Appendix 5 for Math 632: 2025 Winter Notes

Chunye Yang

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1 Review on Dimension

In this section, we finally begin to discuss some geometric notions associated to a scheme. We begin with the notion of dimension and after we are done with it, we will go to smoothness in the next section.

We first give the definition of the dimension of a scheme, which is surprising only topological:

Definition 1.1. Suppose that you have a topological space X, we define it's dimension of X, (denoted dim X) to be the supremum of the length of the chains of closed irreducible subsets and we starting the indexing of this chain with 0(as we going up the chain we are getting larger and larger sets). It's possible that the dimension is infinite and it's sometimes called **Krull dimension**.

You might also saw the definition of the dimension of a ring as the supremum of the chains of **nested** primes ideals (as we go up the chain, we are getting smaller and smaller chains). You can also reverse this by looking at chains of growing primes since you can easily see this is an equivalent notion. Notice that the inclusion reversing bijections between irreducible closed subsets of $SpecA$ and primes of A, we can show that the dimension of a ring is equal to the Krull dimension of $SpecA$. Also, recall that due to the fact of Zariski topology neglects nilpotents, we have $dimA = dimA/\mathfrak{N}(A)$.

Proposition 1.2. Suppose that Z is the union of two closed subsets X, Y, then dim Z is the maximum of dim X and dim Y.

Proof. This is just the basic fact that if you have a closed irreducible subset W, it either lies entirely in X or in Y, it can't happen that there is a point of W in X but not in Y or vice versa. This is because if that happens we can write $W = (W \cap X) \cup (W \cap Y)$ and $W \cap Y$ is not contained in $W \cap X$, which is a contradiction to the fact that W is irreducible.

Thus, if you have any chain of irreducible closed subsets, either it's completely in X, or it's completely in Y, thus the dimension of X is less than or equal to the maximum dimension of X and of Y. On the other hand, as a closed subspace of Z, any irreducible closed subset of a closed subset is still irreducible closed in the ambient space, thus the dimension of X and Y are less than or equal to the dimension of Z, we are done.

Also, we know that this plays well with irreducible closed subsets since an irreducible closed subset can't be written as the union of two proper closed subsets. $\square$

Due to our above proposition, if you have a reducible space, say with two irreducible components, the dimension of the space is the maximal of the two irreducible components, then the dimension of the space is the maximum of these two components. Therefore, this notion of dimension is not a local characteristic of a space, thus sometimes we will just talk about the dimension of irreducible spaces:

Definition 1.3. We say a topological space is **pure dimensional** if each of it's irreducible components have the same dimension. We also say that a space is of pure dimension n and you can guess what it means.

Now recall that we proved in a previous review session that if you have a topological space X and an open subset $U \subset X$, you have a inclusion-preserving bijection between irreducible closed subsets of U and the irreducible closed subset of X that meets U by taking intersections and closures.

Proposition 1.4. A scheme have dimension n iff it admits an affine open cover by affine open subsets of dimension at most n and the equality is achieved for at least one of the affine opens.

Proof. Let's first assume that X has dimension n, and we pick any affine cover of X, by the inclusion preserving bijection we mentioned, the dimension of U is less than or equal to n. Also, since X has dimension n, there is a chain of length n of irreducible closed subsets, and the minimal one meets some open, thus every thing in this chain meets that open, we are done.

Conversely, the same line of reasoning means that the dimension of X is less than or equal to n, one of them is dimension n means that the dimension of X is larger than or equal to n, we are done. □

Remark 1.5. If you look closely at the proof above, you will see that the real content of the above proposition is: If you have an open cover if a topological space, then the dimension is the supremum of the dimension of the opens in the open cover.

As one simple application of this result:

Proposition 1.6. A Noetherian scheme is of dimension 0 iff it has finitely many discrete points(this means that we are using the discrete topology).

Proof. First, suppose that X is zero dimensional, then we know that it's covered by finitely many Noetherian affines of dimension 0. Then we only have to prove that a zero dimensional Noetherian ring A only have finitely many primes. This is due to the fact that being dimension 0 forces every prime to be a maximal prime, then notice that if you consider the set of all finite intersections of primes in A, it's not empty and being Noetherian implies that there is a minimal element, say m, then for any other maximal ideal m' we know that $m' \cap m$ is equal to m, then we know that m' contains m, thus one of the primes in the intersection, then since we are in dimension 0, it's equal to one of them.

Now conversely, suppose that it only have finitely many points with discrete topology, then you cover it with finitely many affines, each is with discrete topology, this means that every ideal is maximal, thus zero dimensional. □

Proposition 1.7. A surjection of integral domains of the same dimension must be an isomorphism.

Proof. If $f : A \to B$ is a surjection of integral domains, we want to show that it is injective if A, B are of the same dimension. This follows from the fact that the kernel is a nonzero ideal if this is not an isomorphism, then $B \cong A/I$, since this is an integral domain, (0) is always a prime ideal. Then, by the inclusion preserving bijection between prime ideals of A/I and the prime ideals of A that contains I, we know that if I is not 0, then the dimension of A/I is at least 1 less than the dimension of A. □

As our first nontrivial example of dimension involving morphisms of schemes:

Proposition 1.8. Suppose that $f : X \to Y$ is an integral morphism of schemes then for any nonempty fiber of f has 0.

Proof. Let's first draw the fiber diagram:

$$\begin{tikzcd} f^{-1}(y) \arrow[r] \arrow[d] & X \arrow[d] \\ Speck(y) \arrow[r] & Y \end{tikzcd}$$

Notice that integral morphisms are preserved by base change, thus we know that $f^{-1}y$ is $SpecA$ where $k(y) \hookrightarrow A$ is integral. Then the desired property hold if we can show that if we have any field $k \hookrightarrow A$ where A is integral over k, then dimension of A is zero. This turns out to be a special case of the proposition below we will prove, but we prove here the special case first. One of the properties of integral morpihsms we proved is that if we have an integral extension the target is a field iff the source is a field, thus you get $f^{-1}y$ is actually the spec of a field.

And if you prefer a more elementary proof, you can consider: $k \hookrightarrow A$ then, if we want to prove A is of zero dimensional, we are trying to prove that if you have an inclusion of primes $p \subset m$ in A, we want to show that $p = m$, then by modding out p, we can assume that A is an integral domain, then if A/p is a field, then $p = m$, thus we are trying to prove that A/p is a field. But remember being integral is preserved by quotients of A, then we can assume that $k \hookrightarrow A$ maps k into an integral domain. Then suppose that x is in A, it's integral over A means that there is a monic polynomial in x such that the polynomial is 0, but since x is not 0 and A is an integral domain, then you can cancel powers of x to get the lowest term in a constant, but remember the coefficients are in a field, thus invertible, thus you have x times something is invertible, thus x is invertible. $\square$

The proof given in the above proposition can be seen as a special case of the following result:

Proposition 1.9. Suppose that $SpecA \to SpecB$ corresponds to an **integral extension** $\varphi : B \to A$, then dim $SpecA$= dim $SpecB$.

Proof. It's really important that this ring hom is an integral extension not just an integral ring homomorphism. **** later* $\square$

And following the same line of reasoning, we got:

Proposition 1.10. Suppose that $v : X \to \widetilde{X}$ is the normalization of a scheme, then dim X=dim $\widetilde{X}$.

Proof. **** later after we reviewed the normalization.* $\square$

The following are two useful exercises we will use later, and after we see the notion of flatness, we will also see an generalizations of that:

Proposition 1.11. Suppose that X is an affine k-scheme and K/k is an algebraic field extension, then suppose that X has pure dimension n, then $X_K = X \times_k K$ has pure dimension n.

Proof. Suppose that $X = SpecA$, let's first show that we can assume that A is an integral domain. Suppose that Q is a minimal prime in A_K, notice that $SpecA_K \to SpecA$ is surjective since surjectivity is preserved, then for Q, I will let you figure out why there is a minimal prime P in A such that the inverse image of Q is P, then we know that the extension of P is contained in Q, then is we denote the extension of P by P', consider $A \otimes_k K/P' \cong (A \otimes_k K) \otimes_A (A/p) \cong A/p \otimes_k K$, this is going to be a closed subset of X_K that contains the irreducible component corresponding to Q. If we can show that it's pure dimensional of dimension n, we are done. Giving A/p is of pure dimension n.

Thus consider the following diagram where X_i is an irreducible component of X_i:

$$\begin{array}{ccccc} X_i \times_k K & \hookrightarrow & X \times_k K & \longrightarrow & K \\ \downarrow & & \downarrow & & \downarrow \\ X_i & \hookrightarrow & X & \longrightarrow & k \end{array}$$

It's not hard to show that these $X_i \times_k K$ cover $X \times_k K$, and they are closed, then by our above, we only have to show that each of these is pure dimensional. Since we know that X_i is just A/p_i for some minimal prime p_i, it's an integral domain.

Then notice that an irreducible component in $X_i \times_k K$ is a minimal prime p in $A_K = A \otimes_k K$, then consider the following ring map:

$$A \to (A \otimes_k K)/p$$

if $a \in Ker$ of the above map, then we first claim that $a = 0$, this is because first it's mapped to the class of $a \otimes 1$, which is in p a minimal prime, thus a zero divisor. Notice that A_K is a free A module since K is free over k, then we know that multiplication by a is injective if a is not zero since A is an integral domain, then we know that $1 \otimes 1$ is not zero, but $a \cdot (1 \otimes 1)$ which is $a \otimes 1$ is zero, thus a is zero.

The above argument shows that $A \to A_K/p$ is injective, now if we can use this is an integral extension we are done by the fact that integral extension preserves dimension. In fact we know that $A \hookrightarrow A \otimes_k K$ is integral, then we use the fact that being integral is preserved by quotient on the target. To show that $A \to A_K$ is integral, we only have to show that every $a \otimes b$ is integral over A (we can assume that a is not 0, otherwise there is nothing to show), but notice that since $k \to L$ is algebraic, we know that there is a monic polynomial f with coefficients in k such that $f(b) = 0$, then we know that $a \otimes f(b) = 0$, but notice that this is equal to $a \cdot f(1 \otimes b)$, again since A in an domain and multiplication by a is injective if $a \neq 0$, thus $f(a \otimes b) = 0$ and use the fact that $k \hookrightarrow A$. $\square$

Not surprisingly, the converse also holds:

Proposition 1.12. Under the above assumption, if X_K is of pure dimension n, then X is also of pure dimension n.

Proof. As we discussed in the previous proposition, suppose that p is a minimal prime in A, then we can consider it's extension, which is contained in some minimal prime Q in A_K, then we following the same argument of the above proposition, we have the ring hom:

$$A/p \to A/p \otimes_k K$$

If we can show that this is injective and notice that the dimension of the right hand side is n, we are done.

Let's denote $A/p = R$, which is a domain, then using the fact that $R \otimes_k K$ is a free R-module, and R is a domain, we know that if $r \in R$ is not 0, then the multiplication is injective, this shows that if r_1, r_2 is mapped to the same element $r_1 \otimes 1 = r_2 \otimes 1$, then $(r_1 - r_2) \otimes 1$, then $r_1 - r_2 = 0$ since this is $(r_1 - r_2) \cdot (1 \otimes 1)$. □

Proposition 1.13. The dimension of $\mathbb{Z}[x]$ is 2.

Proof. **** later after we review the primes in $\mathbb{Z}[x]$* □

Now we introduce another notion which is codimension. This is a notion that is different from dimension in the sense that we want it to be local, at least you can determine the codimension of something irreducible closed by only looking at the irreducible component containing it. But we need extra care when we define it, just for the reason that dimension behaves oddly with disjoint unions. For example, is Y is a closed subset of X, we can define the codimension of X to be $\dim X - \dim Y$. However, this is a bad definition, and you just have to look at the example where if X is the disjoint of a point Y and a curve Z, we already show that $\dim X = 1$ since it's the maximum of the dimensions of Y and the curve. However, if we use our definition above of the codimension, you will get the conclusion of codim $Y = 1$, which doesn't make sense since the irreducible component containing Y is just a point, which should have dimension 0. There is also the question of what should we do if we are trying to define the codimension of some reducible closed subsets, where you can imagine you probably will run into trouble if you are trying to define the codimension of something that can be decomposed as a curve that passes through a surface, to avoid pathology, we define:

Definition 1.14. Suppose that $Y \subset X$ is an irreducible subset of X in the subspace topology, then the codimension of Y is defined to be the supremum of the length of chains of irreducible closed subsets containing Y, denoted by $codim_X Y$, and we start indexing from 0.

Equivalently, since we know that the closure of an irreducible subset is also irreducible, we can also let the chain start with $\overline{Y}$.

Similar our treatment of dimension, suppose that p is a prime ideal in a ring A, it's **codimension** is defined to be the length of chains of prime ideals contained in p, which ends at p, and we start the indexing from 0. You should easily see why this is the equivalent definition of the codimension of $V(p)$ in $Spec A$. This is also called the **height** of the prime ideal.

Finally, you should also see that the codimension of p is the dimension of A_p.

To have an analogy of dimension and codimension in linear algebra, we see that the dimension of a linear space is define to be the supremum of the length of chains of linear subspaces, and the codimension of a subspace is defined to be the supremum of the length of chains of subspaces that contains this subspace. This is better than the definition of dim X - dim Y, since the former definition applies to the case where both X and Y are infinite dimensional. You can also define it to be the dimension of X/Y, which is like the dimension of A_p.

Proposition 1.15. Suppose that Y is an irreducible closed subset of a scheme X, let η be the unique generic point. Then the codimension of Y is the dimension of $\mathscr{O}_{X,\eta}$.

Proof. Suppose that we have a chain of irreducible closed subsets containing Y, take any affine open around η, let's say $Spec A$, by our previous remark about the inclusion-preserving bijection between irreducible closed subsets of X that meet $Spec A$ and irreducible closed subsets of $Spec A$, we get a chain of irreducible close subsets in A that is contained in p, thus we know that the codimension of p in A is larger than or equal to the codimension of Y in X. Conversely, if we have any chain of primes cntained in p, you get chains of irreducible closed subsets of $SpecA$, then you take the closure to get chains of irreducible closed subsets that starts at Y in X, thus the codimension of p in A is larger than or equal to the codimension of Y in X, thus they are equal. Finally, since $A_p \cong \mathscr{O}_{X,\eta}$, we know that the codimension of p in A is equal to the dimension of A_p which is equal to the dimension of $\mathscr{O}_{X,\eta}$. □

Remark 1.16. Still, suppose that X is an scheme, what is a codimension 0 irreducible closed subset Y in X? You look at the definition, then you find that this is just an irreducible component of X since having codimension 0 just means that there is no larger irreducible closed subset.

Then what about a codimension 1 irreducible closed subset Y, what are the geometric intuition for them. First, it's contained in some irreducible components, say Y', and having codimension 1 means that there is no other irreducible closed subsets between Y and Y', this holds for any irreducible components that contain Y.

Then, if you have a closed subset Y of X, all of whose irreducible components are of codimension 1 in X, this is called a **hypersurface** in X.

It's reasonable to imagine that the equality where the dimension of a closed subset plus it's codimension in the ambient space is equal to the dimension of the ambient space. However, this only holds **sometimes** in algeraic geometry when the situation is nice, you will see an example where it fails later and you will also see that when you are working with irreducible varieties, this indeed holds, but we always have the following inequality holds:

Proposition 1.17. Suppose that Y is an irreducible closed subsets in a scheme X, then we have:

$$codim_X Y + dim Y \leqslant dim X$$

Proof. This is just the definition, you first see that if any of the two terms on the left hand side is infinity, the right hand side is easily seen to be infinity and we have an equality.

Now suppose that both of them is finite, then write out the chain, you will see why. □

What you can see from the above proof is an weird phenomenon, if you already have for a specific Y, this strict inequality, the failure of this inequality (when you achieve equality) can only happen if you have a chain that doesn't pass through Y, below is a description of an example where this fail:

Think of the picture where you have a curve passing through a surface only intersecting at one point, then you have p, a closed point one the curve which is not the intersecting point, then the dimension of $\{p\}$ is 0 and the codimension of $\{p\}$ is easy to see 1, whose sum is less than 2. However, if you choose this closed point to be on the surface, this equality holds. Even more peculiar stuff can happen, you think of the point of intersection, it's of codimension 2, which is contained in a codimension 0 irreducible closed subset, which is the curve and you have no codimension 1 things in between. We will have more to say on this later.

Below, we make a short observation for future reference, and this is also the first glimpse that codimension 1 stuff is rather special:

Proposition 1.18. In a UFD A, any codimension 1 prime ideals are principal.

Proof. Suppose that p is a codimension 1 ideal in A, choose any $f \neq 0$ in p, since this is a UFD, there is going to be an irreducible factor of f that belongs to p, say g, then $0 \subsetneq (g) \subset p$, then p being a codimension 1 ideal implies that $(g) = p$. □

The following result helps us understand the dimension 0 stuff:

Proposition 1.19. Suppose that (A, m) is a Noetherian local ring, then the followings are equivalent:

1. A is of dimension 0.
2. $m = \mathfrak{N}(A)$.
3. For some $n > 1$, we have $m^n = 0$,
4. A has finite length.

5. Every descending sequence of ideal in A is eventually constant.

Proof. 1 implies 2 is easy, since A is of dimension 0, then this m is the only prime ideal in A, thus we use the fact that nilradical is the intersection of all primes.

Then 2 implies 3 is also easy since we are in a Noetherian ring, then $\mathfrak{N}(A)$ is finitely generated, then you can choose the power to be a large power.

3 implies 4 is also easy since you can just use the powers of m, as what follows:

$$0 = m^n \subsetneq m^{n-1} \subsetneq \cdots m^1 \subsetneq m^0 = A$$

Since we already proved that length is additive in an exact squence, we only have to show that each consequtive quotient m^i/m^{i+1} is of finite length. However, notice that this is annihilated by m, thus it's an A/m-module, and finitely generated, thus a finite dimension vector space, thus finite-length as an A/m-module. However, there is a bijection between A and A/m submodules of M which is just $N \mapsto N$. Then, I will let you figure out why this implies that this is also of finite length as an A-module.

4 implies 5 is also easy, consider a descending sequence of ideals, if it's not constant, it's obvious that A can't have finite length.

