Appendix 15 for Math 632: 2025 Winter Notes

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In this part of the note series we return to the notion of **flatness**, which is a concept that is very algebraic at first glance. We could have discussed this way before, but a number of examples that shows that this algebraic notion is in fact geometric are very connected to the notion of cohomology. For example, if $\pi : X \to Y$ is a projective flat morphism over a connected Noetherian base Y, then we will see that a lot of numerial invaraints of fibers are constant, including dimension and numbers interpretable in terms of Euler characteristics:

1. The Hilbert polynomial.
2. The degree in projective space.
3. As a special case of 2, if we have a finite morphism π, then it's degree is constant. Recovering our previous fact that the degree of a finite morphism between regular curves are constatn.
4. The arithmetic genus.
5. The degree of a line bundle if the fiber is a curve.
6. Intersections of divisors and line bundles.

Therefore, this flatness seems to be the right notion we need to describe a nice varying family over a base. This is indeed case, and you will see in the next chapter that flatness will be the starting point of the definition of a **moduli space**. For example, you will see, as an example, the **Hilbert scheme** of $\mathbb{P}^n$ parametrizes closed subschemes of $\mathbb{P}^n$. Maps from a scheme B to the Hilbert scheme correspond to finitely presented closed subschemes of $\mathbb{P}^n_B$ flat over B. Another example is the moduli space of smooth projective curves, which is defined by universal property that morphisms into this moduli space correspond to projective flat finitely presented families whose geometric fibers are smooth curves. (However, this modulis space doesn't exist.)

Flatness is also very central in **deformation theory**, which is the key to understand schemes or other geometric objects can deform. Finally, if you further read the notes on etale cohomology, flatness is central in the **faithfully flat descent theory** which tells you how to glue in Grothendieck topologies in the same way as we do in the usual Zariski topolygy.

Finally, since flatness implies a lot of numerical invariants coming from cohomology is constant, you should not be surprised that flatness can also be characterized cohomologically. We will do this first before we head to the pleasant geometric implications coming out of flatness. Besides from the general criterion we give to check flatness, you should also pay attention to how flatness is checked over special kinds of rings, such as over an integral domain, a PID, a DVR and a local ring.

1 Easier facts

Some facts about flatness are easy and immediate, but there are facts out there that requires more work. We begin with the low-hanging fruits of flatness first.

Recall that if M is an A-module, we say M is **flat over** A if $M \otimes_A -$ is an exact functor from Mod_A to itself. We say a ring map $A \to B$ is flat if B is flat as an A-module.

Recall that we have seen that free modules are flat and projective modules are flat (as the consequence of higher Tor vanish and the LES). Moreover, recall that localization is exact, therefore we know that $B \to S^{-1}B$ for any multiplicative system is exact. Some easy examples are: since any module over a field is free, then any module over a field is flat. Since for any ring A, the polynomial ring $A[x_1, \cdots, x_n]$ is also free, then $A \to A[x_1, \cdots, x_n]$ is free.

Example 1.1. Consider the ring map:

$$\mathbb{Q}[x] \to \mathbb{Q}[y], \quad x \mapsto y^2$$

Recall that this is the double cover of the affine line, let's show that this is flat as this will be an important example later.

I claim that $\mathbb{Q}[y]$ is free over $\mathbb{Q}[x]$. Consider the map:

$$\mathbb{Q}[x] \oplus \mathbb{Q}[x] \mapsto \mathbb{Q}[y]$$

where the first 1 maps to 1 and the second 1 maps to y. Then consider the image of $(f(x), g(x))$ we know that it's image is:

$$f(y^2) + yg(y^2)$$

Notice that the degree of each term in these two polynomials $f(y^2)$ and $yg(y^2)$ are either all even or all odd, therefore in order for the sum to be 0, the only way is that $f = g = 0$. This proves that this is an injective map. Notice that any $F(y)$, by separating the even and odd degrees in y, it can be written as the form of $f(y^2) + yg(y^2)$, therefore it's surjective, therefore $\mathrm{Q}[y]$ is free.

