Appendix 3 for Math 632: 2025 Winter Notes

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1 Review on Fibered Products

Now, let's begin the review of fibered product, this is an extremely important construction, not only because the way we construct it will be used over and over again in the later construction of normalization and the relative version of spec and proj. This construction itself is also of extreme importance. We begin straight out with the existence theorem showing that fibered product exists in the category of schemes:

Theorem 1.1. Suppose that $\alpha : X \to Z$ and $\beta : Y \to Z$ are morphisms of schemes, then the fibered product:

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\alpha'"] \arrow[d, "\beta'"'] & Y \arrow[d, "\beta"] \\ X \arrow[r, "\alpha"'] & Z \end{tikzcd}$$

exists in the category of schemes.

When A is a ring, we will always use $\times_A$ instead of $\times_{SpecA}$ to simplify the notation. Also, suppose that B is an A-algebra and X is an A-scheme, we will use $X \times_A B$ instead of $X \times_{SpecA} SpecB$.

Remark 1.2. Before we begin to prove the existence, we here give a few warnings and remarks.

First, we won't use the way we are familiar with to define the fibered product, which is we first specify the underlying set, then make it a topological space, finally give the structure sheaf. Here, you might think that we will take the product set and take the product topology, then figure out what is the structure sheaf. However, this is not the case, since the underlying set of the fibered product is not the product of the sets. You will later be able to verify that for an algebraically closed field k, $\mathbb{A}^1_k \times_k \mathbb{A}^1_k \cong \mathbb{A}^2_k$, which is not the product.

However, recall that if W is a scheme, we can define W-valued points, which are just $Hom(W, X)$, then by the definition of fibered product, we see that:

$$Hom(W, X \times_Z Y) = Hom(W, X) \times_{Hom(W,Z)} Hom(W, Y)$$

Therefore, the reason why the fiber product of topological spaces are products of the underlying sets is just a lucky consequence that we can view a topological space as p-valued points where p is just a point.

Finally, you will later be able to verify that the fibered product of noetherian schemes won't necessarily be noetherian.

Now be begin the proof of Thm 1.1:

Proof. We will first prove that the fibered product exists when X, Y and Z are affine. Then we already know that the fibered product also exists when $\beta : Y \to Z$ is an open embedding, this will be the basic blocks we will use to build fibered products, and you will see repeated later this idea.

To have a review, suppose that we have $SpecB \to SpecA$ and $SpecC \to SpecA$, we want to know what is $SpecB \times_A \times SpecC$. We claim that it it $Spec(B \otimes_A C)$, to see this is the case, notice that we know that maps into affine schemes are in bijection with the ring maps between global sections, then we can switch back and forth between the following diagrams:

$$\begin{tikzcd}
W \arrow[drr, bend left=10] \arrow[dr, bend right=10] & & \\
& Spec(B \otimes_A C) \arrow[r, "\alpha'"] \arrow[d, "\beta'"'] & SpecC \arrow[d, "\beta"] \\
& SpecB \arrow[r, "\alpha"'] & SpecA
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma(W, \mathscr{O}_W) & & \\
& B \otimes_A C & C \arrow[l] \arrow[ull] \\
& B \arrow[u] \arrow[uul] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

Then any morphism $W \to Spec(B \otimes_A C)$ such that the two triangles commutes will correspond to a ring map $B \otimes_A C \to \Gamma(W, \mathscr{O}_W)$ such that the triangles commute. Then we only have to show that the fibered product is the fibered coproduct in the category of rings, we then are done, which shoule be easy to show by viewing this diagram in the category of A-modules and A-algebras and rings back and forth.

Step 1 of the proof comes as: suppose that X and Z are affine and $\beta : Y \to Z$ factors as:

$$Y \hookrightarrow Y' \to Z$$

where the first morphism i is an open embedding and Y' is affine, then we claim that $X \times_Z Y$ exists. This is because the following diagram:

$$\begin{tikzcd}
W \arrow[r] \arrow[d, hook, "open"'] & Y \arrow[d, hook, "open"] \\
W' \arrow[r] \arrow[d] & Y' \arrow[d] \\
X \arrow[r] & Z
\end{tikzcd}$$

The below Cartesian square exists because we are over affine, the above Cartesian square exists since $Y \hookleftarrow Y'$ is an open embeddings, then the stack of Cartesian square is Cartesian.

Step 2 of the proof is by arguing that the fibered product of affine with arbitrary schemes over affine exists. This is the key of this proof, where the gluing part comes in and then you have to realize that thanks to the universal property, to check the gluing works, we don't have to check anything.

To warm up, let's first do a easy special case where $Y = Y_1 \cup Y_2$ and both Y_1 and Y_2 are affine. Let $Y_{12} = Y_1 \cap Y_2$. It's left to you why $X \times_Z Y_1$ and $X \times_Z Y_2$ both exists and we have canonical embeddings $X \times_Z (Y_1 \cap Y_2) \hookrightarrow Y_1$ and $X \times_Z (Y_1 \cap Y_2) \hookrightarrow Y_2$, therefore you can glue the schemes along these two embeddings. (Hint about the canonical open embedding from $X \times_Z (Y_1 \cap Y_2)$, you need to interpret what does the below diagram mean)

$$\begin{tikzcd}[column sep=small]
X \arrow[d] & W_2 \arrow[l] \arrow[dr] & Cartesian \arrow[l, hook'] \arrow[r, dashed, "?"] \arrow[drr] & W_{12} = X \times_Z (Y_1 \cap Y_2) & Cartesian \arrow[l, dashed, "?"'] \arrow[r, hook] \arrow[dl] & W_1 \arrow[r] \arrow[dl] & X \arrow[d] \\
Z & & Y_2 \arrow[ll] & Y_{12} \arrow[l, hook'] \arrow[r, hook] & Y_1 \arrow[rr] & & Z
\end{tikzcd}$$

We claim that the resulting stuff here is the fibered product of X and Y over Z. We check this by showing that the resulting scheme satisfies the universal property of fibered product: First, what is the canonical morphism from W to X, Y. You have $W_1 \to Y_1$ and $W_2 \to Y_2$ which all goes to Y, then you would expect that they glue, and you should check that they do glue and you should also check why the maps to X also glue.

(**Caution!**: You should really check this since this is not that obvious and do requires some insight behind the machinery that works behind! Hint: use the above morphism where you replace $1, 2$ by i, j.)

Now, suppose that you have the defining diagram of fibered product, you want to show that there is a unique map, first you notice that if you consider the the restriction of the inverse image of Y_1, Y_2 and Y_{12}, which by the universal properties of fibered product give you morphisms to W_1 and W_2 and W_{12}, where the restriction should agree again because the universal property of fibered product.

Then, we deal with the case where Y is covered with Y_i's where each is affine, we denote Y_{ij} as $Y_i \cap Y_j$, similarly Y_{ijk} as the intersection of three of them. Here comes the most confusing part of this proof. We make some comments to make it's easier to understand what is going one in the gluing process. You first need to figure out, say, what exactly is the gluing between W_i and W_j:

$$\begin{tikzcd}
 & W_{ij} \arrow[r] \arrow[dl, hook] \arrow[dr, hook] & Y_{ij} \arrow[dl, hook] \arrow[dr] & \\
W_i \arrow[r] & Y_i & W_j \arrow[r] & Y_j
\end{tikzcd}$$

in this diagram, the gluing is the image of W_{ij} in W_j and W_j, and the gluing isomorphism is by the inverse of the restriction. Now, in this interpretation, we see that the intersection of the gluing open of W_i W_j and W_k will be the intersection of the image of W_{ij} and W_{ik} inside W_i. This is really hard to write down what we need here, but I hope the following two diagrams can be helpful in remembering the gluing arguments:

$$\begin{tikzcd}
 & W_{ijk} \arrow[d] \arrow[ddl, hook] \arrow[ddr, hook] & \\
\cdots & Y_{ijk} \arrow[ddl, hook] \arrow[ddr, hook] & \cdots \\
W_{ij} \arrow[r, hook] \arrow[d] & W_i \arrow[d] & W_{ik} \arrow[l, hook'] \arrow[d] \\
Y_{ij} \arrow[r, hook] & Y_i & Y_{ik} \arrow[l, hook']
\end{tikzcd}$$

You can imagine that this diagram has three components in total and each one is showing how to glue in W_i or W_j or W_j, the diagram above is showing how to glue in W_i the other two components are missing in the $\cdots$.

This shows that it glues well to W, and the morphisms to X and Y also glue for the same reason as before since we don't have to check cocycle conditions here for morphisms to glue. Now we still have to show that this glued scheme W satisfies the universal property of fibered product, which is essentially the same as the case when we only have two affine opens in Y, since to show the morphisms glue, we only have to look at two different affines at a time, no cocycle conditions are involved.

Step 3, now when only Z is affine but X, Y are arbitrary, we claim that the fibered product still exists. Here we are basically using the arguments in step 2, with an affine open cover of X, instead of Y, but now we can use the fact that affine and arbitrary over affine exist. The reason we can copy the previous step is because that you can see that the gluing arguments in the previous step only requires one of the factors to be fixed and when the other factor ranges over an open cover, the fibered product of intersection of any two opens in that open cover still exist, we didn't use any other special properties of affine opens besides in the previous step, we only know the existence if both are affine.

Step 4, now, if Z can be embedded into Z' via $i : Z \hookrightarrow Z'$ where Z' is an affine scheme. We claim that $X \times_Z Y$ is the same as $X \times_{Z'} Y$. This follows from the fact that open embeddings are monomorphisms, you need to make sure you understand why monomorphism implies the result.