Finally, let's see 5 implies 1, suppose that $p \subset m$ is a prime ideal, we want to show that $p = m$, one of the ways to do it is by showing the A/p is a field. First, notice that it's a domain, and the fact that descending ideal in A is eventually constant implies that any descending ideal in A/p is eventually constant. Then suppose that $x \in A/p$ is not zero, then consider the descending ideal sequence: (x^n) for all n, then it's eventually constant impleis that there is $y \in A/p$ such that $x^n = x^{n+1}y$, thus we know from A/p is a domain that $xy = 1$, we are done. ☐

Now, we begin to give another interpretation of dimension for irreducible varieties over a field k in terms of transcendence degrees. This is both a power tool for some theoretic results of dimensions, it's also a good chance to introduce the ideal of **Noether Normalization**, which is also useful in other respects, and of interesting geometric interest. We begin by a snap review on transcendence degree:

Recall that an element of a field extension E/F is called algebraic over F if it's integral over F, also a field extension E/F is an algebraic extension if every element in E is algebraic over F, thus this is just E/F is an integral extension. Thus we know that the composition of two algebraic extensions is also algebraic. Now, suppose that E/F is a field extension with F' and F'' being two intermediate field extension over F, we define the compositum of F' and F'', denoted by $F'F''$ by the smallest field extensions of F contained in E that contains F, F''. We will write $F' \sim F''$ if $F'F''$ is algebraic over both F' and F''.

Proposition 1.20. Given E/F, this $\sim$ relation defined above is an equivalent relation on intermediate field extensions.

Proof. Reflexivity is trivial since $F'F' = F'$. Notice that if $F' \sim F''$, then the fact that $F'F'' = F''F'$ implies that this relation is symmetric. Suppose that $K_1 \sim K_2$ and $K_2 \sim K_3$, we claim that $K_1 \sim K_3$, consider K_1K_3, we want to prove that K_1 and K_3 are algebraic over it. Notice that K_1 is algebraic over K_1K_2, and K_2 is algebraic over K_2K_3, we claim this is enough to show that K_1 is algebraic over K_1K_3.

This comes from an important observation, which is also crucial to how we interpret this equivalence relation. Notice that K_1K_2 is algebraic over K_1 and K_2 in particular implies that K_1 is algebraic over K_2 and K_2 is algebraic over K_1. Conversely, suppose that K_1 is algebraic over K_2 and K_2 is algebraic over K_1. We claim that K_1K_2 is algebraic over K_1 and K_2. To show it's algebraic over K_2, notice that we can characterize K_1K_2 as $K_1(K_2)$, thus elements in K_1K_2 are just of the form f/g where f, g are polynomials in variables from K_2 and with coefficients in K_1. Notice that if f, g are both algebraic over K_2, then we know that f/g is also algebraic. But to prove f, g are algebraic, we only have to prove that xy^n is algebraic where $x \in K_1$, $y \in K_2$. But notice that x is algebraic over K_2, we are done. The other direction is similar. Here, notice that since K_1, K_2 are not assumed to have any containment relation a priori, this algebraic we are saying is not in terms of an integral extension, it's just in E, if K_1 is algebraic over K_2, then any element in K_1 satisfies a monic polynomial with coefficents in K_2.

Therefore, what we are actually capturing is the relation of two intermediate field extensions being algebraic over each other in a larger field. This is useful to prove the transitivity, since we know that K_1 is algebraic over K_2 and K_2 is algebraic over K_3, thus K_1 is algebraic over K_3 by transitivity of being algebraic, which you can prove fairly easily. ☐

Now, given the above result, we define:

Definition 1.21. Say E/F is a field extension, then a **transcendence basis** of E/F is a set of elements x_i in E such that $\{x_i\}$ is algebraically independent over F, in the sense that there is no trivial polynomials F in x_i (multi-variable polynomials) with coefficients in F such that $F(x_i) = 0$, and $F(\{x_i\}) \sim E$.

Since we define this set of elements $\{x_i\}$ as a basis, we want to show that if there is another set of transcendence basis, then they have the same number or cardinality, which is another analogy we have for the dimension in linear algebra, but we will need the following intermediate results in order to show this analogy, the first one of which gives the explanation of what our definition of $\sim$ is really capturing:

Lemma 1.22. Suppose that $S \subset E$ where E/F is a field extension, then S is a maximal algebraically independent set iff E is algebraic over $F(S)$.

Proof. Suppose that $E/F(S)$ is algebraic, we want to show that there is no other element α in E such that $S \cup \{\alpha\}$ is still algebraic independent. Suppose that $\alpha \in F(S)$, since we assumed that α is algebraic over $F(S)$, it satisfies the following condition:

$$\frac{p_0}{q_0} + \frac{p_1}{q_1}\alpha + \cdots + \frac{p_r}{q_r}\alpha^r = 0$$

where p_i, q_i are rational polynomials with finitely many variables in S. Then by clearing the denominators we can assume that:

$$r_0 + r_1\alpha + \cdots r_r\alpha^r = 0$$

for polynomials r_i in with finitely many variables in S. This show that α is not algebraically independent to S.

Conversely, suppose that S is maximal, then for any elements in E, it's left to you to show that $E/F(S)$ is algebraic. □

Remark 1.23. Now if we consider our definition above where $F(\{x_i\}) \sim E$, this is really capturing the fact that $\{x_i\}$ is **maximally algebraically independent** according to the above lemma.

Next we state a result that is useful for us to think about our definition of $\sim$ above:

Lemma 1.24. Suppose that E/F has an algebraically independent set A of cardinality $|A|$, then we can embed $F(x_i, i \in A)$ into E.

Proof. Use the definition of algebraic independent and universal property of localization. □

The next lemma will be essential in our proof of the transcendence basis is well behaved:

Lemma 1.25. Suppose that E/F is a field extension and E is algebraic over $F(a_1, \cdots, a_n)$ then if $\{b_i\}$ is an algebraically independent set, then $|\{b_i\}| \leqslant n$.

Proof. Notice that b_1 is algebraic over $F(a_1, \cdots, a_n)$, then use definition of being algebraic and then clearing the denominators, there is a nontrivial polynomial p such that $p(b_1, a_1, \cdots, a_m) = 0$ where b_1 has to appear in p and there are also some a_i in p, say a_1.

This in particular says that a_1 is algebraic over $F(b_1, a_2, \cdots, a_n)$. This implies that $F(b_1, a_1, \cdots, a_n)$ is algebraic over $F(b_1, a_2, \cdots, a_n)$, thus we now have E is algebraic over $F(b_1, a_2, \cdots, a_n)$. Now, this in particular implies that b_2 is algebraic over $F(b_1, a_2, \cdots, a_n)$, then we run the same argument above to have: $q(b_2, b_1, a_2, \cdots, a_n) = 0$ where b_2 has to appear somewhere and there has to be some a_i's appearing since b_2, b_1 are assumed to be algebraically independent. With a re-numbering, we can assume that it's a_2, then a_2 is algebraic over $F(b_2, b_1, a_3, \cdots, a_n)$, thus $F(b_2, b_1, a_2, a_3, \cdots, a_n)$ is algebraic over $F(b_2, b_1, a_3, \cdots, a_n)$, since E is algebraic over $F(b_2, b_1, a_2, a_3, \cdots, a_n)$, then E is algebraic over $F(b_2, b_1, a_3, \cdots, a_n)$. Then you can repeat this process, to obtain that E is algebraic over $F(b_1, \cdots, b_m, a_{m+1}, \cdots, a_n)$. Then if $m > n$, you will get E is algebraic over $F(b_1, \cdots, b_n)$ after n steps, thus we know that $\{b_1, \cdots, b_n\}$ is already a maximal independent set, we are done. □

Theorem 1.26. Suppose that we have a field extension E/F and $x_1, \cdots, x_n$ is a transcendence basis of E/F, then any other transcendence basis also has n elements.

Proof. Notice that E is algebraic over $F(x_1, \cdots, x_n)$, by our remark about $F(x_1, \cdots, x_n) \sim E$, then for any transcendence basis $\{y_i\}$, it's in particular algebraically independent, thus the cardinality is less than or equal to n. $\square$

Remark 1.27. Notice that in the proof of the above theorem, suppose that we start with a infinite transcendence basis, then we claim that any other transcendence basis must also has infinite cardinality. Otherwise, suppose that you have $\{b_1, \cdots, b_n\}$ a transcendence basis, apply the above lemma, you conclude that any algebraically independent set has cardinality less than or equal to n.

But at this point of time, this is the most we can say, you only can tell the difference between finite and infinite. We don't know that if two infinite transcendence basis has the same cardinality, besides we know the fact that they are both infinite! It turns out to be true that any two transcendence basis have the same cardinality, but we won't prove or use it here.

Definition 1.28. Base on the above result, we will call the size of any transcendence basis the **transcendence degree** of E/F, denoted by $trdeg(E/F)$.

Remark 1.29. One quick observation of the above definition is that: if E/F is a finitely generated field extension, then the transcendence degree, $trdeg(E/F)$ is automatically finite. This is just an application of one of the above lemma.

Remark 1.30. Technically speaking, we didn't prove that there is even a maximal algebraically independent set. But it's easy to show using the usual Zorn's lemma argument as what we did when we prove any linear space has a maximal linearly independent subset.

This is what we want to say about transcendence degree in this point, before we have more to say about it's geometrically nature later, let's directly state the main theorem we are interested:

Theorem 1.31. Suppose that A is a finitely generated k-algebra which is a domain over a field k, then $dimA = trdeg(K(A)/k)$, hence if X is an irreducible k-variety, then $dimX = trdeg(K(X)/k)$.

As we promised, the proof of this important result uses Noether Normalization, which we will prove later. But what you should notice here is that since X is an irreducible k-variety, it a finite type integral scheme over k which is separated. Notice that since it's finite type, suppose that it's covered by finitely many $SpecA$ where A is a finitely generated domain over k. Since X is irreducible, this open contains the generic point, and we know that the dimension of $SpecA$ is just the codimension of the generic point in $SpecA$, which according to the result we will prove is just $trdeg(K(A)/k) = trdeg(K(\eta)/k)$ where η is the unique generic point since the function field can be computed by any affine open around η. Thus the dimension of each $SpecA$ is the same, thus according to a previous exercise we proved, the dimension of X is $trdeg(K(A)/k)$ for any of the $SpecA$.

Here the **finite type** assumption is important, which allow to compute dimension using the transcendence degree. If we lose this assumption, the above argument fails.

Example 1.32. Consider the space $Speck[x]_{(x)}$, which is just a two point space that is of dimension 1. This is because that (x) is a irreducible closed point, and the closure of the generic point is also irreducible closed, which is the whole space.

However, since the generic point is open, which is just $Speck(x)$. It has dimension 0, which is not the dimension of X. The reason of why the above theorem doesn't work is that $k(x)$ is not a finitely generated algebra of k.

We can also use the above result to prove Hilbert's Nullstenllensatz, which we used above and promised to prove:

Proposition 1.33. Suppose that $A = k[x_1, \cdots, x_n]/I$, then the residue field of any maximal ideal is a finite extension of k.

Proof. If we want to show that $k[x_1, \cdots, x_n]/m$ is a finite extension of k, we notice that this is a domain and it's obviously finitely generated k-algebra. Then we can apply the above theorem, since the field of fractions of $k[x_1, \cdots, x_n]/m$ is just itself, which should be transcendence 0 since $dim(k[x_1, \cdots, x_n]/m) = 0$.

Now we only have to prove that if a field extension K/k is of transcendence 0 and finitely generated as a k-algebra, then it's a finite extension. But notice that transcendence 0 implies that this is an algebraic extension, and finitely generated means that it's also a finitely generated field extension, thus algebraic plus finitely generated as a field extension implies that it's finite. □

This also gives a geometrically believing result:

Proposition 1.34. Suppose that $f : X \to Y$ is a dominant morphism of irreducible k-varieties, then $dimY \leqslant dimX$, and in terms of geometric intuition, lower dimension stuff can't dominate a higher dimension stuff.

Proof. Consider the field extension $k \hookrightarrow K(Y) \hookrightarrow K(X)$, then we know that any transcendence basis of $K(Y)$ is a transcendence basis in $K(X)$. □

Proposition 1.35. Suppose that $f_1(x_1, \cdots, x_n), \cdots, f_m(x_1, \cdots, x_n) \in k[x_1, \cdots, x_n]$ are polynomial in n variables over a field k with $m \geqslant n$, then the polynomials $f_1, \cdots, f_m$ are algebraically dependent.

Proof. This is just an easy application of the fact that $x_1, \cdots, x_n$ is a transcendence basis, then in particular any set of algebraically independent elements should have cardinality less than n. □

Remark 1.36. If $m = n$, how can you tell if they are independent or not?

The answer if easy when $char(k) = 0$ and you only have to look at the Jacobian determinant of $f_1, \cdots, f_n$, they are algebraically dependent iff the determinant is 0. We will prove this later.

But please pay special attention to the case when $char(k) = p$, the polynomials $f_i = x_i^p$ will be independent while the Jacobian determinant is 0.

Another geometric exercise we can do is:

Proposition 1.37. The following three equations cut out an integral surface S in $\mathbb{A}_k^4$.

Proof. You probably think this maybe a curve since it uses three equations, even though it's really a surface. This is particular easy if you find that on $D(x)$ or $D(y)$ or the other two affine opens **** later.* □

Definition 1.38. We can also define the degree of a rational map between irreducible varieties. **** later after review of rational maps*

Now we begin to prove the Noether normalization theorem:

Theorem 1.39. Suppose that A is a finitely generated k algebra and a domain. If $trdegK(A)/k = n$, there will be elements $x_1, \cdots, x_n \in A$ algebraically independent over k such that A is a finite extension of $k[x_1, \cdots, x_n]$.

Remark 1.40. The geometric content of this lemma is the following: say $X = SpecA$, then we have a surjective finite morphism $X \to \mathbb{A}_k^n$ where n is the transcendence degree of $K(A)/k$. It's sometimes called the **geometric Noether normalization**.

Now let's see Nagata's proof of Noether normalization theorem:

Proof. Say $trdegK(A)/k = n$, since we assume that A is a finitely generated k-algebra and a domain, then we can assume $A = k[x_1, \cdots, x_m]/p$ where p is a prime ideal, then we know that $K(A)$ is generated by $x_1, \cdots, x_m$ as a field extension, then we know that $m \geqslant n$, otherwise we saw that if $K(A)$ is generated by m elements as a field extension, the transcendence degree is less than or equal to m.

Say $m = n$, then we see that $A = k[x_1, \cdots, x_n]/p$, we claim that $p = 0$, which means that $x_1, \cdots, x_n$ are algebraically independent, which implies that $p = 0$. To see this, suppose that $f(x_1, \cdots, x_n) = 0$, then we claim that $k(x_1, \cdots, x_n)$ is algebraic over $k(x_1, \cdots, x_{n-1})$. This is because that we can assume that x_n appears in f then this is just trivial definition. Then by an early results, any algebraically independent sets has cardinality less than or equal to $n - 1$, a contradiction.

Therefore, we can only consider the case where $m > n$ and we can assume that for lower m, the result has been proved. Let's first outline the strategy we will employ in this proof: Say $A = k[y_1, \cdots, y_m]/p$, we will fine $m-1$ elements in A such that A is finite over the subring of A which is generated by these $m-1$ elements, which you can write as $k[z_1, \cdots, z_{m-1}]/q$. So you will have a picture like this:

$$B = k[z_1, \cdots, z_{m-1}]/q \xhookrightarrow[finite]{} A = k[y_1, \cdots, y_m]/p$$

It's a easy consequence that if $B \hookrightarrow A$ is a finite ring extension of domains, then the induced field extension between field of fractions is still finite and in particular, algebraic. Thus the transcendence degree of $K(B)/k$ is the same as the transcendence of $K(A)/k$. Can you figure out why?

Then we can apply our induction hypothesis to conclude that we have the following picture:

$$k[x_1, \cdots, x_n] \xhookrightarrow[finite]{} B = k[z_1, \cdots, z_{m-1}]/q \xhookrightarrow[finite]{} A = k[y_1, \cdots, y_m]/p$$

which will complete the proof. Therefore, we are only left to prove the assertion that we do have $m-1$ elements with the desired property.

Notice that this p in $A = k[y_1, \cdots, y_m]/p$ can not be 0, thus there will be a nontrivial polynomial f such that $f(y_1, \cdots, y_m) = 0$. Here comes the trick that Nagata uses which makes this approach not so easy to come up with. Notice that there will be a nonzero polynomial f such that $f(y_1, \cdots, y_m) = 0$, you can use the fact that we assumed that $trdeg(K(A)/k)$ is $n < m$ but the images of $y_1, \cdots, y_m$ in $K(A)$ will generate $K(A)$ to conclude that p is not 0. Then, notice that if we define $z_1 = y_1 - y_m^{r_1}$, $z_2 = y_2 - y_m^{r_2}$ and so on until $z_{m-1} = y_{m-1} - y_m^{r_{m-1}}$ for positive integers r_i to be chosen. What is our strategy here, we already see that for $y_1, \cdots, y_{m-1}$, they are all integral over the subring generated by $z_1, \cdots, z_{m-1}$, if we can also prove that y_m is integral over $z_1, \cdots, z_{m-1}$, then remember that finitely generated plus integral implies finite, thus our goal now is to create a integral dependence relation of y_m in terms of $z_1, \cdots, z_{m-1}$.

To do this, we first notice that:

$$f(z_1 + y_m^{r_1}, \cdots, z_{m-1} + y_m^{r_{m-1}}, y_m) = 0$$

Then we would like to expand this polynomial out, then each of the monomials in f, upon expanding in terms of $z_1 + y_m^{r_1}, \cdots, z_{m-1} + y_m^{r_{m-1}}, y_m$, will give you something that is a constant times a power of y_m and with no z_i's factors, then I will let you figure out why you can find the r_i's inductively, such that we have:

$$0 \ll r_1 \ll r_2 \ll \cdots \ll r_{m-1}$$

such that each of the terms of y_m powers appearing in the expansion of the monomials doesn't collide with each other, including those terms that have some z_i factors inside it. Thus, by arranging the powers to be distinct, we avoids the case that when you have terms of the same powers, you probably is going to get cancellation. Thus, you obtained an integral dependence relation of y_m in terms of $z_1, \cdots, z_{m-1}$, we are done. $\square$

Remark 1.41. Now, we look at the geometric intuition behind Nagata's trick in the proof of the normalization. To give an explicit example, suppose that you have a n-dimension variety X inside $\mathbb{A}_k^m$ for some $m > n$, say you have the variety defined by $xy = 1$ in $\mathbb{A}_k^2$, then what Nagata's trick is to find a lower dimension variety Y in $\mathbb{A}_k^m$ and a morphism from X to Y such that the morphism is finite, and for Y, we already have the desired result. However, this Y is not chosen to be arbitrary, for example, if you want to project $xy = 1$ to the x or y axis, it will not be finite since the morphism fails to be closed. Thus this implies that $k[x] \hookrightarrow k[x,y]/(xy = 1)$ or $k[y] \hookrightarrow k[x,y]/(xy = 1)$ doesn't work. However, you can see that $k[x - y] \to k[x,y]/(xy = 1)$ work, thus if you project to the line that tilts 45 degrees that passes through the origin, it works.