Here is an important observation we will use later. Notice that if $a \in A$ is a non zero divisor and M is flat over A, then we know that the map $M \stackrel{\times a}{\to} M$ where you multiply every element in M with a is going to be injective. This is because this can be identified with the map we get by tensor $A \stackrel{\times a}{\to} A$ with M. Therefore if there is a non zero divisor such that $M \stackrel{\times a}{\to} M$ is not injective then M is not flat. Therefore this is usually used as a criterion for nonflatness.

Example 1.2. For another example, if A now is a PID, then we will show that a finitely generated module is free iff it's torsion-free iff it's flat.

Recall that we know that any finitely generated module M over A is isomorphic to:

$$A^{\oplus r} \oplus A/(f_1) \oplus A/(f_2) \oplus \cdots \oplus A/(f_r)$$

where $f_i | f_{i+1}$ and none of them is 0. Then we know that M is free iff it's torsion-free. We also know that if M is free it's flat, therefore now we assume M is flat.

Since A is a domain, then we know that any nonzero element is a nonzerodivisor, therefore we know that if M is not free, then for example we know that the following map is not injective:

$$M \stackrel{\times f_1}{\to} M$$

Therefore M is not flat, therefore flatness implies freeness.

Proposition 1.3. If $B \to A$ is a ring map and M is flat over B, then $M \otimes_B A$ is flat over A. **This is saying that flatness is preserved by base change.**

Proof. Notice that we have natural isomorphisms:

$$N \otimes_A (M \otimes_B A) \simeq N \otimes_B M$$

and $- \otimes_B M$ is exact. $\square$

Proposition 1.4. If $B \to A$ is flat and M is flat over A then M is flat over B. This can be summarized to **flatness is preserved by composition**.

Proof. Notice that:

$$N \otimes_B M \simeq N \otimes_B (A \otimes_A M) \simeq (N \otimes_B A) \otimes_A M$$

Then we use the composition of exact functor is exact. □

One good property about flatness is that it can be checked stalk locally:

Proposition 1.5. An A-module M is flat iff M_p if flat over A_p for all prime ideals p. (We can easily prove it suffices to check only all maximal ideals, but we don't need it.)

Proof. If M is flat over A, then M_p is flat over A_p is just the result of M_p is isomorphic $M \otimes_A A_p$ and flatness is preserved by base change.

Now if M_p is flat over A_p for all p, to check flatness of M, we only need to recall that injectiveness can be check stalk locally. □

Now we define flatness on the level of schemes:

Definition 1.6. Let X be a scheme and $\mathcal{F}$ a quasicoherent sheaf on X. We say that $\mathcal{F}$ is **flat at** $p \in X$ if $\mathcal{F}_p$ is flat over $\mathcal{O}_{X,p}$. We say that $\mathcal{F}$ is **flat over** X iff it's flat at any point $p \in X$.

Since $\mathcal{F}$ is quasicoherent, we can check this on an affine cover of X. That is to say that is $\widetilde{M}$ is flat over $SpecA$ for an A-module M iff M is flat over A.

Definition 1.7. Let $\pi : X \to Y$ be a morphism of schemes, we say that it's **flat at** p if $\mathcal{O}_{X,p}$ is flat over $\mathcal{O}_{Y,\pi(p)}$. We will say that π is **flat over** Y if it's flat at all $p \in X$.

These two definitions can actually be made into one definition:

Definition 1.8. Let $\pi : X \to Y$ be a morphism of scheme and $\mathcal{F}$ a quasicoherent sheaf over X. We will say that $\mathcal{F}$ is **flat over** Y **at** $p \in X$ if $\mathcal{F}_p$ is flat over $\mathcal{O}_{Y,f(p)}$. We say that $\mathcal{F}$ is **flat over** Y if it's flat over Y at any $p \in X$.

Then the two definitions above are just when π is the identity morphism of X and when $\mathcal{F} = \mathcal{O}_X$. And you should notice that this can be both checked locally on Y and locally on X.

Example 1.9. One easy example of a class of flat morphisms is the **open embedding**. As we know that the corresponding morphism of the stalks is just the identity.

Example 1.10. Let's see what this notion of flatness looks like in the affine case.

Let $B \to A$ be a ring map, then I claim this ring map is flat iff $SpecA \to SpecB$ is flat. More generally, if M is an A-module, then M if flat over B iff $\widetilde{M}$ is flat over $SpecB$.

