p \to 0$$

I claim that $\mathcal{I}_p$ is exactly $\mathcal{O}_C(-p)$ since we know that p is an effective Cartier divisor as p is a regular point, thus we know that the local ring $\mathcal{O}_{C,p}$ is regular local of dimension 1, thus the maximal ideal is principal, i.e. cur out by one element, say f, then by shrinking this affine open, we may assume that f is an element of A, then I claim by further shrinking the affine open to distinguished affine open if necessary we can assume that f generates m, this is because A is Noetherian, thus we may assume m is finitely generated, then apply geometric Nakayama lemma. Then we know that the SES of closed embedding is just:

$$0 \to \mathcal{O}_C(-D) \to \mathcal{O}_C \to \mathcal{O}|_p \to 0$$

we tensor it with $\mathcal{O}(D)$, and the result is just:

$$0 \to \mathcal{O}_C(D-p) \to \mathcal{O}_C(D) \to \mathcal{O}_C(D) \otimes \mathcal{O}|_p \to 0$$

Then we use the fact that Euler characteristic is additive in SES to conclude that:

$$\chi(X,\mathcal{O}(D-p)) + \chi(X,\mathcal{O}_C(D)\otimes\mathcal{O}|_p) = \chi(X,\mathcal{O}_C(D))$$

however, if you think about what is $\mathcal{O}|_p$, this is just the skyscrper sheaf at p with $k(p)$, and since $\mathcal{O}_C(D)$ is invertible, the tensor is again the skyscraper sheaf. Therefore, we are asking what is the Euler characteristic of a skyscraper sheaf. Since we can view this as the pushforward of a closed embedding, we can compute the cohomology on $Speck(p)$, thus the zero degree is $k(p)$ but degree 1 vanishes, thus it's just the dimension of $k(p)$ over k, thus the degree of p, thus we obtain a relation:

$$\chi(X, \mathcal{O}(D-p)) + \text{deg}p = \chi(X, \mathcal{O}_C(D))$$

Then we will induct on $\sum |a_p|$, and you see that the base case where the absolute values are all 0 are just trivial. Then suppose that we have proved this when the sum is k and we are looking when the sum is $k+1$, we will first assume all a_p that are not 0 are negative, then we know that if we consider $D' = D + p$, we got:

$$\chi(X, \mathcal{O}(D)) + \text{deg}p = \chi(X, \mathcal{O}(D'))$$

Then we need to prove:

$$\chi(C, \mathcal{O}_C(D')) - \text{deg}p = \text{deg}D + \chi(C, \mathcal{O}_C)$$

Notice that $\text{deg}D + \text{deg}p = \text{deg}D'$, we reduced the question to D', where we can use the assumption. Then if there is a a_p that is positive, similar argument with $D' = D - p$ suffices. ☐

Remark 4.4. Recall that if C is further regular, then it's also normal, then we know that the Picard group $Pic(C)$ can be generated by $\mathcal{O}(D)$(recall that the descriptions of $\mathcal{O}(D)$ using effective Cartier divisors and Weil divisors are the same when the scheme is normal), thus the above proposition is powerful in this situatio. We will see applications later.

The following is an important definition motivated by the above result:

Definition 4.5. Suppose that C is a projective k scheme that is pure dimensional 1. Say $\mathcal{L}$ is an invertible sheaf on C, we define the degree of $\mathcal{L}$ by:

$$\text{deg}\mathcal{L} = \chi(C, \mathcal{L}) - \chi(C, \mathcal{O}_C)$$

Thus we have a degree map from $\text{Pic}C$ to $\mathbb{Z}$ and the inverse image of d is called degree d line bundles and denoted by $\text{Pic}^d C$.

The following proposition is a justification of the above definition:

Proposition 4.6. Suppose that $\mathcal{L}$ is invertible sheaf on a projective k-scheme of pure dimension 1, let s be a nonzero rational section of $\mathcal{L}$ on C, which is regular at all singular points of C, then let D be the divisor of zeros and poles of s:

$$D = \sum_{p \in C} v_p(s)[p]$$

Then $\text{deg}\mathcal{L} = \text{deg}D$. In particular, if C is regular, then the degree of a line bundle on C can be computed by counting zeros and poles of any rational section of $\mathcal{L}$ not vanishing on a component of C.

Proof. First, suppose that p is a singular point of C, then since we assume s is regular at p, it has no zeros and poles at p, thus D only consists of regular points of C, therefore, we can use the previous proposition and say that:

$$\text{deg}(D) = \chi(C, \mathcal{O}_C(D)) - \chi(X, \mathcal{O}_C)$$

therefore we only need to show $\mathcal{O}_C(D) \simeq \mathcal{L}$, which is what we proved when we introduced effective Cartier divisors. ☐

The next few propositions gives you some applications:

Proposition 4.7. Recall that we proved before if you have an irreducible regular projective k-variety of dimension 1 and $\mathcal{L}$ is a line bundle with s a nonzero rational section, it has the same number of zeros and poles counted with valuation and degree multiplicities. We can give a new proof using degree of the line bundle.

Proof. First, notice that to compute the zeros and poles of s, we can first restrict it and assume that it's a rational section of the structure sheaf, then it has an inverse s^{-1}, we see that if p is a zero of s then it's a pole of s^{-1}, and vice versa, notice that the degree of $\mathcal{L}$ is the degree of $divs$ and the degree of $divs^{-1}$, and deg$divs$ is the summation of the valuations of p if p is a zero or pole of s, then we know;

$$\deg s = \sum val_p(s)\deg = \sum -val_p(s)\deg p$$

Then we know that $2\sum val_p(s)\deg p = 0$, then $\sum val_p(s)\deg p = 0$, thus we are done. □

Proposition 4.8. Suppose that s_1 and s_2 are two sections of a degree d line bundle on an irreducible regular projective curve C over k, with no common zeros(regular implies being reduced). Then we know that $s_1, \cdots, s_2$ determines a k-scheme morphism π to $\mathbb{P}^1_k$, then π is a finite morphism and it's degree is d.

Proof. To make sense of the degree of π, we need to prove first that π is finite. To avoid trivial cases, for example a constant morphism from $\mathbb{P}^1_k \to \mathbb{P}^k_1$ determined by $1, 1$ of the structure sheaf of $\mathbb{P}^1_k$ satisfies the condition in the proposition but it's not finite and not interesting since everything is mapped to a single point, thus we avoid this kind situations by assuming the result morphism is not constant. Then it's necessarily dominant and finite.

To prove this, recall that finite is projective plus finite fiber, and by the cancellation theorem of projective morphisms we know that this morphism is indeed projective, and it's of finite fiber since first it's projective, thus closed, thus closed point goes to closed point, therefore, the inverse image of the generic point has any thing in that, it's only the generic point of C, then by our assumption, we know that the generic point is necessarily mapped to the generic point, otherwise it's constant, then we know that this is finite.

Surprisingly, the key to solve this problem depends on your understanding of the pullback morphism, notice that the pullback of x_0 is s_1 and the pullback of x_1 is s_2. Then notice that the vanishing locus of x_0 is just a single closed point in $D_+(x_1)$ with residue field k, then suppose you first restrict x_0 to $D_+(x_1)$ and the pull it back, since we know that pullback commutes with restriction, we know that the result is just the restriction of s_1 to the locus where s_2 doesn't vanish. Then since on $D_+(x_1)$, $\mathcal{O}(1)$ is trivial, we know that the pullback is trivial and the pullback can be done via the ring morphism. Recall that the degree of $\mathcal{L}$ can be computed using the rational section s_1 restricted to the locus where s_2 doesn't vanish and is just the poles and zeros of s_1 restricted there counted with appropriate multiplicity, recall that the degree of π can be computed using the pullback of x_0, with appropriate multiplicity, then since the reside field of the point where x_0 vanishes is k and the restriction of x_0 there is just the uniformizer, the computation of degree of π agrees with the computation of the degree of $\mathcal{L}$ using s_1. (Notice that since s_1 is a global section it doesn't have any poles, this is also the key why they agree here.) □

Proposition 4.9. If $\mathcal{L}$ and $\mathcal{N}$ are two line bundles on a regular projective curve, then:

$$\deg\mathcal{L} \otimes \mathcal{M} = \deg\mathcal{L} + \deg\mathcal{M}$$

Proof. To prove this, first notice that we can pick nonzero rational sections of $\mathcal{L}$ and $\mathcal{M}$ defined on the same open, since they are both line bundles. Say the two sections are s, t, then I claim you got a nonzero rational section $s \otimes t$ in $\mathcal{L} \otimes \mathcal{M}$. To see it's not 0, we notice that for a local ring with maximal ideal m, if $x, y \notin m$ then the product is not in m, then the stalk of $s \otimes t$ at a point p correspond to the product of the stalks of s and t viewed as elements in $\mathcal{O}_{X,p}$ since $\mathcal{L}$ and $\mathcal{M}$ are line bundles, thus we can compute the degree of $\mathcal{L}$ using $s \otimes t$.

Notice that $s \otimes t$ vanishes iff s vanishes or t vanishes, and the valuation corresponding to the sum of valuation of s and t, this suffice to show the assertion. □

Remark 4.10. The above proposition show that for a regular projective curve C over k, the degree map $\deg : \mathrm{Pic}C \to \mathbb{Z}$ is a group homomorphism as tensor products turns the degree into summation.

Proposition 4.11. Suppose that $\pi : C \to C'$ is a degree d dominant morphism of integral projective regular curves over k and $\mathcal{L}$ is an invertible sheaf on C', then:

$$\deg\pi^*\mathcal{L} = \deg\pi \cdot \deg\mathcal{L}$$

Proof. Recall that we can compute the degree of $\mathcal{L}$ using any nonzero rational section s, and I claim the pullback π^*s is also a nonzero rational sections of $\pi^*\mathcal{L}$. To show it's not zero, we only need to show that it's stalk is not in the maximal ideal at some point. This again because we can identity the pullback morphism on the stalks via a trivialization to pullbacks of structure sheaves, thus can be computed using the morphism of schemes. And we know that morphisms of rings map invertible elements to invertible elements thus elements not in the maximal ideal is not mapped to to the maximal ideal.

Thus we can compute the degree of $\pi^*\mathcal{L}$ using π^*s. Now, it's just:

$$\deg\pi^*\mathcal{L} = \sum_p val_p(\pi^*s)\deg p$$

Notice that if the image of p is q, then $\deg p = \deg(k(p)/k(q))\deg q$ and notice that if q is a zero of s, then p is a zero of π^*s. This is because if you recall how valuation is computed, you first choose a trivialization and restriction, then p is a zero of s of order n iff the $s = ut^n$ in $K(C')$, then consider the pullback, we know that $\pi^*s = \pi^*u\pi^*t^n$, and similarly we can prove that the pullback of a pole is also a pole. Then we can rewrite the degree as:

$$\deg\pi^*\mathcal{L} = \sum_q \sum_{p,\pi(p)=q} val_p(\pi^*s)\deg(k(p)/k(q))\deg q$$

Also, notice $val_p(\pi^*s) = n \cdot val_p(\pi^*t) = val_q(s) \cdot val_p(\pi^*t)$, thus we can further rewrite the equation as:

$$\deg\pi^*\mathcal{L} = \sum_q \sum_{p,\pi(p)=q} val_q(s) \cdot val_p(\pi^*t)\deg(k(p)/k(q))\deg q$$

Simplify it, we get:

$$\deg\pi^*\mathcal{L} = \sum_q val_q(s)deg q \sum_{p,\pi(p)=q} val_p(\pi^*t)\deg(k(p)/k(q)) = \sum_q val_q(s)deg q \cdot deg\pi = deg\pi \cdot \sum_q val_q(s)deg q = deg\pi \cdot deg\mathcal{L}$$

$\square$

Now we begin to introduce another important notion, called the **arithmetic genus**:

Definition 4.12. The arithmetic genus of a dimension n scheme X is:

$$p_a(X) = (-1)^n(\chi(X, \mathcal{O}_X) - 1)$$

For the well definedness of this notion, since we haven't prove the general dimension vanishing theorem, we just assume that X is projective over k.

Remark 4.13. Recall that for an integral projective (thus proper) scheme over an algebraically closed field, the global sections of the structure sheaf is 1-dimensional, thus $h^0(X, \mathcal{O}_X)$ is 1, then:

$$p_a(X) = h^1(X, \mathcal{O}_X)$$

Therefore, we can restate the Riemann-Roch for line bundles on integral projective curves as:

$$h^0(X, \mathcal{L}) - h^1(X, \mathcal{L}) = \deg\mathcal{L} + 1 - p_a(X)$$

since $1 = h^0(X, \mathcal{O}_X)$ and $h^1(X, \mathcal{O}_X) = p_a(X)$.

This is the most common formulation of the Riemann-Roch formula.

Remark 4.14. The reason we call this p_a genus is partially for the reason that if you work with a regular projective irreducible curve, then the corresponding complex-analytic object is a compact Riemann-Surface, thus it has a notion of genus g which miraculously agrees with p_a. We will discuss the notion of genus further later and we will be able to compute p_a for a lot interesting cases.

Suppose that C is an irreducible reduced curve over a field k (possibly singular), then recall if $\mathcal{F}$ is a coherent sheaf on C, we know that over the generic point, the fiber of $\mathcal{F}$ is finite dimension over $K(C)$, thus we define the rank of $\mathcal{F}$ to be the dimension of the fiber of $\mathcal{F}$ at the generic point over $K(C)$.

Definition 4.15. Now if we assume further C is projective, then we define:

$$\deg\mathcal{F} = \chi(C, \mathcal{F}) - \mathrm{rank}\mathcal{F} \cdot \chi(X, \mathcal{O}_C)$$

This statement is often called the Riemann-Roch for coherent sheaves on a projective curve.

Remark 4.16. Recall that even if C is not irreducible, if the rank of $\mathcal{F}$ at each generic point of the components are the same, we can still makes sense of the rank of $\mathcal{F}$, for example when $\mathcal{F}$ is a line bundle and the definition just reduces itself to the older definition of the Riemann-Roch for line bundles on a projective curve.

Proposition 4.17. The degree of a coherent sheaf is additive in SES.

Proof. We already saw that the Euler characteristic is additive in SES, thus if the rank is also additive, we are done.

If C is integral, we can just apply the stalkification functor at the generic point, which is exact and use the dimension is additive in SES. If C is not irreducible, and η is a generic point of a component, then subtracting other components, we can assume that there is an open U contained in this component C_1 that contains the generic point, notice that $U = U \cap C_1$, which is irreducible, thus since the rank at η can be computed in this U, we have reduced ourselves to the irreducible case. □

Proposition 4.18. If $\mathcal{F}$ is a torsion sheaf on C, then $\deg\mathcal{F} \geqslant 0$ with equality iff $\mathcal{F} = 0$.

Proof. Notice that since $\mathcal{F}$ is torsion, then by definition it's stalk at each generic points of each components is 0, thus the degree of $\mathcal{F}$ is equal to it's Euler characteristic.

Now since $\mathcal{F}$ is a coherent torsion sheaf on a projective curve over k, we know that it's only supported in a finite number of closed points since it's support closed and not the whole space (pass to affine opens and we know the topology there quite well). This implies that $\mathcal{F}$ is a finite direct sum of skyscraper sheaves, and for each of the skyscraper sheaf, we know that it's still coherent. Notice that for a coherent skyscraper sheaf supported at a closed point p, it's Euler characteristic is always nonnegative and is 0 iff it's 0. This is because we can cover C using affines where only one of them contains p, then I will let you see that only H^0 may be non 0 and all the other degrees vanish. This result plus cohomology commutes with direct sums suffices. □

Remark 4.19. Remember that it's not true that all skyscraper sheaves supported at a closed point p is the direct sum of $k(p)$. This is simple because that a $\mathcal{O}_{C,p}$-module is a $k(p)$-module, thus it's not true that each skyscraper sheaf is the pushforward of something on the closed point.

We review a few properties of finite length sheaves in the following remark since we need them to prove a fact we need in the development of intersection theory which we will do later:

Remark 4.20. If you still remember the definition of a simple object in an abelian category and the notion of composition series, length, you will find that we proved that a coherent torsion sheaf on a reduced projective k-scheme of pure dimension 1 is a finite length object.

Recall that simple objects of a scheme are isomorphic to structure sheaves of closed points. It's pretty easy to prove that structure sheaf of closed points (via pushforward of the closed embedding) is simple since it's supported in only one closed point, thus any subobject is supported at that point, then we can use the fact that for any module A, the only simple objects are A/m.

Conversely, suppose that $\mathcal{F}$ is a simple object of quasicoherent sheaves on X, then we can first show that it must support only at a closed point. We first show that it is only supported at one point. If not suppose that p_1, p_2 are two points where the stalk is not zero, then I will let you think about why the canonical morphism of $\mathcal{F}$ to the skycraper sheaf at p_i using $\mathcal{F}_{p_i}$ suffice to prove it has two different subobjects not 0, thus $\mathcal{F}$ is not simple. Then if this point is not closed, then recall that for the support of a quasicoherent sheaf $\mathcal{F}$, if p is in the support, then the closure of p is in the support, thus a contradiction.

Finally we show that the skyscraper sheaf is the structure of the closed point. Since we already know that $\mathcal{F}$ is supported at p, we can just assume it's a A_m-module M for a ring A and a maximal ideal m, then we know that we have $A_m \to M$ where $1 \mapsto x$ for some x not 0 makes $M \simeq A_m/I$, but being is I is not the unique maximal ideal, we know that we have a subobject:

$$J/I \hookrightarrow M$$

since $\mathcal{F}$ is quasicoherent and on A, we can view it as a module. Therefore we are done. This in particular implies finite sums of structures sheaves of closed points are simple.

We can also prove that if a quasicoherent sheaf $\mathcal{F}$ is simple, then it's finite type and only supported at a finitely many collection of closed points. First, consider a composition series of $\mathcal{F}$, since we know that if we have a SES of quasicoherent sheaves, then we know that the two sheaves on the left and right are finite type implies the one in the middle is finite type. Then this composition series also tells you the support of $\mathcal{F}$ only consists of finitely many closed points, from the fact that support is the union in a SES.

Proposition 4.21. If $k = \overline{k}$, and $\mathcal{F}$ is a torsion sheaf on C, then $\deg\mathcal{F} = l(\mathcal{F})$ where $l(\mathcal{F})$ is the length of $\mathcal{F}$.

Proof. Think about the degree of $\mathcal{F}$, we just saw that it's the dimension of the global sections of $\mathcal{F}$.

Now since we know that $\mathcal{F}$ is just a finite direct sum of structure sheaves of closed points, and the number of these structure sheaves is the length. Since we know the field is algebraically closed, the dimension of $k(p)$ as a k-vector space is just 1, thus the length agrees with the degree. □

The following few propositions are what we will use in the development of intersection theory. Suppose that C is a projective curve purely dimension 1 over a field k, possibly not reduced, and $\mathcal{L}$ is a line bundle on C. Suppose that $\mathcal{F}$ is any coherent sheaf on C, we want to prove that the following number:

$$\chi(C, \mathcal{L} \otimes \mathcal{F}) - \chi(C, \mathcal{F})$$

is the sum over all irreducible components C_i of C of the degree of $\mathcal{L}$ on $C_i^{red} \hookrightarrow C$ times the length of $\mathcal{F}$ at the generic point η_i of C_i. Before we prove this, the following lemma is useful:

Lemma 4.22. If $i : Y \hookrightarrow X$ is a closed embedding defined by a quasicoherent sheaf of ideals $\mathcal{I}$, and $\mathcal{F}$ is a quasicoherent sheaf on X such that $\mathcal{I}\mathcal{F} = 0$ (think through how to define it). Then there is a quasicoherent sheaf $\mathcal{G}$ on Y such that $i_*\mathcal{G} \simeq \mathcal{F}$ via a natural isomorphism. Moreover, if $\mathcal{F}$ is coherent, $\mathcal{G}$ is also coherent.

Proof. We will just take $\mathcal{G} = i^*\mathcal{F}$, and we will first show that the natural morphism:

$$\mathcal{F} \to i_*i^*\mathcal{F}$$

is an isomorphism for a closed embedding i. Recall how this morphism is defined, it's defined by the identity $i^*\mathcal{F} \to i^*\mathcal{F}$ and the adjunction property. This will be a pretty hard argument if we want to prove this is an isomorphism for all $\mathcal{O}_X$-modules, but if only for quasicoherent sheaves, it's not that bad.

To prove it, we know that the adjunction morphism between pushforward and pullback is compatible with restrictions. Since we have really proved this before, let's do it here. The first thing we will prove is that suppose that we have the following Cartesian diagram of open embeddings:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, hook, "i"] \arrow[r, "g"] & V \arrow[u, hook, "j"] \end{tikzcd}$$

We will show that $g_*i^* = j^*f_*$ as functors, to see why recall that for a $\mathcal{F}$ defined on X, suppose that $W \subset V$ is open, then $g_*i^*\mathcal{F}(W) = i^*\mathcal{F}(g^{-1}W) = \mathcal{F}(g^{-1}W)$, on the other hand, we know that $j^*f_*\mathcal{F}(W) = f_*\mathcal{F}(W) = \mathcal{F}(f^{-1}W)$, but notice that since W is contained in V and the then $f^{-1}W = g^{-1}W$, thus they are the same, then notice the restrictions are induced by the original restrictions on $\mathcal{F}$, thus they are the same. We will first prove that the universal unit is compatible with restrictions, and the general case follows from this. To prove this, we consider the following commutative diagram and track where the identity goes:

$$\begin{tikzcd}
Hom(f^*\mathcal{F}, f^*\mathcal{F}) \arrow[rr, "adj"] \arrow[d] \arrow[dr] & & Hom(\mathcal{F}, f_* f^*\mathcal{G}) \arrow[d] \\
Hom(f^*\mathcal{F}|_U, f^*\mathcal{F}|_U) \arrow[r, "adj"] \arrow[drr, "adj"'] & Hom(f^*\mathcal{F}, i_* i^* f^*\mathcal{F}) \arrow[r, "adj"'] & Hom(\mathcal{F}, f_* i_* i^* f^*\mathcal{F}) \arrow[d] \\
 & & Hom(\mathcal{F}|_V, g_* i^* f^*\mathcal{F})
\end{tikzcd}$$

two triangles and one square commutes since the square is just the funtoriality of the adjunction and the commutativity of the left triangle is easy to check, finally the commutativity of the bottom triangle follows from how we defined the adjunction when restricted to an open. However, you try to figure out what is the composition of the left two vertical arrows, you will see that if $f : \mathcal{F} \to f_* f^* \mathcal{F}$ then the image is just restriction of this morphism to V and you notice that $g_* y^* f^*$ is exactly $(f_* f^* \mathcal{F})_V$ using what we proved above since $g_* i^* = j^* f_*$, then we are done.