In general, if the field is infinite, you can show that only a finite number of such lines will make the morphism fail to be finite, thus when the field is finite, you can use: $z_i = y_i - r_i y_m$ instead of $y_m^{r_i}$. However, when the field is finite, we probably going to run into the questions that all the directions allowed in the finite field fails to make the morphism finite. Thus, Nagata's trick is to make nonlinear change of y_m, which increase the directions, or rather, since the curves can bend, it will have more freedom.

Remark 1.42. There is still an loose end we didn;t specifically mention in the proof above. To use the induction hypothesis, we have to show that the transcendence degree of the fraction field of the subring generated by $z_1, \cdots, z_{m-1}$ over k is still n.

This will follow from the following general result. Say you have a finite ring extension between integral domains: $A \hookrightarrow B$, then we claim that the induced ring morphism between the fraction fields is also a finite field extension. Once you proved this, notice that if we denote the subring generated by $z_1, \cdots, z_{m-1}$ by B, then we have the following diagram:

$$\begin{tikzcd}
B \arrow[r, hook, "finite"] \arrow[d] & A \arrow[d] \\
Frac(B) \arrow[r, hook, "finite"'] & Frac(A) \\
k \arrow[u, hook] \arrow[ur, hook] &
\end{tikzcd}$$

Since finite extension is in particular algebraic, thus the transcendence degree of $K(B) \hookrightarrow K(A)$ is 0, thus you can easily show that the transcendence degree of $Frac(B)/k$ is the same as the transcendence degree of $Frac(A)/k$.

Finally, our assertion that $K(B) \hookrightarrow K(A)$ is finite is also easy, since the field extension factors through $K(B) \hookrightarrow S^{-1}A \hookrightarrow K(A)$ where S is the nonzero elements in B. Since $K(B) \hookrightarrow S^{-1}A$ is finite since finitely generated here is preserved by localization (in particular integral), thus the source is a field iff the target is a field, thus $S^{-1}A \hookrightarrow K(A)$ is an isomorphism.

As an application of the ideas we used above, we can prove that:

Proposition 1.43. Suppose that X, Y are irreducible k-schemes, then $dim_k(X \times_k Y) = dimX + dimY$. (Technically speaking, we haven't even proved that they even have finite dimensions, but it will be clear after we prove the main theorem, thus we will first assume that both sides have finite dimension.)

Proof. Suppose that $dimX = n$ and $dimY = m$, then by Thm 1.31, we have $trdeg(K(X)/k) = n$ and $trdeg(K(Y)/k) = n$. Then since we know that the functional field of an integral scheme can be computed in any affine opens around the generic point, we know that we have:

$$k[x_1, \cdots, x_n] \hookrightarrow A \quad \text{and} \quad k[y_1, \cdots, y_m] \hookrightarrow B$$

such that both are finite ring extensions if we can show that A, B are integral domains that is finitely generated over k, this is true since we know that X, Y are finite type over k and are integral. Moreover, being finite type allows us to cover X using finitely many A, thus you will have a diagram like below for X:

$$\begin{tikzcd}
 & \Gamma(X, \mathscr{O}_X) \arrow[d, hook] \arrow[dr] & \\
k[x_1, \cdots, x_n] \arrow[ur, dashed] \arrow[r, hook, "finite"] & A_i \arrow[r, hook] & K(X)
\end{tikzcd}$$

I will let you think about why this implies that the image of all the x_i's all in the images of $\Gamma(X, \mathscr{O}_X)$, thus the universal property of the polynomial algebra gives us the desired induced injective map, which induces a morphism:

$$f : X \to X$$

which is first finite since it's local on the target, and it's also surjective since it's even surjective when we consider $SpecA_i \to \mathbb{A}_k^n$. We can also apply the same reasoning to Y to obtain a surjective finite morphism:

$$g : Y \to \mathbb{A}_k^m$$

Then, notice that being finite and surjecrtive are all nice properties, thus closed under products, thus we have an induced finite, surjective morphism:

$$f \times_k g : X \times_k Y \to \mathbb{A}_k^{m+n}$$

Then, notice that the dimension of $\mathbb{A}_k^{m+n}$ is $m+n = dimX + dimY$, we only have to prove this this surjective, finite morphism allows us to prove that the both side have the same dimension.

Recall that we prove that if we have an integral extension of rings, the morphism it induces preverves dimensions. Here, notice that $A_i \times_k B_j$ covers $X \times_k Y$, and we can study what is the map $f \times_k g$ restricted to $A_i \times_k B_i$, it's left for you to check that the map is induced by $k[x_1, \cdots, x_n] \otimes_k k[y_1, \cdots, y_n] \to A_i \otimes_k B_j$, notice that we already see that this is finite, thus in particular integral, and it's also injective since we are working with a field k, thus tensoring with respect to k is faithlully flat, thus the kernel of the map is the tensor product of the two kernels, which is 0.

Therefore, we have proved that the dimension of $SpecA_i \times_k B_i$ is $m + n$, thus the dimension of $X \times_k Y$ is $m + n$. $\square$

In particular, what we just proved can be applied to the case that when A is a finitely generated k-algebra which is a domain, then $A[x]$ has dimension $dimA + 1$ since $dimk[x] = 1$.

Here are also some results, that you may find useful in other parts of the review:

Proposition 1.44. Suppose that $X = Projk[x_0, \cdots, x_n]/I \hookrightarrow Projk[x_0, \cdots, x_n]$ is a irreducible k-variety for a homogeneous prime ideal I. Then show that the affine cone and projective cone of X has dimension $dimX + 1$.

Proof. Let's first consider the affine cone, notice that it's just $Speck[x_0, \cdots, x_n]/I$, notice that it's still irreducible it's dimension is the transcendence degree of it's fraction field over k. Notice that since X is still an irreducible k-variety, we know that if we consider the usual affine cover, there is at least one of them is with dimension n. Say, it's $Speck[\frac{x_i}{x_j}|i \neq j]/I_{(x_j)}$, which is spec of a domain and $k[\frac{x_i}{x_j}|i \neq j]/I_{(x_j)}$ a subring of $(k[x_0, \cdots, x_n]/I)_{x_j}$, then we know that in the induced extension of fraction fields, if $z_1, \cdots, z_m$ is a transcendence basis of $Frac(k[\frac{x_i}{x_j}|i \neq j]/I_{(x_j)})$ over k, they remain algebraically independent in $Frac((k[x_0, \cdots, x_n]/I)_{x_j})$, thus we know that the dimension of $Spec(k[x_0, \cdots, x_n]/I)$ is at least $dimX$. However, we claim that the after we add x_j to the list, it's still algebraic independent in the fraction field of $k[x_0, \cdots, x_n]/I$, this is because suppose that they are algebraic dependent, thus by clearing the denominators, you can assume that the numerators of $z_1, \cdots, z_m$, denoted by $y_1, \cdots, y_m$ (they are homogeneous elements, say of degree d_i) will be algebraically dependent if we add x_j, thus you can find f a polynomial with coefficients in k such that $f(y_1, \cdots, y_m, x_j) \in I$, notice that I is an homogeneous ideal, thus if we group each cluster with the same degree you will get a few monomials with $z_1, \cdots, z_m, x_j$ such that each of them is of the same degree and is in I, since I is homogeneous. Then say they are of degree D, we divive by x_j^D, I claim that we will obtain a equation with coeffiecients in k and variables in some of the $z_1, \cdots, z_m$'s in $I_{(x_j)}$, which is a contradiction to the fact that we assumed them to be algebraically independent in the first place. As for why the claim is true, it's left to you, just write down what is the equation, and you will see why divide by x_j^D will give you what you want.

Now, we see that the dimension of the affine cone is at least $dimX + 1$. Then we claim that it can't be larger than $dimX + 1$, this is because that any elements in the fraction field of $k[x_0, \cdots, x_n]/I$ will be algebraic over the subfield generated by $z_1, \cdots, z_m, y_m$. To see why, we only have to show that any monomial that is not in I will be algebraic.

Suppose that f is such a monomial, with degree d, we first divide by x_j^d, then we know that it's algebraic over $z_1, \cdots, z_m$, thus suppose that there is a single variable polynomial g with coefficients in $k[\frac{x_i}{x_j}|i \neq j]/I_{(x_j)}$ such that $g(\frac{f}{x_j^d}) = 0$, then we are done. $\square$

Proposition 1.45. Any minimal prime in $k[x_0, \cdots, x_n]$ over a homogeneous ideal I is also homogeneous.

Proof. This is equivalent to show that any minimal prime ideal in a graded ring is homogeneous, which is the result of the following construction.

Suppose that p is a prime in a graded ring S, then the ideal q generated by all homogeneous elements in p is also prime, and it's homogeneous by the way we constructed it. First, it's a homogeneous ideal, thus to prove it's prime, we only have to show that for any homogeneous elements f, g, if $fg \in q$, then either f or g is in q. Notice that either f, g is in p, and f, g are homogeneous, we are done.

Then suppose that p is a minimal prime in a graded ring, the following construction gives us a prime contained in p that is homogeneous, thus since p is assumed to be minimal we are done. $\square$

Proposition 1.46. Suppose that $p \subset q$ are two prime homogeneous ideals in $k[x_0, \cdots, x_n]$, such that there is no homogeneous prime ideals lying strictly between them, then there is no prime ideals strictly lying between $p \subset q$.

Proof. Say that $q \subsetneq P \subsetneq q$ where P is a nonhomogeneous prime ideal, then there will be a $f \in P$ while $f \notin p$, then there is also a $g \in q$ such that $g \notin P$. Notice that we can assume that g is a homogeneous element, consider (p, f), then the minimal prime over it, which should be contained in P and homogeneous, we see that it doesn't contain g, since $g \notin P$ and it is contained in P, which also strictly contains p, we are done. □

Enough with the detour for some useful fact, we now turn to prove the main theorem, Thm 1.31:

Proof. Say X is an integral affine k-scheme of finite type, we will show that $dimX$ is equal to the transcendence degree n of it's function field, by induction on n.

Notice that by Thm 1.39, suppose that $trdeg(K(A)/k) = n$, there is a finite, surjective morphism $SpecA = X \to \mathbb{A}^n$ corresponding to $k[x_1, \cdots, x_n] \hookrightarrow A$, finite ring extension. Then this implies that the dimension is the same $dimX = dim\mathbb{A}^n_k$. If $n = 0$, then we know that $dimk = 0$, thus $dimX = 0$. Then we assume that this holds for all transcendence less than n and we want to prove if for n.

Notice that if we can show that $dim\mathbb{A}^n_k = n$, then we are done. First, we know that $dim\mathbb{A}^n_k \geqslant n$, since you can easily find a sequence of primes with length n, which is: $(0) \subset (x_1) \subset (x_{,1}, x_2) \subset \cdots \subset (x_1, \cdots, x_n)$. We only have to show that the dimension is less than or equal to n.

Say we have any chain of prime ideals in $k[x_1, \cdots, x_n]$:

$$0 = p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_r$$

Notice that we can choose any nonconstant polynomial in p_1, then replace p_1 by a irreducible factor f of that polynomial, since this is a UFD, then (f) is going to be prime. Then we can consider $k[x_1, \cdots, x_n]/(f)$, we claim that it's transcendence degree is $n - 1$. This is because that we can look at f more closely. Take out any one of the variables in f, then the last $n - 1$ variables still remains algebraically independent, for trivial reasons. Then we claim that any monomial is algebraic over the subfield generated by the left $n - 1$ variables, since the elements algebraic over the subfield is also a field, we know that we only have to show x_i is algebraic over the subfield, which is true just by f, we are done. □

Remark 1.47. If you examine the proof above, the only thing we proved is just $dim\mathbb{A}^n_k = n$, then everything follows from Noether and the fact that integral extension preserves dimension.

Then the way we prove $dim\mathbb{A}^n_k$ is n is by induction and Noether normalization, then again integral extension preserves dimension. Therefore, as it may seem like a very chaotic proof, the above is how it's logic flow goes.

Now we turn to the discussion of when our intuition of $dimZ = dimX - codim_X Z$ holds: the short answer is it holds for irreducible varieties, but we can state a theorem that is more general:

Theorem 1.48. Suppose that X is a pure dimensional k-scheme locally of finite type, (so an irreducible k-variety is fine in particular), let Y be an irreducible closed subset and η is the generic point of Y, then $dimX = dimY + dim\mathscr{O}_{X,\eta}$ since we know the dimension of the local ring is the codimension of Y. Therefore, the general inequality:

$$codim_X Y + dimY \leqslant dimX$$

is now an equality:

$$codim_X Y + dimY = dimX$$

Before we begin to prove it, we can give an algebraic translation of what this means in particular:

Definition 1.49. A ring A is called **Catenary** if for every nested pair of primes $p \subset q \subset A$, every strictly increasing chain of prime ideals from p to q is contained in a maximal such chain of finite length (i.e. it can be extended to a finite length chain such that in this chain, there is no prime strictly in between any two elements in the chain.) and every such maximal finite length chain have the same length.

You will see in the proof of the next proposition why being catenary related to the theorem above:

Proposition 1.50. If A is a localization of a finitely generated ring over k, then A is catenary.

Proof. The first thing you can show is that since $Speck[x_1, \cdots, x_n]$ is an irreducible (thus pure dimensional) and it's of course finite type over k, thus suppose that $p \subset q$ are two prime ideals. Then if you consider $Speck[x_1, \cdots, x_n]/p$ where you can view $Speck[x_1, \cdots, x_n]/p$ as an irreducible closed subspace inside it. Then, notice that by Thm 1.48, the codimension of $Speck[x_1, \cdots, x_n]/q$ plus the dimension is the dimension of $Speck[x_1, \cdots, x_n]/p$. Thus, you can first show that the maximal prime chains between p, q has to have finite length since $dimSpeck[x_1, \cdots, x_n]$ has finite dimension.

Moreover, any two maximal chains between p, q has to have the same length. To see this, notice that the dimension of $Speck[x_1, \cdots, x_n]/p$ is the length of chains of primes containing p, which can be reached by passing q, then suppose that there are two different length maximal chains between p, q, one of them is with length less than codimension of $Speck[x_1, \cdots, x_n]/q$ inside $Speck[x_1, \cdots, x_n]/p$. Then by induction, we can show that $dimSpeck[x_1, \cdots, x_n]/p = l + dimSpeck[x_1, \cdots, x_n]/q$, thus the codimension has to be the length of any maiximal chain. (The key here is to go one by one!)

Now we can prove that any localization of a catenary ring is catenary. This is because that primes in a localization correspond in an inclusion preserving way to primes in the original ring such that they don't intersect with the multiplicative system you are working with, and suppose that they correspond to $p \subset q$ in the original ring, since q doesn't intersect with the multiplicative system, then the chains between p, q will not intersect with the multiplicative system.

Finally, following the same line of thinking, you can prove that quotient of catenary ring is catenary. □

Now we begin to prove Thm 1.48:

Proof. **Step 1:** We will show that this is equivalent to show that if X is an irreducible affine k-variety and Z is a closed irreducible subset maximal among those irreducible closed subset strictly contained in X, then $dimZ = dimX - 1$.

First, since X is pure dimensional, each of it's components are of the same dimension. Also, Y as an irreducible closed subset, is contained in one of the irreducible components. Thus we can focus on just one irreducible component. Then also notice that we assume that $dimY$ is finite, otherwise there is nothing to prove. Similarly, we can assume that $codim_X Y$ is also finite. Let's assume that it's achieved by the chain $Y = Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_r = X$. Then if we assume that $Y = Y_{r-1}$ is maximal among all those irreducible closed strictly contained in X, we know that $codim_X Y_{r-1} = 1$, and if we prove that $dimX = dimY_{r-1} + 1$, then you move to $dimY_{r-1} = dimY_{r-2} + 1$, finally, after enough inductive steps, you will be done.

Moreover, notice that by focusing on a single irreducible component, we can take the reduced scheme structure on it, and by doing this, we can just focus on integral schemes over k locally of finite type.

Therefore, we need to prove that suppose that you have an integral k-scheme X locally of finite type over k, and Y is maximal among all those irreducible closed subsets in X, then $dimX = dimY + 1$. Then, I will let you figure out why we can further reduce from this case to X is an affine k-variety $SpecA$ where A is finitely generated k-algebra which is a domain, and Y is a maximal irreducible closed subset strictly contained in X. □

Then, we can assume that $dimX = d$ by the above theorem. Notice that we in the first place only assumed that $dimY$ and $codim_X Y$ is finite, but now we can assume that $dimX$ is also finite. Then by Noether normalization, we can find a finite surjective morphism $\pi : X \to \mathbb{A}^d_k$, then since finite morphisms are closed, we know that $\pi(Y)$ is closed in $\mathbb{A}^d_k$, we also claim that it's irreducible, since if not, you can find two opens inside that don't intersect, then you take the inverse image, you get two opens in Y that don't intersect, a contradiction to the irreducibility of Y.

Proof. **Step 2:** Now, we claim that it will suffice if we show that $\pi(Y)$ is a hypersurface, or in other words, the topological space of $\pi(Y)$ is $Speck[x_1, \cdots, x_d]/(f)$ for some degree m polynomial f. This is because that we already showed that the dimension is $d - 1$ for any f, nonconstant. Then, notice that $\pi(Y)$ is irreducible closed, thus you can assume that (f) is a prime ideal, then you can consider the restriction of the morphism

$\pi : X \to \mathbb{A}^d$ under:

$$\begin{tikzcd} SpecA/p \arrow[r] \arrow[d, hook] & Speck[x_1, \cdots, x_d]/(f) \arrow[d, hook] \\ X = SpecA \arrow[r] & \mathbb{A}^n_k \end{tikzcd}$$

where the morphism $SpecA/p \to Speck[x_1, \cdots, x_d]/(f)$ is induced by:

$$\begin{tikzcd} k[x_1, \cdots, x_n] \arrow[r, hook, "\varphi"] \arrow[d] & A \arrow[d] \\ k[x_1, \cdots, x_n]/(f) \arrow[r, hook] & A/p \end{tikzcd}$$

where $(f) = \varphi^{-1}(p)$, thus we know that it's still injective, and being integral is stable under this operation, thus we get another integral extension of rings, thus we are done.