This is most easily proved by consider it's contrapositive. First suppose that M is not flat over B, then there will be an exact sequence of B-modules:

$$0 \to N' \to N \to N'' \to 0$$

such that:

$$N' \otimes_B M \to N \otimes_B M$$

is not injective. Notice that this map can also be viewed as a map of A-modules, then we know that it's not injective, therefore there will be a prime p in A such that:

$$N' \otimes_B M_p \to N \otimes_B M_p$$

is not injective. But note that the above map is identified with:

$$N'_q \otimes_{B_q} M_p \to N_q \otimes_{B_q} M_p$$

and this is not injective where q is the inverse image of p. This is because we have a natural isomorphism:

$$N \otimes_B A_p \simeq N \otimes_B B_q \otimes_{B_q} A_p \simeq N_q \otimes_{B_q} A_p$$

Notice that $N'_q \to N_q$ is injective, therefore M_p is not flat over B_q. Therefore $\widetilde{M}$ is not flat over $SpecB$.

Conversely, let's assume $\widetilde{M}$ is not flat over $SpecB$, we know that there will exist a prime in A such that M_p is not flat over B_q where q is the inverse image of p. Then we know that there is a SES of B_q-modules:

$$0 \to N' \to N \to N'' \to 0$$

such that $N' \otimes_{B_q} M_p \to N \otimes_{B_q} M_p$ is not injective. Using similar argument as above we know that:

$$N' \otimes_B M_p \to N \otimes_B M_p$$

is not injective, thus $N' \otimes_B M \to N \otimes_B M$ is not injective, we are done.

Corollary 1.11. If $\pi : X \to Y$ is an affine morphism and $\mathcal{F}$ is a quasicoherent sheaf over X, then $\mathcal{F}$ is flat over Y iff $\pi_*\mathcal{F}$ is flat over Y.

Proof. Since $\pi : X \to Y$ is affine and flatness can be checked affine locally, we know that we can assume that π is just $SpecA \to SpecB$ and $\mathcal{F} = \widetilde{M}$. Then we know that $\widetilde{M}$ is flat over $SpecB$ iff M if flat over B. However we know that the pushforward of $\widetilde{M}$ to $SpecB$ is just $\widetilde{M}$ where M is regarded as a B-module. □

Example 1.12. Let X be a scheme, then the canonical map $Spec\mathcal{O}_{X,p} \to X$ is flat. To check this, notice that we can assume X is just an affine scheme, say $SpecA$, then the morphism is just $SpecA_p \to SpecA$, then we know that $A \to A_p$ is flat.

The canonical projection $\mathbb{A}^n_A \to SpecA$ is flat, as we know that $A \to A[x_1, \cdots, x_n]$ is flat.

Similarly, if $\mathcal{F}$ is a locally free sheaf of finite rank over X, then $\mathbb{P}\mathcal{F} \to X$ is flat. We check it again locally on X, and we can assume this is just $\mathbb{P}^n_A \to SpecA$, this is because if we want to check it's flat over $p \in \mathbb{P}^n_A$, we only need to assume we work with the canonical projection $\mathbb{A}^n_A \to SpecA$, which is flat, therefore we are done.

Finally $Speck \to Speck[t]/(t^2)$ is not flat. Consider the inclusion:

$$(\bar{t}) \hookrightarrow k[t]/(t^2)$$

Let's tensor with k, which is:

$$(\bar{t}) \otimes_{k[t]/(t^2)} k \to k[t]/(t^2) \otimes_{k[t]/(t^2)} k$$

Notice that the first term is just $(t)/(\bar{t})^2$, but notice that $\bar{t}^2 = 0$, therefore we know that we get:

$$(\bar{t}) \to k, \quad \bar{t} \mapsto t \cdot 1 = 0$$

which is not injective. Therefore we are done.

Proposition 1.13. If $\pi : X \to Y$ is a flat morphism, then $\pi^* : \mathrm{Qcoh}_Y \to \mathrm{Qcoh}_X$ is exact.