For the general case, the hint is to recall how the adjunction morphism can be define using composition of something with the universal unit and recall that restriction can be viewed as applying i^*, then we only have to use one more fact $i^* f^* = g^* j^*$ and I will left the remaining stuff to you.

Then if we restrict to $SpecA/I \to SpecA$, we know the the morphism $\mathcal{F} \to i_* i^* \mathcal{F}$ restricted to $SpecA$ is the image of the identity of $i^*\mathcal{F}$ restricted to $SpecA/I$. Notice that we know that we the morphism on $SpecA$ is induced by the following morphism of A-modules:

$$M \to M \otimes A/I$$

then since M is annihilated by I then this is an isomorphism.

Technically speaking, there are indeed some identifications I didn't mention, hopefully the following diagram is helpful for the details if you really want to know:

$$\begin{tikzcd}
Hom(f^*\mathcal{F}, \mathcal{G}) \arrow[r, "adj"'] \arrow[d, "a^*"] & Hom(\mathcal{F}, f_*\mathcal{G}) \arrow[d, "b^*"'] \\
Hom(a^* f^*\mathcal{F}, a^*\mathcal{G}) \arrow[r] & Hom(b^*\mathcal{F}, b^* f_*\mathcal{G})
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] & Y \\
X' \arrow[u, "a"] \arrow[r, "g"'] & Y' \arrow[u, "b"']
\end{tikzcd}$$

here a, b are isomrophisms, thus in this case, the pullback functor can written in a extremely easy form and it's the inverse of the pullback functor, then using this, hopefully you can figure out the identifications.

The assertion that $i^*\mathcal{F}$ is finite type when $\mathcal{F}$ is finite type is easy to prove. □

Then we are ready to prove the assertion, let's first do this first step:

Proposition 4.23. We can first reduce the case where $\mathcal{F}$ is scheme-theoretically supported on C^{red}.

Proof. The statement of the proposition is not that clear, but you will figure out what that means in the proof. For the notation we will use, we will use $\mathcal{I}$ to denote the quasicoherent ideal sheaf corresponding to C^{red}.

Now suppose that on C, you have a SES of coherent sheaves:

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

Then we know that since tensor with a line bundle is exact, then the quantity $R(C, \mathcal{F}) = \chi(C, \mathcal{F} \otimes \mathcal{L}) - \chi(C, \mathcal{F})$ is additive in SES. For later use, we also prove that $\mathcal{F}_\eta$ for a generic point of the irreducible components is of finite length over $\mathcal{O}_{C,\eta}$. First, we know that $\mathcal{F}_\eta$ is finitely generated over $\mathcal{O}_{C,\eta}$, which is a Noetherian ring, then if we can show that the support of $\mathcal{F}_\eta$ in $\mathcal{O}_{C,\eta}$ is only the maximal ideal, then it's of finite length. This is easy to prove once you understand how does localization affect the support. Suppose that M is a finitely generated module over A and p is a minimal prime, then the support of M_p in A_p is the support of M in A such that the primes is contained in p, thus p is only point in the support, thus it's of finite length.

On the other hand, notice that if we compute:

$$\sum_i \mathrm{deg} i^* \mathcal{L} \cdot l(\mathcal{G}_{\eta_i})$$

Since we know that the stakification is exact and length is additive in SES, then we know that:

$$\sum_i \mathrm{deg} i^* \mathcal{L} \cdot l(\mathcal{G}_{\eta_i}) = \sum_i \mathrm{deg} i^* \mathcal{L} \cdot l(\mathcal{F}_{\eta_i}) + \sum_i \mathrm{deg} i^* \mathcal{L} \cdot l(\mathcal{H}_{\eta_i})$$

Thus it's also additive in SES.

Now consider $\mathcal{I}$, recall that it's defined affine locally by the nilradicals, thus since we know that in a Noetherian ring, products of Nilradical vanishes on a high enough power, we can assume that $\mathcal{I}^n$ vanishes since we know that the product of quasicoherent ideals sheaves can also be defined using distinguished affine base. Similarly, we can prove that a quasicoherent ideal sheaf times a quasicoherent sheaf can also be defined using the affine local base, and the result is a subsheaf of the original quasicoherent sheaf. Then we consider the following filtration of $\mathcal{F}$:

$$0 = \mathcal{I}^r \mathcal{F} \subset \mathcal{I}^{r-1} \mathcal{F} \subset \cdots \subset \mathcal{I}\mathcal{F} \subset \mathcal{F}$$

Notice that the subquotients are annihilated by $\mathcal{I}$ (quotients of quasicoherent sheaves are again computed using distinguished affine base).

Then recall that if you have a quasicoherent sheaf finite type $\mathcal{F}$ on X, it's scheme theoretic support is the intersection of closed subschemes $Y \hookrightarrow X$ such that $\mathcal{F}$ is annihilated by the ideal sheaf of Y. Then we know that since the subquotients are annihilated by $\mathcal{I}$, the scheme theoretic support is contained in C^{red}. Then if you consider the closed embedding:

$$C^{red} \hookrightarrow C$$

As we already proved that the subquotient $\mathcal{I}^{k-1}\mathcal{F}/\mathcal{I}^k\mathcal{F}$ is annihilated by $\mathcal{I}$, there is a coherent sheaf $\mathcal{G}^k$ on C^{red} such that there is a isomorphism between $i_*\mathcal{G}^k$ and $\mathcal{I}^{k-1}\mathcal{F}/\mathcal{I}^k\mathcal{F}$. Then since we know closed embedding induce isomorphism of cohomology, the corresponding dimension will be the same.

On the other hand, we claim that the length of $\mathcal{G}_{\eta_i}$ as a $\mathcal{O}_{C^{red},\eta}$-module is the same as the length of the stalk of the subquotient at η_i as a $\mathcal{O}_{C,\eta_i}$-module. This just follows from the following algebraic fact. If $A \to A/I$ is the usual quotient map and M is an A-module annihilated by I, then I is also a A/I-module, and the length of M as an A-module is the same as the length of M as an A/I-module. Or, informally speaking, quotients doesn't affect the length as long as the action is well defined. This follows again from the fact that any subobject of M is again annihilated by I and thus the subsubtients, then any composition series of M as an A-module is also a composition series of M as an A/I module since A/m is also a simple A/I-module if $m \supset I$. And if M has infinite length, the argument is similar.

Then to prove the identity for $\mathcal{F}$, it suffice to prove the identity for $\mathcal{I}\mathcal{F}$ and the subquotient by additivity in SES, and notice that to prove the identity for the subquotient, it suffice to pass to $\mathcal{G}^k$ on C^{red}. We argue similarly for $\mathcal{I}\mathcal{F}$ and keep going down, finally, we only have to prove that any coherent sheaf on C^{red} satisfies the identity. Therefore, we can assume that □

Proposition 4.24. The desired property is true for a reduced projective curve over a field k.

Proof. Before we prove this, we emphasize why we want the reducedness hypothesis. The reason is that we want to use the fact that on a reduced projective C curve, we have the following property: Say $\mathcal{L}$ is a line bundle on C and $q_1, \cdots, q_m$ are closed points of C, then we can write $\mathcal{L}$ as $\mathcal{O}(\sum n_j p_j)$ for finitely many regular points distinct from q_i. But this result uses the fact that nonregular points on C form a finite set, which is a consequence of C is reduced, where we can define the normalization.

Then on C, we have this coherent sheaf $\mathcal{F}$, we know that the associated points of $\mathcal{F}$ are finitely many and we know that besides the generic points, the remaining stuff are closed points. Then we can apply the result we discuss in the preceding paragraph and say $\mathcal{L} = \mathcal{O}(\sum n_j p_j)$ for finitely many p_j, regular points distinct from the associated points of $\mathcal{F}$. Then we will prove the desired property by a induction on the summation of absolute values $\sum |n_j|$. The base case is when it's 0 and $\mathcal{L}$ is just $\mathcal{O}_C$, then we need to prove the degree of $\mathcal{O}_C$ on each C_i is 0. This is true by definition.

Then we want to prove:

$$\chi(C, \mathcal{F}(\sum n_j p_j)) - \chi(C, \mathcal{F})$$

is of the desired form. We will use a induction that require the following diagram to be exact:

$$0 \to \mathcal{F}(\sum n_j p_j - p_i) \to \mathcal{F}(\sum n_j p_j) \to \mathcal{O}|_{p_i} \to 0$$

To see how we obtain it, notice that let's assume that not all n_j are 0 and one of them, say n_i is actually positive (if negative, we only have to plus p_i instead of subtracting it.) Then we obtain the SES of closed embeddings of an effective Cartier divisor:

$$0 \to \mathcal{O}(-p) \to \mathcal{O}_C \to \mathcal{O}|_{p_i} \to 0$$

, then we tensor with $\mathcal{O}(\sum n_j p_j)$ which is a line bundle that it's still exact, then we want to claim that after we tensor $\mathcal{F}$, it's also exact. However, $\mathcal{F}$ is not a vector bundles, thus we need some thought on this assertion. First, we know that tensor is right exact, thus we only have to prove that after we tensor $\mathcal{F}$, the morphism $\mathcal{F}(\sum n_j p_j - p_i) \to \mathcal{F}(\sum n_j p_j)$ is injective. Notice that we have a SES of vector bundles:

$$0 \to \mathcal{O}(\sum n_j p_j - p_i) \to \mathcal{O}(\sum n_j p_j)$$

We already proved that when everything is a vector bundle, we can apply tensor and get exact sequence again, thus we are done.

Then using Euler characteristic is additive in SES, we have:

$$\chi(C, \mathcal{F}(\sum n_j p_j)) = \chi(C, \mathcal{F}(\sum n_j p_j - p_i)) + deg(p_i) \cdot \text{rank}\mathcal{F}|_{p_i}$$

Then we proved that:

$$\chi(C, \mathcal{F} \otimes \mathcal{L}) - \chi(C, \mathcal{F}) = \chi(X, \mathcal{F}(\sum n_j p_j - p_i)) + deg(p_i) \cdot \text{rank}\mathcal{F}|_{p_i} - \chi(C, \mathcal{F})$$

Then notice that $\deg(\mathcal{O}(\sum n_j p_j)) = \deg(\mathcal{O}(\sum n_j p_j - p_i)) + \deg p_i$ By induction hypothesis, we know that $\chi(X, \mathcal{F}(\sum n_j p_j - p_i)) - \chi(C, \mathcal{F})$ is the sum over irreducible components C_i of degree of $\mathcal{L}'$ on C_i times the rank length of $\mathcal{F}$ at η_i where $\mathcal{L}' = \mathcal{O}(\sum n_j p_j - p_i)$.

Now recall that if $i : C_i \hookrightarrow C$ is the closed embedding then the pullback of $\mathcal{O}(\sum n_j p_j)$ using i only gives you $\mathcal{O}(\sum n_j p_j \cap C_i)$ i.e. those p_j's that is not in C_i doesn't get pullback. To see this is true, recall that $\mathcal{O}(\sum n_j p_j)$ is actually tensor products of single $\mathcal{O}(p_i)$ or $\mathcal{O}(-p_i)$ we only have to prove this for each p_i. Let's first consider the pullback of $\mathcal{O}(-p_i)$. We only have to show that the pullback of the ideal sheaf of $p_i \hookrightarrow C$ is the ideal sheaf of $p_i \hookrightarrow C$ if p_i is in C_i and is the structure sheaf if not. To see this, notice that the pullback of the quasicoherent ideal sheaf is still a quasicoherent ideal sheaf since it's locally free! Thus we only have to show that this ideal sheaf cuts out p_i if p_i is in C_i. This follows from the following algebraic result: If $m \subset A$ is an ideal generated by a single element f, then if you consider the quotient $A \to A/I$ and $m \supset I$, then m/I is also generated by a single element. And the result for p_i not in C_i just follows from the fact that $\mathcal{I}$ away from p_i is isomorphic to the structure sheaf via the inclusion map. And here we are again using the fact that pullback commutes with restriction.

Now to prove it for $\mathcal{O}(p_i)$, we need a little bit thought, since it's the dual of the ideal sheaf. Recall that if $\mathcal{F}$ is locally free, then pullback commutes with dual, since we can argue using transition functions. Say $\mathcal{F}$ is locally free on X with transition functions T_{ij} between trivializing neighborhoods U_i U_j, then the dual have transition functions $(T_{ij}^{-1})^t$, apply the pullback, the transition functions is just the matrix where each entry is the pullback of the entry in $(T_{ij}^{-1})^t$. But if you first apply pullback and then dual, you first do the pullback operation and then take inverse and then transpose. Therefore, to prove the two matrix are the same, we only need to consider the one with out the transpose as pullback commutes with transpose. Therefore we only have to prove that first taking inverse and then pullback times the pullback is identity, which is the following:

$$f(T^{-1}) \cdot f(T) = Id$$

This is because that f defined this way commutes with matrix multiplication, thus we are done.

Now recall that since p_i is a regular point, we know a regular point on a locally Noetherian scheme is only contained in exactly one component, thus we are only trying to prove that the rank of $\mathcal{F}$ at p_i is the same as the length of $\mathcal{F}$ at η_i where η_i is the only component containing p_i. If we can show that $\mathcal{F}_{p_i}$ is free, then we are done, since we know that $\mathcal{F}_{\eta_i}$ is a further localization of $\mathcal{F}_{p_i}$ and $\mathcal{O}_{\eta_i}$ is a field (since C is reduced). Recall that $\mathcal{O}_{C,p_i}$ is a regular local ring of dimension 1, thus a DVR, we know that if $\mathcal{F}_{p_i}$ is torsion free, then it's free. Recall that we assumed that p_i is not an associated point of $\mathcal{F}$, thus we know that suppose that t is the uniformizer of the maximal ideal of $\mathcal{O}_{C,p_i}$, it's not a zero divisor, otherwise, since we know that this prime is locally generated by this single element, in that affine open, p_i now is an associated prime since everything in it is a zero divisor, then it's contained in one of the associated primes, but it's already a maximal ideal, thus it's itself an associated prime, a contradiction. Therefore in $\mathcal{O}_{C,p_i}$ there are no zerodivisors, thus it's free, we are done. □

Remark 4.25. In fact, we will see in a later chapter that all proper curves over k are projective, thus these statements apply to proper curves over k as well.

One consequence of the above result on nonreduced projective curves is the following generalization:

Proposition 4.26. If $\mathcal{L}$ and $\mathcal{M}$ are line bundles on a projective curve C over k, then $\deg(\mathcal{L}\otimes\mathcal{M}) = \deg(\mathcal{L}) + \deg\mathcal{M}$.

Proof. We already saw this result for regular curves since we can compute the degree of the line bundle using the divisor of nonzero rational sections. If C is not even reduced, we use the above result. The key here is that for a line bundle, the length at a generic point is the same as the length of the structure sheaf. Therefore we have the following equation:

$$\chi(C,\mathcal{L}\otimes\mathcal{M}) - \chi(C,\mathcal{M}) = \sum \deg_i(\mathcal{L})\cdot l(\mathcal{O}_{C,\eta_i})$$

Notice that the right hand side is:

$$\deg\mathcal{L} = \sum \deg_i(\mathcal{L})\cdot l(\mathcal{O}_{C,\eta_i})$$

Thus we add $\deg\mathcal{M}$ to both sides to get:

$$\deg(\mathcal{L}\otimes\mathcal{M}) = \deg\mathcal{L} + \deg\mathcal{M}$$

□

Now we use this to see a few properties of ample line bundles on curves:

Proposition 4.27. If C is a projective curve and $\mathcal{L}$ is ample on C, then $\deg\mathcal{L} > 0$.

Proof. Recall that $\mathcal{L}$ is ample on C iff a high enough tensor power of $\mathcal{L}$ is very ample.

Now suppose that $\mathcal{L}$ is very ample the degree is computed by degree of $\mathcal{L}$ to C_i^{red} times the length of the structure sheaf, thus we only have to show this for an integral projective curve C(pullback of $\mathcal{L}$ is still very ample).

Then since $\mathcal{L}$ is very ample, then it's globally generated, we choose a section s of $\mathcal{L}$ that doesn't vanish at the generic point, then consider the following sequence:

$$0 \longrightarrow \mathcal{O}_C \xrightarrow{s} \mathcal{L} \longrightarrow Coker \longrightarrow 0$$

Now using Euler characteristic is additive upon SES, we conclude that $\deg\mathcal{L}$ is $\chi(C, Coker)$. Notice that the stalk of the cokernel is 0 at the generic point, then it's zero on a dense open by the geometric Nakayama's lemma. Therefore, it's supported in a finite number of closed points, thus a torsion sheaf, which is not 0.

It's not 0 since we can choose s to vanish somewhere. If all global sections don't vanish everywhere, $\mathcal{L}$ is $\mathcal{O}$ which is not ample since you can choose any coherent sheaf that is not globally generated. Then by an earlier result, a torsion sheaf has nonnegative degree and it's degree is 0 when the sheaf is 0, we are done. □

Proposition 4.28. Suppose that $\mathcal{L}$ is a base point free invertible sheaf on a projective variety X over k, then we can pick $n+1$ sections which don't have common zeros to obtain a morphism:

$$\phi : X \to \mathbb{P}^n_k$$

for some n. Then $\mathcal{L}$ is ample iff ϕ is finite.

Proof. Recall that if ϕ is finite, then we know that the pullback of $\mathcal{O}(1)$ is ample, and we also know that the pullback of $\mathcal{O}(1)$ is $\mathcal{L}$, we are done.

If ϕ is not finite, we want to show that $\mathcal{L}$ is not ample. First, we know ϕ is projective by the cancellation theorem. Then it's not finite implies that it doesn't have finite fiber. Then there is a point in $\mathbb{P}^n_k$, whose inverse image is closed, but not finite. Since X is a Noetherian space so is the inverse image, thus there is a component of the inverse image which is of dimension 1, let's call it C, and we equip it with the reduced scheme structure, which maps to $\mathbb{P}^n_k$.

Now notice that we consider the restriction of $\mathcal{L}$ to C, via the closed embedding. If $\mathcal{L}$ is ample, since closed embedding is finite, we know that the pullback of ample by a finite morphism is ample, thus it's ample on C.

But on the other hand, this is also the pullback of $\mathcal{O}(1)$ by a constant morphism, notice that pullback of a line bundle along a constant morphism is the same of the pullback of the structure morphism since both of them have the same stalk. Therefore, we know the pullback of $\mathcal{L}$ on C is the structure sheaf, which have degree 0, thus not ample due to the previous result, a contradiction. □

The last part of this section serves as a precursor of some numerical stuff that comes later, especially after we introduce the notion of flatness. We start the discussion by some setups and a definition:

Suppose that X is a proper k-variety and $\mathcal{L}$ is a line bundle on X. If $i : C \hookrightarrow X$ is a one dimensional closed reduced subscheme of X, the degree of $\mathcal{L}$ on C is just $\deg_C \mathcal{L} = \deg i^*\mathcal{L}$.

Definition 4.29. $\mathcal{L}$ is **numerically trivial** if $\deg_C \mathcal{L} = 0$ for all such C.

Proposition 4.30. $\mathcal{L}$ is numerically trivial iff $\deg_C \mathcal{L} = 0$ for all integral curves C in C.

Proof. One direction is trivial. Now suppose that the relation holds for all integral curves. Suppose that $i : C \hookrightarrow X$ is a reduced but not necessarily irreducible curve. Let C_j be the components of C and we can form $j : C_j \hookrightarrow C \hookrightarrow X$. Therefore, we are show that the degree on C is 0 if it's degree on each C_j is 0. Recall that we have a formula to compute the degree of $\mathcal{L}$. It's the sum over all irreducible components where o each component you compute the product of the degree and the length of the structure sheaf, which is 1, thus it's just the sum of degrees on each component, we are done. □

Proposition 4.31. If $\pi : X \to Y$ is a proper morphism and $\mathcal{L}$ is numerically trivial on Y, then $\pi^*\mathcal{L}$ is numerically trivial on X.

Proof. By the previous proposition, we only need to consider integral curves C in X. Now notice that the morphism $C \to Y$ is still proper, thus projective since C is a curve, then it's either constant or finite.

If it's constant, then the pullback of $\pi^*\mathcal{L}$ is the structure sheaf, with degree 0, we are done.

If it's finite, we know that:

$$\chi(C, \tau^*\mathcal{L}) - \chi(C, \mathcal{O}_C) = \chi(Y, \tau_*\tau^*\mathcal{L}) - \chi(Y, \tau_*\mathcal{O})$$

to be completed later □

Proposition 4.32. $\mathcal{L}$ is numerically trivial iff it's numerically trivial on each irreducible component of X.

Proof. Say $i : X_i \hookrightarrow X$ is the closed embedding of the irreducible component of X. Say C is an integral curve on X with the closed embedding $j : C \to X$. We know that C is irreducible closed, thus contained in one X_i, then this closed embedding j factors as:

$$C \hookrightarrow X_i \hookrightarrow X$$

Therefore we know that the restriction of $\mathcal{L}$ to C is the same as first restrict to X_i then to C, therefore, if $\mathcal{L}$ is numerically trivial on X, then it's numerically trivial on X_i.

Obviously the above argument is reversible, thus we know that if $i^*\mathcal{L}$ is numerically trivial on each X_i then it's numerically trivial on X. □

Proposition 4.33. If $\mathcal{L}$ and $\mathcal{L}'$ are both numerically trivial, then so is $\mathcal{L} \otimes \mathcal{L}'$ and $\mathcal{L}^\vee$.

Proof. This basically just the statement that restriction commutes with pullbacks. Now recall that if you consider the following diagram:

$$\begin{tikzcd} C \arrow[r, hook] \arrow[dr] & X \arrow[d] \\ & Speck \end{tikzcd}$$

Since closed embeddings are proper, this is a proper integral curve, thus projective, thus we can use the fact that degree of line bundles on a projective curve is additive upon tensor product.

Finally for the dual, since we know that $\mathcal{O}_X \simeq \mathcal{L} \otimes \mathcal{L}^\vee$ thus again, using degree is additive upon tensor, we can show the degree of $i^*\mathcal{L}^\vee$ is 0. ☐

One of the consequence of the above few propositions is the following definition:

Definition 4.34. Notice that since tensors of numerically trivial line bundles are still numerically trivial, it forms a subgroup of Pic$X$, denoted by $\mathrm{Pic}^\tau X$.

We will say that two line bundles are **numerically equivalent** if they differ by a numerically trivial line bundle in Pic$X$, or they represent the same class in PicX modulo the subgroup $\mathrm{Pic}^\tau X$.

A property of line bundles that is stable under numerical equivalence is said to be a **numerical property**. We will see later that ampleness and nefness are both numerical properties.

Remark 4.35. We will later define the Neron-Severi group $\mathrm{NS}(X)$ of X as Pic$X$ modulo algebraic equivalences, which will be define after we discuss flatness.