But how does this help since $\pi^{-1}(\pi(Y))$ is in general not Y, but it contains Y, thus the dimension of $\pi^{-1}(\pi(Y))$ should be the same as the dimension of Y, otherwise it's the whole space, which is impossible since it means that the whole X is mapped to a proper closed subset (a hyper surface), contradiction to the fact that π is surjective. ☐

Finally, if $\pi(Y)$ is not a hypersurface, we will use the following theorem, usually called **Going-Down** type of theorems, with it's own interest to get a contradiction. But let's prove the following theorem first, where later you will see a very similar version when we discussion flat ring homomorphisms. Be sure you are comfortable with the statement since we will not prove it here, both for the reason that this holds more generally for integral ring extensions and for the fact that prove it in this easier sense use Galois theory, which is a detour we don't want to make here.

Theorem 1.51. Suppose that $\Phi : B \hookrightarrow A$ is a finite extension of rings so A is a finite B-module, then if B is an integrally closed domain and A is an integral domain, then given nested prime ideals $q \subset q'$ in B and a prime p' in A such that it lies over q', then there is a prime p contained in p' such that it lies over q.

Before we prove this theorem, let use it to finish the proof of Thm 1.48:

Proof. **Step 3:** Now if $\pi(Y)$ is not a hypersurface, it's still irreducible and closed, now we claim that it's contained in an irreducible hyperfurface. Say the underlying topological space of $\pi(Y) = Speck[x_1, \cdots, x_n]/P$ for a prime ideal P, take out any nonzero polynomials f inside and take one of it's irreducible factors, we see that $\pi(Y)$ is contained in $Speck[x_1, \cdots, x_n]/(f)$, which is an irreducible hypersurface, then consider $k[x_1, \cdots, x_n] \hookrightarrow A$ where $k[x_1, \cdots, x_n]$ are obviously integrally closed and A is a domain, we have $(f) \subset P$ in $k[x_1, \cdots, x_n]$ where (f) is strictly below P, if we say the prime corresponding to Y in A is Q, then there is a prime Q' lying below Q such that the inverse image if (f), and notice that Q is also strictly below Q' since their inverse image differ, we now get a contradiction since we assume that there is no larger irreducible closed subset strictly over Y! ☐

We end this discussion here with some results that we can prove by Thm 1.48, we begin by a general notion:

Proposition 1.52. Suppose that X is a Noetherian scheme, we define a function $dim_X(\cdot) : X \to \mathbb{Z}_{\geqslant 0}$, by taking $dim_X p$ to be the dimension of the largest irreducible component of X that contains p. We will call this the **dimension of** X **at** p. We can prove that this function is **upper semicontinuous**. Remember that a function from a topological space to $\mathbb{R}$ is upper semicontinuous is for each $x \in \mathbb{R}$, $f^{-1}(-\infty, x)$ is open in X. In formally speaking, functions with this property can only jump up upon taking limits, (think of a function that maps $\mathbb{R}$ to $\mathbb{R}$ may help with your intuition.) Since here our function is mapping everything into $\mathbb{Z}$ then we again embed $\mathbb{Z}$ into $\mathbb{R}$ then use the same definition. Then, what you will observe is that the dimension less than or equal to a positive integer is open.

Proof. As what we mentioned in the above remark, what you need to prove is that the set where $dim_X p \leqslant n$ is open for all nonnegative integer n, since by this suppose that x is any real number, if it's an integer, then we are done, if it's not an integer, then it's in between two integers, then we are also good since we know that the image is in $\mathbb{Z}_{\geqslant 0}$.

Now, suppose that $p \in X$ is any point, the most important ingredient in this proof is that since this is a Noetherian scheme, we can pick an Noetherian affine open $SpecA$ where A is Noetherian, then we know that each irreducible component of X if that intersects with $SpecA$, then the intersection will be an irreducible component of $SpecA$, then since every Noetherian ring has finitely many minimal primes, then only finitely many irreducible component intersect with $SpecA$, then we pick that one with the largest dimension which contains x. Now, since we assumed that the irreducible component containing x has dimension less than n, and since in $SpecA$, except the intersection of the components containing x, if we take the union of other irreducible components in $SpecA$, since only finitely many of them the union is still closed, then we take them out, we know that in the complement, every point has dimension in X less than n almost by definition, thus we are done.

Notice that the above argument also works for locally Noetherian schemes, even though we usually only consider irreducible components in Noetherian spaces, where this notion behave well. But if we only work with Noetherian schemes, this space is in particular a Noetherian space. Then it only have finitely many irreducible components, then suppose that $dim_X p = m < n$, then we can assume that there is an irreducible component of X, denoted by X' such that $dimX' = m$ and $p \in X'$, then we take Y to be the union of all the other irreducible components, which doesn't contain X, then if U is the complement of Y, every point p in U will satisfy $dim_X p = m < n$. □

Remark 1.53. This kind of semicontinuous properties is actually a recurring topic we will see in the future as well. They are an important class of properties you care about in algebraic geometry. Below are some of the example of semicontinuous properties we have saw or will see in the near future:

1. The dimension of a point in a variety is upper semicontinuous, as proved above.
2. Fiber dimension, which we will see very soon. (upper semicontinuous)
3. The rank of a finite type quasicoherent sheaf.
4. Degree of a finite morphism. (upper semicontinuous)

Proposition 1.54. *Suppose that p is a closed point of a locally finite type k-scheme X. Then we have the following three integers are the same: (1): $dim_X p$ (2): $dim\mathscr{O}_{X,p}$ (3) $codim_X p$*

Proof. Notice that since p is a closed point, then the generic point of the closure is itself, thus the latter two integers are the same. To show that they coincide with the first integer, we first consider the codimension $codim_X p$, say it's infinite, then we can easily see that $dim_X p$ is also infinite. The converse also holds, since if $dim_X p$ is infinite, then there is an irreducible component containing p such that the dimension is infinite. Then notice that if we consider this irreducible component, take the reduced scheme structure, we get a locally finite type pure dimensional k-scheme. (In fact that you can show that if X is a locally finite type scheme over k, then any closed embedding to X is also locally of finite type over k since quotients preserves finitely generated as an k-algebra.)

Then we can use the theorem we proved above to conclude that $codim_X p = codim_X p + dimp = dimX$, thus $codim_X p$ is also infinite.

Then if $codim_X p$ is finite, then suppose that we reach the codimension in X_i, an irreducible component of X, then we claim that $dim_X p$ is the dimension of X_i, if not, suppose that it's attained in X_j, which means $dimX_j > dimX_i$, then we give it the reduced scheme structure, since X_j also contains p, then we know that $dimX_j = codim_{X_j} p + dimp = codim_{X_j} p \leqslant codim_{X_i} p = dimX_i - dimp = dimX_i$, thus $dimX_j \leqslant dimX_i$, a contradiction. Then we are done.

Conversely, suppose that $dim_X p$ is finite, then say the largest dimension of the irreducible containing p is X_i, then we know that if $codim_X p = codim_{X_i} p$, we are done. If not, say $codim_X p = codim_{X_j} p$, then following similar arguments as above, we can prove that $dimX_j > dimX_i$, a contradiction, we are done. □

We now prove a result that is of conceptual importance, which generalize the fact that if you have an irreducible variety over $\mathbb{Q}$ of dimension n, then if you do the base change to $\mathbb{R}, \mathbb{C}$, the result may not be irreducible anymore, but fortunately, it's still pure dimensional, and each irreducible component of the base change has dimension of the same dimension to the variety you started with. In particular, we already saw that since $\mathbb{C}$ is algebraically closed, the base change is indeed irreducible, then this tells you that if you have a $\mathbb{Q}$-variety in $\mathbb{A}^n_{\mathbb{Q}}$ cut out by a few equations with coefficients in $\mathbb{Q}$, then it's dimension is the same to the variety if you bow view it as cut out by equations with $\mathbb{C}$-coefficients in $\mathbb{A}^n_{\mathbb{C}}$.

Proposition 1.55. Suppose that X is a locally finite type k-scheme of pure dimension n, and $k \hookrightarrow K$ is any field extension, not necessarily algebraic. Then we claim that X_K is also of pure dimension n.

Proof. We will divide this proof into several steps:

Step 1: This proof uses a result that deserves a quick review. Suppose that X is a scheme locally finite type over k, then we will prove that a point $p \in X$ is a closed point in X iff it's residue field is a finite extension of k, which basically follows from HIlbert's Nullstellensatz. Furthermore this will imply that if we work with schemes locally of finite type over k, then the closed points are dense.

To see our first assertion, first assume that p is a closed point, then it correspond to a maximal ideal of an affine open around p, then the Hilbert's Nullstellensatz will suffice. Now suppose that the residue field of p is a finite extension of k. Then notice that if we cover X by affine opens which are spectra of finitely generated k-algebras, then Hilbert's Nullstellensatz shows that if p is in any of these opens, it's closed in that affine open, then it's closed, globally.

Now why this implies that the closed points are dense? We can prove that if A is a finitely generated k-algebra, then the closed points in $SpecA$ are dense. Suppose that $f \in A$ is a element such that $D(f)$ is nonempty, then we claim that there is a closed point in $D(f)$, this will show that the closed points are dense since $D(f)$'s are a basis.

To show this, consider the localization of A at f, which is still a finitely generated k-algebra. Then we can still apply Hilbert's Nullstellensatz to A_f, thus the closed points of $SpecA_f$ are still those where the residue field is a finite extension of k. Suppose that p_f is such a prime, then if you localize at p_f and then quotient $A_f p_f$, this ring is canonically isomorphic to A_p/pA_p, thus p is a closed point of $SpecA$. Then we only have to prove that there is such a prime. But, we know that it's maximal ideal does the job. To make this argument somewhat more rigorous, consider following diagram:

$$\begin{tikzcd}
A_f \arrow[d, "loc"] \arrow[drr, dashed] & & A \arrow[ll] \arrow[d, "loc"] \\
(A_f)_{p_f A_f} \arrow[rr, dashed, leftrightarrow] & & A_p
\end{tikzcd}$$

I will let you figure out why it induces an isomorphism between $(A_f)_{p_f A_f} \cong A_p$. Then I will let you figure out why under this moprhism, the prime $p_f A_f (A_f)_{p_f A_f}$ is mapped to the prime ideal pA_p. Hint: both $p_f A_f (A_f)_{p_f A_f}$ and pA_p is the extension of p!

Then if you have an arbitrary isomorphism of rings $R \cong S$, then if a prime p in R is mapped to q in S, then the isomorphism canonically descend to $R/p \cong S/q$. Finally, though it's a rather trivial argument, you need to make sure why this implies that if one of the residue field is an finite extension of k, then so is another.

Now, why this implies that closed points are dense in X? Notice that we just proved that closed points in X are exactly those with residue field a finite extension of k, and it can be computed in any affine open. If we want to show that for any open in X there is a closed point, then just notice that for any open, there is an affine open inside it which will be the spec of a finitely generated k-algebra, thus it contains a point whose residue field is a finite extension of k, we are done. ☐

To record for our later use, we state the preliminary result we prove here:

Proposition 1.56. In a scheme X locally of finite type over k a point is a closed point in X iff it's residue field is a finite extension of k. Also, the closed points are dense in X.

Proof. See the above proposition. But you should be aware that if $SpecA$ where a is not a finitely generated k-algebra, the closed points need not to be open. ☐

Let's continue our proof:

Proof. **Step 2:** We claim that it will suffice to prove for the case where $X = SpecA$ where A is a finitely generated k-algebra. Notice that since X is assumed to be pure dimensional, say of dimension n and locally of finite type over k, we claim that can find an open cover of X where each of them is of dimension n and each is the spec of a finitely generated k-algebra. To see this, first, we can definitely find an open cover with specs of finitely generated k-algebras. We claim that each of these affine open is pure dimensional of dimension n.

To see this, consider those irreducible components of X which intersects with $SpecA$ where $SpecA$ is one of those affine opens. Then notice that this A is in particular Noetherian, thus only finitely many irreducible components intersects with $SpecA$, then say X_i is one of them, then we take the complement in $SpecA$ all the other irreducible components of X except X_i, since only finitely many irreducible components intersect with $SpecA$, the union of them is still closed in $SpecA$, thus the complement is still open, thus contains a closed point, say c. This implies that the dimension of $SpecA \cap X_i$ is still n since we can use the fact that $dim(SpecA \cap X_i) = codim_{SpecA \cap X_i}(c) + dim(c) = codim_{SpecA \cap X_i}(c) = n$ since we know that the codimension of c in X_i and $SpecA \cap X_i$ is equal to each other.

Therefore, we can consider the following diagram:

$$\begin{tikzcd} Spec(A \otimes_k K) \arrow[r] \arrow[d] & X_K \arrow[r] \arrow[d] & SpecK \arrow[d] \\ SpecA \arrow[r] & X \arrow[r] & Speck \end{tikzcd}$$

if you can show that $Spec(A \otimes_k K)$ is still pure dimensional of dimension n, then we claim that we are done since first when $SpecA$ ranges over the open cover of X, $Spec(A \otimes_k K)$ covers X_K. If $Spec(A \otimes_k K)$ is pure dimensional n, we claim that this implies that X_K is pure dimensional of dimension n. Since $Spec(A \otimes_k K)$ is all of dimension n, then we know that $dimX = n$. Then since X_K is locally finite type over K, then you can prove that if X_K is not pure dimensional, say there is an irreducible component whose dimension is less than n, then for the $Spec(A \otimes_k K)$ that intersects with that component, the dimension of $Spec(A \otimes_k K)$, since it's pure dimensional, is less than n, which is a contradiction. This proves our assertion that we can assume $X = SpecA$ for a finitely generated k-algebra A.

Step 3: Then we claim that we can assume that A is a domain. To see this, assume that $A = k[x_1, \cdots, x_n]/I$, then we consider the following diagram:

$$\begin{tikzcd} Spec(K[x_1, \cdots, x_n]/I) \arrow[r] \arrow[d] & SpecK \arrow[d] \\ Spec(k[x_1, \cdots, x_n]/I) \arrow[r] & Speck \end{tikzcd}$$

Notice that surjectivity is preserved by base change, thus we know that every prime in $k[x_1, \cdots, x_n]/I$ is the inverse image of some prime in $K[x_1, \cdots, x_n]/I$, then consider a minimal prime P over I, we claim that it's inverse image is also a prime over I, otherwise, say p' is the minimal prime over I that lies below the inverse image, by the surjectivity, there is a prime in $K[x_1, \cdots, x_n]/I$ such that the inverse image is p' which is not equal to the inverse image of P, but it can only lies below P, which is a contradiction. Then if we use p to denote the inverse image of P, we know that the ring homomorphism:

$$k[x_1, \cdots, x_n]/p \to K[x_1, \cdots, x_n]/P$$

is well defined and we have:

$$\begin{tikzcd} Spec(K[x_1, \cdots, x_n]/p^e) \arrow[r, hook] \arrow[d] & Spec(K[x_1, \cdots, x_n]/I) \arrow[r] \arrow[d] & SpecK \arrow[d] \\ Spec(k[x_1, \cdots, x_n]/p) \arrow[r, hook] & Spec(k[x_1, \cdots, x_n]/I) \arrow[r] & Speck \end{tikzcd}$$

is still Cartesian where p^e is the extension of p. Notice that if this $SpecK[x_1, \cdots, x_n]/p^e$ is pure dimension of m, then we are done, since we know that $Speck[x_1, \cdots, x_n]/P$ is an irreducible component of

$SpecK[x_1, \cdots, x_n]/p^e$ since P is a minimal prime over I^e and which is contained in p^e and is contained in P. This proves our assertion that we can always assume that A is a domain.

To proceed to step 4, we need another result in field theory: ☐

Lemma 1.57. Suppose that $k \hookrightarrow K$ is an arbitrary field extension, then it can be factored to a purely transcendental extension and an algebraic extension.

Proof. This proof is basically the fact that any field extension $k \hookrightarrow K$ has a maximal algebraically independent set, then we can use our earlier characterization of a set being maximally algebraically independent set using every other element is algebraic over it.

Therefore, suppose that this maximally independent set is $\{e_i\}_{i \in I}$, we can decompose the field extension as:

$$k \hookrightarrow k(\{e_i\}) \hookrightarrow K$$

where the first one is purely transcendental by definition and the second one is algebraic since $\{e_i\}$ is maximal. ☐

Let's go back to our proof:

Proof. **Step 4:** We claim that we can assume the field extension $k \hookrightarrow K$ is a purely transcendence extension with transcendence basis $\{e_i\}$. Notice that we can consider the diagram:

$$\begin{tikzcd}
X_K \arrow[r] \arrow[d] & X_{k(\{e_i\})} \arrow[r] \arrow[d] & X \arrow[d] \\
SpecK \arrow[r] & Speck(\{e_i\}) \arrow[r] & Speck
\end{tikzcd}$$

Notice that if we can prove that $X_{k(\{e_i\})}$ is of pure dimension n, since we already proved the case for algebraic extension and $X_{k(\{e_i\})}$ is still affine, we are done.

Step 5: We claim that $A \otimes_k K$ is still an integral domain, this is because that K is the faction field of $k[e_i]_{i \in I}$, then notice that we proved that localization commutes with tensor products, thus it's just a localization of $A \otimes_k k[e_i]$, which is in particular of a localization of a domain(Make sure you understand why this is also a domain.), thus it's a domain.

Step 6: Finally, suppose that $K(A)/k$ has a transcendence basis $\{t_1, \cdots, t_d\}$, then we claim that $\{e_i\} \cup \{t_1, \cdots, t_d\}$ is a transcendence basis of $K(A_K)/k$. To show this, we need to show that they are algebraically independent and everything else are algebraic over them. We claim that $K(A_K) = K(A)(e_i)$, in the sense that the following diagram is commutative:

$$\begin{tikzcd}
 & A[e_i] \arrow[d, "loc"] \\
A \arrow[ur] \arrow[r, hook] \arrow[d, hook] & A \otimes_k K = A_K \arrow[d, hook] \\
K(A) \arrow[r] & K(A_K) \cong Frac(A[e_i])
\end{tikzcd}$$

Then since $Frac(A[e_i]) \cong K(A)(e_i)$, we are done. Then we claim that this will suffice to show that t_j, e_i's are a transcendence basis over k. First, notice that we need to show that they are algebraically independent. This should be easy since each t_j is in A, then since $A[e_i]$ is a domain and any polynomial with coefficients in A is 0 iff each coefficient is 0. Then we need to prove that every other element is algebraic over the subfield generated by them, this is the content of the following sequence:

$$k \hookrightarrow k(t_j) \hookrightarrow K(A) \hookrightarrow K(A)(e_i)$$

Step 7: The last step, we only have to prove that $K(A_K)/K$ has a transcendence basis $\{t_j\}$. Then we are done. Still consider the extension:

$$k \hookrightarrow k(t_j) \hookrightarrow K(A) \hookrightarrow K(A)(e_i)$$

I will let you figure out why we are done by it. ☐

This is basically all what we want to prove now for codimension and dimensions right now. Of course, there will be later discussion that uses these results we proved above a great deal, but for now, we will turn to Krull Theorems and how it's related to the notion of dimensions: From now on, we will require that rings are **Noetherian**, we start by **Krull's Principal Ideal Theorem**, which relate functions and the codimension 1 points where they vanish:

Proposition 1.58. Suppose that X is a locally Noetherian scheme and f is a function then the irreducible components of $V(f)$ are of codimension 0 or 1.