Finally suppose that Z is arbitrary. We still want to go back to the case of step 2 where one factor is fixed, and when we take intersections it still exists. To do this, cover Z by affine opes Z_i, suppose that $\alpha : X \to Z$ and $\beta : Y \to Z$, we let $X_i = \alpha^{-1}(Z_i)$ and $Y_i = \beta^{-1}(Z_i)$, we also denote Z_{ij} by $Z_i \cap Z_j$ and similarly X_{ij} and Y_{ij}. In this case, we see that $W_i = X_i \times_{Z_i} Y_i$ exists for all i by step 3, also $W_{ij} = X_{ij} \times_{Z_{ij}} Y_{ij}$ exists for all i, j by the previous step. Also, W_{ij} comes with a canonical open embedding into both W_i and W_j, you need to figure out why this is true, since it's not completely trivial. Now, we claim that W_i satisfies the universal property of $X \times_Z Y_i$, this is because we have the following diagram:

$$\begin{tikzcd}
V \arrow[r, dashed] \arrow[rr, bend left] \arrow[dr, dashed] \arrow[ddr, bend right] & W_i \arrow[r] \arrow[d] & Y_i \arrow[d] \\
& X_i \arrow[r] \arrow[d, hook] & Z_i \arrow[d, hook] \\
& X \arrow[r] & Z
\end{tikzcd}$$

you need to figure out why this diagram suffice, then we can identify W_i with $X \times_Z Y_i$ and similarly we have W_{ij} is identified with $X \times_Z Y_{ij}$.

Now, we are back to step 2, then exactly the same argument there follows. ☐

Remark 1.3. You may already notice that the key element in the above proposition is step 2, where we are under the case of one factor is fixed and the other can vary, also the existence of the fibered product of intersection of the varying factor is guaranteed. Then basically every step later is just a repetition of step 2, under mild modifications. This is not a coincidence, it can be understood better in terms of representable functors as explained below.

Remark 1.4. **** later*

Now that we know the existence of fibered product, before we do some concrete examples, we can extract many other useful information from the proof above, for example, if you want to make sure that a commutative square is Cartesian, you can do it locally, before we prove it, a useful result we will use is the following:

Proposition 1.5. Suppose that you have a Cartesian square of schemes:

$$\begin{tikzcd}
X' \arrow[r, "v"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "u"'] & Y
\end{tikzcd}$$

then for any open $V \subset Y'$ and open $W \subset X$, we have a Cartesian diagram:

$$\begin{tikzcd}
g^{-1}V \cap v^{-1}W \arrow[r, "v"] \arrow[d, "g"'] & W \arrow[d, "f"] \\
V \arrow[r, "u"'] & Y
\end{tikzcd}$$

Proof. Consider the following stack of Cartesian diagram:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & W \arrow[d, hook] \\
\cdots \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & X \arrow[d] \\
V \arrow[r, hook] & Y' \arrow[r] & Y
\end{tikzcd}$$

fill in the blanks in $\cdots$ and remember how open embeddings play with fibered products. ☐

Proposition 1.6. Suppose that you have the following commutative square of schemes:

$$\begin{tikzcd}
X' \arrow[r, "v"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "u"'] & Y
\end{tikzcd}$$

This is Cartesian iff for any open cover of Y, say $\{U_i\}$, for an open cover in the inverse image of U_i in both X and Y', we have the following Cartesian diagram:

$$\begin{tikzcd} g^{-1}V_i \cap v^{-1}W_j \arrow[r, "v"] \arrow[d, "g"'] & W_j \arrow[d, "f"] \\ V_i \arrow[r, "u"'] & U_k \end{tikzcd}$$

Proof. First, suppose that the original square we started with is Cartesian, if the right bottom U_i os replaced by Y in the diagram above, then this is the content of the previous proposition, but notice that $U_i \hookrightarrow Y$ is a monomorphism, then we are done.

The real content of the proof is the converse, now suppose the above diagrams are all Cartesian. We want to show that X' together with the morphisms to Y' and X satisfies the universal property of fibered product. Then suppose that you have $\alpha : T \to Y'$ and $\beta : T \to X$ such that they agree after composed with u, f. Then if you restrict them to the inverse image of the intersection of V_i, W_j, then this local Cartesian diagram implies that you only have a unique morphism into $g^{-1}V_i \cap v^{-1}W_j$, but as i, j, k varies, we know that $g^{-1}V_i \cap v^{-1}W_j$ covers X', thus we only have to show that these morphism agree on intersection. But notice that on the intersections, we still have a Cartesian diagram(what is it, you need to be aware, we omit it here since it makes the notation to messy). Then they must be the unique morphism, thus we are done. $\square$

Now, enough with this abstract construction, we begin to compute fibered product in a more practical way: We will see that there are **four** types of morphisms that are easy to deal with when you take fibered products, and essentailly, fibered products of any morphism can be built from these four atomic parts. The first one is the **open embeddings**, which we already discussed before. The most important thing we should keep in mind is that we can always compute the fibered product locally in the sense of the previous propositions we discussed. Many properties that are presvered by fibered products are checked this way as you will see. As one example, we can first show that affine morphisms are stable under taking fibered products:

Proposition 1.7. Affine morphisms are stable under taking fibered products.

Proof. Consider the following diagram:

$$\begin{tikzcd} Y \times_X Z \arrow[r] \arrow[d] & Z \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

where the bottom morphism is affine, we want to show that the top stuff is affine. By the affine locality of affine morphism on the target, we only have to find an affine open cover of Z such that the inverse image is affine. Choose an affine open cover of X, the inverse image in Z can also be covered by affines, since f is affine, we know that f^{-1} of the affines in X is affine, suppose that U is such an affine in X, then the previous propositions of computing fibered products locally shows that we have the following diagram:

$$\begin{tikzcd} f^{-1}U \times_X V \arrow[r, hook] \arrow[d, hook] & \cdots \arrow[r] \arrow[d, hook] & V \arrow[d, hook] \\ \cdots \arrow[r, hook] \arrow[d] & Y \times_X Z \arrow[r] \arrow[d] & Z \arrow[d] \\ f^{-1}U \arrow[r, hook] & Y \arrow[r, "f"] & X \end{tikzcd}$$

and we also have an canonical isomorphism where we know $f^{-1}U \times_X V \cong f^{-1}U \times_U V$, which is affine. But notice that the $\cdots$ on the top row is the inverse image of V, how should we relate it to $f^{-1}U \times_X V$?

The trick here is that we notice that as an inverse image of V, we know that $\cdots$'s image in $Y \times_X Z$ is mapped to V in Z, then mapped to U in X, thus we know that it's contained in the inverse image of the inverse image of U in Z in $Y \times_X Z$. (It's confusing to say this, so make sure you understand what this means).

But this implies that the image of $\cdot$'s in $Y \times_X Z$ is contained in the inverse image of $f^{-1}U$, thus we can take the intersection and get the inverse image of V, which is exactly what the top left little square is doing. $\square$

This is a typical argument we will see later again and again.

We now begin on the second type which correspond to adding a variable:

Proposition 1.8. There is a natural isomorphism $A \otimes_B B[t] \cong A[t]$, thus we have a Cartesian diagram:

$$\begin{array}{ccc} SpecA[t] & \longrightarrow & SpecB[t] \\ \downarrow & & \downarrow \\ SpecA & \longrightarrow & SpecB \end{array}$$

Proof. Notice that we have the structure ring homomorphism $B \to A$, thus we have a natural morphism from $A \otimes_B B[t]$ as an B-module where $a \otimes f$ is mapped to $a \cdot f$ where now we view the coefficients as in B. We claim that this is an isomorphism since we have an inverse where $A[t] \to A \otimes_B B[t]$ as a morphism of A-algebra, it suffice to determine where t goes, we map it to $1 \otimes t$, then it's left for you to check this is an isomorphism. This is an explicit construction of isomorphisms, you can also show that we have a square of rings, where $A[t]$ satisfies the condition of the fibered coproduct. To show this, you need to show that a B-algebra morphism from $A[t]$ is equivalent of giving a B-algebra morphism from $B[t]$ and from A, which should be easy. $\square$

Also, if you only add finitely many variables, the above argument have their natural generalizations and you have $A \otimes_B B[x_1, \cdots, x_n] \cong A[x_1, \cdots, x_n]$. Or you can iterate the above argument and use induction $A \otimes_B B[x_1, \cdots, x_n] \cong A \otimes_B B[x_1, \cdots, x_{n-1}] \otimes_{B[x_1, \cdots, x_{n-1}]} B[x_1, \cdots, x_n]$. But either way, now, we can define the affine space over any scheme:

Definition 1.9. If X is any scheme, we define $\mathbb{A}^n_X \times_{\mathbb{Z}} Spec\mathbb{Z}[x_1, \cdots, x_n]$. Clearly, we have if $SpecA = X$, by the above argument we know that $\mathbb{A}^n_{SpecA}$ is canonically isomorphism to $\mathbb{A}^n_A$.

Now we turn to the third type, which is the closed embeddings, we also begin fulfill our promise earlier that we can interpret intersections as fibered products. We will discuss in the generality of intersections of locally closed embeddings, as you will see later.

Proposition 1.10. Suppose that $\phi : B \to A$ is a ring morphism, then we have a natural isomorphism:

$$B/I \otimes_B A \cong A/A\phi(I)$$

Proof. Consider the following SES:

$$0 \to I \to B \to B/I \to 0$$

since $- \otimes_B A$ is always right exact, apply it to the above sequence we get:

$$I \otimes_B A \to B \otimes_B A \to B/I \otimes_B A \to 0$$

Then we know that $B/I \otimes_B A$ is isomorphic to $B \otimes_B A$ quotient the image of $I \otimes_B A$ in $B \otimes_B A$, notice we have natural identifications where the image in $B \otimes_B A$ and $B \otimes_B A$ can be identified as $A/A\phi(I)$, we are done. $\square$

As an immediate consequence, we can prove closed embeddings are stable under base change:

Proposition 1.11. Suppose that $Y \hookrightarrow X$ is a closed embedding and $Z \to X$ is any morphism, then the canonical morphism $Y \times_X Z \to Z$ is a closed embedding.