The reason we mention it here is because that it helps us to show that $\mathrm{Pic}X/\mathrm{Pic}^\tau X$, which is usually denoted by $\mathrm{N}^1(X)$, is a quotient of the Neron-Severi group, which is finitely generated due to a highly nontrivial theorem, called **Neron-Severi Theorem**. Therefore, $\mathrm{N}^1(X)$ is also finitely generated. Notice that $\mathrm{N}^1(X)$ is certain abelian and torsion free since if $\mathcal{L}^{\otimes n}$ is trivial, then it's also numerically trivial, then again use the fact that pullback commutes with tensor and on a projective curve, degree is additive upon tensor, we know that $\mathcal{L}$ is numerically trivial. This implies that $\mathrm{N}^1(X)$ is a finite free $\mathbb{Z}$-module, thus has a well defined rank, which is called the **Picard number**, denoted by $\rho(X)$. We will meet this again when we discuss Hodge-Index theorem.

Finally, we will define $\mathrm{N}^1_{\mathbb{Q}}(X) = \mathrm{N}^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, therefore, we know that the Picard number will be the dimension of $\mathrm{N}^1_{\mathbb{Q}}(X)$ and the elements of this group is called **$\mathbb{Q}$-line bundles**.

Definition 4.36. We say that a line bundle $\mathcal{L}$ on X is **numerically effective**, or **nef** if $\deg_C \mathcal{L} \geqslant 0$ for all one dimensional closed curves $C \hookrightarrow X$ for a proper k-variety over k. Clearly, nefness is a numerically property.

The following properties of nefness is easy and resemble what we do for the numerically trivial line bundles, we omit most of the proofs:

Proposition 4.37. $\mathcal{L}$ is nef on X if $\deg_C \mathcal{L} \geqslant 0$ for all integral curves C.

Proposition 4.38. If $\pi : X \to Y$ is a proper morphism and $\mathcal{L}$ is a nef line bundle on Y, then $\pi^*\mathcal{L}$ will be nef on X.

Proposition 4.39. $\mathcal{L}$ is nef iff it's nef on each irreducible component of X.

Proposition 4.40. If $\mathcal{L}$ and $\mathcal{L}'$ are nef, then so is their tensor product, therefore nef elements form a abelian semigroup of Pic$X$.

Proposition 4.41. Ample line bundles are nef.

Proof. Recall that pullback of ample line bundles via finite morphisms are ample, and ample line bundles have nonnegative degree, we are done. ☐

Proposition 4.42. If $n \in \mathbb{Z}^{\geqslant 0}$, then $\mathcal{L}$ is nef iff $\mathcal{L}^{\otimes n}$ is nef.

5 A first Glimpse of Serre Duality

A common version of Riemann-Roch involves Serre duality, which unlike Riemann-Roch, is hard to prove. The following is Serre duality for smooth projective varieties:

Theorem 5.1. Suppose that X is a geometrically irreducible smooth projective k-variety of dimension n, then there is an invertible sheaf ω_X, or simply ω on X such that:

$$h^i(X,\mathcal{F}) = h^{n-i}(X, \omega \otimes \mathcal{F}^\vee)$$

for all $i \in \mathbb{Z}$ and all finite rank locally free sheaves $\mathcal{F}$.

Remark 5.2. We didn't really talked about the notion of geometrically irreducible, thus we make a comment here to say what are they.

The first notion we need to discuss in order to define this notion is the geometric fiber. Geometric fibers are use to remedy the fact that pullbacks of fibers in algebraic geometry doesn't behave as well as how they behave in the category of topological spaces. For example, you can prove that if $X \to Y$ is a map of topological spaces whose fiber at each point of Y is irreducible, then any pullback of this morphism has the same property and the same happens for being connected. However, our old counter example friend:

$$\begin{tikzcd} Spec\mathbb{C} \coprod Spec\mathbb{C} \arrow[r] \arrow[d] & Spec\mathbb{C} \arrow[d] \\ Spec\mathbb{C} \arrow[r] & Spec\mathbb{R} \end{tikzcd}$$

shows that this is false for morphisms of schemes. This is also the example we use to show that injectivity is not preserved by base change and this also tells you that fiber being integral is not preserved by base change.

As for reducedness of fiber is not preserved by base change, let k be a field of characteristic p, then consider the purely inseparable extension $k(u)/k(u^p)$, then consider the diagram:

$$\begin{tikzcd} Speck(u^p) \otimes_{k(u)} k(u^p) \arrow[r] \arrow[d] & Speck(u^p) \arrow[d] \\ Speck(u^p) \arrow[r] & Speck(u) \end{tikzcd}$$

notice that the vertical morphism in the right has reduced fibers, but we will see that it's pullback doesn't. We only have to show:

$$k(u^p) \otimes_{k(u)} k(u^p)$$

is not reduced. For notation simplicity, we will use $t = u^p$, thus $L = k(u^p) = k(t)$, then $K = k(u) = k(t)[u]/u^p - t = L[u]/u^p - t$, therefore we have:

$$K \otimes_L K = L[u]/u^p - t \otimes_L K = K[u]/u^p - t$$

recall that in K there is a v such that $v^p = t$, thus the result is $K[u]/(u-v)^p$ since $u - v$ is not 0, we know this is not reduced.

We introduce geometric points to rectify this problem. A **geometric point** of a scheme X is defined to be a morphism $Speck \to X$ where k is an algebraically closed field. This is a special kind of scheme valued points where the scheme is just $Speck$. This is a enriched version of just a plain point in the underlying set of X since it has the information of a inclusion of the residue field of the point to an algebraically closed field.

Using geometric points, we define a **geometric fiber** of a morphism of scheme $X \to Y$ is just the fibered product with $Speck \to Y$. We say that a morphism has **connected** (resp. **irreducible**, **integral**, **reduced**) geometric fibers it all it's geometric fibers are connected, irreducible, integral or reduced.

In this case, one usually say that a morphism has **geometrically connected** (resp. **irreducible**, **integral**, **reduced**) fibers. A k-scheme X is said to be **geometrically connected** (resp. **irreducible**, **integral**, **reduced**) if the structrue morphism $X \to Speck$ has geometrically connected (resp. irreducible, integral, reduced) fibers.

The following are some properties of this geometric fibers:

Proposition 5.3. The notion of connected (resp, irreducible, integral, reduced) fibers behaves well under base change.

Proof. For integral geoemtric fibers, we only need to show it for irredcuible and reduced geometric fibers. But the rest of them is just a fact that stacks of Cartesian squares are still Cartesian.

$$\begin{tikzcd} X \arrow[d, "\pi"'] & X \times_Y Z \arrow[l] \arrow[d] & \cdots \arrow[l] \arrow[d] \\ Y & Z \arrow[l] & Speck \arrow[l] \end{tikzcd}$$

□

Example 5.4. We give examples to show that geometrically something is not just something. For example, the following is an example of an integral scheme that is not geometrically integral.

We still consider the $\mathbb{R}$-scheme $Spec\mathbb{C}$, which is obviously integral, but we already see that if you base change using $Spec\mathbb{C} \to Spec\mathbb{R}$, you get something not irreducible and not connected. And our example before using $k(u)/k(u^p)$ shows that reducedness is not geometrically reducedness.

Below is another example that shows you that geometric fibers are useful:

Example 5.5. Consider the classic example where we project the parabola to the x-axis. i.e. the morphism $Spec\mathbb{Q}[y] \to Spec\mathbb{Q}[x]$ defined by the ring homomorphism $\mathbb{Q}[x] \to \mathbb{Q}[y]$ where $x \mapsto y^2$. We have seen that the fiber of this morphism over $\mathbb{Q}$-valued points is either 1 point of 2 points. If you are not considering the $\mathbb{Q}$-valued points, things can get more complicated. But you will see that if we consider geometric fibers, the answer is that the fiber is always two points, except at $[(x)]$, the origin. This is basically just the following algebra fact:

$$\mathbb{Q}[x,y]/x - y^2 \otimes_{\mathbb{Q}[x]} k = k[y]/y^2 - a$$

and since k is algebraically closed, this is exactly tow points. To show the above isomorphism, notice that if we consider the ring morphism $\mathbb{Q}[x] \to k$ and if x goes to a, then we know that using the universal property of tensor, there is a natural morphism from left to right and you can easily define a morphism from right to left, then check that they are inverses of each other. Similarly, you see that the geometric fiber over the origin is a single point, which is not reduced. This is closer to our geometric intuition.

However, it seems annoying to check that something is geometrically something, since it seems like we need to consider all the possible algebraically closed fields K and do the base change. Fortunately, if X is a k-scheme, this is not the case as we only have to consider **only the algebraic closure** of the base field k. We do not want to give the whole proof here, since it involves too much algebra that will take us too far. It's not that hard to prove this in some sense, therefore, I only tell you the result here and the detailed proof will be given in other sections where we solve a bunch of related algebra problems once and for all.

Remark 5.6. In the process of proving the above result, we will also prove that this is equivalent to consider all possible field extension of k, thus in particular, if you consider the identity $k \to k$, this implies that if X is geometrically something, then X is something.

Now we go back to Serre Duality:

This invertible sheaf ω_X is called the dualizing sheaf, which will be formally define later, much later, and the above theorem is a consequence of a perfect pairing:

$$H^i(X, \mathcal{F}) \times H^{n-i}(X, \omega_X \otimes \mathcal{F}^\vee) \to H^n(X, \omega_X) \simeq k$$

and the smoothness assumption can be relaxed somehow and for later use, we mark the fact that $H^n(X, \omega_X) \simeq k$ and this implies that $h^n(X, \omega_X) = 1$.

Remark 5.7. The invertible line bundle ω_X in the theorem turns out to be the canonical sheaf or canonical bundle $\mathscr{K}_X$, which is defined to be the determinant of the cotangent bundle $\Omega_{X/k}$ of X.

We will connect the dualizing sheaf and the canonical bundle later when we prove Serre duality. We will not do this for a long time since this will take us on a detour that is too long. Thus, we prove Serre duality at the end of our journey.

If we assume Serre duality for now, we can give another description of Riemann-Roch. Recall that we have already discussed that if C is an integral projective curve over an algebraically closed field k, then we have aa formula for the degree of line bundles on C:

$$h^0(C,\mathcal{L}) - h^1(C,\mathcal{L}) = \deg\mathcal{L} - p_a(C) + 1$$

Now, if we assume Serre duality holds there, the above formular becomes:

$$h^0(C,\mathcal{L}) = h^0(C,\omega_C \otimes \mathcal{L}^\vee) = \deg\mathcal{L} - p_a(C) + 1$$

This is another common version of Riemann-Roch we use a lot.

Proposition 5.8. Suppose that C is geometrically integral, thus in particular geometrically irreducible, which is smooth and projective over k, then the following holds:

1. $h^0(C,\omega_C)$ is the arithmetic genus of C.

2. $\deg\omega_C$ is $2g-2$.

Proof. Notice that here, we assumed that we are working with integral schemes, but this may not be over an algebraically closed field, thus we can not just assume $p_a(C)$ is $h^1(C,\mathcal{O}_C)$, rather, we need to use the original definition:

$$p_a(C) = -h^0(C,\mathcal{O}_C) + h^1(C,\mathcal{O}_C) + 1$$

Recall that by Serre duality, we know that $h^0(C,\mathcal{O}_C) = h^1(C,\omega_C) = 1$ since the dimension of C is 1 and we know that $H^n(X,\omega_X) \simeq k$, then we can apply this to C, thus $p_a(C) = h^1(C,\mathcal{O}_C) = h^1(C,\mathcal{O}_C^\vee)$, therefore, apply Serre duality again we get: $h^1(C,\mathcal{O}_C^\vee) = h^{1-1=0}(C,\mathcal{O}_C^{\vee\vee} \otimes \omega_C) = h^0(C,\omega_C)$.

To show the second assertion, notice that, we can apply Riemann Roch to ω_C and the result is:

$$\deg\mathcal{L} = \chi(C,\omega_C) - \chi(C,\mathcal{O}) = h^0(C,\omega_C) - h^1(C,\omega_C) - h^0(C,\mathcal{O}_C) + h^1(C,\mathcal{O}_C)$$

Notice that $h^0(C,\omega_C) = g$ and $h^1(C,\omega_C) = 1$, then we only have to show that $h^1(C,\mathcal{O}_C) - h^0(C,\mathcal{O}_C) = p_a(C) - 1$, but this is just the definition of $p_a(C)$. ☐

Example 5.9. Recall that we have classified all the line bundles on $\mathbb{P}^1_k$, this is indeed geometrically irreducible as the base change is $\mathbb{P}^n_K$, which is irreducible. We have also seen that $\mathbb{P}^1_k$ is smooth over k, thus Serre duality holds and there is a dualizing invertible sheaf, which is $\mathcal{O}(k)$ for some integer k.

To find this integer k, we know that the degree of $\mathcal{O}(1) = 1$, computed by using the nonzero section x thus the degree of $\mathcal{O}(k) = k$ since we have already proved that degree of line bundles adds up when you apply tensor. Recall that the genus of $\mathbb{P}^1_k$ is 0 since we already know that $\chi(C,\mathcal{O}_C) = \binom{1}{0} = 1$, and thus $\deg\omega_C = -2$, thus this line bundle should be $\mathcal{O}(-2)$.

Besides this, we also have a perfect pairing for each n:

$$H^0(\mathbb{P}^1,\mathcal{O}(n)) \times H^1(\mathbb{P}^1,\mathcal{O}(-2-n)) \to k \simeq H^1(\mathbb{P}^1,\mathcal{O}(-2))$$

We didn't really tell you what is this pairing in general, but I claim that after we prove Serre duality, you can interpret this pairing as follows:

Recall that $H^0(\mathbb{P}^1,\mathcal{O}(n))$ can be thought as the space of all homogeneous degree n polynomial in x,y, $H^1(\mathbb{P}^1,\mathcal{O}(-2-n))$ can be thought as the space of homogeneous degree $-2-n$ Laurent polynomials in $(xy)^{-1}k[x^{-1},y^{-1}]$. Then if you have a polynomial f in $H^0(\mathbb{P}^1,\mathcal{O}(n))$ and a Laurent polynomial g in $H^1(\mathbb{P}^1,\mathcal{O}(-2-n))$, you just need to take the product, which gives you a homogeneous degree 2 Laurent polynomial, then you read off the coefficient of $x^{-1}y^{-1}$.

Assume Serre duality (and also another vanishing theorem due to Serre), we can also prove the following:

Proposition 5.10. Suppose that X is a connected smooth projective $\overline{k}$-variety of dimension at least 2, let D be an effective Cartier divisor whose associated line bundle $\mathcal{O}(D)$ is ample, then D is connected.

Proof. Recall how the relation between an effective Cartier divisor and it's corresponding line bundle. We know that there is a canonical global section of $\mathcal{O}(D)$, called s_D that cuts out D.

Recall that since s_D is an element of $\mathcal{O}(D)$, which cuts out D, then notice that as a closed subset, $V(s_D)$ is the same as $V(s_D^n)$, even though the closed subscheme are not necessarily the same. Then since we are only looking at the topological property of D, we can raise $\mathcal{O}(D)$ to an arbitrarily large power. Since D is an effective Cartier divisor, nD is also effective for any n nonnegative. Finally we can consider the SES of the effective Cartier divisor:

$$0 \to \mathcal{O}(-nD) \to \mathcal{O}_X \to \mathcal{O}_{V(s^n)} \to 0$$

Then we know that the cohomology gives you a LES, recall that $h^1(X, \mathcal{O}(-nD)) = h^{n-1}(X, \omega_X \otimes \mathcal{O}(nD))$, now since we know that $\mathcal{O}(D)$ is ample, due to a theorem we will prove later, called Serre vanishing theorem, we can conclude that $H^{\dim X - 1}(X, \omega_X \otimes \mathcal{O}(nD))$ vanishes since $\dim X$ is at least 2, therefore $h^1(X, \mathcal{O}(nD)) = 0$ and similarly you can prove $h^0(X, \mathcal{O}(nD)) = 0$, therefore this LES tells you that $h^0(X, \mathcal{O}_{V(s^n)})$ is 1 and since this is actually the pushforward of $\mathcal{O}_{V(s^n)}$ via a closed embedding, we know that the cohomology is just the global sections on $V(s^n)$. Suppose that $V(s^n)$ is not connected, then the dimension is at least the number of the components, which is a contradiction. $\square$

Then as what we promised, we will use Serre duality to classify the vector bundles on $\mathbb{P}^1_k$:

Theorem 5.11. If $\mathcal{E}$ is a vector bundle of rank r on $\mathbb{P}^1_k$, then:

$$\mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \cdots \oplus \mathcal{O}(a_r)$$

for a unique nondecreasing sequence of integers.

This theorem is sometimes called **Grothendicek's theorem** since Grothendieck has too few theorems named after him even though this theorem was proved independently by many people.

Remark 5.12. Recall that we have seen that, if we want to ask what happens on $\mathbb{P}^n_k$ for general n, when $r = 1$, this is just the fact that $\mathrm{Pic}\mathbb{P}^n_k$ is $\mathbb{Z}$ generated by $\mathcal{O}(1)$. However, if the rank is not 1, the above theorem is false. After we learn differentials, a counterexample will be given which states that it's not even possible to filter the rank 2 bundle $\Omega_{\mathbb{P}^2_k}$ on $\mathbb{P}^2_k$, not even to split is as direct sums.

One true version of the theorem above on $\mathbb{P}^n_k$ for general n is given by Horrocks which tells you that a finite rank locally free sheaf $\mathcal{E}$ on $\mathbb{P}^n_k$ splits precisely when all the middle cohomology of all of it's twists is 0, i.e. when $H^i(\mathbb{P}^n_k, \mathcal{E}(m)) = 0$ for $0 < i < n$ and all $m \in \mathbb{Z}$.

Now, let's begin to prove the theorem, which involves clever usage of cohomology. We break the proof into a few propositions:

Proposition 5.13. Suppose that $\mathcal{E} = \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \cdots \oplus \mathcal{O}(a_r)$ with $a_1 \leqslant \cdot \leqslant a_r$. Then a_i can be determined from $\dim_k \mathrm{Hom}(\mathcal{O}(m), \mathcal{E})$ as m ranges all integers. This shows the uniqueness part of the above theorem.

Proof. Recall that a morphism from $\mathcal{O}(m)$ to $\mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \cdots \oplus \mathcal{O}(a_r)$ is the same as mrophisms from $\mathcal{O}(m)$ to each of these $\mathcal{O}(a_i)$, or in other words, the hom set splits into a direct sum, thus to determine it's dimension, we only need to consider the dimension $\dim \mathrm{Hom}(\mathcal{O}(m), \mathcal{O}(a_i))$.

First, we notice that tensor by a line bundle is a fully faithful functor, this is basically because you have the dual bundle $\mathcal{L}^\vee$ for a line bundle $\mathcal{L}$. Then if we denote tensor with $\mathcal{L}$ by F and tensor with $\mathcal{L}^\vee$ with G, then $FG \simeq id$ and $GF \circ id$. Then I claim that this suffice to show F, G are both fully faithful, this is because that F, G are now equivalence of categories and this is just a general property. Therefore to figure the dimension of $\mathrm{Hom}(\mathcal{O}(m), \mathcal{O}(a_i))$, we can tensor with $\mathcal{O}(-m)$ and look at $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(a_i - m))$. Therefore, we know that if m is larger than the maximum of a_i, the dimension is 0 and the dimension will jump when we lower m and it reaches each a_i, I will let you figure out why this helps. $\square$

Now for the existence of such a decomposition, we will prove using induction on the rank r. We already know the answer for $r = 1$, thus let's assume $r > 1$ and the first step is the following argument:

Proposition 5.14. For $m \ll 0$, $Hom(\mathcal{O}(m), \mathcal{E}) \neq 0$ and for $m \gg 0$ $Hom(\mathcal{O}(m), \mathcal{E}) = 0$.

Proof. The idea is that we can first show:

$$\mathcal{H}om(\mathcal{O}(m), \mathcal{E}) = \mathcal{E}(-m)$$

This follows from the fact that $\mathcal{H}om$ commutes with tensor with line bundles. Suppose that $\mathcal{L}$ is a line bundle, then:

$$\mathcal{H}om(\mathcal{F}, \mathcal{G}) \otimes \mathcal{L} = \mathcal{H}om(\mathcal{F} \otimes \mathcal{L}, \mathcal{G} \otimes \mathcal{L})$$

This morphism is given by on each U, $f : \mathcal{F}|_U \to \mathcal{G}|_U \mapsto f \otimes \mathcal{L}|_U$, we already see that tensor with a line bundle is an isomorphism, thus we are done. (A delicate point here is that $\mathcal{H}\wr\Updownarrow((\mathcal{F} \otimes \mathcal{L})|_U, (\mathcal{G} \otimes \mathcal{L})|_U)$ is identified with $\mathrm{Hom}(\mathcal{F}|_U \otimes \mathcal{L}|_U, \mathcal{G}|_U \otimes \mathcal{L}|_U)$).

Then we only need to tensor $\mathcal{H}om(\mathcal{O}(m), \mathcal{E})$ with $\mathcal{O}(m)$. Then since we know that when $m \ll 0$ $-m \gg 0$, then Serre's theorem tells us that $\mathcal{E}(-m)$ is globally generated, since it's not 0, there should be at least a nonzero global section. Then when $m \gg 0$ we know that $-m \ll 0$ and we also need to prove $H^0(\mathbb{P}^1_k, \mathcal{E}(-m)) = 0$. This is where we can use Serre duality since we know that the dualizing sheaf on $\mathbb{P}^1_k$ is just $\mathcal{O}(-2)$, then $H^0(\mathbb{P}^1_k, \mathcal{O}(-m)) = H^0(\mathbb{P}^1_k, \mathcal{E}(-m+2) \otimes \mathcal{O}(-2))$, then by Serre duality, this is $H^1(\mathbb{P}^1_k, \mathcal{E}(m-2))$, notice that Serre's vansihing theorem says that for $m \gg 0$ and $i > 0$ the cohomology vanishes, we are done. □

Then we know that there is an integer a_r where $\mathrm{Hom}(\mathcal{O}(a_r), \mathcal{E}) \neq 0$ but $\mathrm{Hom}(\mathcal{O}(a_r+1), \mathcal{E}) = 0$. Therefore, we can choose a nonzero morphism:

$$\phi : \mathcal{O}(a_r) \to \mathcal{E}$$

Proposition 5.15. The above morphism ϕ is injective.

Proof. Consider the kernel of this morphism and we take the stalk at p, then we get:

$$K_p \hookrightarrow k[x]_f \longrightarrow \mathcal{E}_p$$

where K_p is the stalk of the kernel at p since we know that $k[x]_f$ is torsion free, then the kernel is also torsion free. Now, suppose that M is a trosion free module on a reduced ring, then if M is not zero, then M_p is not 0 for any p, this is because if $M_p = 0$, then for any m there is some elements in A but not in p, thus in particular not 0 such that the product with m is 0, which implies that $m = 0$ since M is torsion free.