This is clearly the result of the the following algebraic version:

Proposition 1.59. Suppose that A is a Noetherian ring and $f \in A$. Then every prime p minimal among those containing f has codimension 0 or 1. More precisely, if f is not a zero divisor, then every prime minimal among those containing f has codimension precisely 1.

Before we prove it, let's see how this algebraic version imply the geometric version:

Proof. We want to prove the irreducible components of $V(f)$ have codimension 0 or 1 in X. Suppose that η is the generic point of one of it's irreducible components, then we are trying to consider $dim\mathscr{O}_{X,\eta}$. Take an affine open around η, say $SpecA$, and assume that A is Noetherian. Then notice that the dimension of $\mathscr{O}_{X,\eta}$ is the codimension of η in $SpecA$. Notice that if we still denote the restriction of f to $SpecA$ $f|_{SpecA}$ by f, this η is a minimal prime over f, thus if f is not a zero divisor, the codimension is 1 by the algebraic version of Krull's theorem. If not, then it can be codimension 1 or 0, we are done. ☐

Here are also an example we have about how to apply this Krull's principal theorem to prove some geometric facts:

Example 1.60. Suppose that f is an irreducible homogeneous polynomial in $n+1$ variable over k, then $V(f)$ in $\mathbb{P}^n_k$ is an integral scheme of dimension $n-1$.

To prove it, we work in each $D_+(x_i)$. We only have to prove that in each $D_+(x_i)$, the closed subscheme is of dimension $n-1$. The whole idea here is that if f is irreducible, then if you homogenize it by dividing a power of x_i, you end up with a irreducible polynomial in the variables $\frac{x_j}{x_i}$, this is because that if it's not, then you get some factorization, then you multiply with a power of x_i, you end up with the polynomial you started and it's now reducible! Then, we only have to prove that if you have f in n-variables which is irreducible, then $V(f)$ in $\mathbb{A}^n_k$ is of dimension $n-1$. Notice that the codimension of $V(f)$ is 1, then since $\mathbb{A}^n_k$ is a locally finite type scheme over k, then $codimV(f) + dimV(f) = dimX = n$, thus since $codimV(f) = 1$ since f is not a zerodivisor, thus $dimV(f) = n-1$.

Now let's prove the algebraic version, Thm 1.59:

Proof. We start with a Noetherian ring A and an element $f \in A$. Let's first prove that if the minimal prime over f is of codimension 0, then f is necessarily a zero divisor. Notice that being codimension 0 means that it's a minimal prime, then p is an associated prime, thus in particular it's elements are all zero divisors, then $f \in p$ is also a zero divisor.

Now, let's deal with the general case by showing that $codimp$ is always less or equal to 1 for any prime p minimal over (f). Notice that localization at a prime doesn't change it's codimension, in the sense that if we want to compute the codimension of p in A, we can compute the codimension of pA_p in A_p. Thus, by localizing at p, we can assume that p is a maximal ideal.

If there is no primes strictly smaller than p, we are done, if not suppose that we have $q \subsetneq p$, for a prime ideal q, then $f \notin q$ since we assumed that p is already a minimal prime over f. If we can show that q has codimension 0, then we are done. The key idea is the following clever construction of **symbolic powers**: ☐

Proposition 1.61. Let $\Phi : A \to A_q$ be the localization map, define $q^{(n)} = \Phi^{-1}(q^n A_q)$, the inverse image of the extension of q^n. Notice that $q^{(1)} \supset q^{(2)} \supset \cdots$ is a descending chain of ideals of A. Then we claim that if $ag \in q^{(n)}$, and $g \notin q$, then $a \in q^{(n)}$, or in other words, $q^{(n)}$ is q-primary for every n.

Proof. This result relies on a simple fact about primary ideals, which is powers of maximal ideals are primary. To see this, suppose that m is a maximal ideal in A, then $fg \in m^n$ while $f \notin m$, then $(f) + m = A$, thus there is an $a \in A$ and $b \in m$ such that $af + b = 1$, then notice that $b^n \in m^n$, then we see that $1 = 1^n = fs + b^n$ for some $s \in A$, then multiply by g, you get $g = fgs + gb^n \in m^n$.

Then back to our question, since $ag \in q^{(n)}$, then $\frac{ag}{1} \in q^n A_q$, which is qA_q-primary, then since we assumed that $g \notin q$, then $\frac{g}{1} \notin qA_q$, then $\frac{a}{1} \in q^n A_p$, thus we know that $a \in q^{(n)}$ by definition.

One last remark is that this whole proof works is we have some ideal P such that it's radical is a maximal ideal m, then P is m-primary. □

Now back to the proof of Thm 1.59:

Proof. Now consider $Rad(f)$, the radical of f, which is the intersection of all the primes over (f), which is p by our assumption on p. Then, notice that in $A/(f)$, which is still a Noetherian local ring, whose maximal ideal is the nilradical, then it's of dimension 0 by Prop 1.19, then still by Prop 1.19, any chain of ideal in $A/(f)$ eventually stabilizes, in particular, the images of $q^{(1)} \supset q^{(2)} \cdots$ stabilizes, then we know that there is an n_0 such that $q^{(n)} = q^{(n+1)}$ after we mod out (f), for n sufficiently large.

We next claim that in fact $q^{(n)} = q^{(n+1)} + fq^{(n)}$ for all $n \geqslant n_0$. This is because that suppose that $b_n \in q^{(n)}$, there is some $a \in A$ and $b_{n+1} \in q^{(n+1)}$ such that $b_n = b_{n+1} + af$, notice that $b_{n+1} \in q^{(n+1)} \subset q^{(n)}$, thus $af \in q^{(n)}$, then since $f \notin q$, then $a \in q^{(n)}$, thus we are done.

Finally, by Nakayama's lemma, we claim that $q^n A_q = 0$, which implies that the dimension of A_q is 0 since A_q is also a Noetherian ring, thus the codimension of q is 0, we are done. To show our claim, consider $q^n A_q$ and $q^{n+1} A_q$, by the relation $q^{(n)} = q^{(n+1)} + fq^{(n)}$, we know that $q^{(n)} = q^{(n+1)}$ since the ideal (f) is contained in all the maximal ideals in A (since A has only one maximal ideal p), then this implies that $q^n A_q = q^{n+1} A_q$. □

Remark 1.62. We recap on the important ingredients of the previous proof. The first realization is that you want to prove that if there is a strictly smaller prime under p, it's codimension is 0, which implies that you want to show A_q is of dimension 0, where you can use the characterization Prop 1.19. In the end, we will show that the power of it's maximal ideal stabilizes, which suffices.

Another important ingredient is the symbolic power, since if we can show that if $q^{(n)} = q^{(n+1)}$ for any n large enough then $q^n A_q = q^{n+1} A_q$, this is because that by definition, $q^n A_q$ is the extension of q^n, then notice that the inverse image clearly has q^n, then we know that the extension of $q^{(n)}$ is also $q^n A_q$.

But how can we prove $q^{(n)} = q^{(n+1)}$, well, this kind of argument is always done by Nakayama's lemma, which works best in local conditions, this is why we assumed in the beginning that the ring is local with unique maximal p.

Finally, the symbolic power allows us to create q-primary ideals for a not necessarily maximal ideal q.

Now, we start using Krull's Principal theorem to prove some geometric facts. The first result we prove is not related to using Krull's Theorem, but it's an essential exercise both in the sense that we will use it repeatedly later and in the sense that it's the preparation for something that uses Krull's Theorem:

Proposition 1.63. *Suppose that k is a field, possibly finite, let $\{p_1, \cdots, p_m\} \in \mathbb{P}^n_k$ be a collection of points, not necessarily closed and none of which is the generic point of $\mathbb{P}^n_k$. Then there is a large $N >> 0$ such that a nonempty hypersurface of degree N in $\mathbb{P}^n$ such that it doesn't contain any of the points p_i.*

Proof. Let's do this by induction on m, notice that for $m = 1$, suppose that we have a point $p \in \mathbb{P}^n$, not necessarily closed, we want to find a hyperfurface that doesn't contain p. Notice that we can assumed that p is closed, since if it's not, we can consider it's closure, we know that it's closure must contain some closed points since a closed subscheme of a locally finite type over k-scheme is also locally finite type over k, thus closed points are dense, in particular, it has some. Then if H is a hypersurface that doesn't pass the closed point in the closure, it can't pass p, we are good.

Now, suppose that this closed point, in $D_+(x_i)$ correspond to a closed point in $\mathbb{A}^n_k$, then since it's still a proper ideal, there is a polynomial f in the n variables $\frac{x_j}{x_i}$ such that f is not in that maximal ideal. Then turn it into homogeneous polynomial in x_i such that it's degree is the lowest, the the hyperplane it determines doesn't contain p, otherwise, in $D_+(x_i)$, it will contain p, a contradiction.

Now, let's do induction for higher m, say now we know it holds for m, we want to ask whether if holds for $m+1$ or not. Consider the point p_{m+1}, since we already have a hypersurface defined by some f such that it doesn't contain $p_1, \cdots, p_m$, if it also doesn't contain p_{m+1}, then we are done. If it contains p_{n+1}, we will call this hypersurface f_{n+1}. Now, we consider the set of n points consisting of $\{p_1, \cdots, \hat{p_i}, \cdots, p_{m+1}\}$ where we omit p_i where $1 \leqslant i \leqslant m$. For each i, we have f_i such that it doesn't contain $\{p_1, \cdots, \hat{p_i}, \cdots, p_{m+1}\}$. Most importantly, if in the procedure of doing this, if we happen to encounter one such that p_i is not in the hypersurface defined by f_i, we are done since we can just take f_i to be the hypersurface we want, thus we can assume that all p_i is in f_i.

Then we consider the homogeneous polynomial:

$$F = \sum_i f_1 \cdots \hat{f_i} \cdots f_n$$

You might have noticed that F at this point have no reason to be homogeneous, but we will first treat the case when F is indeed homogeneous then explain in the end what to do if it's not homogeneous. We claim that the hypersurface defined by F will suffice. If not, we know that it will contain some p_j, then suppose that p_j is in one of the standard affine covers, say $D_+(x_k)$, then we divide F by a power of x_k to obtain a summation of $m+1$ polynomials in $\frac{x_i}{x_k}$. Here is the key point: for those terms in the summation where f_j is not excluded, after we divide by a power of x_k, it's in the maximal ideal corresponding p_j in $D_+(x_k)$, then let's look at the term where f_j is excluded, it's must also in that maximal ideal by noticing we can subtract the summation of the others, but notice that none of

$$\{f_1, \cdots, \hat{f_j}, \cdots, f_{m+1}\}$$

contains p_j, thus their union won't, thus a contradiction! Therefore, we must encounter some f_i where p_i is not in f_i in the process of finding f_i's ,we are done.

Now if F is not homogeneous, then notice that the degree of each of F's terms is:

$$d_1 + \cdots + \hat{d_i} + \cdots + d_{m+1}$$

The key point here is that if we have f_i satisfying our requirement, then any power of f_i still satisfies our assumption. Since if f_i doesn't vanish at $\{p_1, \cdots, \hat{p_i}, \cdots, p_{m+1}\}$ but vanish at p_i, and each of them are primes, thus insensitive to powers. Now notice that powers of f_i will translate to multiples of degree, in the sense that $deg(f_i^n) = n(deg f_i)$. Then it's left to you that can replace f_i by suitable powers of f_i to make F homogeneous.

Finally we need to make sure that F is not 0. But this is automatically true, since we already showed that F is not in any p_i without assuming F is not 0. $\square$

Proposition 1.64. If we can assume that the field k is infinite, then we can replace hypersurface in the previous proposition by hyperplanes and obtain the same result.

Proof. **** The below proof is wrong! Please see the proof later in Appendix 10 for the right version!*

Notice that we can still assume that every one of the points we are considering is closed. Now suppose that we have $p_1, \cdots, p_m$ distinct closed points in $\mathbb{P}^n$, we want to find a hyperplane defined by $a_i x_i = 0$ such that none of the p_i's is in the hyperplane. We can assume that these points in any $D_+(x_k)$ will correspond to the closed point that of the form k^n, this is because that if not, we can assume that the maximal ideal doesn't contain any linear polynomials (check this and make sure you understand!), then we claim that none of the hyperplanes contain this point(also make sure you understand this!).

Now, suppose that you have $p_1, \cdots, p_m$ and suppose that for p_i is in $D+(x_k)$ and it correspond to the maximal ideal $(\frac{x_i}{x_k} - a_i)$, now we claim that a hyperplane defined by $h_j x_j$ contains that point iff it vanishes at $[a_0 : \cdots : a_{k-1} : 1 : a_{k+1} : \cdots : a_n]$. It's easy to see that if it vanishes here then it contains that point. Also, if it contains that point, then $h_j \frac{x_j}{x_k}$ vanishes at a_i, then multiply by x_k, you get that $h_j x_j$ vanishes at $[a_0 : \cdots : a_{k-1} : 1 : a_{k+1} : \cdots : a_n]$. Then we can do the same procedure for all p_i's, then the conclusion is that the hyperplane defined by $h_j x_j$ doesn't contain $p_1, \cdots, p_m$ iff it doesn't vanish at several $[a_{0,j} : \cdots : a_{k-1,j} : 1 : a_{k+1,j} : \cdots : a_{n,j}]$ where $1 \leqslant j \leqslant m$.

Now the question becomes that if k is infinite, is it possible to find a hypersurface defined by $h_j x_j$ such that it doesn't vanish at a collection of points. Let's first consider those hyperplanes that vanishes. This is asking what h_j can make $h_j x_j$ vanish at $[a_{0,i} : \cdots : a_{k-1,i} : 1 : a_{k+1,i} : \cdots : a_{n,i}]$ when you replace x_j by $a_{j,i}$, notice that this gives you a hypersurface in k^n defined by $h_j a_{j,i} = 0$. Then we see that if we consider the union, which is a finite union of hypersurfaces, if it's not the whole space, then there is going to be a n-tuple of h_j such that the hyperplane defined by it doesn't vanished at those finitely many points.

Here notice that the hyperplanes all contain 0, and also, they are not all of k^n, thus we are just trying to show that when k is infinite, any nontrivial k-vector spaces are not a finite union of it's proper subspaces. This is sometimes called the **avoidance lemma for vector spaces**: we will prove that if V is a union of n proper subspaces, then $|k| < n$. To see this we can suppose that $V = W_1 \cup \cdots \cup W_n$, we can also assume that not one of them is contained in the union of the others, then take $x \in W_1 - W_2 \cup \cdots \cup W_n$. Also, take one $y \notin W_1$, then consider $y + kx$, we claim that this is disjoint from W_1, otherwise, you will get that $y \in W_1$, which is a contradiction. Then we also see that this set $y + kx$ intersects with $W_2, \cdots, W_n$ at most one point, since otherwise you will have $y + k_1 x \in W_i$ and $y + k_2 x \in W_i$, then $x \in W_i$, which is a contradiction. Therefore, we have an injection from k to $W_2, \cdots, W_n$, thus $|k| < n$, we are done. □

Similar to the avoidance lemma for vector spaces we also have the prime avoidance lemma, whose idea is very similar to the proof of the previous two propositions:

Proposition 1.65. Suppose that A is a ring and $p_1, \cdots, p_n$ are primes of A, then if I is another ideal of A not contained in any p_i. Then I is not contained in the union of p_i's, i.e there is a $f \in I$ such that $f \notin p_i$ for all i.

Proof. Let's first prove it and we will give it geometric interpretation and also explain why it's called prime avoidance.

Let's try to do induction on n, first if you only have 1, there is nothing to prove. Then suppose that it holds of n, we want to consider $n+1$. Notice that we can assume that it's not contained in the union of any n out of $n+1$ primes, then by our hypothesis, there is $f_1, \cdots, f_{n+1}$ such that f_i is not in any $\{p_1, \cdots, \hat{p_i}, \cdots, p_n\}$. If in the process of finding f_i's we find some one that is not contained in these $n+1$ primes, we are done, thus we can assume that this doesn't happed. Then consider the summation, of $F = \sum f_1 \cdots \hat{f_i} \cdots f_n$ we claim that it's not in any of these p_i's, to see this, we are trying to figure out is it in p_j, notice that the summation of all terms without f_j being excluded is in p_j, then we know that if the summation where f_j is exclude is also in p_j, but this is a contradiction.

Notice that the summation F can't be 0 since we already proved it it not in any p_j's.

We can give it geometric interpretation, under our assumption, which can be translated to be $V(I)$ avoids $p_1, \cdots, p_m$ since I is not contained in any p_i, then we claim that there is a closed subset of the form $V(f)$ for some f containing $V(I)$ such that it avoids any of the primes $p_1, \cdots, p_n$.

Sometimes, the contrapositive is also useful, which means that if I is contained in a finite union of primes, then I is necessarlily contained in one of the primes. □

Now we use this result to prove something more interesting and actually uses Krull's Theorem Thm 1.59, it will be a series of results:

Proposition 1.66. Suppose that X is a closed subset of $\mathbb{P}^n_k$ of dimension at least 1 and let H be a nonempty hypersurface in $\mathbb{P}^n_k$, then $H \cap X$ is not empty(set-theoretically).

Proof. This is very different to the case in the affine space, where you can easily find a closed point and a hypersurface such that they don't intersect.

Let's first make the observation that we can always assume that X to be irreducible since it's only has finitely many irreducible components and if it's holds for some of it's irreducible components, it of course holds for X. Also, since we assumed that the dimension of X is at least 1, there is at least an irreducible component whose dimension is 1, we will use that one. Thus, we can assume that I is a homogeneous prime.