Proof. Consider the diagram:

$$\begin{tikzcd} Y \times_X Z \arrow[r] \arrow[d] & Z \arrow[d] \\ Y \arrow[r, hook] & X \end{tikzcd}$$

We want to show that the top morphism is closed embedding, we recycle the diagram when we prove affine morphism is stable under taking fiber product. You will see that we have the following diagram again:

$$\begin{tikzcd} f^{-1}U \times_X V \arrow[r, hook] \arrow[d, hook] & \cdots \arrow[r] \arrow[d, hook] & V \arrow[d, hook] \\ \cdots \arrow[r, hook] \arrow[d] & Y \times_X Z \arrow[r] \arrow[d] & Z \arrow[d, dashed] \\ f^{-1}U \arrow[r, hook] & Y \arrow[r, "f"] & X \end{tikzcd}$$

Then the morphism from the inverse image of V to the inverse image of V is just the composition of the top two morphisms. Moreover, since we know that the morphism from $f^{-1}U$ to X and from V to X all factor through U, we have:

$$\begin{tikzcd} f^{-1}U \times_X V \arrow[r] \arrow[d] & V \arrow[d] \\ f^{-1}U \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} f^{-1}U \times_X V \arrow[r] \arrow[d] & V \arrow[d] & \\ f^{-1}U \arrow[r] & U \arrow[r, hook] & X \end{tikzcd}$$

Thus the moprhisms in the left square makes $f^{-1}U \times_X V$ satisfy the universal property of $f^{-1}U \times_U V$, thus using isomorphism of $f^{-1}U$ and $V \cong SpecB$, apply the previous algebraic proposition, we know the left diagram is isomorphic to the right diagram:

$$\begin{tikzcd} SpecB/IB \arrow[r] \arrow[d] & SpecB \arrow[d] \\ SpecA/I \arrow[r] & SpecA \end{tikzcd} \qquad \begin{tikzcd} f^{-1}U \times_X V \arrow[r] \arrow[d] & V \arrow[d] \\ f^{-1}U \arrow[r] & U \end{tikzcd}$$

We are done. □

Now we begin to deal with how to interpret intersections as fibered products:

Proposition 1.12. Suppose that $Y \hookrightarrow X$ and $Z \hookrightarrow X$ are two closed embeddings, then their intersection $Y \cap Z$ we defined earlier using the sum of ideal sheaves defining Y and Z respectively is isomorphism to the fibered product of these two morphisms over X.

Proof. First consider the Cartesian diagram:

$$\begin{tikzcd} Y \times_X Z \arrow[r, hook] \arrow[d, hook] & Z \arrow[d, hook] \\ Y \arrow[r, hook] & X \end{tikzcd}$$

by our discussion above, we know that each morphism in this diagram is a closed embedding, thus the composition is an embedding. We claim that $Y \times_X Z \hookrightarrow X$ is the intersection we defined earlier. To show this, we only have to show that the quasicoherent defining ideal sheaf is the same. Recall how we get the quasicoherent ideal sheaf. We know that on each affine open, it's defined by $SpecA/I \to SpecA$, here given an affine open $SpecA$ in X, repeating our argument above, we first see that the inverse image of the inverse image in both Z and Y are $SpecA/I$ and $SpecA/J$, then the inverse image in the Cartesian product is just $Spec(A/I \otimes_A A/J) \cong SpecA/(I+J)$, this can be easily checked by showing that $A/I+J$ satisfies the universal property of the fibered coproduct of A/I and A/J over A, or you can construct an explicit isomorphism. Recall how we get I and J, they are just $\mathscr{I}(SpecA)$ and $\mathscr{J}(SpecA)$ under some isomorphisms. This hold in general as if you replace $SpecA$ by $SpecA_f$, you will obtain I_f and J_f.

In conclusion, the above argument shows that the ideal sheaf $\mathscr{I}+\mathscr{J}$ and the ideal sheaf defining $Y\times_X Z$ agree on a distinguished affine base on X, then notice that the associated sheaf has a natural sequence into $\mathscr{O}_X$, which will have exactly the same image open by open set, thus they agree. ☐

Remark 1.13. This is a very good example of how equivalent definitions, despite the fact that they are equivalent, due to the realization of the definitions are drastically different, your same approach to the same problem but interpreted in different definitions can go well or got stuck due to the procedure you need to do to jump back and forth between different definitions.

Then we prove locally closed embeddings are preserved by base change:

Proposition 1.14. Locally closed embeddings are preserved by base change.

Proof. Suppose that $f : Y \hookrightarrow X$ is a locally closed embedding which decomposes as $f : Y \hookrightarrow Z \hookrightarrow X$ where the first is a closed embedding and the second is an open embedding, then consider the diagram:

$$\begin{tikzcd} Y\times_Z (W\times_X Z) \arrow[r, hook, "closed"] \arrow[d] & W\times_X Z \arrow[r, hook, "open"] \arrow[d] & W \arrow[d] \\ Y \arrow[r, hook, "closed"] & Z \arrow[r, hook, "open"] & X \end{tikzcd}$$

Then use the fact that stacks of Cartesian squares is Cartesian. ☐

Now, what about we have n locally closed embeddings, how can we define it's scheme theoretic intersection? One natural thinking is to take intersection **step by step**, since we only have finitely many such stuff. However, there is the question of whether the order by which we take intersections will affect the final result or not. We will sketch the ideal by which you can show that it's independent of the order. The idea is simple, we will show that no matter which order you use to take the intersection, it will satisfy a certain universal property: any morphism from W to X_i such that the composition with $X_i \hookrightarrow X$ agrees corresponds to a unique morphism from W to the intersection. This can be proved by an iteration argument of the universal property of fibered product of two element. Then there will be two canonical morphisms going between the intersection defined by taking fibered products in different orders, then the above property will show that they are isomorphisms.

Definition 1.15. Suppose that we have finitely many locally closed embeddings $X_i \hookrightarrow X$, then we define the scheme theoretic **intersection of finitely many locally closed embeddings** to be the fibered products taken in any order.

This is a good place to see some examples, we first see an algebraic exercise:

Proposition 1.16. Suppose that A is a ring, then if $I\subset A[x_1,\cdots,x_m]$ and $J\subset A[y_1,\cdots,y_n]$ are ideals, then $A[x_1,\cdots,x_m]/I\otimes_A A[y_1,\cdots,y_n]/J \cong A[x_1,\cdots,x_m,y_1,\cdots,y_n]/(I,J)$ where (I,J) is the ideal generated by elements in I,J.

Proof. This follows from several isomorphism of rings we proved before, we only sketch the proof here: First write both quotients on both side as a tensor product, then use the isomorphism $A[x_1,\cdots,x_m]\otimes_A A[y_1,\cdots,y_n]\cong A[x_1,\cdots,x_m,y_1,\cdots,y_n]$, the the remaining part should be easy. ☐

The purpose of this algebraic exercise is the following geometric result:

Proposition 1.17. Suppose that X, Y are locally of finite type A-schemes, then $X\times_A Y$ is also locally of finite type, the same holds for finite type.

Proof. The difference here in this problem is that we don't have to care about affineness of the inverse image, instead, we only have to show that it can be covered by (finitely many) appropriate affine opens. After you figure out this, this is just the previous algebraic result. ☐

The last type, the 4th type of morphism we want to emphasize here is localizations and how it plays with fibered products: again we first see an algebraic result:

Proposition 1.18. Suppose that $\Phi : B \to A$ is a ring homomorphism and $S \subset B$ is a multiplicative subset of B, which implies that $\Phi(S)$ is a multiplicative of B, then we have a natural isomorphism:

$$\Phi(S)^{-1}A \cong A \otimes_B (S^{-1}B)$$

Proof. The morphism is still defined by universal properties we will omit this. Instead, you will see that the explicit map is going to be $\frac{a}{\Phi(s)^n} \mapsto a \otimes \frac{1}{s^n}$ and $a \otimes \frac{b}{s^n} \mapsto \frac{a\Phi(b)}{\Phi(s)^n}$, which is a B-algebra homomorphism. $\square$

Notice that this proposition holds for general localizations, not just the localization of an element in the ring. You will see the application of this later, but one surprising fact about this localizations is that, the morphism of schemes it induces $SpecS^{-1}A \to SpecA$ is a **monomorphism** of schemes:

Proposition 1.19. In the category of schemes, $SpecS^{-1}A \to SpecA$ is a monomorphism.

Proof. Suppose that we have $f, g : X \to SpecS^{-1}A$ such that after compose $SpecS^{-1}A \to SpecA$, they are equal, notice that by the bijection between ring homs from A to the global section of X, we know that the two ring homs coming out of A factor through $S^{-1}A$ and is the same. This implies that first, the elements of S is mapped to invertible elements in the global section of X, then consider the following diagram:

$$\begin{tikzcd} A \arrow[r] & S^{-1}A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \Gamma(X, \mathscr{O}_X) \end{tikzcd}$$

We know that the image of $\frac{a}{s}$ will be the multiplication of the image of a with the inverse of the image of s, we are done. $\square$

Since compositions of monomorphisms remain a monomorphism, we know that locally closed embeddings or maps from the spec of stalks at a point of X to X are all monomorphism. The second is a monomorpism since it can be decomposed to a localization map composed with a open embedding.

Remark 1.20. Here we insert a small parenthesis reviewing how did we define the canonical morphism from $\mathscr{O}_{X,x} \to X$, the basis ideal is that we pick an affine open around x and define it to be the composition of $Spec\mathscr{O}_{X,x} \to U \hookrightarrow X$. The only problem is if this well defined in the sense that if we have U, V two affine opens around x, then we have two definitions, do they agree?

The answer is yes, and you can see this by picking a smaller affine open in the intersection, thus the only thing we need to prove is that, the two morphisms below agree:

$$\begin{tikzcd} Spec\mathscr{O}_{X,x} \arrow[r] \arrow[dr] & V \arrow[d, hook] \\ & U \end{tikzcd}$$

since we already know that the open inclusion of V into X factors through embeddings into U. Then notice that the morphisms from U to $Spec\mathscr{O}_{X,x}$ is the morphism determined by the map from the sections on U to $\mathscr{O}_{X,x}$, notice that the factors through the map form the sections on V to $\mathscr{O}_{X,x}$, which determines the morphism from V to $Spec\mathscr{O}_{X,x}$, thus we are done.

The above discussion about fibered product of localizations is useful afterwards when you try to compute the scheme theoretic fibers.

Recall that $\mathbb{A}^n_A$ defined by the Spec of the polynomial ring is canonically isomorphic to $\mathbb{A}^n_{\mathbb{Z}} \times_{\mathbb{Z}} SpecA$, similarly we got:

Proposition 1.21. There is an isomorphism $\mathbb{P}^n_A \cong \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} SpecA$.