Also, for a reduced ring A a module M is torsion iff it's torsion free at each p.

Then we know that since: $\phi : \mathcal{O}(a_i) \to \mathcal{E}$ is not 0, then either it's restriction to $D_+(x)$ is not zero or it's restriction to $D_+(y)$ is not 0. Say it's restriction to $D_+(x)$ is not 0, then we know that on $D_+(x)$ is correspond to a $k[x]$-module morphism:

$$k[x] \to k[x]^r$$

this is because $k[x]$ is a PID and $\mathcal{E}(D_+(x))$ is a torsion free $k[x]$-module, thus a finite sum, the rank is determined by taking the stalk.

Knowing this, we know that since this is not 0, then it's not zero at each point in $D_+(x)$ by our knowledge of morphism of quasicoherent modules on affines, especially at the generic point. This also implies that it's not 0 on $D_+(y)$ since it's it's 0 on $D_+(y)$, then the corresponding morphism at the generic point is 0. Therefore, it's not 0 at each stalk, this implies hat if you take the stalk of:

$$Ker \to \mathcal{O}(a_i) \to \mathcal{E}$$

the stalk of the fiber is 0 since $k[x]_f$ is also a PID. I will leave the left details to you, but this tells you that the kernel is 0. □

Now let $\mathcal{F}$ be the cokernel of the injective morphism ϕ:

Proposition 5.16. $\mathcal{F}$ is locally free.

Proof. To prove this, notice that $\mathbb{P}_k^1$ is a regular curve, the for any coherent sheaf on $\mathbb{P}_k^1$, we have a canonical SES:

$$0 \to \mathcal{F}_{\text{tor}} \to \mathcal{F} \to \mathcal{F}_{\text{lf}} \to 0$$

where $\mathcal{F}_{\text{tor}}$ is torsion and $\mathcal{F}_{\text{lf}}$ is locally free. Notice that $\mathcal{E} \to \mathrm{F}_{\text{lf}}$ is surjective since it's the composition of two surjective morphsims. Then let $\mathcal{L}$ be the kernel of this surjective morphism, and we consider the sequence of morphisms:

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F}_{\text{lf}} \to 0$$

We notice that this is a SES and $\mathcal{E}$ and $\mathcal{F}_{\text{lf}}$ are both locally free of finite rank, then we already proved that this implies that $\mathcal{L}$ is also locally free of finite rank. Moreovre, I claim that the rank of $\mathcal{L}$ is 1 thus it's $\mathcal{O}N$ for some N and to see this, notice that we can only consider the stalk at the generic point, then we see that the dimension of the stalk of $\mathcal{F}_{\text{lf}}$ is the same as that of $\mathcal{F}$, which is the rank of $\mathcal{E}$ minus 1 by our definition, thus we are done as rank is additive in SES.

Then if we consider the following diagram:

$$\begin{tikzcd}
\mathcal{L} \arrow[dr, hook] & & & \\
\mathcal{O}(a_i) \arrow[u, dashed] \arrow[r, hook] & \mathcal{E} \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}_{\text{lf}}
\end{tikzcd}$$

then the universal property of the kernel gives a unique morphism from $\mathcal{O}(a_i)$ to $\mathcal{O}(N)$. I claim that this is not the 0 morphism, and to show this we only need to find a point and the morphism on the stalk is not 0, but recall that if you have a similar diagram of modules:

$$\begin{tikzcd}
Ker \arrow[dr, hook] & & & \\
A \arrow[u, dashed] \arrow[r, hook] & B \arrow[r] & C \arrow[r] & D
\end{tikzcd}$$

then the unique morphism to the kernel is not 0 if $A \to B$ is not 0. This holds in our original diagram since $\mathcal{O}(a_i)$ is not 0 and the morphism is an injection. Let's call this nonzero morphism $\psi : \mathcal{O}(a_i) \to \mathcal{O}(N)$. In particular, this implies that $a_i \leqslant N$.

We also know that this morphism is injective using the same idea as the previous proposition. But notice that there is a morphism $\mathcal{O}(N) \simeq \mathcal{L} \to \mathcal{E}$ therefore by how we choose a_i we know that $N \leqslant a_i$, thus $N = a_i$, but notice that this implies that the morphism, after we tensor with $\mathcal{O}(-N)$, is just a morphism $\mathcal{O} \to \mathcal{O}$, which is not 0, thus this is just mapping 1 to $t \in k^*$, the units in k, and this is obviously an isomorphism. This implies that $\mathcal{O}(a_i)$ is also the kernel of $\mathcal{E} \to \mathcal{F} \to \mathcal{F}_{\text{lf}}$, and $\mathcal{F} \to \mathcal{F}_{\text{lf}}$ is already surjective, this implies that it's actually an isomorphism. I will let you see why this is true, but the hint is to take stalks and reduced to the corresponding statements for modules. □

Now since $\mathcal{F}$ is locally free, whose rank is 1 less than the rank of $\mathcal{E}$, by our inductive hypothesis, it's of the form:

$$\mathcal{F} \simeq \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \cdots \oplus \mathcal{O}(a_{r-1})$$

where $a_1 \leqslant \cdots \leqslant a_{r-1}$, so in the end we have a SES of the following form:

$$0 \to \mathcal{O}(a_i) \to \mathcal{E} \to \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \cdots \oplus \mathcal{O}(a_{r-1}) \to 0$$

One observation is that $a_r \geqslant a_i$ for $i \leqslant r-1$ and the reason is that we can tensor this SES with $\mathcal{O}(-a_r-1)$ and take the associated LES, and you will notice that part of the LES is:

$$H^0(\mathbb{P}^1, \mathcal{E}(-a_r-1)) \to H^0(\mathbb{P}^1, \mathcal{F}(-a_r-1)) \to H^1(\mathbb{P}^1, \mathbb{O}(-1))$$

but recall that $H^1(\mathbb{P}^1, \mathcal{O}(-1)) = 0$ as we already computed the cohomology, and recall that by our definition of a_r, we know that:

$$\mathrm{Hom}(\mathcal{O}(a_r+1),\mathcal{E})=0$$

thus we know that $H^0(\mathbb{P}^1,\mathcal{E}(-a_r-1))=0$, this implies that $H^0(\mathbb{P}^1,\mathcal{F})=0$ therefore since cohomology commutes with direct sums, we know that $H^0(,\mathbb{P}^1_k,\mathcal{O}(a_i-a_r-1))=0$, thus we know that $a_i-a_r-1<0$, thus $a_r>a_i-1$, thus $a_r\geqslant a_i$.

Finally, we wish to show that the SES:

$$0\to\mathcal{O}(a_i)\to\mathcal{E}\to\mathcal{O}(a_1)\oplus\mathcal{O}(a_2)\oplus\cdots\oplus\mathcal{O}(a_{r-1})\to 0$$

allows us to express $\mathcal{E}$ as a direct sum. Let's first focus on the case where $r=2$ and we will return to the general case later. When r is 2, we know that the SES is of the form:

$$0\to\mathcal{O}(a_2)\to\mathcal{E}\to\mathcal{O}(a_1)\to 0$$

Then recall that in this case we know how to use transition functions of $\mathcal{O}(a_1)$ and transition functions of $\mathcal{O}(a_2)$ to compute the transition function in the middle. First, recall that we already saw that all vector bundles on $\mathbb{A}^1_k$ are trivial, thus we now that if we restrict it to any open, it's still trivial, then we know that the transition function from $D_+(x)$ to $D_+(y)$ is:

$$\begin{bmatrix} t^{-a_1} & \alpha(t) \\ 0 & t^{-a_2} \end{bmatrix}$$

where t is actually a change of variable of $\frac{y}{x}$ and $\alpha(t)$ is a Laurent polynomial in t. Now if $\alpha(t)$ is 0, we see that the transition matrix of $\mathcal{E}$ is the same as that of $\mathcal{O}(a_1)\oplus\mathcal{O}(a_2)$, thus it's isomorphic to the direct sum. Therefore, if we show that it's possible to make $\alpha(t)=0$, we are done.

To do this, let's first think about how we obtained this matrix. There is something interesting going on here. First, in order to obtain this matrix, we chose local trivialization such that it locally looks like the tautological morphism but the sacrifice is that there might be a nonzero α. If we can choose a different trivialization and we give up the tautological morphism, situation can be improved. Let's consider the following morphism:

$$\begin{tikzcd}
0 \arrow[r] & k[t,t^{-1}] \arrow[r] & k[t,t^{-1}]\oplus k[t,t^{-1}] \arrow[r] & k[t,t^{-1}] \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}(a_2)(D_+(x)\cap D_+(y)) \arrow[r] \arrow[u, "\simeq"] \arrow[d, "\simeq"] & \mathcal{E}(D_+(x)\cap D_+(y)) \arrow[r] \arrow[u, "\simeq"] \arrow[d, dashed, "\simeq"] & \mathcal{O}(a_1)(D_+(x)\cap D_+(y)) \arrow[r] \arrow[u, "\simeq"] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & k[t,t^{-1}] \arrow[r] & k[t,t^{-1}]\oplus k[t,t^{-1}] \arrow[r] & k[t,t^{-1}] \arrow[r] & 0
\end{tikzcd}$$

this is the diagram that gives you the transition function. Notice that we can use any automorphism to alter the transition function, this makes us to alter the transition function by any any automorphism, but the sacrifice is that after you use the automorphism to alter it, it's not the tautological diagram any more and you can think of this automorphism as a change of basis. This automorphism is easy to find, since you want an invertible 2×2 matrix M such that:

$$\begin{bmatrix} t^{-a_1} & \alpha(t) \\ 0 & t^{-a_2} \end{bmatrix}\cdot M=\begin{bmatrix} t^{-a_1} & 0 \\ 0 & t^{-a_2} \end{bmatrix}$$

Then you only need to multiply both side with the inverse of $\begin{bmatrix} t^{-a_1} & \alpha(t) \\ 0 & t^{-a_2} \end{bmatrix}$, which is upper triangular and since product of upper triangular matrices is upper triangular. We are done.

Finally, let's consider what happens for an arbitrary r. The situation is quite similar, since we know that this time if we use the tautological diagram to compute the transition function of $\mathcal{E}$, we obtain the following matrix:

$$\begin{bmatrix} t^{-a_r} & A \\ 0 & T \end{bmatrix}$$

Where the matrix T is: the transition function of $\mathcal{O}(a_{r-1}) \oplus \mathcal{O}(a_{r-2}) \cdots \mathcal{O}(a_1)$ then we can basically use the same idea as $r = 2$ to convert A to 0 and thus prove the general case.

6 Hilbert Functions, Hilbert Polynomials and Genus

Suppose that X is a projective scheme over k with a closed embedding $X \hookrightarrow \mathbb{P}^n_k$, if $\mathcal{F}$ is coherent on X, we define the **Hilbert function** of $\mathcal{F}$ by:

$$h_{\mathcal{F}}(m) = h^0(X, \mathcal{F}(m))$$

where we interpret $\mathcal{F}(m)$ as the twisting by the pullback of $\mathcal{O}(m)$ on $\mathbb{P}^n$. One quick observation of this definition is that, it depends on the embedding of X, if we embedding into $\mathbb{P}^N$ for a different N, then the pullback of $\mathcal{O}(m)$ can differ. Although this dependency is always suppressed in the notation.

The Hilbert function of X is the Hilbert function of the structure sheaf. As what is known by people back a long time aga, the Hilbert function is always eventually a polynomial for large m. As what we will see below, this polynomial contains a lot of interesting geometric information. In modern language, we expect this eventual polynomiality comes from the fact that the Euler characteristic is a polynomial and higher cohomology vanishes for $m \gg 0$.

Theorem 6.1. If $\mathcal{F}$ is a coherent sheaf on a projective k-scheme $X \hookrightarrow \mathbb{P}^n_k$, then $\chi(X, \mathcal{F}(m))$ is a polynomial of degree equal to $\dim \mathrm{Supp} \mathcal{F}$. Hence by Serre vanishing theorem, for $m \gg 0$, $h^0(X, \mathcal{F}(m))$ is equal to a polynomial $p_{\mathcal{F}}(m)$ of degree $\dim \mathrm{Supp} \mathcal{F}$.

In particular, we know that for $m \gg 0$, $h^0(X, \mathcal{O}(m))$ is a polynomial with degree equal to $\dim X$.

Definition 6.2. The polynomial $p_{\mathcal{F}}(m)$ defined in above theorem is called the Hilbert polynomial and we will define $p_X(m) = p_{\mathcal{O}_X}(m)$ for a projective k-scheme with closed embedding $X \hookrightarrow \mathbb{P}^n_k$.

Remark 6.3. The proof will use associated points of coherent sheaves on locally Noetherian schemes and also supports of a coherent sheaf, since we didn't really had a section to fully discuss these important algebraic notions that pops up again and again, here will be a good place to review them. This will also help us understand rational functions better which is also a subject that shows up before.

6.1 Associated primes, Support and More

This discussion will be mainly around the notion of **associated points**, both because of it's own importance, but also for the fact that this is the notion that refines the notion of support. This will also be useful later when we discuss regular sequences, depth and Cohen-Macaulayness and more.

We don't want to mention to much on the motivation or anything of associated points before we develop the basic theory. But one thing that might motivates you is that the notion of associated points is **crucial** for our understanding of finitely generated module M over a Noetherian ring A, even we will discuss the notion in more generality.

You can try to keep this example in mind when read the discussions below.

$$M = A = k[x, y]/(y^2, xy)$$

There are a zillion facts of associated points that are really useful, and we start building them right now.

More on the notion of support: As we have said, the notion of associated points refines the notion of support when M is finitely generated over a Noetherian ring A. But in what follows, we make **no assumption on Noetherianity or finite generation** until we need to. Recall that the support of an element $m \in M$ is defined by;

$$\mathrm{Supp} m = \{p \in \mathit{Spec} A : m_p \neq 0\}$$

If you recall the notion of the support of a global section of a sheaf on X, this definition correspond to the support of m in the global section of $\widetilde{M}$ on $\mathit{Spec} A$ thus it's a closed subset, so it's of the form $V(I)$ for some ideal I. We will soon give the best I. Now we define the annihilator ideal of $m \in M$ is:

$$\text{Ann}_A m = \{a \in A : am = 0\} = \ker(A \overset{\times m}{\to} M)$$

Proposition 6.4. Supp$m = V(\text{Ann}_A m)$

Proof. Recall that if $p \supset \text{Ann}m$, then m_p is not 0 in M_p since if $\frac{m}{1} = 0$, then there is something a not in p, thus not in Ann$m$ such that $am = 0$, a contradiction.

On the other hand, if p is in the support of m, then this implies that any $a \in p$ satisfies the property that $am \neq 0$, thus $p \subset \text{Ann}m$. □

Remark 6.5. This Ann$m$ is the best ideal in the sense that any thing in the support contains $Ann_A m$

Recall that we have define the notion of:

$$\text{Supp}\widetilde{M} = \{p \in SpecA : \widetilde{M}_p \neq 0\}$$

Since we have canonical isomorphism that $\widetilde{M}_p = M_p$, we know that the support of $\widetilde{M}$ is the same as the support of M:

$$\text{Supp}M = \{p \in SpecA : M_p \neq 0\}$$

Therefore Supp$\widetilde{M}$ = Supp$M$. And if M is principal and generated by m, then Supp$M$ = Supp$m$ = $V(\text{Ann}m)$.

Support is a good notion because it behaves well in SES and localizations, thus it's a notion that is geometric in nature:

Proposition 6.6. Suppose that:

$$0 \to M' \to M \to M'' \to 0$$

is a SES of A-modules, then:

1. SuppM = Supp$M' \cup$ SuppM''.
2. If M is finitely generated over A (A don't have to be Noetherian), then Supp$M$ is closed.

Proof. The first assertion comes from localization is exact.

The second assertion comes from induction on the generators. We know this is true if M is generated by 1 generator, then if it's genearted by $m_1, \cdots, m_n$, consider the following SES:

$$0 \to am \to M \to M/am \to 0$$

Then use the fact that the support is the union of two closed subsets. □

Remark 6.7. If M is not finitely generated, Supp$M$ may not be closed. For example, let $A = \mathbb{Z}$ and let $M = \oplus_{p \in \mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$. As localization commutes with direct sums, we now that for every prime number p localize $\mathbb{Z}/p\mathbb{Z}$ at (p) gives you $\mathbb{Z}/p\mathbb{Z}$. But if you localize at (0), you find that everything is 0, thus the support is the complement of $\{(0)\}$. Then for any ideal $I \subset \mathbb{Z}$, since it's a PID, it's (m), then we know that $V(I)$ consist of the prime divisors of m, only finitely many. When $I = 0$, we know that $V(I) = Spec\mathbb{Z}$, but we know that $\{0\}$ is not in the support, thus the support is not closed.

Proposition 6.8. Suppose that M is a finitely generated A-module and $x \in A$ is an element that vanishes at each point in the support of M, then some power x^n of x annihilates every element of M.

Proof. By assumption, we know that $x \in p$ if $p \in$ Supp$M$. Let $m_1, \cdots, m_n$ generate M, I claim that there is a n such that $x^n m_1 = 0$, this is because that the support of M is the union of the support of one m_1 and something else, therefore we know that x is in the annihilator of m_1 since we know that $V(\text{Ann}m_1) = \text{Supp}m_1$, then we know that x is in the radical of Annm_1, then we know that some power of x annihilates m_1.

We can run this argument for each m_i and we take the power of x to be the maximum. □

Proposition 6.9. Suppose that M is a finitely generated A-module and $x \in A$, then $\mathrm{Supp}(M/xM)$ is $\mathrm{Supp}M \cap V(x)$, where M/xM is defined by the exact sequence:

$$M \xrightarrow{\times x} M \to M/xM \to 0$$

Proof. Again, since localization is exact, we can localize the exact sequence at any prime and obtain an exact sequence. We are trying to find those p such that after you localize M/xM is not 0.

If it's not 0, first observation is that M_p is not 0, otherwise as the cokernel, it's definitely 0. But only the fact that p is in $\mathrm{Supp}M$ is not enough, since it can happen that after you localize xM_p is M_p, but recall that if M is finitely generated over A, M_p is finitely generated over A_p, by assumption xM_p is not M_p, then we know that $x \in p$, otherwise x is a unit after you localize, and $xM_p = M_p$. This prove that if p is in the support, then $p \in \mathrm{Supp}M \cap V(x)$.

On the other hand, if $p \in \mathrm{Supp}M \cap V(x)$, we know that since $M_p \neq 0$ and M_p is also finitely generated, then $pA_pM_p \neq M_p$, thus since $(x) \subset p$ we know that $xM_p \neq M_p$, we are done. □

Now we stop discussing the notion of support for a while and turn to the main material, i.e. associated points, as you will see, they also help us understand the notion of support.

Definition 6.10. We define the **associated points**, or **associated primes** of an A-module M to be the set:

$$\{p \in SpecA : p = \mathrm{Ann}m \text{ ,for some } m \in M\}$$

It's denoted by $\mathrm{Ass}_A M \subset SpecA$

Definition 6.11. The **associated primes of a ring** A is the associated points of A as an A-module over itself.

Example 6.12. If A is an integral domain, then $\mathrm{Ass}A = \{0\}$. This is because $\mathrm{Ann}x = 0$ for each nonzero $x \in A$ as $xa = 0$ implies $a = 0$ for nonzero elements and if $x = 0$, then the annihilator is the whole ring, thus not prime.

Example 6.13. If $f \in k[x_1, \cdots, x_n]$, then the associated primes $k[x_1, \cdots, x_n]/(f)$ as a $k[x_1, \cdots, x_n]$-module are those principal ideals generated by the prime factors of f.

This argument will apply to all UFD. Say the f unique factorization of f into irreducible factors is:

$$f = ug_1^{a_1} \cdots g_n^{a_n}$$

Now consider $\mathrm{Ann}m$ for $m \in k[x_1, \cdots, x_n]/(f)$, say m is represented by the polynomial m, we can also obtain a unique factorization of m, we know that if a is in the annihilator of m, then am is in (f), then the unique factorization tells that a always has a factor of some powers of g_i, on the other hand, those elements with the prescribed powers of g_i is in the annihilator, therefore the annihilator is:

$$\mathrm{Ann}m = (g_1^{b_1} \cdots g_n^{b_n})$$

And you see that it's prime iff it's (g_i) for some g_i. Therefore we are done.

Remark 6.14. A key observation of associated primes is that p is an associated prime of M iff there is an injection of A-modules:

$$A/p \hookrightarrow M$$

This is because that if this injection exists, we know that it's defined by $1 \mapsto m$, then we know that p is the annihilator of m, thus an associated point. On the other hand, if $p - \mathrm{Ann}m$, we can define the map by mapping 1 to m and it's injective.

Proposition 6.15. Suppose that $M' \subset M$ is a submodule of M, then $\mathrm{Ass}M' \subset \mathrm{Ass}M$.

Proof. Say $p \in \mathrm{Ass}M'$, then there is an injection:

$$A/p \hookrightarrow M' \hookrightarrow M$$

Therefore p is also an associated prime of M by the above remark. □

Remark 6.16. Notice that if $M' \subset M$, we have already seen that $\mathrm{Supp}M' \subset \mathrm{Supp}M$ since support is the union upon SES.

Proposition 6.17. $\mathrm{Ass}M \subset \mathrm{Supp}$

Proof. For any p that is an associated point, consider the injection:

$$A/p \hookrightarrow M$$

Then notice that if we localize at p, we get an inejection:

$$k(p) \hookrightarrow M_p$$

where $k(p)$ is the residue field at p, then we know that M_p is not 0 since $k(p)$ is not 0. □

Remark 6.18. Notice that when M is finitely generated, then $\mathrm{Supp}M$ is closed, thus $\overline{\mathrm{Ass}M} \subset \mathrm{Supp}M$. We will prove later that the equality holds when A is Noetherain and M is finitely generated.

We have said a lot about the associated primes but we didn't prove that the existence, or in other words, the question is does any nonzero module over A have at least one associated prime? The answer is yes, if A is Noetherian, but we don't have to assume M is finitely generated. To prove this, the first observation is if $m \in M$ is not 0, then $\mathrm{Ann}m \subset \mathrm{Ann}xm$ which are strictly contained in A for every nonzero $xm = n$.

Proposition 6.19. Suppose that A is Noetherian and M is a nonzero module over A. Say m is nonzero in M, then there is a nonzero multiple of m, say $xm = n$, such that any nonzero multiple of n, say yn satisfies $\mathrm{Ann}yn = \mathrm{Ann}n$ and moreover $\mathrm{Ann}n$ is prime, thus M has an associated prime.