Proof. Let:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

be an exact sequence of quasicoherent sheaves over X. We want to show that $\pi^*\mathcal{F}' \to \pi^*\mathcal{F}$ is exact. Recall that if $p \in X$, we know that the induced morphism on stalk $(\pi^*\mathcal{F}')_p \to (\pi^*\mathcal{F})_p$ can be identified with:

$$\mathcal{F}'_{\pi(p)} \otimes_{\mathcal{O}_{Y,\pi(p)}} \mathcal{O}_{X,p} \to \mathcal{F}_{\pi(p)} \otimes_{\mathcal{O}_{Y,\pi(p)}} \mathcal{O}_{X,p}$$

where $\mathcal{F}'_{\pi(p)} \to \mathcal{F}_{\pi(p)}$ is injective. Then since $\mathcal{O}_{X,p}$ is flat over $\mathcal{O}_{Y,\pi(p)}$, we know that after we tensor with $\mathcal{O}_{X,p}$, it's still injective. □

Proposition 1.14. Suppose that $\pi : X \to Y$ is a morphism of scheme and $\mathcal{F}$ is a quasicoherent sheaf on X flat over Y. Let $\psi : Y \to Z$ be a flat morphism, then $\mathcal{F}$ is flat over Z.

Proof. We need to show that for each $p \in X$, $\mathcal{F}_p$ is flat over $\mathcal{O}_{Z,\psi\pi(p)}$ where we have a ring map:

$$\mathcal{O}_{Z,\psi\pi(p)} \to \mathcal{O}_{Y,\pi(p)} \to \mathcal{O}_{X,p}$$

Since we know that ψ is flat, we know that $\mathcal{O}_{Z,\psi\pi(p)} \to \mathcal{O}_{Y,\pi(p)}$ is flat. Moreover $\mathcal{F}_p$ is flat over $\mathcal{O}_{Y,\pi(p)}$, then we know from the fact that flat module is preserved by composition, we are done. □

Proposition 1.15. Suppose that $\pi : X \to Y$ is a morphism and $\mathcal{F}$ is a quasicoherent sheaf on X that is flat over Y. Let $\rho : Y' \to Y$ be any morphism of schemes with $\rho' : X \times_Y Y' \to X$ being the induced morphism. Then $(\rho')^*\mathcal{F}$ is flat over Y'.

Proof. The diagram looks like:

$$\begin{tikzcd}
(\rho')^*\mathcal{F} \arrow[d, dash] & \mathcal{F} \arrow[d, dash] \\
X \times_Y Y' \arrow[r, "\rho'"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\
Y' \arrow[r, "\rho"'] & Y
\end{tikzcd}$$

We need to show that for each $p \in X \times_Y Y'$, $(\rho')^*\mathcal{F}_p$ is flat over $\mathcal{O}_{Y',\pi'(p)}$. Recall that this Cartesian diagram induces a diagram:

$$\begin{tikzcd}
(\rho')^*\mathcal{F} \arrow[d, dash] & \mathcal{F} \arrow[d, dash] \\
\mathcal{O}_{X \times_Y Y',p} & \mathcal{O}_{X,\rho'(p)} \arrow[l, "\rho'"'] \\
\mathcal{O}_{Y',\pi'(p)} \arrow[u, "\pi'"] & \mathcal{O}_{Y,q} \arrow[l, "\rho"] \arrow[u, "\pi"']
\end{tikzcd}$$

where $q \in Y$ is $\pi\rho'(p) = \rho\pi'(p)$. Recall that $\mathcal{O}_{X \times_Y Y',p}$ is a further localization of $\mathcal{O}_{X,\rho'(p)} \otimes_{\mathcal{O}_{Y,q}} \mathcal{O}_{Y',\pi'(p)}$ (see the remark below), which is flat. Then We have the following diagram:

$$\begin{tikzcd}
\mathcal{O}_{X,\rho'(p)} \otimes_{\mathcal{O}_{Y,q}} \mathcal{O}_{Y',\pi'(p)} \arrow[drr, dashed] & & (\rho')^*\mathcal{F} \arrow[d, dash] & \mathcal{F} \arrow[d, dash] \\
& & \mathcal{O}_{X \times_Y Y',p} & \mathcal{O}_{X,\rho'(p)} \arrow[l, "\rho'"'] \arrow[ulll] \\
& & \mathcal{O}_{Y',\pi'(p)} \arrow[u, "\pi'"] \arrow[uull] & \mathcal{O}_{Y,q} \arrow[l, "\rho"] \arrow[u, "\pi"']
\end{tikzcd}$$