Proof. First, we consider how to get a morphism from $\mathbb{P}^n_A$ to the fibered product. Obviously, we would like to use the universal property, which means that we only have to give a morphism from $\mathbb{P}^n_A$ to $\mathbb{P}^n_{\mathbb{Z}}$ and to $SpecA$ such that the composition to $Spec\mathbb{Z}$ agrees. Then the morphism from $\mathbb{P}^n_A$ to $\mathbb{P}^n_{\mathbb{Z}}$ is just defined to be the morphism defined by the natural ring map $\mathbb{Z}[x_0, \cdots, x_n] \to A[x_0, \cdots, x_n]$, you can check that this indeed defines a global morphism. Then the morphism from $\mathbb{P}^n_A$ to $SpecA$ is just the structure morphism. Recall that the structure morphism can be described locally by morphisms from $D_+(x_i) \to SpecA$, then you can use this to check that the composition really agrees.

This will give the morphism f from $\mathbb{P}^n_A$ to $\mathbb{P}^n_{\mathbb{Z}}$. We now only have to show this is an isomorphism. Recall how this morphism can be described locally, we take the affine open $D_+(x_i)$ in $\mathbb{P}^n_{\mathbb{Z}}$, then by some general arguments involving the fibered product we know that the restriction of the morphism of f to the inverse image of $D_+(x_i)$ in $\mathbb{P}^n_A$ to the inverse image of $D_+(x_i)$ in the fibered product is just indicated in the below diagram:

$$\begin{tikzcd}
 & \mathbb{P}^n_A \arrow[r, dashed] \arrow[rr] \arrow[dr] & & \\
D_+(x_i) \arrow[ur, hook] \arrow[r, dashed] \arrow[dr] & D_+(x_i) \times_{\mathbb{Z}} SpecA \arrow[r, hook] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} SpecA \arrow[r] \arrow[d] & SpecA \arrow[d] \\
 & D_+(x_i) \arrow[r, hook] & \mathbb{P}^n_{\mathbb{Z}} \arrow[r] & Spec\mathbb{Z}
\end{tikzcd}$$

Make sure you understand why the morphism in the above diagram from $D_+(x_i)$ to the fibered product is induced by $D_+(x_i) \to D_+(x_i)$ and $D_+(x_i) \to SpecA$, which corresponds to the a certain diagram of rings, which involves the isomorphism $A[\frac{x_j}{x_i} | i \neq j] \cong \mathbb{Z}[\frac{x_j}{x_i} | i \neq j] \otimes_{\mathbb{Z}} A$, thus this is an isomorphism, we are done. (Make sure you understand this statement.) □

Now, we consider another example of fibered product which is called **extending the base field**, when you are dealing with a scheme over a non algebraically closed fields, you will use this repeatedly to reduce the problem you are interested in to the case of an algebraically closed field:

Example 1.22. Suppose that X is a k-scheme and l is a field extension of k, and as you can guess, l is always the algebraic closure of k. Then we know that $X \times_k l$ is going to be a l-scheme.

The reason we are interested in this construction is that we can sometimes check properties of X by checking properties on $X \times_k l$ instead, and since we always take l to be the algebraic closure of k, it's likely that working with $X \times_k l$ is going to be easier. This is called the **descent property**, we will see three examples below.

We first see that the property of being the spec of a normal integral domain domain descends, we start by an algebraic result:

Proposition 1.23. Suppose that A is an k-algebra and l is an field extension of k that is at least assumed to be . Then if $A \otimes_k l$ is a normal integral domain, then A is an normal integral domain.

Proof. We will need a special case of flatness in this proof, which will be covered in the lecture notes and will also be reviewed later.

Now to show our assertion, we first observe that we have an injection $A \hookrightarrow A \otimes_k l$, this is since over a field, everything is flat, thus tensor is left exact, which boils down to the fact that free objects are flat. Then the following diagram is essential:

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, hook] & K(A) \arrow[d, hook] \\
A \otimes_k l \arrow[r, hook] & K(A) \otimes_k l
\end{tikzcd}$$

We claim that the field of fractions of $A \otimes_k l$ is $K(A) \otimes_k l$, to see this, we first notice that $K(A) \otimes_k l$ is a field. In general tensor product of fields are not in general fields, but here it is since we are tensoring over k and one of the factor is the fraction field. To be more precise: to show it's a field, we need to check $\sum k_i \otimes l_i$ is invertible, then by clearing the denominators of k_i since its' a fraction $\frac{a_i}{b_i}$, we can assume that the left factor all in k, thus we only have to prove $1 \otimes \sum l_i$ where $\sum l_i$ is not zero is invertible, which should be obvious. Therefore, since you can easily see that this field contains $A \otimes_k l$, it will contain the fraction field on $A \otimes_k l$ since this is characterized by the smallest field containing $A \otimes_k l$. On the other hand, there is a natural ring morphism from $K(A) \otimes_k l$ to $K(A \otimes_k l)$ defined by the universal property of tensor, since it goes out of a field thus injective. This morphism defined above can be checked to be the inverse of the natural morphism out of $K(A \otimes_k l)$.

Then we claim that $A \otimes_k l \cap K(A) = A$, where the intersection is taken in $K(A) \otimes_k l$ **** details to be figured out later* □

Then we see that morphisms being equal can also be checked by extending base fields:

Proposition 1.24. Suppose that $\pi : X \to Y$ and $\rho : X \to Y$ are morphisms of k-schemes, then if l/k is a field extension, consider the induced morphism $\pi_l : X \times_{Speck} Specl \to Y \times_{Speck} Specl$ and the induced morphism $\rho_l : X \times_{Speck} Specl \to Y \times_{Speck} Specl$. If $\pi_l = \rho_l$, then $\pi = \rho$.

Proof. We will quote a later proposition 1.35 stating that surjectivity is preserved by base change.

First, let's explain how we get the induced morphism in the statement:

$$\begin{tikzcd}
Y \times_k l \arrow[rrrd] \arrow[rddd] & & & \\
& X \times_k l \arrow[lu, dashed] \arrow[r] \arrow[d] & X \arrow[r, "\pi"] \arrow[d] & Y \arrow[d] \\
& Specl \arrow[r] \arrow[d] & Speck \arrow[r, "id"] & Speck \\
& Specl \arrow[rru] & &
\end{tikzcd}$$

This diagram above gives the π_l which is the dotted morphism, and if you switch π to ρ, you get ρ_l in the as the dotted morphism. Let's first show that π and ρ equal to each other on the level of sets. To show this, we first notice that since $Specl \to Speck$ is surjective, $X \times_k l \to X$ is also surjective, thus any $x \in X$ comes from $x' \in X \times_k l$ then goes to the same point in $Y \times_k l$ under the same morphism $\pi_l = \rho_l$ to the same point in Y. Then we have to show that the morphism on the level of locally ringed spaces is equal to each other. Then this can be check affine locally in the sense if we choose an affine cover of Y, consider it's inverse image, we only have to show that the morphism of locally ringed spaces equal after we restrict to this locally setting. Moreover, to check this, we can also check by choosing an affine open cover of the inverse image, therefore we reduced to the case where X and Y can be assumed to be affine(you should realize that you can do this because that fibered products play well with open embeddings).

However, when X, Y are affine, we are only trying to prove that if two morphisms equal to each other $B \otimes_k l \to A \otimes_k l$ which are induced by ring homomorphisms $\varphi : B \to A$, then the ring hom is equal to each other. This has nothing to do with ring homs, we can just prove this for maps as k-vector spaces, we take a basis of A, B and l, which contains 1, then if the map from A to B differs, then there will be a basis of A which is mapped to different linear combinations, then since tha map $A \otimes_k l \to B \otimes_k l$ is defined by $b \otimes l \mapsto \varphi(b) \otimes l$, then consider $e \otimes 1$, since e is mapped to different basis, then the image is also different basis, thus not equal to each other since we have an explicit basis of $A \otimes_l k$. □

The last easy example of descent property is that closed embeddings descends:

Proposition 1.25. Suppose that $\pi : X \to Y$ is an affine morphism over k and also l/k is a field extension, then π is a closed embedding iff π_k defined in the previous proposition is.

Proof. First, suppose that π is a closed embedding, then you can easily show that if $SpecB$ is an affine open in Y with the inverse image under π being $SpecA$, then the corresponding morphism will be $SpecA \times_k l \to SpecB \times_k l$, which is induced by $B \to A$, since tensor product preserved surjectivity, we are done.

Conversely, if $B \otimes_k l \to A \otimes_k l$ is surjective, while $B \to A$ is not surjective, this implies that if you view $B \to A$, as a k-vector space morphism, the cokernel is not 0, thus tensor with l over l won't give you zero, a contracition to $B \otimes_k l \to A \otimes_k l$ is surjective. □

We will see more examples of certain properties are preserved after you do some constructions involving fibered products in the future, but before we proceed to these properties, let's discuss **pullback of families** and fiber of morphism:

We first see some motivating examples: How would you consider a family of curves in the form:

$$y^2 = x^3 + tx$$

which is parametrized by t. One way to consider it is to view it as a morphism:

$$Speck[x,y,t]/(y^2 - x^3 - tx) \to Speck[t]$$

which is induced by the obvious map. Then we can pullback this family to the u,v-plane by setting $t = uv$, or we can set $t = 3$ to get curves like $y^2 - x^3 - uvt = 0$ or $y^2 - x^3 - 3x = 0$, this is formally described by two fibered products:

$$\begin{tikzcd} Speck[x,y,u,v]/(y^2 - x^3 - uvx) \arrow[r] \arrow[d] & Speck[x,y,t]/(y^2 - x^3 - tx) \arrow[d] \\ Speck[u,v] \arrow[r] & Speck[t] \end{tikzcd}$$

and:

$$\begin{tikzcd} Speck[x,y]/(y^2 - x^3 - 3x) \arrow[r] \arrow[d] & Speck[x,y,t]/(y^2 - x^3 - tx) \arrow[d] \\ Speck \arrow[r] & Speck[t] \end{tikzcd}$$

where the bottom map is defined by $t \mapsto uv$ and $t \mapsto 3$. You should make sure you understand why the diagrams above are Cartesian. Now we make this precise:

Definition 1.26. Suppose that $Y \to Z$ is a morphism of schemes, we will interpret this as a family of schemes parametrized by Z, which is called the **base scheme** or just **base**. If we have another morphism $X \to Z$, we can interpret the Canonical morphism in the fibered product diagram $X \times_Z Y \to$ to be the pullback of $Y \to Z$. This is viewed as we know have a family of scheme parametrized by X, over a different base X, thus this is also called the **base change**.