Proof. Since we work with Noetherian rings, we know we every nonempty collection of ideals has a maximal element. Now consider the collection:

$$\{\mathrm{Ann}(zm) : zm \neq 0\}$$

Pick x such that $\mathrm{Ann}xm$ is maximal, then any y such that $yxm \neq 0$, we know that $\mathrm{Ann}xm \subset \mathrm{Ann}yxm$, but by maximality, this is an equality.

This is prime since if $ab \in \mathrm{Ann}xm$, then $abxm = 0$, then if $axm \neq 0$, we know that $\mathrm{Ann}axm = \mathrm{Ann}xm$, thus $b \in \mathrm{Ann}xm$, thus $\mathrm{Ann}xm$ is prime. □

Therefore, we have proved the following result:

Proposition 6.20. If A is a Noetherian ring and M is a nonzero module over A, then for any nonzero element $m \in M$, there is a prime $p = \mathrm{Ann}xm \supset \mathrm{Ann}m$, which is an associated prime for some $x \in A$ and $xm \neq 0$.

Recall that if A is a ring and M is an A-module, we have seen that the canonical morphism: $M \to \prod_{p \in SpecA} M_p$ is an injection, but actually, as you will see, if we work with Noetherian rings, we only need to consider the associated primes:

Proposition 6.21. The canonical morphism $M \to \prod_{p \in \mathrm{Ass}M} M_p$ is injective for a module over a Noetherian ring A.

Proof. Say m vanishes at all associated primes of M, we want to show that $\mathrm{Ann}m = A$, or equivalently $m = 0$. To see this, we have just proved that for any nonzero m, there is an associated prime $p \supset \mathrm{Ann}m$. Or in other words, m_p is not 0, otherwise, there is some a not in p, thus not in $\mathrm{Ann}m$ such that $am = 0$, contradiction. □

Remark 6.22. For the above result to hold, we actually only have to consider the maximal associated primes since for the nonmaximal associated primes, it's just a further localization of the maximal stuff.

Now we begin to investigate the relation between zero divisors and associated primes:

Proposition 6.23. If A is a Noetherian ring and $f \in A$, then for any module M over A, f is a zero divisor of M if f vanishes at one associated primes of M.

In other words, we proved that the set of zero divisors is the union of all associated primes.

Proof. Notice that after we prove this, when applying this result, we only need to use the maximal associated primes.

It's easy to see that if f is in one associated prime, then it's a zero divisor.

To prove the converse, consider the following commutative diagram:

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d, "f"'] & \prod_{p\in \mathrm{Ass}M} M_p \arrow[d, hook, "f"'] \\ M \arrow[r, hook] & \prod_{p\in \mathrm{Ass}M} M_p \end{tikzcd}$$

the right vertical arrow is injective since we assume f doesn't vanish at any p, thus it's a unit after localization at p. Then this diagram implies that the vertical arrow on the left is also injective, which is equivalent to f is not a zero divisor. ☐

Remark 6.24. By this point, you probably have notice that the right assumptions we want to assume for the notion of support or the notion of associated primes are different.

If we want the support to be nice, the module should better be finitely generated but we don't have to assume the ring is Noetherian.

For associated primes, the nice property that that for any nonzero m, there is an associated prime $p \supset \mathrm{Ann}m$ or even the existence of an associated prime need the ring to be Noetherian, but the module don't have to be finitely generated.

Then, we can expect that if you want a good interplay between support and associated points, we should assume we work with finitely generated module over a Noetherian ring.

Now we begin to study how associated points behave in SES. They still behave quite well, but not as well as the support:

Proposition 6.25. Suppose that:

$$0 \to M' \to M \to M'' \to 0$$

is a SES, then:

$$\mathrm{Ass}M' \subset \mathrm{Ass}M \subset \mathrm{Ass}M' \cup \mathrm{Ass}M''$$

Notice that we are not make any assumptions on Noetherianity or finite generation.

Proof. We already saw $\mathrm{Ass}M' \subset \mathrm{Ass}M \subset$, therefore, we only need to show the second assertion.

This is the first complicated proof of the discussion of associated points. Say we have an associated prime p of M, which is $\mathrm{Ann}m$, now consider the submodule Am, which has an associated prime p, then notice that we have a SES:

$$0 \to M' \cap Am \to Am \to Am/Am \cap M' \to 0$$

Then if the proposition is true, we know that either $Am \cap M'$ or $Am/Am \cap M'$ has p as an associated prime, also, notice that we have the following commutative diagram of A-modules:

$$\begin{tikzcd} 0 \arrow[r] & Am \cap M' \arrow[r] \arrow[d, hook] & Am \arrow[r] \arrow[d, hook] & Am/Am \cap M' \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0 \end{tikzcd}$$

then since we know how associated primes behave with submodules, we only need to prove the assertion of the top row. Now, notice that for any $x \in Am \cap M'$ or $y \in Am/Am \cap M'$, it's annihilated by elements in p, then if we can find such an element that it's only annihilated by elements in p, we are done. But notice that we have a commutative diagram of A-modules:

$$\begin{tikzcd} 0 \arrow[r] & I/p \arrow[r] \arrow[d, dashed, "\simeq"] & A/p \arrow[r] \arrow[d, dashed, "\simeq"] & A/I \arrow[r] \arrow[d, dashed, "\simeq"] & 0 \\ 0 \arrow[r] & Am \cap M' \arrow[r] & Am \arrow[r] & Am/Am \cap M' \arrow[r] & 0 \end{tikzcd}$$

where I is the annihilator of m in $Am/Am \cap M'$. Then recall that we can now think of the top row as A/p modules, that are also A-modules via $A \to A/p$, then we can localize at the nonzero elements of A/p to obtain another SES whose term in the middle is $k(p)$, therefore either the left or right term is not 0, thus there is a nonzero element say of the form $\frac{x}{t}$, then $\frac{x}{1}$ is also not 0, then this implies that x is only annihilated by p, otherwise, suppose that s is something not in p such that $sx = 0$, this implies that $\overline{s}x = 0$, a contradiction. ☐

Example 6.26. We give an example that tells you that its not true that $\mathrm{Ass}M = \mathrm{Ass}M' \cup \mathrm{Ass}M''$.

Consider the following SES of $\mathbb{Z}$-modules:

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

Recall that the associated points of domains are just the zero ideal (0). But the associated point of $\mathbb{Z}/2\mathbb{Z}$ is $\{(0), (2)\}$.

Therefore, even in extremely good situations, we can not expect this union to hold.

Now we begin to bring the finite generation assumption and Noetherian assumption together. The first result we show is that finitely generated modules over Noetherian ring has finitely many associated pts:

Proposition 6.27. Suppose that M is a finitely generated nonzero module over a Noetherian ring A, then M has a finite filtration:

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$$

such that subquotients are isomorphic to A/p_i for some prime p_i of A.

Proof. Pick any nonzero elements of m, we already know that there is a nonzero multiple of m, say xm, we know that $\mathrm{Ann}xm = p_1$, then we define $M_1 = Axm$, this isomorphic to A/p_1. Then consider the module M/M_1, again a module over A, if it's 0, then we are done. If it's not, we again pick an element $\overline{x_2}$ using similar argument, which you can lift to an element x_2 in M, then define M_2 to be the one generated by x_1, x_2, then we know that we have an isomorphism:

$$M_2/M_1 \simeq A\overline{x_2}$$

This isomorphism is induced by $M \to M/M_1$, and you can see that since x_1, x_2 are mapped inside $A\overline{x_2}$, we get the induced $M_2 \to A\overline{x_2}$ which is obviously surjective and the kernel is M_1, then we have the desired isomorphism. This way, we get the desired $A/p_2 \simeq M_2/M_1$. Of course, M_2 strictly contains M_1 and if $M/M_2 = 0$, we are done.

If not keep going using the above argument, we get a sequence of increasing chains of submodules of M, since a finitely generated module over a Noetherian ring is Noetherian, then the chain terminates. ☐

Proposition 6.28. With no hypothesis on finite generation or Noetherianity, if an A-module M has a finite filtration like above, then each associated prime of M is one of those p_i's.

In particular, since we prove that finitely generated module over a Noetherian ring has such a finite filtration, then it has finitely many associated points.

Proof. We use the fact that associated points behave fairly well in SES and also notice that the associated points, of A/p_i, is only p_i since obviously every elements in A/p_i is annihilated by p_i and if x annihilates $\overline{y}$, then since p_i is prime $x \in p_i$, thus we are done. ☐

Remark 6.29. One caveat is that not all primes arsing in the filtration is an associated point of M. One example is again:

$$0 \to 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

You could possibly argue that this is because we didn't choose the filtration wisely. But the following example show that there are situations where better choices don't solve the problem.

Example 6.30. Consider the module $M = (x, y) \subset A = k[x, y]$, then I claim that any filtration of M in the form of the above proposition gives you a prime p_i which is not an associated prime.

To see this, notice that the associated primes of M is only the (0), then we only have to show that any filtration of M, the subquotients can't be isomorphic to A. To see this, suppose that:

$$M_i/M_{i-1} = A/p_i$$

is such a filtration, notice that this is just the quotient of two ideals:

$$J/I \simeq A/p_i$$

this is induced by $A \to J/I$ where $1 \mapsto f$, whose kernel is p_i, since everything in I is in the kernel, we know that $p_i \supset I$ and if I is not 0, this implies that p_i is not 0. Therefore we can assume I is 0, then we know that:

$$A/p_0 \to J$$

is an isomorphism, this implies that $p_0 = 0$, but this means that J is not (x, y) since it's easy to prove that (x, y) is not isomorphic to A for degree reasons, thus we know that the next subquotient K/J is isomorphic to A/p_i for some p_i not 0, we are done.

Remark 6.31. However, these p_i's showing up in the filtration still can tell us something: each of the p_i's contain an associated prime of M and we will prove this later.

Let's go back and keep discussing associated points and SES:

Proposition 6.32. Suppose that A is Noetherian and we have a SES of A-modules:

$$0 \to M' \to M \to M'' \to 0$$

the suppose that $p \in \mathrm{Ass}M''$ but $p \notin \mathrm{Supp}M'$ which is a hypothesis stronger than $p \notin \mathrm{Ass}M'$, then $p \in \mathrm{Ass}M$.

Proof. Let's choose $m'' \in M''$ such that $\mathrm{Ann}m'' = p$, we lift m'' back to M and say m maps to m, then we can consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & Ker \arrow[r] \arrow[d, hook] & Am \arrow[r] \arrow[d, hook] & Am'' \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

notice that the support and associated points of submodules are contained in the support and associated points of the original module, then we can show the claim for the top row.

Let's write it as:

$$0 \to p/I \to A/I \to A/p \to 0$$

then from our hypothesis, we know that p is not in the support of p/I, well, this is the clever point of the proof, since even though we didn't assume finiteness conditions for M, M', M'', we have converted the question to that of ideals of A, which is **finitely generated**! Therefore the support of p/I is closed, say is $V(J)$, and we can take $J = \cap_{q \in \mathrm{Supp}p/I}$, then the assumption that p is not in $\mathrm{Supp}p/I$ means that there is some $x \in J$ not in p. But notice that $x \in J$ means that

Now notice that since p/I is finitely generated, then we already saw the fact that if an element vanishes at all the points in the support, some powers of that element annihilates the module, then we apply this to x and p/I. We know x^n annihilates all elements in p/I but x^n is not in p.

I claim that this implies that p is an associated point of A/I since if you consider the class of $\overline{x^n}$ in A/I then we know that p is contained in the annihilator of $\overline{x^n}$, on the other hand, for any element y not in p, $\overline{yx^n}$ is not in p/I since we know that x^n is not in p, therefore we know that $x^n y$ is not in p, and in particular not 0, thus we know that y is not in the annihilator of $\overline{x^n}$, we are done. ☐

Using this result, we begin further study of associated points and the first result we show is that minimal primes in the support are always associated primes, at least for finitely generated modules over a Noetherian ring:

Proposition 6.33. Suppose that M is a finitely generated module over a Noetherian ring A, and $p \subset A$ is a prime ideal corresponding to an irreducible component of $\mathrm{Supp}M$, then $p \in \mathrm{Ass}M$.

Proof. Recall that for finitely generated modules over a Noetherian ring, the support is closed, and we know that minimal primes will correspond to irreducible components, thus say p is a minimal prime in the support, first we know that it must appear as one of the primes in the filtration.

Say it corresponds to M_i/M_{i-1}, then we first localize the filtration at p, then after the localization, it become a filtration of M_p, and possibly some subquotients will be 0, we fine those nonzero subquotients, and we pick the first nonzero one say it's (M_i/M_{i-1}), then we consider the original filtration:

$$0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0$$

we claim that p is not in the support of M_{i-1}. This is because we can consider the previous filtration::

$$0 \to M_{i-2} \to M_{i-1} \to M_{i-1}/M_{i-2} \to 0$$

Then we already know that p is not in the support of M_{i-1}/M_{i-2}, thus if it's in the support of M_{i-1}, it's in the support of M_{i-2}, then we keep going, finally reach a contradiction. □

Remark 6.34. The above proposition implies that for any Noetherian ring, all minimal primes are associated points. Therefore we know that each element in a minimal prime is a zero divisor.

This holds even without Noetherian hypothesis since if $s \in p$ where p is a minimal prime, we know that pA_p is the unique prime in A_p so we know that $\frac{s}{1}$ is a nilpotent, then there are some $t \in A - p$ such that $ts^n = 0$, this implies s is a zero divisor.

Another corollary of the above proposition:

Corollary 6.35. If M is finitely generated over a Noetherian ring A, then:

$$\overline{\mathrm{Ass}M} = \mathrm{Supp}M$$

It also has some interesting applications:

Proposition 6.36. Suppose that A is Noetherian, then those $p \in SpecA$ such that A_p is nonreduced is the support of $N(A)$, the nilradical of A, therefore the reduced points of a locally Noetherian scheme is open.

Proof. I claim that the nilradical of A_p is just $\mathfrak{N}(A)_p$. One direction is easy and for the converse, suppose that $\frac{a}{b}$ is nilpotent, if it's 0, then it's equal to $\frac{0}{b}$, we are done. Let's assume that it's not 0, thus every y not in p satisfies that $ay \neq 0$. Then using $\frac{a}{b}$ is nilpotent, we know that $a^n y = 0$ for some y not in p. But ay is not 0, thus we know that $\frac{a}{b} = \frac{ay}{by}$ and ay is nilpotent.

Therefore, we know that A_p is nonreduced iff p is in the support of $\mathfrak{N}(A)$, therefore since $\mathfrak{N}(A)$ is finitely generated over a Noetherian ring, we know that it's support is closed, thus the complement is open. □

The following proposition gives you a simple way to think about associated points of a Noetherian ring A:

Proposition 6.37. A prime ideal p of a Noetherian ring is an associated point if and only iff there is a $f \in A$ such that $\mathrm{Supp}f = V(p)$.

Proof. Say p is an associated point, then $p = \mathrm{Ann}f$ for some $f \in A$, and we know that $\mathrm{Supp}f = V(\mathrm{Ann}f)$.

Conversely, if $V(p) = \mathrm{Supp}f$, p is a the minimal prime of support of (f), therefore we know that p is an associated prime of (f), and $(f) \hookrightarrow A$ gives you that p is also an associated prime of A. □

Proposition 6.38. Recall that we said before that for finitely generated modules over a Noetherian ring, the primes showing up in the filtration will contain an associated prime, we now prove it

Proof. Notice that if p_i appears in the filtration, say $M_i/M_{i-1} = A/p_i$, we localize at p_i, then since $(A/p_i)_{p_i}$ is the residue field, it's not 0, this in particular implies that $(M_i)_{p_i}$ is not 0, thus p_i is in the support, and it clearly contains a minimal element in the support, which is an associated prime. □

Now we want to show that the notion of support and associated points are geometric in nature since they commutes with localizations. Let S be a multiplicative set of A and $p \subset A$ is a prime ideal not meeting S, so (abusing notations a little bit), we know $p \in SpecS^{-1}A \subset SpecA$:

Proposition 6.39. For any A-module M, $\mathrm{Supp}M \cap SpecS^{-1}A = \mathrm{Supp}_{S^{-1}A}S^{-1}M$

Proof. Notice that for any p not meeting S, if we first localize at S, then at p, we are actually doing a further localization and thus it's the same as localizing at p_i directly.

Then if we assume that p is in support of M and it also doesn't intersect S, then M_p is not 0 and of course $S^{-1}M$ localize at $S^{-1}p$ is the same as directly localizing at p, thus not 0, thus it's in the support of $S^{-1}M$, conversely, this also holds. □

We will also show associated points also commutes with localizations, but for this, we need to assume we work with finitely generated modules over Noetherian rings:

Proposition 6.40. Let M be a finitely generated module over a Noetherain ring, then:

$$\mathrm{Ass}M \cap SpecS^{-1}A = \mathrm{Ass}_{S^{-1}A}S^{-1}M$$

Proof. Let's first prove $\mathrm{Ass}M \cap SpecS^{-1}A \subset \mathrm{Ass}_{S^{-1}A}S^{-1}M$. Say p is an associated prime of M, then we have an injection:

$$A/p \hookrightarrow M$$

since localization commutes with quotients and is exact:

$$S^{-1}A/S^{-1}p \hookrightarrow S^{-1}M$$

since p is in $SpecS^{-1}A$, $S^{-1}p$ is not the whole ring, thus still prime, we are done.

Conversely, suppose that $S^{-1}p = q$ is an associated prime, and it's $\mathrm{Ann}\frac{a}{s}$, since s is a unit, it's just $\mathrm{Ann}\frac{a}{1}$. Therefore, we know that q is in $V(\mathrm{Ann}\frac{a}{1}) = \mathrm{Supp}S^{-1}(a)$, thus we know that p is in $\mathrm{Supp}(a)$ since support commutes with localization. Moreover, we know that q is minimal in $\mathrm{Supp}S^{-1}(a)$, therefore, we know that p is minimal in $\mathrm{Supp}M$ since if there is a smaller prime, then it also doesn't intersect S, thus it will give you a smaller prime in $\mathrm{Supp}S^{-1}(a)$. □

Definition 6.41. Say M is finitely generated module over a Noetherian A, then we call the associated primes of M that are not minimal primes in $\mathrm{Supp}M$ the **embedded points** or **embedded primes**.

Remark 6.42. We have showed that the associated primes of $k[x_1, \cdots, x_n]/(f)$ are the prime factors of f. Since the support of $k[x_1, \cdots, x_n]/(f)$ are those primes containing f, we see that the associated points are precisely the minimal primes of the support. Therefore, this can be translated to the statement that hypersurfaces have no embedded points. And this holds for any UFD.

Proposition 6.43. Reduced rings have so embedded primes. Even more, we can prove that if a is annihilated by an embedded prime, a is a nilpotent.

Proof. Say $p = \mathrm{Ann}a$ is an embedded prime, we claim that a is in all the minimal primes. Since p is a embedded, we know that it's not minimal in $\mathrm{Supp}A$, but $\mathrm{Supp}A$ is $SpecA$, therefore we know that the embedded prime is not contained in the union of all mininmal primes due to prime avoidance.

Say $b \in p$ is not in any minimal prime such that $ab = 0$, then a is in all minimal primes, thus it's a nilpotent, we are done.

This is we assume A is Noetherian, but we can also do this even if we don't assume A is Noetherian. Say $p = \mathrm{Ann}a$ for some a, then we know that $pa = 0$, then since p is not in every minimal primes, then this implies that a is since 0 is in every prime. Therefore, again, we obtain that a is nilpotent. □

Remark 6.44. The converse of the above proposition is wrong: Rings without embedded primes may have nilpotents. And an easy example is:

$$k[x]/(x^2)$$

Notice that the support of this ring is a single point, thus the associated prime is a single point, no embedded points, but x is a nilpotent.

Proposition 6.45. If p is an embedded prime of A, then A_p is nonreduced.

Proof. Since p is an embedded prime of A, then pA_p is also an embedded prime of A_p as we already saw that if p is an associated prime that doesn't meet S, then $S^{-1}p$ is an associated prime. $\square$

Now let's do some computations:

Example 6.46. Consider the ring $k[x,y]/(y^2,xy)$. We want to compute it associated primes. It's obviously a Noetherian ring and therefore we don't have to worry about what result we established above we can use.

Now, say $f \in k[x,y]/(y^2,xy)$, let's first compute the support, which means that we need to compute the annihilator of f. Let's say that f is represented by $a_0 + a_1x + \cdots + a_nx^n + b_1y$, then say:

$$(a_0 + a_1x + \cdots + a_nx^n + b_1y) \cdot (c_0 + c_1x + \cdots + c_nx^n + d_1y) = (f(x) + b_1y) \cdot (g(x) + d_1y) = 0$$

this product is:

$$a_0d_1y + c_0b_1y + fg(x)$$

It's 0 iff $a_0d_1 + c_0b_1 = 0$ and $fg = 0$. But notice that if f is not 0, then this is 0 if $g = 0$, in this case the result is 0 if $d_1 = 0$, thus the annihilator is $\{0\}$, the support is the whole space.

If $f = 0$, then this is 0 if $c_0b_1y = 0$, but since $f(x) + b_1y \neq 0$, we know that $b_1 \neq 0$, thus $c_0 = 0$, therefore we know that for all the elements of the form:

$$c_1x + \cdots + c_nx^n + d_1y$$

the product is 0, thus the annihilator is (x,y), thus the support is the origin. But if $f = 0$, then the support is empty space since the annihilator is the whole ring.

Then recall that for a Noetherian ring, p is an associated prime iff there is a f such that $V(p) = \text{Supp} f$. Then for different f, the support is either $V(x,y)$ or $V(0)$ or empty, thus the annihilator is only two points, $(x,y),(0)$, one embedded point.

This also tells us that the zero divisor is all elements in (x,y) (since it's the union of associated primes) or in other words those f such that $f(0,0) = 0$. This agrees with our computation earlier that when a_0 is not 0, no nonzero element will be able to make the product to 0.

Let's see another example:

Example 6.47. Suppose that $X = Spec\mathbb{C}[x,y]/I$ with associated points $(y-x^2),(x-1,y-1)$ and $(x-2,y-2)$. We will verify that such a scheme exists so for now just use the assumptions.

In this example, we try to recover the geometric picture of X just using the information of associated primes. Recall that all the minimal element in the support of $\mathbb{C}[x,y]/I$ will be associated points, therefore we know that the irreducible components are just $V(y-x^2)$ and $V(x-2,y-2)$ as $(x-1,y-1)$ contains $(y-x^2)$, therefore we know that as a subset of $\mathbb{A}^2_{\mathbb{C}}$, X is the parabola plus a single point.

However, we know that X is not reduced since we have embedded points $(x-1,y-1)$. However, notice that $(x-2,y-2)$ is contained in no other primes and notice that X is a disjoint union, then from the fact that associated primes of the product ring is computed in each ring, then we know that the point $(x-2,y-2)$ has no embedded prime, thus it could have reduced structure, but also could have nonreduced structure since only from no embedded primes, we don't know much of it.