Here the dotted map is induced by the universal property of tensor products of algebra, which is identified with a localization map. (Again, see the remark below) We know that the stalk of $(\rho')^*\mathcal{F}$ at p is $\mathcal{F}_{\rho'(p)} \otimes_{\mathcal{O}_{X,\rho'(p)}} \mathcal{O}_{X \times_Y Y',p}$. Here we know that: $\mathcal{F}_{\rho'(p)} \otimes_{\mathcal{O}_{Y,q}} \mathcal{O}_{Y',\pi'(p)}$ is flat over $\mathcal{O}_{Y',\pi'(p)}$, then $\mathcal{O}_{X \times_Y Y',p} \to \mathcal{O}_{X,\rho'(p)} \otimes_{\mathcal{O}_{Y,q}} \mathcal{O}_{Y',\pi'(p)}$ is flat. Moreover, we can think of $\mathcal{F}_{\rho'(p)} \otimes_{\mathcal{O}_{Y,q}} \mathcal{O}_{Y',\pi'(p)}$ as:

$$\mathcal{F}_{\rho'(p)} \otimes_{\mathcal{O}_{X,\rho'(p)}} \mathcal{O}_{X,\rho'(p)} \otimes_{\mathcal{O}_{Y,q}} \mathcal{O}_{Y',\pi'(p)}$$

Then we know that since $\mathcal{O}_{X,\rho'(p)} \otimes_{\mathcal{O}_{Y,q}} \mathcal{O}_{Y',\pi'(p)} \to \mathcal{O}_{X \times_Y Y',p}$ is flat, then consider:

$$\begin{tikzcd}
& \mathcal{F}_{\rho'(p)} \otimes_{\mathcal{O}_{Y,q}} \mathcal{O}_{Y',\pi'(p)} \arrow[d, dash] & \mathcal{F}_{\rho'(p)} \otimes_{\mathcal{O}_{Y,q}} \mathcal{O}_{X \times_Y Y',p} \arrow[d, dash] \\
\mathcal{O}_{Y',\pi'(p)} \arrow[r] & \mathcal{O}_{X,\rho'(p)} \otimes_{\mathcal{O}_{Y,q}} \mathcal{O}_{Y',\pi'(p)} \arrow[r] & \mathcal{O}_{X \times_Y Y',p}
\end{tikzcd}$$

We know that $\mathcal{F}_{\rho'(p)} \otimes_{\mathcal{O}_{Y,q}} \mathcal{O}_{X \times_Y Y',p}$ is flat over $\mathcal{O}_{X \times_Y Y',p}$ as $\mathcal{O}_{X,\rho'(p)} \otimes_{\mathcal{O}_{Y,q}} \mathcal{O}_{Y',\pi'(p)} \to \mathcal{O}_{X \times_Y Y',p}$ is flat. Then once we prove $\mathcal{O}_{Y',\pi'(p)} \to \mathcal{O}_{X \times_Y Y',p}$ is flat, we are done, i.e. we need to prove that $\mathcal{O}_{Y',\pi'(p)} \to \mathcal{O}_{X,\rho'(p)} \otimes_{\mathcal{O}_{Y,q}} \mathcal{O}_{Y',\pi'(p)}$ is flat, but this is again the fact that flatness is preserved by change of the base ring.

In conclusion, this proposition is proved by noting $\mathcal{O}_{X,\rho'(p)} \otimes_{\mathcal{O}_{Y,q}} \mathcal{O}_{Y',\pi'(p)} \to \mathcal{O}_{X \times_Y Y',p}$ is flat and use flatness is preserved by change of the base ring. $\square$

Remark 1.16. We see a similar remark in the note series of etale cohomology about why the stalk of the pullback localizes to the tensor product of algebras. Here we prove it again just for completeness.