With this in mind, we now turn to study the fiber of a morphism: we start by noticing that in the category of topological spaces, fiber of a continuous map is naturally identified with a fibered product:

Proposition 1.27. Suppose that $f : Y \to Z$ is a continuous map, X is a point in Z, then the fiber of f over X is naturally identified with $X \times_Z Y$.

Proof. You can check this by checking $f^{-1}X$ satisfies the universal property of the fibered product or by our explicit description of the fibered product in the categroy of topological spaces. In either way, the moral is from now on we shoule view fiber as characterized by the universal property that if you have another object going to Y such that it's image lies in $\{x\}$, it should factor through the fiber, this is precisely what the fibered product is doing. $\square$

More generally, if you have two continuous map $\pi : X \to Z$ and $\tau : Y \to Z$, we can first form the fibered product $X \times_Z Y \to X$, and for a point p in X, we know that we can identify the fiber of the morphism $X \times_Z Y \to X$ over p as another fibered product:

$$\begin{tikzcd} f^{-1}p \arrow[r] \arrow[d] & X \times_Z Y \arrow[r, "g"'] \arrow[d, "f"] & Y \arrow[d, "\tau"'] \\ \{p\} \arrow[r, hook] & X \arrow[r, "\pi"] & Z \end{tikzcd}$$

However, you can quickly notice that since the larger rectangle is also Cartesian, $f^{-1}p$ can also be identified as the fiber of τ over $\pi(p)$. We will see a similar result later in terms of schemes after we officially define what is the fiber as a scheme.

This is enough motivation for the definition of fibers in the category of schemes, which should be an object that you can define by a fibered product. If $\pi : Y \to Z$ is a morphism of schemes, and $z \in Z$ is a point, we would like to define it as something like the fibered product of the below diagram:

$$\begin{tikzcd} Y \times_Z \{z\} \arrow[r] \arrow[d] & \{z\} \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

but what is this $\{z\}$? Since we are working in the category of schemes, this should be a scheme that represents the point. One natural though is to define it by $Spec\mathscr{O}_{Z,z}$ and the natural morphism $Spec\mathscr{O}_{Z,z} \to Z$. However, this turns out not to be a good definition, which is mainly because the local ring $\mathscr{O}_{Z,z}$ still carries information of a neighborhood around z, not just the information of the point, this is why we should use $Speck(z)$ where $k(z)$ is the residue field of z, which is just the point and nothing else. And this morphism is defined by the composition:

$$Speck(z) \to Spec\mathscr{O}_{Z,z} \to Z$$

One key observation is that this induces isomorphism on the residue field. Then notice that the first morphism is a closed embedding and the second one is already showed to be a monomorphism, thus the composition is still a monomorphism. With this:

Definition 1.28. Suppose that $g : Y \to Z$ any morphism of scheme, the scheme theoretic fiber of g above p for a point not necessarily closed in Z, denoted by $g^{-1}(p)$, is defined by the fiber product:

$$\begin{tikzcd} g^{-1}(p) \arrow[r] \arrow[d] & Speck(p) \arrow[d] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

Let's first determine what is the topological space underlying $g^{-1}(p)$:

Proposition 1.29. The topological space underlying the scheme theoretic fiber above p is naturally identified with the topological fiber of $X \to Y$ above p.

Proof. This is still done affine locally. Notice that if we choose any affine open in Z that contains p, then you see that $g^{-1})(p)$ can be computed via this affine open. To be precise, if the affine open is denoted by U, the following diagram is also Cartesian:

$$\begin{tikzcd} g^{-1}(p) \arrow[r] \arrow[d] & Speck(p) \arrow[d] \\ g^{-1}U \arrow[r, "g"] & U \end{tikzcd}$$

then let's consider an affine open cover of $g^{-1}U$, then we can consider:

$$\begin{tikzcd} V \times_U Speck(p) \arrow[r, hook] \arrow[d] & g^{-1}(p) \arrow[r] \arrow[d] & Speck(p) \arrow[d] \\ V \arrow[r, hook] & g^{-1}U \arrow[r, "g"] & U \end{tikzcd}$$

we want to see that is the morphism $V \times_U Speck(p)$, it's left to you to see that this is the morphism induced by the ring morphism: $A \to A \otimes_B (B_p/pB_p)$c where p in here is the prime corresponding to the point p. The key is to understand this morphism, notice that by our proposition before, if we denote the image of the complement the prime p in B by S, we can see that this is the ring morphism:

$$A \to S^{-1}A/(pB_p)$$

which can be decomposed into $A \to S^{-1}A \to S^{-1}A/((pB_p))$, notice that each of the morphism induces and injective morphism on the level of topological spaces, thus we know that understand $g^{-1}(p)$ as a subset of $g^{-1}U$, which can be done locally on each V in this affine open cover of $g^{-1}U$. To be more precise, in the category of topological spaces, if we have a morphism $f : X \to Y$, a subset of X is the fiber of $y \in Y$ iff there is an open cover of X such that the intersection with each of the open is the fibered of f viewed as restricted to that open. Now if we have $SpecA \to SpecB$ and p a prime of B, we are trying to find those primes in A such that it's inverse image in B is just p. First, this primes should be contain the extension of p in A, also they should not contain the image of the complement of p in A, this is precisely captured by $A \to S^{-1}A/(pB_p)$. This is just a general algebraic argument about the behavior of primes under localizations and quotients. $\square$

Proposition 1.30. Suppose that $\pi : Y \to Z$ and $\tau : X \to Z$ are two morphism of schemes and $p \in X$ is a point. Then the fiber of $X \times_Z Y \to X$ over p is isomorphic to the base change to p of the fiber of $\pi : Y \to Z$ over $\tau(p)$.

Proof. First, let's write out the fibered product and the fiber in the statement:

$$\begin{tikzcd}
f^{-1}p \arrow[r] \arrow[d] & X \times_Z Y \arrow[r, "g"'] \arrow[d, "f"] & Y \arrow[d, "\pi"'] \\
Speck(p) \arrow[r] & X \arrow[r, "\tau"] & Z
\end{tikzcd}$$

Now, first notice that the morphism τ gives us a induced morphism on the residue fields $k(\tau(p)) \to k(p)$, which induces a morphism $Speck(p) \to Speck(\tau(p))$. Now, we have the following diagram:

$$\begin{tikzcd}
 & \pi^{-1}(\tau(p)) \arrow[drr] \arrow[ddd, bend right] & & \\
f^{-1}p \arrow[ur, dashed] \arrow[r] \arrow[d] & X \times_Z Y \arrow[rr, "g"'] \arrow[d, "f"] & & Y \arrow[d, "\pi"'] \\
Speck(p) \arrow[r] \arrow[dr] & X \arrow[rr, "\tau"] & & Z \\
 & Speck(\tau(p)) \arrow[urr] & &
\end{tikzcd}$$

This is some general property of Cartesian diagrams which has nothing to do with the schemes: suppose that we have the following diagram:

$$\begin{tikzcd}
V \arrow[r] \arrow[d] \arrow[rrr, dashed, bend left, "h"] & W \arrow[r] \arrow[d] & Y \arrow[d] & A \arrow[l] \arrow[d] \\
U \arrow[r, "f"] \arrow[rrr, bend right, "b"'] & X \arrow[r, "g"] & Z & B \arrow[l, "a"]
\end{tikzcd}$$

where each of the smaller squares are Cartesian and there is a morphism b such that $a \circ b = g \circ f$, then we claim that there is a induced morphism $h : V \to A$ such that the outer square consisting of V, U, A, B and morphisms h, b is Cartesian.

Let's see what is this induced morphism h, this is induced by the universal property of A, where we have $V \to Y$ and $V \to B$ which agree after we compose them with $Y \to Z$ and $B \to Z$. We claim that this h will make the outer square Cartesian. Suppose that you have a F and morphisms to U and A such that they agree after you go to B. But it's left to you to see why this also gives morphisms to Y and U such that after you go to Z somehow, you get they agree, thus by the universal property of V, you get a unique morphism, completing the proof. $\square$

These two propositions above help us completely regain the picture we have in the category of topological spaces, though when we try to identify the fiber of the fibered product with the fibered of another morphism, we need to do a base change. You should not be surprised by this result since if you try to directly resemble the result we had in topological spaces, you see that even though you have $Speck(p) \to Z$, which agrees with our topological picture, you still have to consider the scheme structure here, where you find this $Speck(p)$ is not the scheme point of $\tau(p)$, which you don't have to worry about in the category of topological spaces since you don't have other structures beyond the point. Therefore here, a natural thinking here should be to use $Speck(p) \to Speck(\tau(p))$ to get the right scheme strcture, then everything should be natural.

Here is a concrete example where we can compute fibers:

Example 1.31. Consider $\mathbb{Q}[x] \to \mathbb{Q}[y]$ where $x \mapsto y^2$, this induces a morphism $Spec\mathbb{Q}[y] \to Spec\mathbb{Q}[x]$. This should be thought as the projection from the parabola to the x-axis. If you write $\mathbb{Q}[y]$ as $\mathbb{Q}[x,y]/(x-y^2)$ via the obvious isomorphism, you can see this geometric picture clearer.

Now, consider the preimage of 1 in $Spec\mathbb{Q}[x]$, which correspond to the closed point $(x-1)$. Notice that the residue field is just $\mathbb{Q}[x]/(x-1)$, this follows from the general algebra fact that $A/m \cong A_m/mA_m$. Then the fiber should be the spec of $\mathbb{Q}[x,y]/(x-y^2) \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/(x-1)$, but by our discussion above on how to compute this kind of tensor products, we see it's just $\mathbb{Q}[x,y]/(x-1, x-y^2) \cong \mathbb{Q}[y]/(y^2-1) \cong \mathbb{Q}[y]/(y-1) \times \mathbb{Q}[y]/(y+1)$ by Chinese Remainder theorem. Then we got the fiber is just two points.