From the associated primes, we also know all the zero divisors, for example $x+y-2, x+y-3$ and $y-x^2$ are all zero divisors.

Finally, let's give such an I, for example $(y-x^2)^3 \cap (x-1,y-1)^{15} \cap (x-2,y-2)$. It's clear from $V(I) \cup V(J) = V(I \cap J)$, we know that $(y-x^2)$ and $(x-2,y-2)$ are associated points as they are the generic points of the components. Also, we know that $(x-1,y-1)$ is also an associated point as we can consider the element $(x-1)^{14}$, this is not in the intersection, therefore not 0, but notice that it's annihilator contains $(x-1,y-1)$ but not x, thus we know that it's annihilator is just $(x-1,y-1)$, thus this prime is also an associated prime. We are done.

We now come to some geometric definitions related to the algebraic facts we talked about:

Definition 6.48. Suppose that X is a locally Noetherian scheme and $\mathcal{F}$ is a coherent sheaf on X, we say x is an **associated point** of $\mathcal{F}$ if in the local ring $\mathcal{O}_{X,x}$ the maximal ideal is an associated point of $\mathcal{F}_x$. We say that x is an **embedded point** of $\mathcal{F}$, if the maximal ideal of $\mathcal{O}_{X,x}$ is an embedded point of $\mathcal{F}_x$.

Again, we define the associated points of X to be the associated points of $\mathcal{O}_X$ and similar for embedded points.

Remark 6.49. Thank to the fact that we are working with coherent sheaves over locally Noetherian schemes, this definition we give can also be defined affine locally as we already proved that for finitely generated modules over Noetherian rings, associated points commute with localization.

Many results we know for associated points of modules over a ring has it's generalization on schemes:

Proposition 6.50. Say X is locally Noetherian and U is an open subscheme, then the natural map:

$$\Gamma(U, \mathcal{O}_X) \to \prod_{p \in \operatorname{Ass} X \cap U} \mathcal{O}_{X,p}$$

is injective.

Proof. We already saw a similar result for modules over Noetherain rings. Therefore since we work with locally Noetherian schemes, is U is an affine open which is the spec of a Noetherian ring, the assertion holds.

Then again, thanks to we are working with locally Noetherian schemes, we can cover U by affine opens which are specs of Noetherian rings, let's call them U_i. Then if $x \in \Gamma(U, \mathcal{O}_X)$ is mapped to 0, then for each U_i, we know that x is again mapped to 0 via:

$$\Gamma(U, \mathcal{O}_X) \to \prod_{p \in \operatorname{Ass} X \cap U_i} \mathcal{O}_{X,p}$$

Notice that this map can be factored through the following diagram:

$$\begin{tikzcd}
\Gamma(U, \mathcal{O}_X) \arrow[rr] \arrow[dr] & & \prod_{p \in \operatorname{Ass} X \cap U_i} \mathcal{O}_{X,p} \\
& \Gamma(U_i, \mathcal{O}_X) \arrow[ur, hook] &
\end{tikzcd}$$

therefore x restricted to each U_i is 0, the by the sheaf axioms, we conclude x is zero on U. □

Definition 6.51. An open subscheme U of a scheme X is called **scheme-theoretic dense** if any function on any open subset V is 0 if it's restrict to 0 on $U \cap V$.

Notice that this is stronger than the usual topological dense (U has to be dense, i.e. intersects each nonempty open nontrivially).

Remark 6.52. If X is locally Noetherian, the result of the above proposition shows that U is scheme-theoretic dense if it contains all associated points of X.

Say U is scheme theoretic dense and p is an associated point of X, then let V be an affine open around p, then we know that f on V is 0 if it restricts to 0 on $U \cap V$. But if p is not in U, we know that $U \cap V$ doesn't contain p and we can assume that p is actually a minimal prime in V since if U contains the minimal prime below p it also contains p. Then let f be a function on V that vanishes elsewhere, but not on p. Such an element exists as the minimal prime is the annihilator of some f and you can just pick that f, this gives you a contradiction.

The converse is just the content of the previous proposition.

Associated points and support also helps us to under stand the notion of length:

Proposition 6.53. Suppose that A is Noetherian, then any finitely generated module whose support is just finitely many closed points of $SpecA$ has finite length.

Proof. Recall that any finitely generated module over a Noetherian ring has a filtration we discussed in this section and say p_i appears in that filtration, then p_i is in the support, this implies that p_i is maximal by the assumption, therefore we know that M has finite length. □

Remark 6.54. We already saw that over any ring if M is of finite length, then it's support is just finitely many closed points, thus for finitely generated modules over a Noetherian ring, these two are equivalent.

Proposition 6.55. Suppose that $\mathcal{F}$ is a coherent sheaf on a locally Noetherian scheme X that is supported in finitely many closed points, then $\mathcal{F}$ has finite length.

Proof. First, again, we decompose $\mathcal{F}$ into a finite direct sum of skyscraper sheaves, each supported at a closed point. Then we only have to show that each one of the is of finite length.

For each one of them, we only have to show that the $\mathcal{O}_{X,p}$-module is of finite length. According to the preceding proposition, we only have to it's only supported at the unique maximal ideal. I will let you show this. □

Remark 6.56. We have also saw that if $\mathcal{F}$ is finite length, then it's supported only on finitely many closed points and finite type. Therefore, for coherent sheaves on a locally Noetherian scheme, finite type and finite support at closed points is equivalent to be of finite length.

Proposition 6.57. If X is a finite length scheme, then it only have finitely many points with discrete topology.

Proof. Notice that if $\mathcal{O}_X$ is finite length, it's supported at finitely many closed points, then we are done. □

Proposition 6.58. If X is locally finite type over $k = \overline{k}$ and of finite length l, then $\dim_k \Gamma(X, \mathcal{O}_X) = l$.

Proof. Omitted. □

Associated points also helps us refine our definition of rational functions to locally Noetherian schemes, not just integral schemes:

Definition 6.59. Suppose that X is a locally Noetherian scheme, a **rational function** is an element of the union of images of the following injective map:

$$\Gamma(U, \mathcal{O}_X) \to \prod_{p \in \mathrm{Ass} X \cap U} \mathcal{O}_{X,p}$$

where U should contain all the associated points of X, or in other words, U should be **scheme-theoretic dense**, not just dense.

Proposition 6.60. Rational functions on a locally Noetherian scheme is the colimit of the functions over all open subsets containing all the associated points of X:

$$\mathrm{colim}_{\{U : U \supset \mathrm{Ass} X\}} \mathcal{O}(U)$$

with the obvious maps between different $\mathcal{O}(U)$.

Proof. Say $V \subset U$ and $V \supset \mathrm{Ass} X$, we have the following diagram:

$$\begin{tikzcd} \Gamma(U, \mathcal{O}_X) \arrow[rr, hook] \arrow[dr] & & \prod_{p \in \mathrm{Ass} X} \mathcal{O}_{X,p} \\ & \Gamma(V, \mathcal{O}_X) \arrow[ur, hook] & \end{tikzcd}$$

then we see that the restriction must be injective as well, similar to what we saw on integral schemes. And the colimit is index by all the open subsets of X that contains all associated points with open inclusion.

Then the universal property of colimits gives a natural map to the original definition of rational functions, which is obviously surjective. It's also easy to show it's injective, we are done. □

Using the above proposition, we can think a rational function f as something that comes from a actually function f on $\Gamma(U, \mathcal{O}_X)$ if f, g, two actually functions on U agrees as rational functions, they agree as actually function on U since every morphism is injective. Moreover, every rational function has a maximal **domain of definition** as if a regular function f comes from regular functions on U, V, then the two regular functions must agree on $U \cap V$ since every map is injective, therefore, they actually comes from something on $U \cup V$. Using this interpretation, we also find that the rational functions forms a ring, called the **total fraction ring** or **total quotient ring** of X.

Definition 6.61. We say that a rational function f is **regular at** p is the maximal domain of definition contains p. A rational function is called **regular** if it's regular at all points of X. This implies that it actually is a global function.

The complement of the regular locus of f is called the **indeterminacy locus** of f.

Remark 6.62. I will let you see that this theory of rational functions of locally Noetherian schemes works perfectly for integral schemes even though it's not locally Noetherian, we use the definition of associated points using the local ring, and you find that the only associated point is the generic point and everything else can be carried out here as well.

6.2 Back to Hilbert Functions and Polynomials

Let's go back to prove Thm 6.1, for the notation, we will define the dimension of the empty subset and the degree of the 0 polynomial to be -1.

Proof. Define $p_{\mathcal{F}}(m) = \chi(X, \mathcal{F}(m))$ and we will show it's a polynomial of desired degree.

First, we would like to reduce to the case where k is algebraically closed. Consider the following diagram:

$$\begin{tikzcd} X_K \arrow[r, hook] \arrow[d] & \mathbb{P}^n_K \arrow[r] \arrow[d] & SpecK \arrow[d] \\ X \arrow[r, hook] & \mathbb{P}^n_k \arrow[r] & Speck \end{tikzcd}$$

Notice that we know that $\chi(X, \mathcal{F}(m))$ are actually determined by cohomlogy and we know that:

$$H^i(X, \mathcal{F}) \otimes_k K \simeq H^i(X_K, \mathcal{F} \otimes K)$$

for all quasicoherent sheaves on X and any field extension K. We will take $K = \overline{k}$ and we know that:

$$\dim_{\overline{k}}(V \otimes_k K) = \dim_k V$$

for any V vector space, therefore we know that $\chi(X, \mathcal{F}(m)) = \chi(X_K, \mathcal{F}(m) \otimes K)$ where the tensor $\mathcal{F}(m) \otimes K$ just means that pullback. Since we know that in the morphism:

$$\mathbb{P}^n_K \to \mathbb{P}^n_k$$

$\mathcal{O}(m)$ pulls back to $\mathcal{O}(m)$, $\mathcal{F}(m) \otimes K$ is the same as $(\mathcal{F} \otimes K)(m)$, thus we will first consider what happens on X_K. In particular we want K to be an infinite field, and this is one example that even you are only concerned with arithmetic informations, you sometimes have to go to geometric setting.

Recall that a coherent sheaf $\mathcal{F}$ on a Noetherian scheme has only finitely many associated points. We can use this fact to prove a result we will use again, thus we state it separately: $\square$

Proposition 6.63. Say X is a projective k-scheme with k infinite and $\mathcal{F}$ is a coherent sheaf on X, then if $\mathcal{L}$ is a very ample line bundle, then there is an effective Cartier divisor on X such that $\mathcal{L} = \mathcal{O}(D)$ and moreover D doesn't contain any associated point of $\mathcal{F}$.

Proof. Since $\mathcal{L}$ is very ample, then we know that we have the following diagram:

$$\begin{tikzcd} X \arrow[rr, hook, "i"'] \arrow[dr] & & \mathbb{P}^n_k \arrow[dl] \\ & Speck & \end{tikzcd}$$

with $i^*\mathcal{O}(1) \simeq \mathcal{L}$. Recall that the associated points of $\mathcal{F}$ is only finitely many, which goes to finitely many points in $\mathbb{P}^n_k$. Then notice that if you can show that there is a hyperplane not containing these points, say defined by the linear equation f, then we call this hyperplane D' and we can pull it back, which is still effective, and it will not contain the associated points of $\mathcal{F}$.

The existence of this hyperplane D' is where we use the fact that k is infinite. We have already seen something similar before and in general if k is not infinite, we can find a hypersurface, but not a hyperplane. We prove it again here just to remind you why k being infinite is important. The idea is simple, suppose that these finitely many points are all k-valued points, i.e. $(a_0, \cdots, a_n)$, then it's easy and everything is just linear algebra recall that a k-valued point $(a_0, \cdots, a_n)$ is in the hyperplane $\sum_i b_i x_i$ iff $\sum_i b_i a_i = 0$. Then we consider equations:

$$\sum_i b_i a_i = 0$$

with variables a_i, then we know that since not all a_i's are 0, the solution space is n-dimensional, and in particular proper, then the assertion is just the statement that for an infinite field k, and any k-vector space V, finitely many proper subspaces of V couldn't cover V. This is also the classical proof when you work with $\mathbb{PR}^n$ or $\mathbb{PC}^n$.

For the general case, even though we can always assume that these p_i's are closed, but they may not be k-valued, so we may need a different point of view. Say $p_1, \cdots, p_n$ are these points we want to avoid, and say each p_i is generated by a few homogeneous polynomials. Then consider the hyperplane:

$$D' = x_0 + \cdots + x_{n-1} + bx_n$$

We claim that we can modify the coefficient b such that it avoids all these points. Notice that p_i is in the hyperplane if any $n+1$-tuple of solutions to the equation D' is a solution for each of the generating polynomials in p_i. Then notice that we can set $x_0 = \cdots = x_{n-1} = 1$ and we know that x_n will be to take a specific value depending on b such that D is 0. But after you set $x_0 = \cdots = x_{n-1} = 1$, the condition that the generating polynomial of p_i to be 0 become the condition that several polynomial of x_n is 0 when x_n take that specific value. But notice that each of the polynomial has only a finite number of solutions, thus we know that if these solutions don't have a common one, we are done, but if there are some common solutions, it's again finitely many, then we can take b to be a specific element in k to avoid that since k is infinite.

Therefore, we have found a hyperplane D' in $\mathbb{P}^n_k$ such that it avoids the associated points of $\mathcal{F}$. Moreover, we see that D' is cut out by an element in $\mathcal{O}(1)$, therefore we know that $\mathcal{O}(1)$ is isomorphic to $\mathcal{O}(D')$, then if we can prove that the pullback of $\mathcal{O}(D')$ is isomorphic to $\mathcal{O}(D)$ where D is the pullback of the closed embedding, we are done. Again, as pullback commutes with dual, we only need to prove that the pullback of the ideal sheaf of D' is the ideal sheaf of D'. Consider the following diagram:

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d, hook] & D' \arrow[d, hook] \\ X \arrow[r, hook] & \mathbb{P}^n_k \end{tikzcd}$$

we know that D is also locally principal by the local description, and it's locally $(\overline{s}) \subset A/I$ if D' is locally defined by $(s) \to A$, then notice that the pullback of this morphism is:

$$(\overline{s}) \to A/I$$

thus the ideal of D is the same with the pullback of $(s) \to A$, thus they are indeed the same, therefore the pullback of $\mathcal{O}(-D)$ is $\mathcal{O}(D')$, then so is the dual, thus we are done. $\square$

Remark 6.64. Here when we try to pullback the effective Cartier divisor, we use the fact that $X \hookrightarrow \mathbb{P}^n_k$ is a closed embedding, which we understand pretty well locally, but this argument is not true for general morphisms $X \to \mathbb{P}^n_k$.

Then let's get back to the proof of Thm 6.1:

Proof. We first prove a simple case, that is when the support of $\mathcal{F}$ is zero dimensional. That is saying that $\mathcal{F}$ is supported at a finite collection of closed points. Then If we compute the cohomology of $\mathcal{F}$, you can choose the cover carefully such that all higher cohomology vanishes for all $\mathcal{F}(m)$. Then if you compute the degree 0-cohomology of $\mathcal{F}$, it's just the sum of dimensions of $\mathcal{F}_{p_i}$ where p_i runs over all point where $\mathcal{F}$ is supported. But notice that for such a sheaf, tensor with a line bundle doesn't change it, thus the function:

$$\chi(X, \mathcal{F}(m)) = h^0(X, \mathcal{F})$$

which is a constant function, or a polynomial of degree 0.

Then we would like to use induction in some way, this propbably means that give a $\mathcal{F}$ on X, we need to somehow decrease the dimension of the support, to do this, say D on X is an effective Cartier divisor cut out by $x \in \Gamma(X, \mathcal{O}(1))$ and D doesn't contain any associated point of $\mathcal{F}$.

The reason we want D to be away from the associated points of $\mathcal{F}$ is because that we can first consider the morphism of sheaves on X:

$$\mathcal{O}(-1) \to \mathcal{O}(-1) \otimes \mathcal{O}(1) \simeq \mathcal{O}$$

where m is mapped to $m \otimes x$, then we can tensor it with $\mathcal{F}$ to obtain:

$$\mathcal{F}(-1) \to \mathcal{F}$$

Notice that locally on the standard affine open, we know that this is just a module map(we just use the local knowledge of what tensor with a quasicoherent sheaf does):

$$M \xrightarrow{\times x} M$$

or rather, it's:

$$M \otimes A \to M \otimes A \otimes A$$

where $m \otimes a \mapsto m \otimes a \otimes x$.

But notice that since x doesn't vanish on any associated point of M by definition of D, it's not a zero divisor, thus it's injective, thus $\mathcal{F}(-1) \to \mathcal{F}$ is injective. Let $\mathcal{G}$ be the cokernel:

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{G} \to 0$$

I claim that the support of $\mathcal{G}$ is $V(x) \cap \mathrm{Supp}\mathcal{F}$ as we already saw this is locally true, and we know that the cokernel of morphism of quasicoherent sheaves on affine open doesn't need to be sheafified. Then we know that $V(x)$ meets all the positive dimensional components of $\mathrm{Supp}\mathcal{F}$ since we are working in projective spaces and $V(x)$ as a hyperplane, meets everything of positive dimension.

Finally, by Krull's principal theorem, if the dimension of $\mathrm{Supp}\mathcal{F}$ is positive, the dimension of $\mathrm{Supp}\mathcal{G}$ is at least one less as one can first pass to the affine case and prove that in each of the standard affine, the dimension of $\mathrm{Supp}\mathcal{F} \cap V(x)$ drops by 1, notice that since x doesn't vanish at any associated points, we know that it doens't contain any minimal primes in $\mathrm{Supp}\mathcal{F}$, thus the dimension of the intersection at least drops by 1, unless $\mathcal{F} = 0$. But notice that when the support of $\mathcal{F}$ is zero dimensional, we know that the support of $\mathcal{G}$ is either 0 or -1, but either way, the Euler characteristic of $\mathcal{G}$ is still a polynomial then we can twist by $\mathcal{O}(m)$ and obtain:

$$0 \to \mathcal{F}(m-1) \to \mathcal{F}(m) \to \mathcal{G}(m) \to 0$$

use the fact that Euler characteristic is additive in SES, we obtain:

$$\chi(X, \mathcal{F}(m)) - \chi(X, \mathcal{F}(m-1)) = \chi(X, \mathcal{G}(m))$$

then we can use the inductive hypothesis, finally, the result comes from the following easy result for polynomials in one variable: $\square$

Proposition 6.65. Suppose f, g are functions on integers with $f(m) - f(m-1) = g(m)$ for all m and g is a polynomial of degree d, then f is a polynomial of degree $d+1$.

Proof. Consider the binomial coefficients:

$$\binom{x}{r}$$

for all $0 \leqslant r \leqslant d$, I claim that these form a $\mathbb{Z}$ basis for degree d polynomials with rational coefficients. To see why, notice that if you write out this binomial coefficient:

$$\binom{x}{r} = \frac{x(x-1)\cdots(x-r+1)}{r!}$$

These binomial coefficients only make sense when x is at least r, but these polynomial make sense for arbitrary x. Notice that for $0 \leqslant r \leqslant m$, the degree of the $\binom{x}{r}$ polynomial is r, thus we know that these are linearly independent, and they span all the polynomials with degree at most d since they span x^r for $0 \leqslant i \leqslant d$.

These polynomials satisfies an important property:

$$\Delta\binom{x}{r+1} = \binom{x+1}{r+1} - \binom{x}{r+1} = \binom{x}{r}$$

Then what is important for us is that:

$$f(m) - f(m-1) = \sum c_r \binom{m}{r} = g(m)$$

But notice that $\sum c_r \binom{m}{r} = \sum c_r \binom{m+1}{r+1} - \sum c_r \binom{m}{r+1}$ Then if we define:

$$F(m) = C + \sum c_r \binom{m+1}{r+1}$$

We know that $F(m) - F(m-1) = \sum c_r \binom{m}{r} = g(m)$, this implies that:

$$F(m) = F(0) + \sum_{i=0}^{m} g(i) \qquad f(m) = f(0) + \sum_{i=0}^{m} g(i)$$

Therefore we know f and F only differs by a constant, then since F is a polynomial, so is f. □

This finishes the proof of the theorem.

Remark 6.66. There is one point that we didn't made really clear when proving this. We made a base change and pulled back $\mathcal{F}$ to $\mathbb{P}^n_{\overline{k}}$, but we didn't prove that the dimension of the support of the pullback of $\mathcal{F}$ is the same as the dimension of the support of $\mathcal{F}$. The reason we we didn't mention it there is because this will use some algebra notions we haven't fully discussed yet, i.e. flatness. But we still want to mention it here so that after you learned flatness, you can come back and understand the proof.

The tool we use it that if k is a field and K is it's algebraic closure, then the canonical morphism:

$$k[x_1, \cdots, x_n] \to K[x_1, \cdots, x_n]$$

is faithfully flat so is localizing at a prime of $K[x_1, \cdots, x_n]$.

We first consider what is the support of the pullback $\mathcal{F} \otimes K$. Well, we know that the morphism:

$$\mathbb{P}^n_{\overline{k}} \to \mathbb{P}^n_k$$

can be locally described by:

$$\mathbb{A}^n_{\overline{k}} \to \mathbb{A}^n_k$$

induced by the canonical inclusion, then we know that if locally $\mathcal{F}$ is an $k[y_1, \cdots, y_n]$-module on $\mathbb{A}^n_k$, the pullback is just $M \otimes \overline{k}[y_1, \cdots, y_n]$, then we know that localizing at a prime p of $\overline{k}[y_1, \cdots, y_n]$, i.e. $(M \otimes \overline{k}[y_1, \cdots, y_n])_p$ is isomorphic to $M_q \otimes (\overline{k}[y_1, \cdots, y_n]_p)$ (this can be proved in much more generality), then you use fully faithfulness, to conclude that the result is 0 iff M_q is 0.

Then this implies that the support of the pullback is the inverse image of the support of $\mathcal{F}$. Notice that this is a coherent sheaf on a Noetherian scheme, therefore, the support is closed, we endow it with reduced scheme structure and get the following diagram:

$$\begin{tikzcd}
\bullet \arrow[r, hook] \arrow[d] & \mathbb{P}^n_K \arrow[r] \arrow[d] & SpecK \arrow[d] \\
\mathrm{Supp}\mathcal{F} \arrow[r, hook] & \mathbb{P}^n_k \arrow[r] & Speck
\end{tikzcd}$$

this implies that Supp$\mathcal{F} \times K$ is the inverse image, and we can compute the dimension using a proposition we proved long ago when discussing dimension: dimension of pure dimensional locally finite type over k schemes are preserved by base change.

Therefore, we are good if Supp$\mathcal{F}$ is pure dimensional, if not, we take out the component of the largest dimension, then form the Cartesian diagram again, then we know that the pullback is again pure dimensional, therefore the dimension of the pullback of the support is at least the same as that of the support. However, if the dimension is larger, you will get the conclusion that the largest dimension component is the union of a few stuff with lower dimension, a contradiction.