We only need to consider the affine case, and we can assume that we are considering a diagram:

$$\begin{tikzcd} SpecA \otimes_C B \arrow[r, "\rho'"'] \arrow[d, "\pi'"] & SpecA \arrow[d, "\pi"'] \\ SpecB \arrow[r, "\rho"] & SpecC \end{tikzcd}$$

suppose that p is a prime ideal in $A \otimes_C B$, again we use $\rho'(p), \pi'(p)$ and q. Notice that there is a multiplicative system S in $A \otimes_C B$, which consists of element $x \otimes y$ where $x \notin \pi'(p)$ and $y \notin \rho'(q)$ (you can easily check this is indeed a multiplicative system). Then notice that this is contained in the multiplicative system $A \otimes_C B - p$.

Then notice that we have a natural isomorphism of $A \otimes_C B$-algebras: $S^{-1}(A \otimes_C B) \simeq A_{\rho'(p)} \otimes_{C_q} B_{\pi'(p)}$ where the isomorphisms are indicated below:

$$\begin{tikzcd} & A \otimes_C B \arrow[dl] \arrow[dr] & \\ S^{-1}(A \otimes_C B) \arrow[rr, dashed, shift left] & & A_{\rho'(p)} \otimes_{C_q} B_{\pi'(p)} \arrow[ll, dashed, shift left] \end{tikzcd}$$

The map from $S^{-1}(A \otimes_C B)$ is given by the universal property of localization. It's inverse is given by the universal property of tensors of C_q-algebras. We first define:

$$A_{\rho'(p)} \to S^{-1}(A \otimes_C B), \quad \frac{x}{y} \mapsto \frac{x \otimes 1}{y \otimes 1}$$

Similarly we define:

$$B_{\pi'(p)} \to S^{-1}(A \otimes_C B), \quad \frac{x}{y} \mapsto \frac{1 \otimes x}{1 \otimes y}$$

Then you can check that after precomposing $C_q \to A_{\rho'(p)}$ and $C_q \to B_{\pi'(p)}$, they agree, therefore we are done.

In particular, taking $\mathcal{F}$ to be $\mathcal{O}_X$, you get the fact that flat morphsims are preserved by base change.

Remark 1.17. We have already said that flat morphisms can be checked locally both on the source and target. The previous two propositions shows that flat morphisms also behave well with respect to composition and base change. Therefore flat morphism is a nice class.

Another good property of flat morphism is that it will send associated points to associated points:

Proposition 1.18. Suppose that $\pi : X \to Y$ is a flat morphism of locally Noetherian schemes, then any associated point of X will be mapped to an associated point of Y. More generally, if $\mathcal{F}$ is a coherent sheaf of X (still with the Noetherian hypopthesis) flat over Y, then any morphism $\pi : X \to Y$ maps associated point of $\mathcal{F}$ to associated point of Y.

Proof. First, suppose that $\pi^\sharp : (B, n) \to (A, m)$ is a local morphism of Noetherian local rings, such that A is flat over B. If n in B is not an associated prime of B, then I claim there is an $f \in n$ such that it's not in any associated prime of B and hence a nonzerodivisor. To see why, we use prime avoidance lemma, since n is not an associated prime, then we know that if all elements in n are zerodivisors, it's contained in the finitely many associated primes, then prime avoidance tells you that n is such an associated prime, a contradiction. Then since $B \to A$ is flat, we know that consider tensor $B \xrightarrow{\times f} B$ with A, which gives $A \xrightarrow{\times \pi^\sharp f} A$, which is injective, therefore $\pi^\sharp f \in m$ is also a nonzerodivisor. Therefore m is not an associated prime of A.

Now consider $\pi : X \to Y$, if $x \in X$ is an associated point, we want to show that $y = \pi(x)$ is also an associated point. Consider the induced map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, recall that we know that the maximal ideal of $\mathcal{O}_{X,x}$ is an associated prime, as it's preserved under localization. Recall that the notion of associated primes commutes with localization, then y is an associated prime of Y iff the maximal ideal of $\mathcal{O}_{Y,y}$ is an associated prime. But we proved that if y is not, then x is not. Since x is, then y is.