What about fibers over other points? What about the fiber of 0, you should verify that it's just $\mathbb{Q}[y]/(y^2)$, which is not reduced. Also the preimage of 1 is $Spec\mathbb{Q}[y]/(y^2+1) = Spec\mathbb{Q}[i] = Spec\mathbb{Q}(i)$, which is still a reduced point. But this can be viewed as degree 2 over $\mathbb{Q}$. What about the preimage of the generic point, it's $Spec\mathbb{Q}[x,y]/(x-y^2) \otimes_{\mathbb{Q}[x]} \mathbb{Q}(x)$. By repalcing x by y^2, we get it's $Spec\mathbb{Q}[y]_{\mathbb{Q}[y^2]}\mathbb{Q}(y^2)$. Our job is to understand this ring. The first thing we need to observe is that any $f(y)$ polynomial in y is invertible in this ring, as the inverse is $1 \otimes \frac{f(-y)}{f(y)f(-y)}$, thus this is inside $Q(y)$. Also, for any $\frac{f(y)}{g(y)} = \frac{f(y)g(-y)}{g(y)g(-y)}$, it also goes to $Spec\mathbb{Q}[y]_{\mathbb{Q}[y^2]}\mathbb{Q}(y^2)$, it's easy to see these two are isomorphisms.

Remark 1.32. It's very interesting to see that the above preimages are all related to the number 2 in some sense. It's either two points, or $\mathbb{Q}[y]/(y^2)$ or a degree 2 filed extension $\mathbb{Q}(x) = \mathbb{Q}(y^2) \hookrightarrow \mathbb{Q}(y)$. This is not coincidence, we will discuss these kind of behaviors later in **finite flat morphisms**.

Another example of computing fibered product is the following:

Example 1.33. Consider the morphism of schemes: $X = Speck[t] \to Speck[u] = Y$ which is defined by $u \mapsto t^2$, suppose that characteristic of k is not 2, then we will show that $X \times_Y X$ has two irreducible components.

This is still the problem of understanding a certain fibered product, which is $k[t] \otimes_{k[u]} k[t]$, via $u \mapsto t^2$. To reduce this to the case we discussed above, we write $k[t] \cong k[x,u]/(u^2-x)$, then we see that the map $k[u] \to k[t]$ is now $k[u] \to k[x,u]/(u^2-x)$ where $u \mapsto u$ similarly, we do the same procudure to another factor in the tensor. In the end, we get:

$$k[x,u]/(u^2-x) \otimes_{k[u]} k[y,u]/(u^2-y) \cong k[x,y,u]/(u^2-x, u^2-y)$$

Notice that u^2-x and u^2-y are both irreducible, thus prime since polynomial ring over a field is a UFD. Then notice that $V(u^2-x)$ and $V(u^2-y)$ are irreducible. We claim that there are the irreducible components, which means that they are minimal primes in this ring. This is exactly the case since we are over a noetherian setting, thus any smaller primes, say in (u^2-x) is generated by $f_i(u^2-x)$, thus not irreducible, not prime, contradiction.

The last example which is of extreme importance is the first description of the blow up. We will come back to this notion later, but this is a good place to develop some familarity of below-up through one concrete example:

Example 1.34. In this example, let's first consider the fibered product: $\mathbb{A}_k^2 \times_k \mathbb{P}_k^1$, it has an open cover consisting of two affine opens, which are both $\mathbb{A}_k^2 \times_k \mathbb{A}_k^1 \cong \mathbb{A}_k^3$. You can see this by the fact that two $\mathbb{A}_k^1$ covers $\mathbb{P}_k^1$. Now, suppose that coordinate on $\mathbb{A}_k^2$ is xy and the projective coordinate on $\mathbb{P}_k^1$ is uv, we will define a closed subscheme of $\mathbb{A}_k^2 \times_k \mathbb{P}_k^1$ defined by a single equation $xv = yu$.

Let's now explain what we mean by the closed subscheme defined by $xv = yu$. We will define it locally, suppose that one of the affine cover of $\mathbb{P}_k^1$ is $k[\frac{u}{v}]$, then the corresponding affine cover of $\mathscr{A}_k^2 \times_k \mathbb{P}_k^1$ is just $Speck[x, y, \frac{u}{v}]$, then the closed subscheme in defined by $x - y\frac{u}{v} = 0$, similarly, in another affine cover it's defined by $y - x\frac{v}{u} = 0$. In other words, this is the closed subscheme defined by the quasicoherent ideal sheaf lcoally defined by $(x - y\frac{u}{v})$ and $(y - x\frac{v}{u})$ on two affine covers. Just like how we defined hypersurfaces in $\mathbb{P}_k^n$, once we show that they agree on intersections, we are done. It's left to you to see that the intersection is just $SpecA[x, y, \frac{u}{v}, \frac{v}{u}]$ and since it's affine, we have a very well understanding of the restriction of the ideal, which is still defined by $(x - y\frac{u}{v})$ and $(y - x\frac{v}{u})$. However, we see that since $\frac{u}{v}$ is invertible, the two ideals are equal, thus we are done.

The procedure we use to identify the intersection is indicated in the following diagram below:

$$\begin{tikzcd}
\mathbb{A}_k^2 \arrow[d] & \mathbb{A}_k^2 \times_k \mathbb{P}_k^1 \arrow[l] \arrow[d] & \mathbb{A}_k^2 \times_k U_1 \arrow[l, hook'] \arrow[d] & \mathbb{A}_k^2 \times_k (U_1 \cap U_2) \arrow[l, hook'] \arrow[d] \\
k & \mathbb{P}_k^1 \arrow[l] & U_1 \arrow[l, hook'] & U_1 \cap U_2 \arrow[l, hook']
\end{tikzcd}$$

The above construction is denoted by $Bl_{(0,0)}\mathbb{A}_k^2$, called the **blow-up** of the affine 2-plane at the origin, which is a closed subscheme of the fibered product, and you can consider the diagram below:

$$\begin{tikzcd}
Bl_{(0,0)}\mathbb{A}_k^2 \arrow[r, hook] & \mathbb{A}_k^2 \times_k \mathbb{P}_k^1 \arrow[r] \arrow[d] & \mathbb{P}_k^1 \arrow[d] \\
& \mathbb{A}_k^2 \arrow[r] & k
\end{tikzcd}$$

to obtain morphisms into $\mathbb{A}_k^2$ and $\mathbb{P}_k^1 \to \mathbb{P}_k^1$.

Then we consider the fiber of $Bl_{(0,0)}\mathbb{A}_k^2$ over any closed points, which will give us some geometric intuition as to what this construction is doing. We first do some general arguments, consider suppose that you have $f : X \to Y$ and $g : Y \to Z$, you want to know what is the fiber of $g \circ f$ over $z \in Z$. You can consider the following diagram:

$$\begin{tikzcd}
X \times_Y g^{-1}z \arrow[r] \arrow[d] & g^{-1}z \arrow[r] \arrow[d] & Speck(z) \arrow[d] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

again since stacks of Cartesian product is Cartesian, we know that $X \times_Y g^{-1}z$ will be the fiber. We are in a particularlly good situation in this problem since $X \to Y$ in our problem is just $Bl_{(0,0)}\mathbb{A}_k^2 \hookrightarrow \mathbb{A}_k^2 \times_k \mathbb{P}_k^1$, a closed embedding. Again, notice that if $p \in \mathbb{P}_k^1$ is a closed point, we can assume that it's contained in one of the affine opens, say U_1, thus we can just consider the following diagram:

$$\begin{tikzcd}
Speck[x,y]/(y - cx) \arrow[r, hook] \arrow[d, hook] & Speck[x, y, \frac{u}{v}]/(\frac{u}{v} - c) \arrow[r] \arrow[d, hook] & Speck(p) = Speck[\frac{u}{v}]/(\frac{u}{v} - c) \arrow[d] \\
Speck[x, y, \frac{u}{v}]/(y - x\frac{u}{v}) \arrow[r, hook] \arrow[d, hook] & Speck[x, y, \frac{u}{v}] \arrow[r] \arrow[d, hook] & U_1 = Speck[\frac{u}{v}] \arrow[d, hook] \\
Bl_{(0,0)}\mathbb{A}_k^2 \arrow[r, hook] & \mathbb{A}_k^2 \times_k \mathbb{P}_k^1 \arrow[r] & \mathbb{P}_k^1
\end{tikzcd}$$

Every square in the above diagram is Cartesian and ff the point is in another affine cover U_2, you repeat the above computation to get the fiber is $Speck[x, y]/(x - cy)$.

You see that over a closed point, the fiber is isomorphism to $k[x,y]/(y-cx)$ or $Speck[x,y]/(x-cy)$, which is a line that passes through 0. This gives you an intuition that this blow up is parametrized by $\mathbb{P}^1_k$, and each closed point in $\mathbb{P}^1_k$ corresponds to a straight line passing through the origin.

We can also study the morphism $Bl_{(0,0)}\mathbb{A}^2_k \to \mathbb{A}^2_k$, we claim that this morphism is an isomorphism over the complement of $(0,0)$. How are we going to prove it?

This still comes from some general arguments, suppose that you have a morphism $f : X \to Y$ and $U \subset Y$ is an open subscheme, how can we show that $f : f^{-1}U \to U$ is an isomorphism. We first need to understand what is the morphism $Bl_{(0,0)}\mathbb{A}^2_k \to \mathbb{A}^2_k$. We know that the blow-up has an open covering consisting $Speck[x,y,\frac{u}{v}]/(x-y\frac{u}{v})$ and $Speck[x,y,\frac{v}{u}]/(y-x\frac{v}{u})$. We use $D(x)$ and $D(y)$ to cover the complement of $(0,0)$, we show that this is an isomorphism by showing that the morphism is an isomorphism when restricted to the inverse image of $D(x)$ and $D(y)$. First, it's left for you to figure out why the morphism is induced by $x \mapsto x$ and $y \mapsto y$. Then there is a subtle question of how to determine the inverse image of $D(x)$. You can's just do this locally since there might be overlappings. Here, we claim that the inverse image of $D(x)$ is contained in $Speck[x,y,\frac{v}{u}]/(y-x\frac{v}{u})$ and the inverse image of $D(y)$ is contained in $Speck[x,y,\frac{u}{v}]/(x-y\frac{u}{v})$. After we showed this the rest of showing this is an isomorphism is easy. Now, you need to show that the inverse image of $D(x)$ in $Speck[x,y,\frac{u}{v}]/(x-y\frac{u}{v})$ is contained in $D(y)\cap D(x)$ and similarly for another one, then we are done.