Example 6.67. Let's see a few examples.

1. We have already saw before that the Hilbert polynomial of $\mathbb{P}^n_k$ is:

$$p_{\mathbb{P}^n_k} = \binom{m+n}{m}$$

as the Euler characteristic $\chi(\mathbb{P}^n_k, \mathcal{O}(m)) = \binom{m+n}{m}$ viewed as the polynomial.

2. Let H be a degree d hypersurface in $\mathbb{P}^n_k$, notice that the closed embedding $H \hookrightarrow \mathbb{P}^n_k$ is a effective Cartier divisor, and it's cut out by $f \in \mathcal{O}(d)$, then we consider the SES of the effective Cartier divisor, which is:

$$0 \to \mathcal{O}(-d) \to \mathcal{O} \to i_*\mathcal{O}_H \to 0$$

Recall that we have the following isomorphism:

$$i_*\mathcal{O}_H \otimes \mathcal{O}(m) \simeq i_*(\mathcal{O}_H \otimes i^*\mathcal{O}(m))$$

This is the projection formula since a closed embedding is always qcqs, then recall that closed embedding gives you isomorphisms of cohomology, then we know that:

$$\chi(H, \mathcal{O}(m)) = \chi(\mathbb{P}^n_k, i_*\mathcal{O}_H \otimes \mathcal{O}(m))$$

then since Euler characteristic is additive in SES and tensor with $\mathcal{O}(m)$ is exact, we know that so is the Hilbert polynomial, then we got the following relation:

$$p_{\mathbb{P}^n_k}(m) = p_{\mathcal{O}(-d)}(m) + p_H(m)$$

since $p_{\mathcal{O}(-d)}(m) = p_{\mathbb{P}^n_k}(m-d)$, we know that:

$$p_H(m) = \binom{m+n}{n} + \binom{m+n-d}{n}$$

Actually, we are able to compute more Hilbert polynomials:

Example 6.68. Recall that on $\mathbb{P}^1_k$, we have a complete linear series $\mathcal{O}(3)$, and it's given by x^3, x^2y, y^2x, y^3 which gives a morphism into $\mathbb{P}^3_k$. This is a closed embedding as you can check locally. Say the coordinate on $\mathbb{P}^3_k$ is x, y, z, w, then we know that on $D_+(x)$, it's inverse image is $D_+(x)$ as well and the ring morphism is given by:

$$k[\frac{y}{x}, \frac{z}{x}, \frac{w}{x}] \to k[\frac{y}{x}]$$

where $\frac{y}{x}$ is mapped to $\frac{y}{x}$ and $\frac{z}{x}$ is mapped to $\frac{y^2}{x}$ finally $\frac{w}{x}$ is mapped to $\frac{y^3}{x^3}$, obviously surjective. And you can verify similar stuff happens on $D_+(y)$ as well.

Recall that this is an example of Veronese embedding. We would like to compute the Hilbert polynomial of the closed embedding:

$$\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$$

By definition, we know that the Hilbert polynomial of the closed embedding is:

$$\chi(\mathbb{P}^3_k, i_*\mathcal{O}_{\mathbb{P}^1_k}(m))$$

Again, we use the projection formula to conclude that the stuff above is:

$$\chi(\mathbb{P}^1_k, \mathcal{O}(3m)) = \binom{3m+1}{1} = 3m+1$$

as we know that the pullback of $\mathcal{O}(1)$ to $\mathbb{P}^1$ is $\mathcal{O}(3)$.

In general we can compute the Hilbert polynomial of all the degree d-Veronese embeddings:

$$v_d : \mathbb{P}^n_k \hookrightarrow \mathbb{P}^N_k$$

where $N = \binom{m+d}{d} - 1$, recall that this closed embedding is defined by the complete linear series $|\mathcal{O}(d)|$, then we know that by definition:

$$p_{v_d}(m) = \chi(\mathbb{P}^N_k, v_d^* \mathcal{O}_{\mathbb{P}^n_k}(m))$$

By projection formula, this Euler characteristic is:

$$\chi(\mathbb{P}^n_k, \mathcal{O}(dm)) = \binom{dm+n}{n}$$

Remark 6.69. One possible confusion that may show up is that, for example, we consider:

$$\mathbb{P}^1_k \to \mathbb{P}^3_1$$

and we want to compute $p_X(m)$, then by definition we are computing $\chi(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1_k}(m))$, but be careful that here this twisting by m is not twisting by $\mathcal{O}(m)$ on $\mathbb{P}^1$, rather it's twisting by the pullback of $\mathcal{O}(m)$ on $\mathbb{P}^3_k$, which is why we say that this Hilbert polynomial depend on the closed embedding.

Here is a degree that is useful when studying curves:

Proposition 6.70. Suppose that we have a sequence of closed embeddings:

$$X \xhookrightarrow{i} Y \hookrightarrow \mathbb{P}^n_k$$

then $p_X(m) \leqslant p_Y(m)$ for $m \gg 0$.

Proof. Consider the SES of closed embedding $i : X \hookrightarrow Y$ and then tensor it with the pullback of $\mathcal{O}(m)$ to Y:

$$0 \longrightarrow \mathcal{I}_{X/Y}(m) \longrightarrow \mathcal{O}_Y(m) \longrightarrow (i_* \mathcal{O}_X)(m) \longrightarrow 0$$

Then we know that by definition:

$$p_Y(m) = \chi(Y, \mathcal{O}_Y(m)) = \chi(Y, \mathcal{I}_{X/Y}(m)) + \chi(Y, (i_* \mathcal{O}_X)(m))$$

The second equation follows from the fact that Euler characteristic is additive in SES. However, notice that using projection formula:

$$\chi(Y, (i_* \mathcal{O}_X)(m)) = \chi(X, \mathcal{O}_X(m)) = p_X(m)$$

Therefore, we get the relation:

$$p_Y(m) = p_X(m) + \chi(Y, \mathcal{I}_{X/Y}(m))$$

By Serre vanishing theorem, we know that for $m \gg 0$, higher cohomology of $\mathcal{I}_{X/Y}(m)$ will vanish, therefore we know that for larger m, the Euler characteristic is just $h^0(Y, \mathcal{I}_{X/Y}(m)) \geqslant 0$, then we are done. $\square$

Proposition 6.71. Suppose that $X \hookrightarrow \mathbb{P}^n_k$ is a projective k-scheme with a coherent $\mathcal{F}$ on X, then if the Hilbert polynomial of $\mathcal{F}$ is 0 then $\mathcal{F}$ is 0.

In particular, if you apply this result to the above proposition, you find that if $p_X(m) = p_Y(m)$ for $m \gg 0$, then $X = Y$.

Proof. Recall that we already see that for any coherent sheaf $\mathcal{F}$ on X, for large enough m, we know $\mathcal{F}(m)$ is globally generated and all the higher cohomology vanishes, therefore we know that $\chi(X, \mathcal{F}(m)) = h^0(X, \mathcal{F}(m))$. Then it's 0 if all global sections are 0, but this, plus $\mathcal{F}(m)$ is globally generated, implies that $\mathcal{F}(m)$ is 0. $\square$

From the Hilbert polynomial, we can extract many important invariants, but we will focus on two of them, **degree** and **arithmetic genus**. Let's deal with the degree first:

Definition 6.72. The **degree** of a projective k-scheme $X \hookrightarrow \mathbb{P}^N_k$ of dimension n is defined to be the leading coefficient of $p_X(m)$ times $n!$.

For example we see that if we use $id : \mathbb{P}^n_k \to \mathbb{P}^n_k$ then we see that the Hilbert polynomial of $\mathbb{P}^n$ is $\binom{m+n}{n}$, then you see that the degree is 1. And if you use the Veronese embedding: the degree of $\mathbb{P}^1$ in $\mathbb{P}^3_k$ is 3.

Proposition 6.73. The degree of $X \hookrightarrow \mathbb{P}^N_k$ is always an integer.

Proof. Let's first see what we need to prove. Suppose that $p_X(m)$ is the Hilbert polynomial of X, we want to show that it's leading coefficient is of the form $\frac{a}{n!}$ for some integer a.

What we know about the polynomial is that it's of degree n since we know that the support of $i_*\mathcal{O}_X$ is X, which is of dimension n. Therefore we know the top degree term is something times m^n. Another thing we know is that $p_X(m)$ is a polynomial with rational coefficients that takes on integer values.

I claim that the information above is enough. To be precise, we will prove that any polynomial f of degree k with rational coefficients that takes on only integer values must have coefficient of m^k an integral multiple of $\frac{1}{k!}$.

We will prove this by induction, and the base case is when f is of degree 0. It's clear that in this case f is just a constant integer. Now consider $g(x) = f(x) - f(x-1)$, we know that this is again a polynomial of degree $k-1$, then by inductive hypothesis, we know that for $g(x)$, the coeffcient for x^{k-1} is $\frac{a}{(k-1)!}$. But suppose that:

$$f(x) = a_k x^k + \cdots + a_0$$

we know that $f(x) - f(x-1) = (-a_k k)x^{k-1} + \cdots$, therefore we know:

$$a_k = \frac{a}{(k-1)!}$$

This implies that:

$$a_k = \frac{a}{k!} = \frac{a}{k!}$$

Therefore, we are done. $\square$

Proposition 6.74. A hypersurface $H \hookrightarrow \mathbb{P}^n_k$ of degree d indeed has degree d in our new definition, preventing a notational crisis.

Proof. Recall that we already saw what is the Hilbert polynomial of H is:

$$p_H(m) = \binom{n+m}{n} - \binom{m+n-d}{n}$$

If you write this out, you will find that it's:

$$p_H(m) = \frac{(m+1)\cdots(m+n)}{n!} - \frac{(m-d+1)\cdots(m+n-d)}{n!}$$

Notice that the top degree terms of the two polynomials cancels each other, then we only need to look at lower degree terms. The degree $n-1$ term of the first and second polynomial are:

$$\frac{n(n+1)}{2n!} \qquad \frac{n(n-2d+1)}{2n!}$$

Therefore, the difference is:

$$\frac{n}{2n!} \cdot (2d) = \frac{d}{(n-1)!}$$

therefore we know that the degree is $d = \frac{d}{(n-1)!} \cdot (n-1)!$. $\square$

Proposition 6.75. Suppose that $C \hookrightarrow \mathbb{P}^n_k$ is a regular curve embedded in the projective space via a degree d line bundle, then the degree of C using this embedding is d, thus preventing another notational crisis.

Proof. Recall that since C is proper, thus projective, therefore, we can use Riemann-Roch on C, which looks like:

$$\deg \mathcal{L} = \chi(C, \mathcal{L}) - \chi(C, \mathcal{O}_C)$$

Well, let's assume that $\mathcal{L} = \mathcal{O}(D)$ for some divisor $D = \sum a_p p$ on C, this is possible since C is regular, then we can assume the degree of $\mathcal{L}$ is $\sum a_p \deg p = d$. Again, this holds because C is regular.

Notice that $p_C(m) = \chi(\mathbb{P}^n_k, i^*\mathcal{O}_C(m))$, notice that the pullback of $\mathcal{O}(1)$ on $\mathbb{P}^n_k$ is $\mathcal{L} \simeq \mathcal{O}(D)$ on C, thus we are computing:

$$\chi(C, \mathcal{O}_C(mD))$$

Notice that if we instead consider:

$$\chi(C, \mathcal{O}_C(mD)) - \chi(C, \mathcal{O}_C)$$

this is the degree of $\mathcal{O}_C(mD)$, which is md, therefore we know that:

$$p_C(m) = md + \chi(C, \mathcal{O}_C)$$

therefore the degree is $d \times 1 = d$. $\square$

Proposition 6.76. The degree of the d-th Veronese embedding $v_d : \mathbb{P}^n_k \hookrightarrow \mathbb{P}^N_k$ is d^n.

Proof. Recall that we already computed the Hilbert polynomial of the Veronese embedding, which is:

$$\binom{dm+n}{n} = \frac{(dm+1)(dm+2)\cdots(dm+n)}{n!}$$

and you see that the top degree term is $\frac{(dm)^n}{n!}$, thus the degree is d^n. $\square$

One interesting application of the notion of degree is to prove Bezout's theorem:

Proposition 6.77. Suppose that $X \hookrightarrow \mathbb{P}^n_k$ is a projective scheme of dimension at least 1, and H is a hypersurface not containing any associated points of X. Then $\deg(H \cap X) = (\deg H) \cdot (\deg X)$.

Proof. Suppose that the degree of H is d, then we are trying to show:

$$\deg X \cap H = d \cdot \deg X$$

Recall that $H \cap X$ is cut out by an element in $\mathcal{O}(d)$ on X, then we know that if we consider the following SES:

$$0 \to \mathcal{O}(-d) \to \mathcal{O}_X \to \mathcal{O}_{X \cap H} \to 0$$

then we use projection formula again to get:

$$p_{X \cap H}(m) = p_X(m) - p_X(m-d)$$

Then since we know that H doesn't contain any irreducible component of X, then $X \cap H$ has dimension precisely one less than X, say $\dim X = t$, then $\dim(X \cap H) = t - 1$, therefore in $p_X(m) - p_X(m-d)$, if we assume that:

$$p_X(m) = a_t m^t + a_{t-1} m^{t-1} + \cdots$$

then we know that $p_X(m-d) = a_t(m-d)^t + a_{t-1}(m-d)^t + \cdots$ and you can compute that in the difference $p_X(m) - p_X(m-d)$, the $t-1$ term is with coefficient $ta_t d$, therefore we know that the degree is:

$$\frac{ta_t d}{(t-1)!} = \deg X \cdot d$$

we are done. $\square$

Remark 6.78. We still want to explain a little bit on dimension of the intersection. If X is just of dimension 0, i.e. a finite collection of closed points, then $H \cap X$ is empty, thus the degree is 0. On the other hand, the degree of X is not 0, thus the conclusion of the above proposition fails, this is why we need the dimension of X to be at least 1.

Then if X has dimension at least one, we know that since H doesn't contain any generic points of X, the dimension of $X \cap H$ drops at least by 1. But we also know that by Krull's principal ideal theorem, in each of the standard affine, components of $X \cap H$ has codimension 0 or 1, since it's cut out by a single equation, and since we work with finite type over k schemes, the dimension of this component either drops by 1 or remains the same, but if it remains the same, we can conclude that the dimension of $X \cap H$ doesn't drop, which is a contradiction, thus in each of the affine component, the dimension is at most 1 less than the dimension of X, therefore we conclude that the dimension at least and at most drops by 1, thus exactly drops by 1.

Remark 6.79. Another point that is worth mentioning in the above proof is the SES:

$$0 \to \mathcal{O}(-d) \to \mathcal{O}_X \to \mathcal{O}_{X \cap H} \to 0$$

To totally understand this SES, we first need compute explicitly how $X \cap H$ is defined in X locally in the standard affine open. Then we need to this closed embedding is the same as the one defined by the element in $\mathcal{O}(d)$, which is obtained by pullback the corresponding defining element in $\mathcal{O}(d)$ on $\mathbb{P}^n_k$. After you show that these two closed embedding are the same, you still need to show that this is an effective Cartier divisor to make sense of this SES, but this is precisely the assumption that H doesn't contain any associated points of X, therefore, you know that it's locally defined by a single nonzero divisor, thus effective, we are done.

Example 6.80. Using the above result, we obtain **Bezout's Theorem for plane curves**.

If C, D are projective plane curves with no common components (recall that projective plane curves are hypersurfaces inside $\mathbb{P}^2_k$) whose degree are m, n respectively. Then C, D meet at mn points, counted with appropriate multiplicities.

To apply the result above, we need to prove that projective plane curves don't have embedded points, and this is done by a previous proposition where we showed that a hypersurface in an affine space doesn't have embedded points.

Proposition 6.81. Suppose that C is a degree 1 pure 1 dimensional closed subscheme of $\mathbb{P}^3_k$ with no embedded points, then C is a line.

Proof. What we want to show is that C is contained in the intersection of two distinct hyperplanes.

To start with, let's reduce the problem to the case where k is algebraically closed. We consider the following diagram:

$$\begin{tikzcd} C_K \arrow[r, hook] \arrow[d] & \mathbb{P}^3_K \arrow[r] \arrow[d] & SpecK \arrow[d] \\ C \arrow[r, hook] & \mathbb{P}^3_k \arrow[r] & Speck \end{tikzcd}$$

We already know that C_K is pure dimensional of dimension 1, and I also claim that C_K has no embedded points and this again comes from the fact of a faithfully flat extension, and we don't plan to discuss that in detail here.

Then, I claim that the degree of C_K is still 1, this is because by definition, the degree of C_K comes out of the Euler characteristic and I will let you think why this is true using the fact that base change by a field extension doesn't affect the dimension of cohomology groups.

Finally, notice that a hyperplane also gets pulled back to a hyperplane, then I will let you think through the remaining reasoning.

Then after we are reduced to the algebraic closed case, or job to to find two closed points in C_K such that any hyperplane containing these two points contains C_K. First, we notice that:

$$\deg(C_K \cap H) = \deg C_K \cdot \deg H = 1$$

if H doesn't contain any generic points of components of C_K. If we assume further that $C_K \cap H$ has at least

We know that as a hyperplane, H intersects with anything of dimension at least 1, and pick two distinct closed points p, q on X and suppose that H is a hyperplane containing these two points.

Now suppose that some components of C is contained in the line $\overline{pq}$, then notice that since both the line and the component of C is 1 dimensional, then the line is contained in C, but it you consider the line, which has degree 1, then you will find that since the degree of C is larger than the line L is L is not C(using a previous proposition and notice that the line and D are of the same dimension), we conclude that $L = C$, therefore, if some component of C lies in L, then C is equal to L.

Now suppose that no component of C is contained in the line p, q, then we can use Bezout's theorem. However, notice that the degree of $X \cap H$ which is a finite collection of closed points, and the collection at least have two distinct points, then we know that the degree is the summation of degrees of these closed points (which will be proved in a later proposition), then since the degree of a closed point is 1 for an algebraically closed field, then the degree of the intersection is at least 2, a contradiction. □

Remark 6.82. There is something worth mentioning in the above proof.

For example, what if C in $\mathbb{P}^n_k$ only has one closed point before we pass to the algebraic closure. The answer is that this never happens, to see why, notice that since C is of dimension 1, then any hypersurface will intersect with C. Then suppose that p is a closed point of $\mathbb{P}^n_k$, then we can find a hypersurface H of high enough degree such that H doesn't contain p, but we know that H still meets C, therefore, they must meet at other closed points of C, we are done. And this can be used to prove that c actually have infinitely many distinct closed points. This goes back to something we didn't really mention and thus probably not realized by people. Even if you have a finite field k, the ring $k[x]$ also have infinitely many irreducible polynomials, thus infinitely many closed points. To be more precise, we can even produce irreducible polynomials of certain degrees.

Another thing that is worth mentioning is that in $\mathbb{P}^3_k$, for an algebraically closed field k, there are indeed hyperplanes passing any given two distinct closed points. This is from the fact that closed points are all k-valued points, and we can produce this hypersurface by considering the solution space of a certain system of linear equations.

Similar methods can also be used to prove that in $\mathbb{P}^N_k$, say $N \geqslant d$, then there are some hypersurface passing any given $d+1$ distinct closed points for an algebraically closed field k. As the rank of the matrix formed by the distinct closed points is at most $d+1 \leqslant N+1$, therefore, the system of linear equations always has a solution.

Finally, we also use the fact that in $\mathbb{P}^3_k$, two hyperplanes containing p, q is enough and any further intersection only gives you the same thing. This following from the following argument. Say the equation of the third one is not independent to the first two, then the intersection can be ignored, but if the equation is independent of the first two, we know this hyperplane is not in the solution space, thus not passing p, q.

Again, similar arguments can be used to prove that $d+1$ hyperplanes defined by independent equations is enough in $\mathbb{P}^N_k$.

Proposition 6.83. Let k be an algebraically closed field and C is an integral degree d curve in $\mathbb{P}^N_k$ with $N \geqslant d$, then C is contained in a linear space spanned by $d+1$ independent linear equations, therefore, in particular, if $N > d$, C is degenerate since it's in the intersection of hyperplanes.

Proof. This proposition is a little bit similar to what we got in the previous proposition.

This time we consider $d+1$ distinct closed points in C (we can again prove this using similar methods as in the above proposition.), then consider all the hyperplanes containing these $d+1$ closed points, notice that this time, not like in $\mathbb{P}^3_k$, two hyperplanes determine a line connecting p, q.

Now if C is not contained in the intersection, we know that there are at most d hyperplanes that are defined by independent equations containing C(otherwise, if there are $d+1$, we already proved that this measn that C is in the interesction of all the hyperplanes), therefore, we can assume that there is a hyperplane H (again, using k is infinite) not containing the generic point, then Bezout's theorem tells you that:

$$\deg(H \cap C) = d$$

But again, we find that $H \cap C$ contains $d+1$ distinct closed points and we know that $H \cap C$ is of dimension 0 therefore we know that the degree of the intersection is at least $d+1$, therefore a contradiction. □

Now let's see how classically Bezout's theorem plays with the notion of degrees, by classically we mean we will work with the complex numbers to avoid any field extension issues:

Proposition 6.84. Recall that by a Bertini type argument, we can prove that if $X \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$ is a complex variety of dimension n, then we can consider the intersection of X with n general hyperplane, then the intersection is a finite number of reduced points, people classically define that the degree of X is the number of the points.

The proposition claims that this agrees with $\deg X$.

Proof. Suppose that the n hyperplanes are chosen to be $H_1, \cdots, H_n$. Notice that if H_1 contains some generic point of some component of X with dimension n, then intersection will still be dimension n, similar arguments apply to all H_i, therefore for dimension reasons we showed that H_{i+1} doesn't contain any generic points of $H_1 \cap \cdots \cap H_i \cap X$ with dimension $n - i$.

Therefore we can apply Bezout's theorem in each step and use the fact that degree of hyperplanes are all 1 to conclude that

$$\deg X = \deg(X \cap H_1 \cap \cdots \cap H_n)$$

But notice that the degree of the intersection is the sum of the degree of these reduced closed points, thus is exactly the number of the points, according to a proposition which we will prove below, or you may just do this by hand since you know that the Euler characteristic is just the dimension of degree 0 cohomology which is equal to it's degree, therefore the degree is equal to the number of closed points. □

Remark 6.85. Therefore, we find that the classical definition which involves making a choice on the hyperplanes, then showing it's independent of the choice can be replaced by the cohomological version which uses the Euler characterization.

This is very similar to the notion of the degree of a line bundle, which was traditionally defined by the degree of a divisor cut out by any nonzero rational section, but then people found that it can be defined using Euler characteristics.

Proposition 6.86. The degree of the d-fold Veronese embedding of $\mathbb{P}^n$ is d^n could be shown using Bezout's theorem.