Notice that if H is defined by a homogeneous polynomial F of degree d, then consider the affine cone of $Projk[x_0, \cdots, x_n]/(F)$, which is $Speckk[x_0, \cdots, x_n]/(F)$, notice that since when all x_i's are zero, F is zero (F is homogeneous !), then we know that the origin is contained in $Speckk[x_0, \cdots, x_n]/(F)$. Now suppose that X is defined by the ideal I in $k[x_0, \cdots, x_n]$, then consider the affine cone: $Speck[x_0, \cdots, x_n]/I$. Now we can assume that F is not in I, otherwise if F is in I, X contains H. Then, consider $V(F)$ in $Speck[x_0, \cdots, x_n]/I$, we know that the codimension is either 0 or 1, which is $Speck[x_0, \cdots, x_n]/(I, F)$.

Notice that since X is at least of dimension 1 and X can be viewed as an irreducible k-variety, then it's affine cone is at least of dimension 2 by Prop 1.44. Thus, notice that by Thm 1.48, we know that the dimension of $Speck[x_0, \cdots, x_n]/(I, F)$ in $Speck[x_0, \cdots, x_n]/I$ is at least 1, thus it's can't just contain the origin, which is a closed point whose dimension is 0, thus there has to be some points besides the origin. Then consider the natural morphism:

$$Speck[x_0, \cdots, x_n] - \{0\} \to \mathbb{P}^n_k$$

Notice that there is a point in $Speck[x_0, \cdots, x_n]/(I, F)$ that is not 0, thus it's mapped to some point in $\mathbb{P}^n_k$, then consider the following diagram:

$$\begin{tikzcd}
 & Speck[x_0, \cdots, x_n]/I - \{0\} \arrow[r] \arrow[d] & Projk[x_0, \cdots, x_n]/I \arrow[dr, hook] & \\
Speck[x_0, \cdots, x_n]/(I, F) - \{0\} \arrow[ur] \arrow[r] \arrow[dr] & Speck[x_0, \cdots, x_n] - \{0\} \arrow[rr] & & Proj[x_0, \cdots, x_n] \\
 & Speck[x_0, \cdots, x_n]/(F) - \{0\} \arrow[r] \arrow[u] & Projk[x_0, \cdots, x_n]/(F) \arrow[ur, hook] &
\end{tikzcd}$$

We claim that is commutative, then it's left to you to show why this suffice for the assertion that X intersects with H.

To check that this is commutative, we first deal with the right half of the map. The key is to figure out what is the morphism $Speck[x_0, \cdots, x_n]/I - \{0\} \hookrightarrow Speck[x_0, \cdots, x_n] - \{0\}$. It comes from the following Cartesian diagram:

$$\begin{tikzcd}
Speck[x_0, \cdots, x_n]/I \arrow[r, hook, "i"] & Speck[x_0, \cdots, x_n] \\
i^{-1}(Speck[x_0, \cdots, x_n] - \{0\}) \arrow[r] \arrow[u, hook] & Speck[x_0, \cdots, x_n] - \{0\} \arrow[u, hook, "open"']
\end{tikzcd}$$

associated to the open embedding $Speck[x_0, \cdots, x_n] - \{0\} \to Speck[x_0, \cdots, x_n]$, then you can just check this locally if you still remember how we defined these kind of morphisms.

Similarly, the left half of the diagram is commutative because all the morphisms are defined similarly by an alike Cartisian diagram as above, and you can check it locally. Or if you want another way, you can consider the following diagram:

$$\begin{tikzcd}
Speck[x_0, \cdots, x_n]/(I, F) \arrow[r, hook] & Speck[x_0, \cdots, x_n]/(F) \arrow[r, hook] & Speck[x_0, \cdots, x_n] \\
Speck[x_0, \cdots, x_n]/(I, F) - \{0\} \arrow[r, hook] \arrow[u, hook] & Speck[x_0, \cdots, x_n]/(F) - \{0\} \arrow[r, hook] \arrow[u, hook] & Speck[x_0, \cdots, x_n] - \{0\} \arrow[u, hook]
\end{tikzcd}$$

and I will leave it to you to see why this diagram implies that the diagram holds. $\square$

Remark 1.67. We are using an important fact in this proof, that is any closed embedding to $\mathbb{P}^n_k$ is defined by a homogeneous ideal, which is proved in other sections. And implicitly reduction of schemes and other facts, make sure you understand all of them, and if not go to other parts of the review sections of the appendix and review them!

Proposition 1.68. Suppose that $X \hookrightarrow \mathbb{P}^n$ is a closed subset of dimension r, then for any codimension r linear space, it meets X.

Proof. Suppose that H is a linear space cut out by r linearly independent linear homogeneous polynomials, say $f_1, \cdots, f_r$, then we claim that $X \cap H$ is not empty. To see this, again we assume that X is irreducible, then say X is cur put by a prime I, then we can also assume that $r < n$, otherwise, a irreducible closed subset of $\mathbb{P}^n$, which is also irreducible of dimension n is the whole space, thus there is nothing to prove. Then let's consider the affine cone of X, which is $Speck[x_0, \cdots, x_n]/I$, which has dimension $r + 1$, then we also consider the affine cone H, which is $Speck[x_0, \cdots, x_n]/(f_1, \cdots, f_r)$, again we know that both such spaces contains the origin, but since $r + 1 \geqslant 1$, we know that it can't just contain the origin.

Then let's deal with $Speck[x_0, \cdots, x_n]/(f_1, \cdots, f_r)$, which is the affine cone of H. Let's make a special announcement here, as you can see, this codimension here is not necessarily a topological concept, since we don't know at this point if H is irreducible and how can we define it's codimension. Here what we mean is that it's defined by a codimension r linear subspace in the space of all homogeneous linear polynomials with coefficients in k. We will review this afterwords to connect with a notion we introduced earlier on linear spaces of the projective space. But here let's just take the interpretation just mentioned. Now, if we can prove that the dimension of $Speck[x_0, \cdots, x_n]/(f_1, \cdots, f_r)$, it won't just contain the origin, thus we are done by the same reason in the previous proposition. Notice that we know that the irreducible component C_0 of $Speck[x_0, \cdots, x_n]/(f_1, \cdots, f_r)$ has codimension $0, 1$ in $Speck[x_0, \cdots, x_n]/(f_1, \cdots, f_{r-1})$ since f_r is not zero in this quotient $Speck[x_0, \cdots, x_n]/(f_1, \cdots, f_{r-1})$, take the irreducible components of $Speck[x_0, \cdots, x_n]/(f_1, \cdots, f_{r-1})$ containing C_0, denoted by C_1, it of codimension $0, 1$ in $Speck[x_0, \cdots, x_n]/(f_1, \cdots, f_{r-2})$, and you take the irreducible component again and keep going, finally, if we can prove that the irreducible component of $Speck[x_0, \cdots, x_n]/(f_1, \cdots, f_r)$ in $Speck[x_0, \cdots, x_n]$ is of codimension no larger than n, thus we know from Prop 1.48 that the dimension is at least 1, we are done.

To show the dimension assertion, we see that $dimC_0 = dimC_1 - codim_{C_1}C_0 \geqslant dimC_1 - 1$, then you keep going, in the end find that $dimC_0 \geqslant n + 1 - r - 1 = n - r$, thus we are done. Also, notice the techniques used here are not restricted to linear subspaces, here we are using the linear space because it's especially easy for us to conclude that f_r is not in $(f_1, \cdots, f_{r-1})$, and you can keep going. ☐

Proposition 1.69. Still, we work in the setup as in the previous proposition, then there is a nonempty intersection of $r + 1$ hypersurfaces that miss X.

Proof. Suppose that the generic points of irreducible components of X is a finite collection, then by Prop 1.63, we can find a hypersurface H_1 that miss all of them.

Then consider the closed subset $X \cap H_1$, we claim that the dimension of $X \cap H_0$ is at most $r - 1$, this is because we missed all the generic points of irreducible components of X, then you keep going, until you find $X \cap H_1 \cap \cdots \cap H_r$ is of dimension at most 0, thus the stuff left is only a finite collection of points, then H_{r+1} can be chosen to avoid all of them by Prop 1.63.

Therefore, the only one last step we need to show is that dimension zero closed subset in $\mathbb{P}^n$ is indeed a finite collection of points. Notice that if a closed subset in $\mathbb{P}^n$ is of dimension 0, then let's consider it's irreducible components, which are $V_+(p)$ for a homogeneous ideal p, since it's dimension 0, we know that there is no larger homogeneous polynomial above p, thus $V(p) = \{p\}$, thus it only has finitely many points. ☐

Proposition 1.70. If further more, we can assume that k is infinite, then we can take the intersection of $r + 1$ hyperplanes, and still got what we want in the previous proposition.

Proof. Notice that the key step is also to show that we can find a hyperplane that misses all the generic points of X, which is a result of Prop 1.64. ☐

After the above few propositions, you probably can see that identifying the irreducible closed subsets of $\mathbb{P}^n$ is an important ingredient we used, thus it's a good time to review some basic result we mentioned without proof earlier when we discussed the proj constructions, let's fix any graded ring S, we start with a very general result that works for any schemes, this is not as trivial as it looks:

Proposition 1.71. Suppose that X is a scheme, then we have a bijection between irreducible closed subsets of X and the closure of points in X. In particular, we can define the generic point of any irreducible closed subsets in a scheme.

Proof. First notice that the closure of any point is irreducible (check this!). Then we only have to show that if C is a irreducible closed subset of X, then there is a unique point p in C such that $\overline{\{p\}} = C$.

To see this, we first pass to U, any affine open subset of X such that $U \cap C$ is not empty. Notice that we already showed before that taking intersections of open subsets with irreducible closed subsets gives you an irreducible closed subsets of the open. Therefore, since this is an affine open, there is a unique generic point for $U \cap C$. Now, suppose that you have another affine open V, there is also a unique generic point of

$V \cap C$. We claim that they are the same point. To see this, use the affine communication lemma to find a distinguished open in $U \cap V$ that is distinguished both in U and V. Let's call this W, we can also choose this W such that W intersects with C, this because that $U \cap V$ intersects with C (because C is irreducible). Then we know that there is also a generic point of $W \cap C$, then notice that in the affine case, if $U \cap C$ correspond to p in $U = SpecA$, then suppose that W is $D(f)$, then notice that $W \cap C$ as a set is just $D(f) \cap V(p)$, and p is still the generic point of $W \cap C$. This argument applies to V, thus we showed that no matter what affine open you use, the generic point of the intersection is always the same, we will call this unique point p.

Finally, we claim that the closure of p is C. To show this, we only have to show that any open in C contains p. But notice that any open in C is a $C \cap O$ for some open subset O in X, therefore we can choose an affine open subset of O such that it also intersects with C (it's possible to do this because that affine open is a basis of X), then you can show that p is in $O \cap C$, we are done. You can also see that this generic point has to be unique, otherwise, you choose an affine open, take the intersection, then you see now that you have two different generic point of the intersection, which is a contradiction, we are done.

Finally, to wrap up, we define a map from X to the irreducible closed subsets of X by mapping p to it's closure, then the above argument shows that the map that maps irreducible closed subsets to X by finding the unique generic point. Then you can find the composition is identity for both directions, we are done. ☐

Remark 1.72. The idea of the above proof is that we want to first find the candidate of the generic point if it exists. And we already know that if p is the generic point, then it's also going to be the generic point of the intersection with an open. Then we have to prove that the way we choose the candidate is well defined. Then we have to show that this candidate really works. Finally we have to show that the candidate is unique.

Then since the closure of a point is irreducible, then we can define two maps then show that they are irreducible, which is our assertion that you have a bijection between point and irreducible closed subsets by taking the closure.

Then we begin to use this result to understand the irreducible closed subsets of $ProjS$, whihc means that we want to understand what is

Proposition 1.73. Suppose that I is any homogeneous ideal of S not containing S_+(we don't actually have to assume this for the statement of the proposition, but since we are working with projs, we can exclude those we are not caring about), then the radical of I is also homogeneous. In particular, for any homogeneous element f, $f \in Rad(I)$ iff $f^n \in I$ for some n.

Proof. This is equivalent of showing that the radical of I is the intersection of all homogeneous primes containing p. To show this, notice that we already proved that any minimal prime over a homogeneous ideal is also homogeneous, this will suffice.

To see why this is equivalent, notice that $f^n \in I$ is equivalent to $f \in Rad(I)$, thus f is in all the homogeneous primes containing I. The converse is easy. ☐

Let $Z \subset ProjS$, then we can define $I(Z)$ to be the intersection of all the homogeneous primes in Z, which doesn't contain S_+ and is still prime, thus still in $ProjS$, if you work with those geometric objects over a filed, you only care about those contained in S_+, thus some people will define this to be the intersection of all the primes in Z and then intersect with S_+, which is still a homogeneous ideal, contained in S_+ but not necessarily prime anymore. These two definitions won't differ that much in terms of practical uses and proofs we give, but we will adopt the first one:

Proposition 1.74. For any two subsets Z_1, Z_2 of $ProjS$, we have $I(Z_1 \cup Z_2) = I(Z_1) \cap I(Z_2)$.

Proof. This is almost by definition, you should be able to see that this is just the distributive law of intersections. ☐

Finally, we can identify all the irreducible closed subsets in $ProjS$:

Proposition 1.75. For any subset $Z \subset ProjS$, the closure of Z is $V(I(Z))$. In particular, the closure of a point $p \in ProjS$ is just $V(p)$.

Proof. First, you see that $V(I(Z))$ is a closed subset that contains Z just by definition, thus it contains the closure of Z. Then to prove the converse inclusion, we want to show that any closed subset that contains Z contains $V(I(Z))$, suppose that $V(J)$ contains Z, then we know that any point in Z, which is a homogeneous prime, contains J, thus J is contained in $I(Z)$, then you know that any prime that contains $I(Z)$ must contain J, then $V(I(Z)) \subset V(J)$, we are done. $\square$

We continue our result with dimension and intersection in $\mathbb{P}^n$:

Proposition 1.76. Suppose we still assume that X is of dimension r and k is an infinite field, then we can find r hyperplanes meeting X at only finitely many points.

Proof. Notice that if $r = 0$, then we know that it's a zero dimensional closed subset of $\mathbb{P}^n$, which is a finite number of points. There is nothing to prove.

Now, suppose that r is at least 1, we prove in a previous result that for any nonempty hypersurface in $\mathbb{P}^n$, it will intersect with X, so in particular a nonempty hyperplane also intersects with X, then we can arrange this hyperplane also by a result we just proved that X avoids all the generic point of X, but still intersects with X at a nonempty subset. Then $X \cap H$ is nonempty and it at most has dimension $r-1$. Notice that if $X \cap H$ is 0 dimensional, then it's already a finite number of points, then we can take other $r-1$ hyperplane to be the same as H_1. Otherwise, we can iterate this process, we are done. $\square$

After our discussion of intersectio and dimension in projective spaces, we can take a look at what we can say about intersection and dimension in affine k-space $\mathbb{A}_k^n$:

Proposition 1.77. Let $k = \overline{k}$ be an algebraically closed field. Suppose that X, Y are irreducible closed subvarieties of codimension m, n respectively in $\mathbb{A}_n^d$. Then every component of $X \cap Y$, the set-theoretic intersection, has codimension at most $m + n$ in $\mathbb{A}_k^d$.

Proof. Let's first consider the diagonal of $\mathbb{A}_k^d \times_k \mathbb{A}_k^d$, which is $\mathbb{A}_k^d \hookrightarrow \mathbb{A}_k^d \times_k \mathbb{A}_k^d$. We claim that this is a regular embedding of codimension d (recall we reviewed regular embedding of codimension r in the section of closed embeddings). To show this, notice that the morphism $\mathbb{A}_k^d \hookrightarrow \mathbb{A}_k^d \times_k \mathbb{A}_k^d$ is induced by:

$$k[x_1, \cdots, x_d, y_1, \cdots, y_d] \to k[z_1, \cdots, z_d]$$

where x_i, y_i are both mapped to z_i. Then we know that this can also be written as:

$$k[x_1, \cdots, x_d, y_1, \cdots, y_d] \to k[x_1, \cdots, x_d, y_1, \cdots, y_d]/(x_i - y_i)$$

for all $1 \leqslant i \leqslant d$. Then notice that in the local rings, the ideal of the diagonal is still generated by $(x_i - y_i)$, then we only have to prove that this forms a regular sequence. This is easy to see since we know that $x_1 - y_1$ is not a zero divisor in $k[x_1, \cdots, x_d, y_1, \cdots, y_d]_p$, then if you quotient it out, notice that we have $k[x_1, \cdots, x_d, y_1, \cdots, y_d]_p/(x_1 - y_1)_p \cong (k[x_1, \cdots, x_d, y_1, \cdots, y_d]/(x_1 - y_1))_p$, then if $x_2 - y_2$ is not a zero divisor in $k[x_1, \cdots, x_d, y_1, \cdots, y_d]/(x_1 - y_1)$, then it won't be a zero divisor in the localization. But this is easy to show. Then the later steps are also easy to show, we omit it here, and in conclusion the diagonal is indeed a regular embedding of codimension r.

Recall that for any locally closed embeddings $U \hookrightarrow X$ and $V \hookrightarrow X$ over A, the intersection in X can be identified as the intersection of Δ and $U \times_A V$ in $X \times_A X$. We only proved this before for open embeddings, but this holds more generally for locally closed embedding, which is the consequence of the base change diagram. Then if you consider the diagram:

$$\begin{tikzcd}
X \cap Y \arrow[r] \arrow[d] & \mathbb{A}_k^d \arrow[d] \\
X \times_k Y \arrow[r] & \mathbb{A}_k^d \times \mathbb{A}_k^d
\end{tikzcd}$$

you see that via this we can view $X \cap Y$ as a closed subscheme of $X \times_k Y$, and notice that since X, Y are irreducible of codimension m, n in $\mathbb{A}_k^d$, we know that the dimension of X and Y are $d - m$ and $d - n$ respectively, then we have already showed that $X \times_k Y$ has dimension $2d - m - n$, and it's also pure dimensional. Then if you can show that component have dimension at least $d-m-n$, then we are in good shape since you can use Prop 1.48.