Finally, what is the fiber over $(0,0)$, we still compute it locally, and you will see that this is an effective Cartier Divisor in the sense that it's a closed subscheme locally defined by a nonzerodivisor. To see this, we consider:

$$\begin{tikzcd}
 & f^{-1}(0,0) \arrow[r] \arrow[d] & Speck[x,y]/(x,y) \arrow[d] \\
Speck[x,y,\frac{v}{u}]/(y-x\frac{v}{u}) \arrow[r, hook] & Bl_{(0,0)}\mathbb{A}^2_k \arrow[r, "f"] & \mathbb{A}^2_k
\end{tikzcd}$$

the fibered product is $Speck[x,y,\frac{u}{v}]/(x,y,y-x\frac{u}{v}) = Speck[\frac{u}{v}]$, notice that this can be simplified by $x=0$, thus cut out by a single equation. Similarly, the other one is cut out by $y=0$.

Proposition 1.35. Surjectivity is preserved by base change.

Proof. □

2 Review of Curly Spec Construction

In this section, we begin to review the the curly spec construction, the plan is to make our notion of *Spec* we know well over a ring to work more generally over a scheme, the main issue comes in is still the problem of gluing argument. The slick way of doing it is still by universal property arguments just like what we did for fiber products. Finally, the upshot is the you will see every affine morphism will be described by this curly spec construction $\mathcal{S}pec$.

For the sake of completeness, we include several definitions here that is useful:

Definition 2.1. Let $(X,\mathscr{O}_X)$ be a ringed space, a **presheaf of $\mathscr{O}_X$-algebra** is a presheaf of abelian groups $\mathscr{F}$ on X such that for each $U \subset X$ open, $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$-algebra. The restriction maps should be morphisms of rings, and the action of $\mathscr{O}_X(U)$ should also make $\mathscr{F}$ a presehaf of $\mathscr{O}_X$-modules. A **sheaf of $\mathscr{O}_X$-algebra** is similarly define.

Definition 2.2. We also have the notion of sheaf of graded rings, suppose that X is a topological space, a **sheaf of graded rings** $\mathscr{R}$ is a sheaf of rings together with subsheaves of abelian groups $\mathscr{R}_n$ suhc that there is an isomorphism:

$$\oplus_n \mathscr{R}_n \cong \mathscr{R}$$

via the inclusion $\mathscr{R}_n \hookrightarrow \mathscr{R}$, such that under the identification of elements in $\mathscr{R}_n$ via $\mathscr{R}_n \hookrightarrow \mathscr{R}$ as element in $\mathscr{R}_n$, we have $\mathscr{R}_n \cdot \mathscr{R}_m$ lies in $\mathscr{R}_{m+n}$.

Remark 2.3. It's easy to see that after the sheafification, if we start with a presheaf of $\mathscr{O}_X$-algebra, the sheafification is also an $\mathscr{O}_X$-algebra. Also, if you start with a presheaf of graded rings, after the sheafification, it's still a sheaf of graded rings. To be precise: from $(\oplus_i \mathscr{F}_i)^\# \cong \oplus_i (\mathscr{F}_i)^\#$, we guess that $(\mathscr{F}_i)^\#$ should be the subsheaves of $(\oplus_i \mathscr{F}_i)^\#$ which gives it the graded structure. You first can check that that this is indeed a subsheaf by looking at stalks and then check that the multiplication works.

Also, you can add additives to this definition above, for example, this is a commutative or noncommutative $\mathscr{O}_X$-algebra. When, X is a scheme, you can also say that this is quasicoherent or coherent when you view it as an $\mathscr{O}_X$-module.

Now, let $(X, \mathscr{O}_X)$ still be a ringed space and $\mathscr{F}$ is any $\mathscr{O}_X$-module:

Definition 2.4. The **tensor algebra** of $\mathscr{F}$ over X is the sheaf associated to the presheaf where on each $U \subset X$, we define it to be $T^\bullet(\mathscr{F}(U))$ where $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$-module.

Remark 2.5. Notice that before sheafification, we started with a presheaf of graded rings, thus according to the remark above, the associated sheaf is also a sheaf of graded rings. Then, what is it's degree n piece, still by the same remark, it's the sheafification of the degree n-piece of the presheaf. We will denote it by $T^n(\mathscr{F})$. Notice that the component in each degree are also $\mathscr{O}_X$-modules.

Remark 2.6. Similarly, you have the **symmetric algebra** structure and the **exterior algebra structure of** $\mathscr{F}$. Their component in each degree is similarly determined. They are denoted by $Sym^\bullet(\mathscr{F})$, $\bigwedge^\bullet(\mathscr{F})$ respectively, and their degree r components are $Sym^r(\mathscr{F})$ and $\bigwedge^r(\mathscr{F})$.

Proposition 2.7. Suppose that not $\mathscr{F}$ is locally free of rank n, then we claim that $T^r\mathscr{F}$, $Sym^r(\mathscr{F})$ and $\bigwedge^r(\mathscr{F})$ are all locally free, of rank n^r, $\binom{n+r-1}{n-1}$ and $\binom{n}{r}$ respectively.

Proof. *** later $\square$

Now, we go back to our construction of $\mathcal{S}pec$. We have already seen before that, if we have R is an A-algebra, then we have a scheme $SpecR$ sitting above $SpecA$. We begin to use universal properties to globalize this notion. Consider an arbitrary scheme X and a sheaf of algebras $\mathscr{R}$ on it. We want to take spec of this sheaf of algebras $\mathscr{R}$, which will give us a morphisms that is "affine" on X, i.e. the structure morpihsm is an affine morphism.

There are two ways you can think of this, notice that for any affine open $SpecA \subset X$, $\Gamma(SpecA, \mathscr{R})$ is an A-algebra, let's call it R, then we hope that above $SpecA$, this $\mathcal{S}pec\mathscr{R}$ will be $SpecR$.

Moreover, it should satisfy a universal property. Recall that for this A-scheme $SpecR$, morphisms of A-schemes to $SpecR$, $W \to SpecR$ correspond to an A-algebra homomorphism $R \to \Gamma(W, \mathscr{O}_W)$. We would like the universal property of $\mathcal{S}pec\mathscr{R}$ to generalize this. Though you might find it's not so clear how it generalizes right after we state it below. But don't worry, we will explain it very soon.

We will describe a **universal property** for this morphism $\beta : \mathcal{S}pec\mathscr{R} \to X$ together with an isomorphism $\Phi : \mathscr{R} \to \beta_* \mathscr{O}_{\mathcal{S}pec\mathscr{R}}$. Be careful, this is **not the usual universal property** we see that is given by mapping properties in the same category, this is given by mapping properties in two different categories. However, this will suffice for our purpose.

Suppose that we are given a morphism $\mu : W \to X$ together with a morphism of $\mathscr{O}_X$-algebras $\alpha : \mathscr{R} \to \mu_* \mathscr{O}_W$, there will be a unique morphism f such that the following diagram commutes:

$$\begin{tikzcd} W \arrow[rr, "f"] \arrow[dr, "\mu"'] & & \mathcal{S}pec\mathscr{R} \arrow[dl, "\beta"] \\ & X & \end{tikzcd}$$

such that $\alpha : \mathscr{R} \to \mu_* \mathscr{O}_W$ is the composition:

$$\mathscr{R} \xrightarrow{\Phi} \beta_* \mathscr{O}_{\mathcal{S}pec\mathscr{R}} \longrightarrow \beta_* f_* \mathscr{O}_W = \mu_* \mathscr{O}_W$$

We will discuss why $SpecR \to SpecA$ is a special case of this universal property we described above later, now we want to interpret this universal property in a different way. Notice that once we fix μ, then given a f such that the above, then it's left to you that we get a morphism α by composition and in particular, it satisfies the composition above. Conversely, given any α we have a unique f such that α satisfies the composition we mentioned above. We can also interpret this universal property as a bijection between f such that the diagram above commutes and α such that the below diagram commutes:

$$\begin{tikzcd} & \mathscr{O}_X \arrow[dl] \arrow[dr] & \\ \mathscr{R} \arrow[rr, "\alpha"] & & \mu_* \mathscr{O}_W \end{tikzcd}$$

This bijection between $Mor_X(W, \mathcal{S}pec\mathscr{R})$ and $Hom_{\mathscr{O}_X,ag}(\mathscr{R}, \mu_*\mathscr{O}_W)$ where the first one is X-scheme morphisms and the second is $\mathscr{O}_X$-algebra morphisms, is natural in W, in the sense that given any $W' \to W$ we have a commutative diagram:

$$\begin{tikzcd} Hom_{\mathscr{O}_X,ag}(\mathscr{R}, \mu_*\mathscr{O}_W) \arrow[d] & Mor_X(W, \mathcal{S}pec\mathscr{R}) \arrow[l] \arrow[d] \\ Hom_{\mathscr{O}_X,ag}(\mathscr{R}, \mu_*\mathscr{O}_{W'}) & Mor_X(W', \mathcal{S}pec\mathscr{R}) \arrow[l] \end{tikzcd}$$

**** this discussion need to be improved later, on the right way to state the universal property*

Then by the "universal property we defined above, suppose that $\pi : Y \to X$ and the isomorphism $\Psi : \mathscr{R} \to \pi_*\mathscr{O}_Y$ also satisfies the same property, there is a "unique" morphism between them. Here the unique is inside the quotation mark since you need to figure out it's unique in which sense.

Now, we begin to construct it, where we will first do the affine case, which will explains why this universal property generalize the one we have for $SpecR \to SpecA$.

Proposition 2.8. Suppose that $X = SpecA$ and $\mathscr{R} = \widetilde{R}$ where R is an A-algebra, then the natural morphism $SpecR \to SpecA$ satisfies the universal property above.

Proof. Now, notice that the pushforward of the structure sheaf on $SpecR$ will be $\widetilde{R}$. On the other hand, suppose that we have an A-scheme morphism $\mu : W \to SpecR$, then by we know that this correspond to an A-algebra homomorphism $R \to \Gamma(W, \mathscr{O}_W)$. Now, in order to check the universal property, suppose that we are given:

$$\begin{tikzcd} W \arrow[dr, "\mu"'] & & SpecR \arrow[dl, "\beta"] \\ & X = SpecA & \end{tikzcd}$$

together with an $\mathscr{O}_X$-algebra morphism $\widetilde{R} \to \mu_*\mathscr{O}_W$. Notice that this is uniquely determined by it's global map which is an A-algebra homomorphism $R \to \Gamma(W, \mathscr{O}_W)$, this correspond to a A-scheme morphism as we discussed above, thus it satisfies the universal property. $\square$

Proposition 2.9. Suppose that $\beta : \mathcal{S}pec\mathscr{R} \to X$ satisfies the universal property for $(X, \mathscr{R})$ and $i : U \hookrightarrow X$ is an open embedding , then $\beta|_U : \mathcal{S}pec\mathscr{R} \times_X U = (\mathcal{S}pec\mathscr{R})|_U \to U$ satisfies the universal property for $(U, \mathscr{R}|_U)$.