Proof. Say the embeddeding is $v_d : \mathbb{P}^n \hookrightarrow \mathbb{P}^N_d$, then recall that we can define the degree by intersecting the image with n general hyperplanes, and then compute the number of the closed points times the degree of the field extension, or in other words, the degree of the intersection in $\mathbb{P}^N_k$.

Now consider the closed embedding Cartesian diagram:

$$\begin{tikzcd}
\cdots \arrow[r, hook] \arrow[d, hook] & H_1 \cap \cdots \cap H_n \arrow[d, hook] \\
\mathbb{P}^n_k \arrow[r, hook, "v_d"] & \mathbb{P}^N_k
\end{tikzcd}$$

what really matters here is that this diagram is actually compressed since we can actually do the intersection step by step the pullback the intersection step by step, therefore this intersection can be thought as the intersection of the pullback of H_i's in $\mathbb{P}^n$, as shown in the following diagram:

$$\begin{tikzcd}
 & \cdots \arrow[r, hook] \arrow[d, hook] & H_1 \arrow[d, hook] & \\
\cdots \arrow[ur, hook] \arrow[dr, hook] \arrow[rrr, hook] & \mathbb{P}^n_k \arrow[r, hook, "v_d"] & \mathbb{P}^N_k & H_1 \cap H_2 \arrow[ul, hook] \arrow[dl, hook] \\
 & \cdots \arrow[r, hook] \arrow[u, hook] & H_2 \arrow[u, hook] &
\end{tikzcd}$$

then you only need to inductively perform the operation in this diagram to obtain the pullback of $H_1 \cap \cdot \cap H_n$ in $\mathbb{P}^n$.

Finally, if you want to compute the degree of this intersection, we use Bezout's theorem since we know that each intersection doesn't contain the generic point, otherwise the dimension won't match in the end. But recall that the pullback of a hyperplane in $\mathbb{P}_k^N$ is a degree d hypersurface in $\mathbb{P}_k^n$, therefore you are computing d^n, we are done. $\square$

The last property of degree we want to show is that sometimes the degree of the union is additive:

Proposition 6.87. Let X, Y be two d-dimensional closed subschemes of $\mathbb{P}_k^n$, with no d-dimensional irreducible components in common, then:

$$\deg X \cup Y = \deg X + \deg Y$$

Proof. I claim that there is a SES of coherent sheaves on $\mathbb{P}_k^n$:

$$0 \to p_* \mathcal{O}_{X \cup Y} \to i_* \mathcal{O}_X \oplus j_* \mathcal{O}_Y \to q_* \mathcal{O}_{X \cap Y} \to 0$$

To see why this is true, a hint is that for any ring A and ideals I, J we have the following SES:

$$0 \to A/(I \cap J) \to A/I \oplus A/J \to A/I + J \to 0$$

where the reason it's exact in the middle is because if $x + y = i + j$, then $x - i = j - y$. Then the SES of the coherent sheaves can be checked using this local result.

Then we know that $p_{X \cup Y} = p_X + p_Y - p_{X \cap Y}$. Since we know that X, Y don't have any common dimension d irreducible components, we know that the intersection has dimension at least 1 less, therefore since we know that the degree is nonnegative, i.e. the leading coefficient is not negative, we know that in $p_X + p_Y - p_{X \cap Y}$, $p_{X \cap Y}$ doesn't contribute to the leading term, therefore we conclude that the degree is the summation. $\square$

As we said before, arithmetic genus is another important invariant you can extract from Hilbert polynomials since we know that by definition $p_X(0) = \chi(X, \mathcal{O}_X)$ which is the same information as $(-1)^{\dim X}(\chi(X, \mathcal{O}_X - 1))$.

Remark 6.88. In history, people was just counting the dimension of functions of various degrees and what they found was that as the degree gets larger, the dimension eventually agrees with a mysterious polynomial, and if they plug 0 into that polynomial, they get a magic invariant they don't understand what that is, as people are mainly still focusing on functions and don't really know higher cohomologies, thus not the Euler characteristic.

One application of this is that we can now prove there are a large family of curves over an algebraically closed field that is not $\mathbb{P}^1$.

Recall that the Hilbert polynomial of $\mathbb{P}^1$ in itself is $m + 1$, therefore we know that $\chi(\mathbb{P}^1, \mathcal{O}) = 1$. Then consider a degree d projective plane curve C in $\mathbb{P}^2$, whose Hilbert polynomial is

$$p_{\mathbb{P}^2}(m) - p_{\mathbb{P}^2}(m - d) = \frac{(m+1)(m+2)}{2} - \frac{(m-d+1)(m-d+2)}{2}$$

Then you plug in 0, you find $\chi(C, \mathcal{O}_C) = \frac{-(d^2-3d)}{2}$. Then there are plenty of d where this is not 1, for example, when $d^3 - 3d$ is positive or in other words, when $d > 2$. Recall that we have proved that for any degree d, we can consider the curve cut out by $x^d + y^d + z^d$, which is regular, therefore we can even found a family of regular degree d plane curves that is not isomorphic to $\mathbb{P}^1$.

Now, consider the SES of the closed embedding:

$$0 \to \mathcal{O}(-d) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_C \to 0$$

use the induced LES and the fact that $h^1(\mathbb{P}^2, \mathcal{O}(-d)) = 0$, we know that $h^0(C, \mathcal{O}_C) = 1$, then we get:

$$h^1(C, \mathcal{O}_C) = (d-1)(d-2)/2$$

Therefore, we know that h^1 of C is precisely the genus, and as you plug in different values, you find that you get curves of genus $0, 1, 3, 6, \cdots$, which brings even more questions, for example, are there curves of other genera and in $\mathbb{P}^2$, is there other genus 0 curves besides degree $0, 1$ plane curves and the same question applies to genus $3, 6, \cdots$ curves. We will discuss all these questions later.

However, the following example shows that non-isomorphic stuff could have the same Euler characteristics, thus this is not complete invariant:

Example 6.89. Let X be a degree d hypersurface in $\mathbb{P}^n$, and we can prove that if $d \leqslant n$, then the Euler characteristic is 1.

This is because we know that the Hilbert polynomial is:

$$p_X(m) = \binom{m+n}{n} - \binom{m+n-d}{n}$$

and if you plug in 0, you just got $1 - 0$ in the end.

For example, we have seen that $\mathbb{P}^1 \times \mathbb{P}^1$ is a quadric hypersurface in $\mathbb{P}^2$, therefore the Euler characteristic is 1, which is the same as $\mathbb{P}^2$, but these two schemes are not isomorphic since we know that one has Picard group $\mathbb{Z} \oplus \mathbb{Z}$ and another one has Picard group $\mathbb{Z}^2$.

In the end, we give a brief discussion of **complete intersections** in $\mathbb{P}^n_k$.

Recall that by definition, we say that a closed subscheme $X \hookrightarrow Y$ is a complete intersection of codimension r if X is a scheme-theoretic intersection of r effective Cartier divisors $D_1, \cdots, D_r$ such that at each point of X, the equations defining D_i for a regular sequence of length r.

Remark 6.90. Here is a remark about the codimension r terminology we used in the complete intersection.

Recall that when we prove Krull's Principal theorem, we mentioned in particular if f is not a zero divisor, then components of $V(f)$ exactly has codimension 1.

Then we begin to prove Krull's height theorem, where we are interested in the codimension of components of $V(f_1, \cdots, f_r)$ for $f_1, \cdots, f_r$ in a Noetherian ring A. We said that the codimension is at most r, but we didn't say when the codimension is precisely r. We make a comment here to deal with the problem.

If you remember how we proved Krull's height theorem, one lemma is of crucial importance, which states that for any strict chain of primes:

$$p_0 \subset p_1 \subset \cdots \subset p_n$$

is a Noetherian ring and $f \in p_n - p_0$, we can produce another strict chain of primes:

$$p_0 = q_0 \subset q_1 \subset \cdots \subset q_n$$

such that $f \in q_1 - q_0$.

Then using this we can prove that in a Noetherian local ring, if $f \in m$, then:

$$\dim A/f \geqslant \dim A - 1$$

and this is because we can assume that the following sequence of primes: $p_0 \subset p_1 \subset \cdots \subset p_n = m$ gives the dimension of A, and if $f \in p_0$, then we know that after you mod out f, this chain of primes still makes sense, thus we know that the dimension agrees with each other.

However, if you assume f is not in p_0, then you can use the lemma and conclude that $\dim A/f \geqslant \dim A - 1$. However, if you assume something even stronger, which is f is not a zero divisor, then obviously f is not in p_0 since it's a minimal prime, then you can conclude that $\dim A/f = \dim A - 1$ since we know that by Krull's Principal ideal theorem, the codimension of each irreducible components of $V(f)$ is 1 and we know that:

$$\dim A \geqslant \dim A/f + \operatorname{codim} V(f)$$

Therefore, combining this with $\dim A/f \geqslant \dim A - 1$, we obtain $\dim A/f = \dim A - 1$. Be careful that this doesn't imply all Noetherian rings are Catenary since we are only considering a special case where the closed subset is cut out by a single equation that is not a zero divisor.

But this is enough for us to give a sufficient condition for when $V(f_1, \cdots, f_r)$ is of pure codimension r. I claim $V(f_1, \cdots, f_r)$ is of pure codimension r if $f_1, \cdots, f_r$ forms a regular sequence of length r. To see why, we want to prove the codimension of each irreducible component of $V(f)$ is r, say p is a minimal prime in $V(f)$, then you can first localize A at p and prove the dimension of A_p is r. Notice that $f_1, \cdots, f_r$ are in p, therefore each of them is in the maximal ideal of A_p, and they are still a regular sequence in A_p, with one minor caveat that after we localize at p, it can happen that some $(f_1, \cdots, f_i)$ is the whole ring for some i, but notice that if localize $A/(f_1, \cdots, f_r)$ at p, it's the same as localization at the extension of p in $A/(f_1, \cdots, f_r)$, which is not zero since the extension is also a prime in the quotient. Therefore we can assume we work with a Noetherian local ring in the beginning and assume that in the maximal ideal, we have a regular sequence of length r and the maximal ideal is minimal above this sequence, and we want to prove the dimension is r.

First, we know that $\dim A/f_1 = \dim A - 1$ by the discussion above, then notice that f_2 is still not a zero divisor in the quotient by definition of regular sequence, therefore we know that $\dim A/(f_1, f_2) = \dim A/f_1 - 1$, if $(f_1, f_2) \neq A$,

then keep going, we know we only need to show:

$$\dim A/(f_1, \cdots, f_r)$$

is of dimension 0. Notice that m is minimal over $(f_1, \cdots, f_r)$, thus the dimension is indeed 0, we are done.

Apply this to complete intersection of codimension r, we will show that it's called codimension r because $D_1 \cap \cdots \cap D_r$ is of pure codimension r in Y. To see why, suppose that η is the generic point of the intersection, we wish to show that the dimension of the local ring is n. Notice that we can first work affine locally and assume that $D_1, \cdot, D_r$ are cut out by $f_1, \cdots, f_r$, we don't know if this is a regular sequence yet, but we know that after you localize at p for every, which is the ring, then after you localize at p, which is a minimal prime in $V(f_1, \cdots, f_r)$, you know that $f_1, \cdots, f_r$ is regular in A_p and is contained in pA_p which is minimal over $(f_1, \cdots, f_r)$, then we have proved that the dimension is r in the above discussion, thus we know that every irreducible component of $D_1 \cap \cdots \cap D_r$ is of codimension r, thus of pure codimension r.

Remark 6.91. Recall that if $f \in A$ is a function in a ring and f is not a zerodivisor in A_p for all primes in A, then f is not a zerodivisor in A.

In fact we can prove that f is a zero divisor iff f is a zero divisor in some A_p for some primes p in A. First, if f is a zero divisor in A_p for some p, then we know that there is a nonzero $\frac{a}{b}$ such that the product is 0, therefore $fat = 0$ for some $t \notin p$ and $at \neq 0$, therefore f is a zerodivisor.

Conversely, suppose that f is a zerodivisor, and $fa = 0$, then we know that $f \in \mathrm{Ann}a$, and since $\mathrm{Ann}a$ is not the entire ring, we can pick a minimal prime p over $\mathrm{Ann}a$, then consider f in A_p, notice that a is not 0 in A_p by our choice of prime, then we are done.

Therefore, actually, we can see that $D_2 \cap D_1$ is also an effective Cartier divisor in D_1 since we can assume that over $SpecA$, D_1 is cut out by f_1 and D_2 is cut out by f_2, we know that f_2 is not a zero divisor in $A/(f_1)$ after we localize at primes of A for all primes, then f_2 is not a zerodivisor for $A/(f_1)$, this argument can be applied to any $D_i \cap \cdots \cap D_1$ in $D_{i-1} \cap \cdots \cap D_1$.

Then let's consider complete intersections in $\mathbb{P}^n_k$, which requires us to understand all the effective Cartier divisors on $\mathbb{P}^n_k$. Recall that if D is a nontrivial effective Cartier divisor(i.e. not the empty divisor and not the whole space), we can consider the corresponding line bundle $\mathcal{O}(D) \simeq \mathcal{O}(m)$ for some m, then notice that this in particular implies that $m \geqslant 0$ since we know know that there is a canonical global section of $\mathcal{O}(D)$ cutting out D, and if the only global section is 0, then the divisor is empty, then we know that D is cut out by a single global section of $\mathcal{O}(m)$ for some nonnegative m, and since D is not the whole space, we know that $m \neq 0$, thus it's strictly positive, thus D is a hypersurface, therefore we have actually proved the following statement:

Proposition 6.92. All nontrivial effective Cartier divisors in $\mathbb{P}^n_k$ are hypersurfaces.

Therefore, complete intersections of codimension r in $\mathbb{P}^n_k$ is just a complete intersection of r hypersurfaces, and we have proved that it will have pure codimension r, thus pure dimensional of dimension $n - r$.

Proposition 6.93. Suppose that X is a dimension m complete intersection in $\mathbb{P}^N_k$, then $h^i(X, \mathcal{O}_X(n)) = 0$ for $0 < i < m$ and all n.

Proof. Recall that complete intersections in $\mathbb{P}^N_k$ are always pure dimensional, therefore we know that X will be a complete intersection of codimension $N - m = r$, say these hypersurfaces are $H_1, \cdots, H_r$ each of degree d_i.

Now recall that we know for $\mathbb{P}_k^N$ and $0 < i < N$, we have the vanishing:

$$h^i(\mathbb{P}_k^N, \mathcal{O}(n)) = 0$$

Then for H_1, we know we have the SES of effective Cartier divisor:

$$0 \to \mathcal{O}_{\mathbb{P}_k^N}(n - d_1) \to \mathcal{O}_{\mathbb{P}_k^N}(n) \to \mathcal{O}_{H_1}(n) \to 0$$

Then take the induced LES of cohomology, we know that $\mathcal{O}_{H_1}(n)$ vanishes for $0 < i < N-1$ since we know the LES looks likes:

$$H^{N-1}(\mathcal{O}_{\mathbb{P}_k^N}(n-d_1)) \to H^{N-1}(\mathcal{O}_{\mathbb{P}_k^N}(n)) \to H^{N-1}(\mathcal{O}_{H_1}(n)) \to H^N(\mathcal{O}_{\mathbb{P}_k^N}(n-d_1))$$

Then let's consider $H_1 \cap H_2$ in H_1, we already saw that $H_1 \cap H_2$ is also a effective Cartier divisor in H_1 and we know that it's cut out by $\mathcal{O}(d_2)$ on H_1 using the usual closed embedding:

$$H_1 \hookrightarrow \mathbb{P}_k^N$$

therefore, you have the following SES:

$$o \to \mathcal{O}_{H_1}(n - d_2) \to \mathcal{O}_{H_1}(n) \to \mathcal{O}_{H_1 \cap H_2}(n) \to 0$$

Then take the associated LES, we conclude that $H^i(H_1 \cap H_2, \mathcal{O}(n))$ vanishes for $0 < i < N-2$.

We keep using this argument until we get $H_1 \cap \cdots \cap H_r$, and get H^i vanishes for $N - r = m$. $\square$

Example 6.94. Let's consider a complete intersection of two quadric surfaces Q_1, Q_2 in $\mathbb{P}_k^3$, I claim that the genus is still 1.

Recall that the degree of this intersection is $2 \times 2 = 4$, and this is because complete intersections implies that Q_2 doesn't contain any generic points of Q_1. Recall that the Hilbert polynomial of Q_1 is:

$$\binom{m+3}{3} - \binom{m+1}{3} = \frac{(m+1)(m+2)(m+3) - (m-1)m(m+1)}{3!}$$

Then the SES of the complete intersection (or the corresponding effective Cartier divisor) means that:

$$p_{Q_1 \cap Q_2} = p_{Q_1}(m) - p_{Q_1}(m-2)$$

Notice that if we plug in 0, we get the Euler characteristic of the intersection is $\frac{6}{6} - \frac{6}{6} = 0$ and we know that the genus is:

$$(-1)^1(-1) = 1$$

More generally, we can compute the genus of the complete intersection of a degree m hypersurface H_m and a degree n hypersurface H_n. And the argument is basically the same as above, you first compute the Hilbert polynomial $p_{H_m}(t)$ and deduce from this the Euler characteristic is:

$$p_{H_m}(0) - p_{H_m}(-n)$$

Since we know that:

$$p_{H_m}(t) = \frac{(t+1)\cdots(t+3)}{6} - \frac{(t+1)t(t-1)}{6}$$

you can compute the genus using this, and if you plug in $m = 2, n = 3$, you obtain that the Euler characteristic of the intersection is $1 - 4 = -3$, thus the genus is $(-1)(-3-1) = 4$.

Moreover, if we take the degree m hypersurface H_m to be smooth, then a Bertini type argument tells us that there is a degree n hypersurface that doesn't contain any component of H_m and the intersection is smooth and thus in particular regular. This is necessarily a complete intersection since hyperplanes in the affine k space don't have embedded points, thus not containing the components means that the defining function of the second hyperplane is not a zero divisor on the first hyperplane, another condition of being a regular sequence is automatic as for primes containing two polynomials, localizing the quotient at that prime doesn't gives you 0.

Proposition 6.95. A rational normal curve C of degree d in $\mathbb{P}^d$ is not a complete intersection if $d > 2$.

Proof. Recall that by definition, C is the image of the d-th Veronese embedding:

$$v_d : \mathbb{P}^1 \hookrightarrow \mathbb{P}^d$$

You can check that this definition indeed agrees with what we saw in this section.

I claim that it's not degenerate. This is due to a proposition we saw before, notice that this Veronese embedding is determined by the complete linear series $|\mathcal{O}(d)|$, and we know that if a morphism $X \to \mathbb{P}^n$ correspond to a base point free series $\Gamma(\mathbb{P}^n, \mathcal{O}(1)) \to \Gamma(X, \mathcal{L})$ via the pullback morphism, then it's nondegenerated iff the above pullback morphism is injective, therefore, in particular, a complete linear series is nondegenarate.

Therefore, if we assume for the sake of contradiction that C is a complete intersection, then it's the intersection of $d-1$ hyperplanes of degree at least 2, thus the degree of C is at least $2d-2$, and when $d > 2$, $2d-2 > d$, a contradiction to the fact that C is of degree d. □

Proposition 6.96. Complete intersections in $\mathbb{P}^n_k$ of positive degree will be connected.

Proof. Recall that we already proved, assuming Serre duality that ample divisors in a smooth projective k-scheme is connected. Here we show that complete intersections D are always connected.

Again, we only have to show that $h^0(D, \mathcal{O}_D) = 1$. Again, we consider the SES of the closed embedding:

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^n_k} \to \mathcal{O}_D \to 0$$

Then we consider the associated LES, since we assume the complete intersection is of positive dimension, then we can rule out $\mathbb{P}^1_k$, therefore, the H^1 cohomology of $\mathcal{O}_{\mathbb{P}^n_k}$ vanishes, therefore we get a SES:

$$0 \to H^0(\mathcal{I}) \to H^0(\mathbb{P}^n_k) \to H^0(\mathcal{O}_D) \to H^1(\mathcal{I}) \to 0$$

Since we know that the dimension of $H^0(\mathbb{P}^n_k)$ is 1, we know also know that $H^0(\mathcal{O}_D) \geqslant 1$, then if we can prove $H^1(\mathcal{I})$ vanishes, we can also prove $h^0(\mathcal{O}_D) \leqslant 1$ thus completing the proof.

To show the vanishing of $H^1(\mathcal{I})$, recall that $\mathcal{I}$ is defined to be the sum of $\mathcal{O}(-d_i)$'s inside $\mathcal{O}_{\mathbb{P}^n_k}$, then we have a SES:

$$0 \to Ker \to \oplus_i \mathcal{O}(-d_i) \to \mathcal{I} \to 0$$

Then consider the LES associated to this SES, part of it looks like:

$$0 = \oplus_i H^1(\mathcal{O}(-d_i)) \to H^1(\mathcal{I}) \to H^2(Ker)$$

Therefore, the question is reduced to prove $H^2(Ker)$ vanishes, then recall that there is again a surjective morphism:

$$\oplus \mathcal{O}(-m) \to Ker \to 0$$

for positive m and this direct sum is finite, and we know that if we want we can take m to be arbitrarily large and in particular $-m$ is not $-n-1$. This is because we get this surjection from the fact that $\mathcal{O}(1)$ is ample, thus we know that:

$$\mathcal{F}(m)$$

for large enough m will be globally generated, then we twist with $\mathcal{O}(-m)$ to get the surjection. This implies that $H^i(\mathcal{O}(-m))$ can always be assumed to vanish. And we know that for $\mathbb{P}^n_k$, cohomology H^i with $i \geqslant n+1$ vanishes for all coherent sheaves, therefore we can repeat the above argument to show that in the end the vanishing is automatic, therefore we are done. □

Remark 6.97. Notice that this statement doesn't require X to be reduced, for example if we consider the hypersurface x^2 in $\mathbb{P}^2$, even though there are indeed local nonzero nilpotents, there are no such elements as global functions.

Finally, before we end this section and start talking about something new, we see an application of the above proposition:

Proposition 6.98. The union of two planes in $\mathbb{P}^4_k$ meeting at one point is not a complete intersection.

Proof. First, notice that the union is connected since each hyperplane is connected and they meet at one common point.

Now assume for contradiction that this is indeed a complete intersection, if we choose a hyperplane that misses the common meeting point and the associated points of this closed subscheme, the triple intersection will be a complete intersection of dimension at least 1, in particular, of positive dimension. But notice that the intersection is not connected as it can be viewed as:

$$(H_1 \cup H_2) \cap H_3 = H_1 \cap H_3 \cup H_2 \cap H_3$$

both are closed, not empty, and disjoint, thus a contradiction to the connectivity of $(H_1 \cup H_2) \cap H_3$. □

References