Then by using the fact that we can identify X, Y by $Speck[x_1, \cdots, x_d]/p$ and $Speck[y_1, \cdots, y_d]/q$ respectively, we know that $X \cap Y$ is isomorphic to $Speck[x_1, \cdots, x_d, y_1, \cdots, y_d]/(p, q, x_i - y_i)$, thus we only have to prove that $Speck[x_1, \cdots, x_d, y_1, \cdots, y_d]/(p, q, x_i - y_i)$ have dimension at least $d-m-n$. We do this step by step, notice that if we only consider $x_d - y_d$, then each component of $Speck[x_1, \cdots, x_d, y_1, \cdots, y_d]/(p, q, x_i - y_i)$ has codimension at most 1 at $Speck[x_1, \cdots, x_d, y_1, \cdots, y_d]/(p, q, x_i - y_i)$ where $i < d$, then you take out the component of $Speck[x_1, \cdots, x_d, y_1, \cdots, y_d]/(p, q, x_i - y_i)$ where $i < d$ it has codimension at most 1 in $Speck[x_1, \cdots, x_d, y_1, \cdots, y_d]/(p, q, x_i - y_i)$ where $i < d-1$, then you keep going then use the fact that when we work with irreducible schemes locally of finite type over k, we have Thm 1.48, then we are done.

Or you can use the generalized Krull's Theorem we are about to prove later. □

Remark 1.78. First, let's see where we used the hypothesis of k is algebraically closed, first, we want $X \times_k Y$ is still a variety, but this will require k to be perfect, which may not be true, but we always know that if k is algebraically closed, then it's automatically perfect. Also, notice that in order to use the results of Thm 1.48, we at least need $X \times_k Y$ to be pure dimensional, which we are not sure, but if k is algebraically closed, since X, Y are irreducible, the product is irreducible.

Moreover, the requirements in the previous result can be relaxed a little bit if we recall some of the results we proved before. First, what if X, Y are assumed to be pure dimensional but not necessarily irreducible. There are two places in the argument above that need to be altered. The first place is that we still want $X \times_k Y$ to be pure dimensional in order to use Thm 1.48.

This is true since if we assume that X is a pure dimensional variety over k with components $X_1, \cdots, X_r$ and Y is pure dimensional with components $Y_1, \cdots, Y_n$, then you see that $X_i \times_k Y_j$ will cover $X \times_k Y$, and notice that each of them is irreducible closed and it of the same dimension where we use the result before that product of irreducible variety will produce something with dimension the sum of the two you started with. Then notice that if X is a topological space which is a union of irreducible closed subsets with the same dimension n, then each component of X has dimension n. This is suppose that X' is a component of X, then notice that $X' = \cup(X' \cap X_i)$, then since it's closed, there is one X_i such that $X' = X_i \cap X'$, thus $X_i = X'$, therefore we are done. Another point is that we still want is the dimension of each component of $X \times_k Y$ is $dimX + dimY$, but this is just the argument above.

What if k is not algebraically closed. Recall that we proved before that for affine k-varieties, if K/k is an algebraic extension, then if X_K is pure dimension of dimension n iff X is also pure dimension of dimension n. Then we can consider the algebraic extension $k \to \overline{k} = K$ and first pass to X_K, Y_K which is pure dimensional. We also have $\mathbb{A}^d_k \times_k K \cong \mathbb{A}^d_K$, and we have natural embeddings $X_K \hookrightarrow \mathbb{A}^d_K$:

$$\begin{tikzcd} X_K \arrow[r, hook] \arrow[d] & \mathbb{A}^d_K \arrow[r] \arrow[d] & SpecK \arrow[d] \\ X \arrow[r, hook] & \mathbb{A}^d_k \arrow[r] & Speck \end{tikzcd}$$

However, when k is not perfect, it can happen that X_K is not reduced anymore, then we take it's reduction, X_K^{red}, which is a closed subvariety of $\mathbb{A}^n_K$ and it's also pure dimensional since dimension is just topological data and take the reduction doesn't affect it. Then via the same property as above, we can prove that $X_K^{red} \cap Y_K^{red}$ has dimension at least $d-m-n$. Now we claim that $X_K \cap Y_K \cong (X \cap Y)_K$. This is because if we assume $X = Speck[x_1, \cdots, x_d]/I$ and $Y = Speck[x_1, \cdots, x_d]/J$, then we have:

$$X_K \cap Y_K \cong Spec(K[x_1, \cdots, x_d]/(I) \otimes)_{K[x_1, \cdots, x_d]} K[x_1, \cdots, x_d]/(J)) \cong Spec(K[x_1, \cdots, x_d]/(I, J))$$

And on the other hand, we have $(X \cap Y)_K = Spec(k[x_1, \cdots, x_n]/I + J) \otimes_k K \cong SpecK[x_1, \cdots, x_n]/(I, J)$. Then notice that we know that $X_K \cap Y_K$ as a topological space is the same as $(X_K \cap Y_K)^{red}$, then if we can prove that we have $(X_K \cap Y_K)^{red} \cong X_K^{red} \cap Y_K^{red}$ we are done. Notice that $X_K^{red} = Spec(K[x_1, \cdots, x_d]/\sqrt{(I)})$ and similarly we have $Y_K^{red} = Spec(K[x_1, \cdots, x_d]/\sqrt{(J)})$, therefore we know that the intersection of the reduction is defined by $\sqrt{(I)} + \sqrt{(J)}$. On the other hand, the reduction of the intersection is defined by the ideal $\sqrt{(I, J)}$. Notice that the summation of two radical ideals in general may not be radical, thus it may be too much to ask for scheme theoretical isomorphisms between $X_K^{red} \cap Y_K^{red}$ and $(X_K \cap Y_K)^{red}$, but at least we can try topologically, which means that we can try to prove:

$$\sqrt{\sqrt{(I)} + \sqrt{(J)}} = \sqrt{(I) + (J)}$$

Clearly since $(I) + (J)$ is contained in $\sqrt{(I)} + \sqrt{(J)}$ the right hand side is contained in the left hand side. To show the reverse containment, we know $\sqrt{(I)} \subset \sqrt{(I) + (J)}$ and similarly $\sqrt{(J)} \subset \sqrt{(I) + (J)}$, thus we know that $\sqrt{(I)} + \sqrt{(J)} \subset \sqrt{(I) + (J)}$, then taking the radical, we get what we want.

You should not be surprising that this result about affine varieties implies results in the projective space:

Proposition 1.79. Suppose that X and Y are pure dimensional closed subvarieties of $\mathbb{P}_k^n$ of codimension d and e respectively and $d + e \leqslant n$, then X and Y intersect in $\mathbb{P}_k^n$.

Proof. Suppose that X is $Projk[x_0, \cdots, x_n]/I$ for a homogeneous radical ideal I, then we know that it's irreducible components correspond to minimal primes over I, and they are of the same dimension, then since taking the affine cone of irreducible projective variety increase the dimension by 1, then we know that the affine cone of X is still pure dimensional with dimension one larger than X, so is the affine cone of Y. Notice that we want to prove X and Y intersect, which can be reduced to the intersection of the affine cone of X and Y contains more than the origin, notice that the codimension of the affine cone of X and Y are now still d, e, then we know that the codimension of $X \cap Y$ is at most $d + e \leqslant n$. Recall this is comes from that for each irreducible components of $X \cap Y$, the dimension of that irreducible component is at least $n + 1 - e - d \geqslant 1$, thus it contains something more than the origin, we are done. □

Proposition 1.80. Suppose that A is a Noetherian ring and $f \in A$ which doesn't vanish at any codimension 0 or 1 primes, then f is invertible.

Proof. Notice that we proved before that if f doesn't vanish anywhere then f is invertible. We assumed that f is not in any codimension 0 or 1 primes. However, notice that $V(f)$, by Krull's Theorem, any irreducible component of $V(f)$ is of codimension 0 or 1, or in other words, the codimension of minimal primes of $V(f)$, if there is any, should be codimension 0 or 1, but we assumed that f is not in any codimension 0 or 1 primes, thus it's not contained in any primes, thus f is invertible. □

Before we proceed to the discussion of the inductive use of Krull's theorem, we make a small detour to a useful characterization of UFD, which is still a result of Krull's Theorem 1.59:

Proposition 1.81. Suppose that A is a Noetherian domain then it's a UFD iff all of it's codimension 1 primes are principal.

Proof. Recall that we proved that if A is a UFD then in general, it's codimension 1 primes are principal.

Therefore, we assume that conversely, all codimension 1 ideals of A is principal, we want to show that A is a UFD. Notice that each codimension 1 prime, being principal, is generated by a prime element, thus also irreducible since any prime element in an integral domain is irreducible by the usual argument. Now, suppose that a is irreducible, then the minimal primes over (a), by Krull's Theorem is all codimension 0 or 1, but in an integral domain, a is not a zerodivisor, thus all the minimal primes of (a) are codimension 1, which are generated by a prime element p, thus we can assume that $a = kp$, since p is not a unit, we know that k is a unit since a is irreducible, thus $(a) = (p)$, thus a is a prime element.

Therefore, this discussion above shows that irreducible elements are prime and prime elements are irreducible. Then, this suffice for a integral domain to be a UFD by general arguments. □

Now we turn to see how to use Krull's Theorem inductively, which starts with the following useful lemma:

Lemma 1.82. Suppose that you have a chain of primes $p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_n$ in a Noetherian ring A, and an $f \in p_n - p_0$. Then there exist a strict chain of primes:

$$p_0 = q_0 \subsetneq q_1 \subsetneq \cdots \subsetneq q_{n-1} \subsetneq q_n = p_n$$

with $f \in q_1 - q_0$.

Remark 1.83. Before we prove it, you should have a picture of what's going on in this statement.

Geometrically speaking we have a chain of irreducible closed subsets and a hypersurface such that the smallest one in the chain is contained in the hypersurface but the largest one is not contained in the hypersurface. What the statement says is that you can create a new chain of irreducible closed subset with the same length and start, end point such that this time only the largest one in the chain is not contained in the hypersurface while everything else is in the hypersurface.

Proof. Unsurprisingly, you do this kind of arguments by decreasing induction. Let's see what happens if we only have:

$$p_{j-2} \subsetneq p_{j-1} \subsetneq p_j$$

with $f \in p_j - p_{j-2}$, then we are seeking a prime q_{j-1} such that we have: $p_{j-2} \subsetneq q_{j-1} \subsetneq p_j$ with $f \in q_{j-1}$.

Notice that since $f \notin p_{j-2}$, then we can consider $(f, p_{j-2}) \supset p_{j-2}$, we take q_{j-2} to be a prime containing (f, p_{j-2}), such that it's also minimal among those contained in p_j. It's clear that q_{j-1} strictly contains p_{j-2}, but why q_{j-1} is strictly contained in p_j. Notice that since q_{j-1} by definition is a minimal prime over $V(f)$ in A/p_{j-2}, which is an integral domain, thus the codimension of q_{j-1} is 1, but notice that p_j is at least on codimension 2 in A/p_{j-2}, thus we know that q_{j-1} is strictly contained in p_j.

Then for the general case, notice that when $n = 0, 1$, there is nothing to prove. Then suppose that we have $f \in p_n - p_1$, we may assume that $f \notin p_{n-1}$, otherwise we can just take $q_{n-1} = p_{n-1}$ and keep going until we hit some p_j such that f is not in it. Then we can apply the above discussion and find q_{n-1}. Now we have a new chain that starts at p_0, but ends with p_{n-2}, q_{n-1}, q_n, such that $f \in q_{n-1}$ while $f \not inp_{n-2}$, then you keep do the discussion above, until you get what you want. □

Proposition 1.84. Suppose that (A, m) is a Noetherian local ring and $f \in m$, then $dimA/(f) \geqslant dimA - 1$

Proof. Notice that the primes in $A/(f)$ correspond to primes containing f, then suppose that you have a chain of primes in A and we can assume that the maximal element is just m and the minimal element in the chain is a minimal prime. Notice that if f is in the minimal prime, then the above discussion shows that we can find a chain of primes with length 1 less in $A/(f)$, thus $dimA/(f) \geqslant dimA - 1$, or if f is not in the minimal prime, then it's also possible that we can find $dimA/(f) \geqslant dimA$, but in general, if we don't assume more about f (for example f is not a zerodivisor), we only know that $dimA/(f) \geqslant dimA - 1$. □

We can finally state and prove the general version of Krull's Principal ideal theorem. As you will see below, it looks like an inductive use of Thm 1.59, but it's actually more subtle:

Theorem 1.85. Suppose that $X = SpecA$ where A is Noetherian and Z is an irreducible component of $V(f_1, \cdots, f_r)$, then $f_1, \cdots, f_r \in A$, then the codimension of Z is at most r.

Proof. Let's first observe that we can always assume that A is a Noetherian local ring. To see this, codimension of $V(f_1, \cdots, f_r)$ correspond to codimension of minimal primes over $(f_1, \cdots, f_r)$, bu notice that codimension of a prime doesn't change if we localize at the prime. Moreover, suppose that p is such a minimal prime over $(f_1, \cdots, f_r)$, pA_p is also going to be minimal over $(\frac{f_1}{1}, \cdots, \frac{f_r}{1})$, we are just trying to prove in this setting, the codimension of pA_p is less than r, thus we can assume p is maximal and it's minimal over $(f_1, \cdots, f_r)$.

Then suppose that we have any chain of primes that ends with m:

$$p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq m = p_k$$

We claim that $k \leqslant r$, to see this, notice that there has to be at least one elements in $f_1, \cdots, f_r$ such that it's not in p_0, since we assumed that m is minimal over $(f_1, \cdots, f_r)$, then we can create another chain of primes with strict containment, such that only p_0 doesn't contain f_i, by a relabeling we can assume that this is f_1. If we still write this chian as $p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq m = p_k$, we claim that there is at least one element in $f_2, \cdots, f_r$ such that it's not in p_1, then we can repeat our argument above.

Now if k is larger than r, then we will in the end get $p_r \subsetneq m$, where p_r contains every $f_1, \cdots, f_r$ while $m \neq p_r$, which is a contradiction to m is minimal over $(f_1, \cdots, f_r)$, thus k has to be less than or equal to r.

You should pay special attention to the case where k is actually less than r and be able to say why this case is OK. Simply speaking, if k is less than r, you will get $p_k = m$ contains certain k elements in $f_1, \cdots, f_r$, thus if we don't assume further properties of $f_1, \cdot, f_r$ and of the ring, nothing goes wrong. □

Remark 1.86. As in the statement of the Krull Principal ideal theorem 1.59, if f is not a zero divisor, then we proved that the codimension of irreducible components of $V(f)$ are exactly one.

Here for the Krull's Height Theorem 1.85, we would like something similar here. And this means that we need make sense of a sequence $(f_1, \cdots, f_r)$ being nonzero divisors, and you know the right notion here is regular sequence. We claim that if $(f_1, \cdots, f_r)$ is a regular sequence, then the codimension of $V(f_1, \cdots, f_r)$ is exactly r.

Let's see how this works. Notice that if we have a regular sequence $(f_1, \cdots, f_r)$ and m is minimal over it, we can still apply the techniques in the above proof, and the same argument still works with nothing wrong happens until you hit the maximal ideal. The issue here is that we know being a non zerodivisor is preserved under localization, thus if we assume that we start by a regular sequence, it is still going to be nonzero divisor after we localize, then we see that if k is less than r, your argument above gives m is minimal over $f_1, \cdots, f_k$, but since k is smaller than r, we know that there are other f_j's not in $f_1, \cdot, f_k$.

Then recall that we can assume that everything $f_1, \cdot, f_r$ are in the maximal ideal after the localization, then the proof in a local ring any permutation of a regular sequence is still regular can also be used to show that if $f_1, \cdots, f_r$ are all in the maximal ideal, then every permutation of the sequence still satisfies the non zero divisor condition. Therefore, we can assume that f_{k+1} is not a zerodivisor after we quotient out $(f_1, \cdots, f_r)$, but this means that f_{k+1} is not in $m = p_k$ since everything in the minimal prime is a zerodivisor.

Using Krull's Theorem, we can prove a few important results:

Proposition 1.87. Any Noetherian local rings will have finite dimension.

Proof. This is basically just using the definition since if (A, m) is Noetherian local, m is firstly finitely generated, thus it's codimension is finite. Also, notice that for a local ring, dimension is the codimension of the unique maximal ideal. □

Proposition 1.88. Again, let $d = dim A$ where (A, m) is Noetherian local, then there exists $g_1, \cdots, g_d \in A$ such that $V(g_1, \cdots, g_d)$ only contains m. Or in other words, m is minimal over $(g_1, \cdots, g_d)$.

Proof. We know that if m is minimal over $(g_1, \cdots, g_k)$ then k is at least d otherwise the Krull's Theorem implies that the dimension is lower than d.

Now, let's try to find these parameters, and you should not be surprised that we will find then inductively and we will find these d elements such that any prime over $g_1, \cdots, g_i$ will be codimension at least i, then by Krull's theorem the dimension is exactly i.

To begin with, let's notice that if dimension of A is just 0, then any elements in m will suffice, and there is nothing to prove. Let's assume that $d \geqslant 1$ at least. Then notice that we can assume that m is not contained in the union of all minimal primes, otherwise the prime avoindance lemma (the contrapositive of what we proved in Lemma 1.65) will imply that m is actually a minimal prime, a contradiction. Then let's choose g_1 in m but not in the union of all the minimal primes. Notice that now suppose that $(g_1) \subset p$, then p is not the minimal primes, thus it at least have codimension 1.

To continue, suppose that $g_1, \cdots, g_i$ has already been chosen, we want to have g_{i+1}. Notice that we can consider $A/(g_1, \cdots, g_i)$, and consider the minimal primes over it, we know that they are of codimension exactly i, which is less than d, then we know that the union of them is not m again because of Prime Avoidance Lemma 1.65. Then we can choose g_{i+1} such that it's not in the union of these minimal primes in $A/(g_1, \cdots, g_i)$. We claim that now any prime over $(g_1, \cdots, g_i, g_{i+1})$ is at least of codimension $i + 1$ since if $p \supset (g_1, \cdots, g_i, g_{i+1})$, then you see that $p/(g_1, \cdots, g_i)$ is not the minimal prime in $A/(g_1, \cdots, g_i)$, then notice that any minimal prime over $(g_1, \cdots, g_i)$ is of codimension at least i, then p contains some of those minimal primes, thus codimension is at least $1 + i$, we are done. □

Based on the above result, we have the following very important definition and it's also going to be useful in later treatments of dimension of fibers:

Definition 1.89. The geometric interpretation of the above result is that if you work with locally Noetherian spaces, then you will see that the stalk of any point of the structure sheaf is a Noetherian local ring, then notice that the maximal ideal can be set-theoretically cut out by exactly d local equations around p.