Proof. First, we see that the pushforward of the structure sheaf of $\mathcal{S}pec\mathscr{R} \times_X U$ is indeed isomorphic to $\mathscr{R}|_U$, which can be seen easily.

The key idea of this proposition is below, which is stated in a simple way, but needs alteration when you want to use it to show the universal property. I will let you do it, which can make you think of what exactly is the universal property. Suppose that W is an U-scheme with the structure morphism $v : W \to U$, now consider the following diagram, I will let you figure out why

$$\begin{tikzcd} W \arrow[ddr, bend right, "v"'] & & \\ & Spec\mathscr{R} \times_X U \arrow[r, hook] \arrow[d] & Spec\mathscr{R} \arrow[d, "\beta"] \\ & U \arrow[r, hook, "i"'] & X \end{tikzcd}$$

Notice that giving a U-scheme moprhism $W \to Spec\mathscr{R} \times_X U$, by the universal property of fibered product, is the same as giving a morphism $f : W \to \mathcal{S}pec\mathscr{R}$ such that $\mu = i \circ v = \beta \circ f$, then, we see that this is the same as an X-scheme morphism, but not the structure morphism from W to X is μ.

Then, by the universal property of $\mathcal{S}pec\mathscr{R}$, this is equivalent of the information of $\mathscr{R} \to \mu_* \mathscr{O}_W$, then by the adjointness of (μ^*, μ_*), it's the same as $\mu^* \mathscr{R} \to \mathscr{O}_W$, notice that $\mu = (i \circ v)$, then $\mu^*(\mathscr{R}) = v^*(i^* \mathscr{R}) = v^*(\mathscr{R}|_U)$, thus this is the same as a morphism $v^*(\mathscr{R}|_U) \to \mathscr{O}_W$, again, using adjointness, this is the same as a morphism $\mathscr{R}|_U \to v_* \mathscr{O}_W$. There is still the question of the commutativity of the diagram, which together with the alteration of this argument is left to you. □

In conclusion, now we have proved that $\mathcal{S}pec\mathscr{R}$ exists for quasicoherent $\mathscr{O}_X$-algebras when X is affine and when Y is an open subscheme of an affine X and $\mathscr{R}$ on Y is the restriction of an quasicoherent $\mathscr{O}_X$-module on X.

Now, we begin one of the main theorems of this section, which is the existence of $\mathcal{S}pec\mathscr{R}$ for arbitrary schemes X and arbitrary quasicoherent $\mathscr{O}_X$-algebras:

Theorem 2.10. *$\mathcal{S}pec\mathscr{R}$ exists for arbitrary schemes X and arbitrary quasicoherent $\mathscr{O}_X$-algebras.*

Proof. We follow the philosophy of the construction of fibered products, where the main problem is still gluing and the realization that via universal properties, we have little to check.

Let's cover X by X_i, affine opens, notice that $\mathscr{R}|_{X_i}$ is still a quasicoherent $\mathscr{O}_{X_i}$-algebra. Then, by our discussion above, we know that $\mathcal{S}pec\mathscr{R}_i$ exists for X_i where $\mathscr{R}_i$ is $\mathscr{R}|_{X_i}$, it also comes with the morphism $\beta_i : \mathcal{S}pec\mathscr{R}_i \to X_i$.

Now, let's deal with the gluing, notice that on $X_i \cap X_j$, by our proposition above, $\mathcal{S}pec\mathscr{R}_{ij} \to X_{ij}$ exists together with canonical open embeddings $\mathcal{S}pec\mathscr{R}_{ij} \hookrightarrow \mathcal{S}pec\mathscr{R}_i$ and $\mathcal{S}pec\mathscr{R}_{ij} \hookrightarrow \mathcal{S}pec\mathscr{R}_j$(Caution! You should realize that in the above argument where we constructed the canonical open embedding), we will use this to do the gluing. This requires us to check the cocycle condition, on $X_i \cap X_j \cap X_k = X_{ijk}$, what is the gluing morphism, following the same idea in fibered product, you will find that the gluing is determined by the following diagram:

$$\begin{tikzcd}
\mathcal{S}pec\mathscr{R}_i \times_{X_i} X_{ijk} \arrow[r, hook] \arrow[d] & \mathcal{S}pec\mathscr{R}_i \times_{X_i} X_{ij} \arrow[r, hook] \arrow[d] & \mathcal{S}pec\mathscr{R}_i \arrow[d, "\beta"] \\
X_{ijk} \arrow[r, hook] & X_{ij} \arrow[r, hook] & X_i
\end{tikzcd}$$

therefore the cocycle condition is checked exactly the same as the fibered product, we don't draw the three component gluing diagram in the fibered product, but if you are familiar with it, you need to go through this argument again. We will call this glued object S.

Moreover notice that the morphism β_i's also glue to a global morphism β, the reason they glue is the same as the reason the morphisms glue in the Cartesian product. One important insight is that this morphism is affine since by construction we have an affine open over of X by X_i such that the inverse image is affine. This in particular implies that the morphism is qcqs.

Finally, we need to show that S satisfies the universal property of $\mathcal{S}pec\mathscr{R} \to X$. First, the pushforward of the structure sheaf of S should be $\mathscr{R}$, which is indeed the case since we know that the morphism is qcqs, thus the pushforward is still quasicoherent, thus we only have to check locally they are isomorphism, which is handy by our construction.

Now, suppose that we are given $\mu : X \to X$ and we have an $\mathscr{O}_X$-algebra morphism $\alpha : \mathscr{R} \to \mu_* \mathscr{O}_W$ such that the following diagram commutes:

$$\begin{tikzcd}
 & \mathscr{O}_X \arrow[dl] \arrow[dr] & \\
\mathscr{R} \arrow[rr, "\alpha"] & & \mu_* \mathscr{O}_W
\end{tikzcd}$$

we want to show that this correspond to a unique X-scheme morphism $W \to S$, which shows that S is

$\mathcal{S}pec\mathscr{R}$. This is still a gluing argument, notice that for each X_i affine open, the following diagram commutes:

$$\begin{tikzcd} & \mathscr{O}_{X_i} \arrow[dl] \arrow[dr] & \\ \mathscr{R}_i \arrow[rr, "\alpha_i"] & & \mu_*(\mathscr{O}_{\mu^{-1}(X_i)}) \end{tikzcd}$$

where now we view $\mu : \mu^{-1}X_i \to X_i$, $\mathscr{R}_i = \mathscr{R}|_{X_i}$ and $\alpha_i = \alpha|_{X_i}$, then locally by the universal property, we have f_i such that:

$$\begin{tikzcd} \mu^{-1}(X_i) \arrow[rr, "f_i"] \arrow[dr] & & \mathcal{S}pec\mathscr{R}_i \arrow[dl] \\ & X_i & \end{tikzcd}$$

commutes and α is the corresponding composition. If these f_i's glue on the intersections, then we are done. But this turns out to be the most easy part of the proof, notice that on the intersections, the maps come from the same α_{ij}, and we already know that $\mathcal{S}pec\mathscr{R}_{ij}$ also satisfies the universal property as which is basically the content of the previous proposition, thus they must be the same, thus agree. ☐

Remark 2.11. We tie some loose ends here: notice that if μ is affine, the way we defined the morphism from W to $\mathcal{S}pec\mathscr{R}$ implies that this is an isomorphism if α is an isomorphism. This doesn't hold is μ is not affine since in this case $\mu^{-1}X_i$ is not affine, and we know that a morphism from a general scheme to an affine maynot be an isomorphism even if the map on global section is an isomorphism. But conversely, if we start with an isomorphism $W \to \mathcal{S}pec\mathscr{R}$, then the unique α is going to be an isomorphism. This is because if you remember how we got this α, it is a composition of an isomorphism and $\beta_*\mathscr{O}_{\mathcal{S}pec\mathscr{R}} \to \mu_*\mathscr{O}_W$, then when f is an isomorphism, $\beta_*\mathscr{O}_{\mathcal{S}pec\mathscr{R}} \to \mu_*\mathscr{O}_W$ is an isomorphism.

Then, we claim that there is a unique isomorphism $\psi : \mathscr{R} \to \beta_*\mathscr{O}_{\mathcal{S}pec\mathscr{R}}$ such that the following diagram commutes:

$$\begin{tikzcd} & \mathscr{O}_X \arrow[dl] \arrow[dr] & \\ \beta_*\mathscr{O}_{\mathcal{S}pec\mathscr{R}} \arrow[d, leftrightarrow, "\Phi"] & & \beta_*\mathscr{O}_{\mathcal{S}pec\mathscr{R}} \\ \mathscr{R} \arrow[urr, "\psi"] & & \end{tikzcd}$$

It comes from the identity $\mathcal{S}pec\mathscr{R} \to \mathcal{S}pec\mathscr{R}$, it's an isomorphism because the previous paragraph.

As what we promised earlier, $\mathcal{S}pec$ construction gives us a complete way to understand affine morphism:

Proposition 2.12. If $\mu : Z \to X$ is an affine morphism then we have a canonical isomorphism $Z \cong \mathcal{S}pec\mu_*\mathscr{O}_Z$.

Proof. By the previous remark, there is a unique morphism $\mu_*\mathscr{O}_Z \to \mu_*\mathscr{O}_Z$ such that the corresponding diagram commutes, then this correspond to a unique morphism from $Z \to \mathcal{S}pec\mathscr{R}$, we claim that this is an isomorphism. This is still because of the above remark, where we notice $Z \to X$ is affine. ☐

The following proposition gives a useful characterization of quasicoherent and finite type quasicoherent sheaves on $\mathcal{S}pec\mathscr{R}$ using information on X.

Proposition 2.13.

References