Formally, we will say that if (A, m) is a Noetherian local ring with dimension d, then elements in A, $f_1, \cdots, f_d$, are called a **system of parameters** for (A, m) if $\sqrt{(f_1, \cdots, f_d)} = m$ or in other words m is minimal over $(f_1, \cdots, f_d)$ or in other words set-theoretically m is cur out by d equations. If X is a locally Noetherian scheme, elements $f_1, \cdots, f_d$ in $\mathscr{O}_{X,p}$ such that m_p is minimal over $f_1, \cdots, f_d$ are called a **system of parameters at** p.

The next result is going to be essential when we discuss smoothness next chapter:

Theorem 1.90. For any Noetherian local ring (A, m, k), we have $dimA \leqslant dim_k(m/m^2)$.

Proof. Say m is the maximal ideal, which is finitely generated, then m/m^2 will also be finitely generated over k, say with dimension n, then choose a basis $x_1, \cdots, x_n$, by Nakayama's lemma it lifts to a minimal generating set of m, thus by Krull's theorem, $dimA = codim(m) \leqslant n$. ☐

1.1 Dimensions of Fibers of Morphisms of Varieties

In this section, we prove that the dimension of fibers of morphisms of varieties behave in a way that you expect from general geometric intuition. The reason why we have waited to this point to discuss this is because we will use the theorem that tells you that if you want to compare the difference of the dimension of a pure dimensional locally finite type k scheme with an irreducible closed subset, then the difference is the dimension of the irreducible closed subset.

We begin our discussion with a key property of fiber dimensions that holds very generally, which tells you that fiber dimension can never be lower than you expected:

Proposition 1.91. Suppose that $\pi : X \to Y$ is a morphism of locally Noetherian schemes, let $p \in X, q \in Y$ be points such that $q = \pi(p)$, then:

$$codim_X p \leqslant codim_Y q + codim_{\pi^{-1}(q)} p$$

Proof. The codimension of the point in the total space is bounded by the codimesion of the image in the source plus the codimension of the point in the fiber.

The proof uses the notion of system of generators which is the reason why we assumed that we are working with locally Noetherain schemes. Recall that the codimension of the point is just the dimension of the local ring, thus we are trying to prove:

$$dim\mathcal{O}_{X,p} \leqslant dim\mathcal{O}_{Y,q} + codim_{\pi^{-1}(q)} p$$

Moreover, this is a local statement since the codimension is just a local definition, in the sense that suppose that we have an open subset U of p, then $codim_U p = codim_X p$, similar statement works for q and we have the following diagram:

$$\begin{tikzcd} X \arrow[d, "\pi"'] & \pi^{-1}V \arrow[l, hook'] \arrow[d, "\pi"] & \pi^{-1}q \arrow[l] \arrow[d] \\ Y & V \arrow[l, hook'] & Speck(q) \arrow[l] \end{tikzcd}$$

which tells you that taking fiber is also local. Then we can just we are dealing with $\pi : X = SpecA \to SpecB = Y$ induced by $B \to A$ and the inverse image of the prime $p \subset A$ is $q \subset B$. Then, you will find that the induced morphisms on the level of stalks are:

$$\begin{tikzcd} A_p \otimes_{B_q} B_q/qB_q = A_p/(qB_q) & A_p \arrow[l] \\ B_q/qB_q \arrow[u] & B_q \arrow[l] \arrow[u] \end{tikzcd}$$

Then we use the knowledge of system of parameters to conclude that if qB_q has a system of parameter like $f_1, \cdots, f_r$ and $pA_p/(qB_q)$ has a system of parameters like $g_1, \cdots, g_s$, this implies that $codim_{\pi^{-1}(q)} p = s$ and $codim_Y q = r$. But notice that if we lift $g_1, \cdots, g_s$ back to A_p, we claim that $f_1, \cdots, f_r, g_1, \cdots, g_r$ cut out

pA_p in the sense that $V(f_1, \cdots, f_r, g_1, \cdots, g_r) = \{pA_p\}$ (of course, here we are using f_i to represent the image of f_i). To see this claim, notice that if any prime p' contains $f_1, \cdots, f_r, g_1, \cdots, g_s$, we know that the image of p' in $A_p/(qB_q)$ contains $g_1, \cdots, g_s$, thus the image of p' is just the image of p by the definition of system of generators. This implies that:

$$p'/(qB_q) = p/(qB_q)$$

thus we know that for any $x \in p$, there is a $y \in p'$ such that $x - y \in (qB_q)$, on the otherhand, notice that the inverse image of p' contains $f_1, \cdots, f_r$, thus is just qB_q, which implies that $(qB_q) \subset p'$, thus we know that $x \in p'$, thus $p \subset p'$, which proves the assertion.

Then by the general version of Krull's Theorem, we obtain the dimension of A_p is at most $r + s$, thus the codimension of p is at most $r + s$, we are done. □

Remark 1.92. There are a few special cases related to this diagram that is worth mentioning.

First, recall that the codimension of a point in the space is always less than or equal to the dimension, thus we have the first special case:

Suppose that we are working with a morphism $f : X \to Y$ of locally Noetherian schemes, then for any $p \in X$, we have the following inequality:

$$codim_X p \leqslant dim Y + dim(f^{-1} fp)$$

In particular, suppose X is an irreducible k-variety and when p is a closed point, we have the inequality:

$$dim X \leqslant dim Y + dim(f^{-1} fp)$$

At a much later point, we will prove that the above inequality is an equality for sufficiently nice morphisms, for example, flat morphism. In practice, for example, when we work with morphism between varieties, we will always take p, q to be closed points and conclude from the above proposition some information of the dimension of the total space.

However, even without nice conditions, we can prove right now that the equality holds for **almost** all points in Y:

Theorem 1.93. Suppose that $\pi : X \to Y$ is a finite type dominant morphism of integral schemes such that the induced field extension $K(Y) \hookrightarrow K(X)$ is finitely generated with transcendence degree r, then there is a nonempty open $U \subset Y$ such that for all $q \in U$, the fiber over q is nonempty and has pure dimension r.

Before we prove it, one corollary is:

Corollary 1.94. Suppose that $\pi : X \to Y$ is a morphism of irreducible k-varieties with $dim X = m$ and $dim Y = n$, then there exists a nonempty hence dense open U in Y such that for all $q \in U$, the fiber over q has pure dimension $m - n$ or empty.

Proof. Recall that any morphism between two k-varieties are finite type, also if we assume the variety of the source and target are irreducible, then if the morphism $f : X \to Y$ is not dominant, then the topological image is contained in a proper closed subset of Y, thus you can pick $U = Y - \overline{f(X)}$, such that for any $q \in U$, the inverse image is empty.

Now if f is dominant, you can apply the above theorem, notice that because f maps the generic point of X to the generic point of Y we have the following diagram:

$$\begin{tikzcd} K(Y) \arrow[rr] & & K(X) \\ & k \arrow[ul, "trdeg=n"] \arrow[ur, "trdeg=m"'] & \end{tikzcd}$$

here we know the transcendence degree since the transcendence degree is equal to the dimension. Then the transcendence degree of $K(X)/K(Y)$ is $n - m$ since the degree is additive, thus the theorem will suffice. □

Now we begin to prove the theorem:

Proof. First, we claim we can reduce the theorem to the case where X, Y are affine. To see why, notice that if you pick an affine open $SpecB$ of the generic point of Y and consider the affine open $SpecA$ in the inverse image that contains the generic point of X, this operation doesn't affect the function fields, thus the assumption of the theorem still holds. Then if you can prove that there is a nonempty open $U \subset SpecB$ such that for each q in U, the inverse image is of pure dimension q, U is still open in Y and the inverse image of q under $SpecA \to SpecB$ is just the intersection of $f^{-1}q \cap f^{-1}SpecB$, then if $f^{-1}q \cap f^{-1}SpecB$ is of pure dimension r, we claim that $f^{-1}q$ is of pure dimension n. This because that each irreducible component of $f^{-1}q \cap SpecB$ is just the intersection of that component with $SpecB$, and we know that intersecting with an open subset doesn't change the dimension, thus we are done.

In order to motivate the proof, we describe our goal. We will produce a nonempty distinguished open subset U of $SpecB$ so that $\pi^{-1}U \to U$ factors through $\mathbb{A}^r_U$ via a finite surjective morphism such that the following diagram commutes:

$$\begin{tikzcd}
X = SpecA \arrow[d] & \pi^{-1}U \arrow[l, hook', "open"'] \arrow[d, "finite{,}surjective"] \\
\mathbb{A}^r_B = SpecB[t_1, \cdots, t_r] \arrow[d] & \mathbb{A}^r_U \arrow[l, hook'] \arrow[d] \\
Y = SpecB & U \arrow[l, hook', "open"']
\end{tikzcd}$$

where the morphism from $SpecA \to SpecB[t_1, \cdots, t_r]$ is induced by a injective ring morphism $B[t_1, \cdots, t_r] \to A$ and the composition $X \to \mathbb{A}^r_B \to SpecB$ equals to $\pi : X \to Y$.

we claim that this suffices to prove the theorem.

The reason behind this is that finite surjection is preserved by base change, thus you can consider the following diagram:

$$\begin{tikzcd}
Spec\kappa \arrow[d] & \mathbb{A}^r_\kappa \arrow[l] \arrow[d] & \pi^{-1}p \arrow[l] \arrow[d] \\
U & \mathbb{A}^r_U \arrow[l] & \pi^{-1}U \arrow[l]
\end{tikzcd}$$

which consists of two Cartesian diagrams and the residue field of p is κ, I will let you figure out why this diagram makes sense, but it implies that $\pi^{-1} \to \mathbb{A}^r_\kappa$ is still finite surjective, then we claim that every irreducible component of $\pi^{-1}p$ is of dimension at most r.

Indeed, suppose that you have an irreducible variety that maps finitely into $\mathbb{A}^r_\kappa$ has dimension at most k. To prove this, you first notice that we only mentioned that a variety and didn't specify the field it lies over, thus you should find that the base field is not that important. Suppose that $X \to \mathbb{A}^r_\kappa$ is finite, then by definition, X should be $SpecC$ where C is a finite $\kappa[x_1, \cdots, x_r]$ algebra. And the morphism is induced by $\kappa[x_1, \cdots, x_r] \to C$, consider the kernel of the morphism, say it's I, then we have an injective ring morphism: $\kappa[x_1, \cdots, x_r]/I \hookrightarrow C$, which corresponds to $X \to V(I) \hookrightarrow \mathbb{A}^n_\kappa$, notice that $X \to V(I)$ correspond to a finite extension, thus integral, thus preserves dimension, thus since $V(I)$ is at most dimension r, we are done.

This proves that every irreducible component of the fiber is of dimension at most r. We still need to show the reverse inequality, to do so, we consider the finite surjective morphism $SpecR = \pi^{-1}p \to \mathbb{A}^r_\kappa$, recall that since this is surjective, there is a prime in $SpecR$, say n, such that under the ring morphism:

$$\kappa[t_1, \cdots, t_r] \to R$$

the inverse image of p is 0, then if we consider $\kappa[t_1, \cdots, t_r] \to R/n$, you find that (0) pulls back to (0), thus this is an injective ring morphism, from a integrally closed domain (since it's a UFD) to a domain, thus the hypothesis of the going down theorem holds, thus we know that $r \leqslant dim(R/n) \leqslant dim(R) = dim(\pi^{-1}p)$, thus we are done.

Now the only thing left is to show that we can build such factorization. First, notice that we have the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[rr, hook, bend left] & A \otimes_B K(B) \arrow[r] & K(A) \\
B \arrow[u, hook] \arrow[r] & K(B) \arrow[u] \arrow[ur, hook] &
\end{tikzcd}$$

you should be able to find the morphisms in the diagram, then since $A \hookrightarrow K(A)$ and $K(B) \hookrightarrow K(A)$ are injective, we know that $A \to A \otimes_B K(B)$ and $K(B) \hookrightarrow A \otimes_B K(B)$ are injective.

Then notice that $A \otimes_B K(B)$ is a finitely generated $K(B)$-algebra since A is finitely generated over B, moreover there is a canonical identification of $A \otimes_B K(B)$ with the localization of A at those nonzero elements in B, then the fraction field of $A \otimes_B K(B)$ is just $K(A)$, thus we know from Thm 1.39 that there are $t_1, \cdots, t_r \in A \otimes_B K(B)$ such that they are algebraically independent, and we have a finite extension $A \otimes_B K(B)/K(B)[t_1, \cdots, t_r]$.

Now, we are thinking of $t_i \in A \otimes_B K(B)$ as fraction, with numerators in A and denominators in B, then by multiplying the product of all the denominators we can assume that $t_i \in A$ with t_i's are algebraically independent over $K(B)$ and $A \otimes_B K(B)$ is finite over $K(B)[t_1, \cdot, t_r]$, which is the following diagram:

$$\begin{tikzcd}
A \arrow[r, hook] & A \otimes_B K(B) \\
B[t_1, \cdots, t_r] \arrow[u, hook] \arrow[r, hook] & K(B)[t_1, \cdots, t_r] \arrow[u, hook] \\
B \arrow[u, hook] \arrow[r, hook] & K(B) \arrow[u, hook]
\end{tikzcd}$$

Recall that A is finitely generated over B, thus also over $B[t_1, \cdots, t_r]$, but we are not sure if A is finite over $B[t_1, \cdots, t_r]$. As you will see, this is the reason why we need to find an open subset of B, but not B.

Now, suppose that A is generated over B by $u_1, \cdots, u_q$, again the Noether Normalization 1.39 tells you that each u_i satisfies a monic equation $f_i(u_i) = 0$ with the coefficients of f_i in $K(B)[t_1, \cdots, t_r]$, however, we can still multiply each f_i with the product of some elements so that after we do the multiplication, all f_i's lie in $B[t_1, \cdots, t_r][t]$, but possibly not monic anymore. To avoid this, say that product is f, we can localize B at f and consider B_f, then f_i's can be thought as in $B_f[t_1, \cdots, t_r][t]$, thus we know that $u_1, \cdots, u_q$ are integral over $B_f[t_1, \cdots, t_r]$, thus we know that A_f is finite over $B_f[t_1, \cdots, t_r]$, which is summarized in the following diagram:

$$\begin{tikzcd}
A_f & A \arrow[l, hook'] \arrow[r, hook] & A \otimes_B K(B) \\
B_f[t_1, \cdots, t_r] \arrow[u, hook] & B[t_1, \cdots, t_r] \arrow[l, hook'] \arrow[u, hook] \arrow[r, hook] & K(B)[t_1, \cdots, t_r] \arrow[u, hook] \\
B_f \arrow[u, hook] & B \arrow[l, hook'] \arrow[u, hook] \arrow[r, hook] & K(B) \arrow[u, hook]
\end{tikzcd}$$

then we take $U = D(f)$ and $\pi^{-1}U = D(f) \subset A$, this proves our assertion. □

Below is a useful criterion of irreducibility:

Proposition 1.95. Suppose that $f : X \to Y$ is a proper morphism to an irreducible variety such that all the fibers are nonempty, irreducible and of the same dimension. Then X is irreducible.

Proof. By definition, a proper morphism is finite type, then X is finite type over a variety, thus a Noetherain scheme, thus it has finitely many irreducible components Z_i. From assumption, we know that f is surjective, then since Y is irreducible, we know that there is an irreducible component of X, say Z_0, such that $f(Z_0) = Y$. (A proper morphism is closed, thus we don't have to take closure.)

Let X_y denote $\pi^{-1}y$, which is irreducible by assumption. But we know that $X_y = \bigcup Z_i \cap X_y$, which is a union of closed subsets of X_y, thus we know that X_y is contained in Z_i for some Z_i. It follows that if $x \in X$ is a point in Z_i but not in Z_j for $j \neq i$, then we know that $X_{f(x)}$ is contained in Z_i.

Now, we can endow Z_0 with the reduced closed subscheme structure and obtain $Z_0 \hookrightarrow X \to Y$, which is a dominant morphism of integral schemes, moreover, notice that $Z_0 \to Y$ is proper, and in particular, separated, thus Z_0 is also a k-variety, integral, then suppose that the dimension of Z_0 is m and the dimension of Y is n. Notice that the generic point η_0 of Z_0 is only in Z_0 but not in any other Z_0, and the image of which is the generic point of Y since $Z_i \to Y$ is dominant between integral schemes, then you can consider the following diagram:

$$\begin{tikzcd}
Z_0 \arrow[r, hook] & X \arrow[r, "f"] & Y \\
g^{-1}f\eta \arrow[u] \arrow[r, hook] & f^{-1}f\eta \arrow[u] \arrow[r] & Speck(f\eta) \arrow[u]
\end{tikzcd}$$

where g is the composition of f with $Z_0 \hookrightarrow X$. Here we know that $g^{-1}f\eta$ is topologically the same as $f^{-1}f\eta$ as we know that the inverse image of $f\eta$ under f should completely lies in Z_0.

Moreover, if you apply the corollary to $g : Z_0 \to Y$, you know that there is an nonempty open of Y such that for each point in that open, the fiber dimension is $dimY - dimZ_0$. Recall that the generic point is in that open, thus $dimY - dimZ_0$ is just d by assumption. However, recall that by our remark above, for any $y \in Y$, we pick a $x \in Z_0$ such that $g(x) = y$, then we have the following inequality:

$$dimg^{-1}y \geqslant dimZ_0 - dimY = d$$

However, we know that the dimension of $g^{-1}y$ is less than or equal to $f^{-1}y$, which is d, and $f^{-1}y$ is irreducible, thus we know that $g^{-1}y$ is $f^{-1}y$, thus for each $y \in Y$, we know the fiber of that point under the morphism f is in Z_0, thus $X = Z_0$, we are done.

Now, suppose that there is a $x \in X$ such that is's not in Z_0 but in some other Z_i, then we can consider the fiber of $f(x)$ in X, which by assumption

To proceed further, we need the following result, suppose that $Y \hookrightarrow X$ is a closed subscheme of X, $p \in Y$ is a point in Y, then we claim that the following diagram is a Cartesian diagram:

$$\begin{tikzcd}
Y \arrow[r, hook] & X \\
Speck(p) \arrow[u] \arrow[r, "id"'] & Speck(p) \arrow[u]
\end{tikzcd}$$

or in other words, the fiber of a closed embedding is the point itself. The key reason behind this is we have a commutative diagram for an affine open of p:

$$\begin{tikzcd}
Y \arrow[r, hook] & X \\
SpecA/I \arrow[u, hook, "open"] \arrow[r, hook] & SpecA \arrow[u, hook, "open"]
\end{tikzcd}$$

then you only need to recall how we defined the morphism $Speck(p) \to X$ or $Speck(p) \to Y$ to check this diagram above satisfies the universal property of fibered product. $\square$

References