Immediately, this argument can be extended to coherent sheaf $\mathcal{F}$ on Y. Here we will use the fact that if $\pi^\sharp : (B, n) \to (A, m)$ is a local map between local Noetherian rings and suppose that M is a finitely generated module over B that is flat with n is not an associated point of B. Then we know that there is an $f \in n$ such that $B \xrightarrow{\times f} B$ is injective, then $M \xrightarrow{\times f} M$ is also injective since M is flat, therefore we know that m is not an associated point of $\mathcal{F}$. The rest of the proof looks similar. □

The above result can help us determine what morphisms are not flat:

Example 1.19. Let's first show that the morphism $Speck[x,y]/(xy) \to Speck[x]$, i.e. the morphism that project the x-axis and y-axis down to the x-axis is not flat. Notice that (x) and (y) are associated primes of $Speck[x,y]/(xy)$, while (x) is mapped to (x) in $Speck[x]$, which is not an associated prime since x is not an zerodivisor.

Then consider the morphism $Speck[x,y]/(y^2,xy) \to Speck[x]$, which you can think of as project the x-axis with a fuzz at the origin to the reduced x-axis. Notice that again we know (x) is an associated prime of $Speck[x,y]/(y^2,xy)$, whose image is (x), not an associated prime.

Finally, consider the canonical map $\mathrm{Bl}_{(0,0)}\mathbb{A}^2_k \to \mathbb{A}^2_k$. I claim that this is not flat. To see why, consider the following pullback diagram:

$$\begin{tikzcd} Projk[X,Y,x]/(xY) \arrow[r] \arrow[d] & Projk[X,Y,x,y]/(Xy-xY) \arrow[d] \\ Speck[x] \arrow[r] & Speck[x,y] \end{tikzcd}$$

Since flat morphism is preserved by base change, we know that if $\mathrm{Bl}_{(0,0)}\mathbb{A}^2_k \to \mathbb{A}^2_k$ is flat, then $Projk[X,Y,x]/(xY) \to Speck[x]$ is flat. Since flatness is local on the source, we know that the following morphism should be flat:

$$Speck[\frac{Y}{X},x]/(x\frac{Y}{X}) \to Speck[x]$$

But we already know this is not flat from the first example.

Remark 1.20. Case 1 and 3 in the previous example will be quite important as they are examples showing that there is a jumping in the fiber dimension. In the case of $Speck[x,y]/(xy) \to Speck[x]$, we know that the fiber of the origin is dimension 1 while all the other fibers of k-points is of dimension 0. Similarly, in the morphism $\mathrm{Bl}_{(0,0)}\mathbb{A}^2_k \to \mathbb{A}^2_k$, the fiber of $(0,0)$ is $\mathbb{P}^1$ which is dimension 1 while other fibers of k-points have dimension 0.

We will later show that flat morphisms won't have such jumpings in fiber dimensions.

Then let's show that cohomology plays well with flat base change:

Theorem 1.21. Suppose we have the following Cartesian diagram:

$$\begin{tikzcd} X' \arrow[r, "\rho'"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ Y' \arrow[r, "\rho"'] & Y \end{tikzcd}$$

let π be quasicompact and separated. Suppose that ρ is flat and $\mathcal{F}$ is a quasicoherent sheaf on X, then the natural push-pull morphism:

$$\rho^*(\mathrm{R}^i\pi_*\mathcal{F}) \to \mathrm{R}^i\pi'_*(\rho'^*\mathcal{F})$$

is an isomorphism.

Proof. Recall that we assumed that π and quasicompact and separated, then so is π'. Then Cech cohomology and Cech higher pushforward makes sense. Also recall that we have proved a similar result when ρ is affine. We defined the push-pull morphism above using the Cech definition and FHHF theorem. So to check this is an isomorphism, we only need to do it locally.

Let's review again how we defined push-pull map locally. Notice that if we pick an affine open of Y, say $SpecC$. Then we can cover the inverse image $\rho^{-1}SpecC$ using affines. Let one of be $SpecB$. Now we want to give a map:

$$\rho^*(\mathrm{R}^i\pi_*\mathcal{F})(SpecB) \to \mathrm{R}^i\pi'_*(\rho'^*\mathcal{F})(SpecB)$$

which will be a map:

$$H^i(X,\mathcal{F}) \otimes_C B \to H^i(X', \rho'^*\mathcal{F})$$

Therefore we can assume that ρ is affine. Recall that we need to choose an affine open cover of X to compute $H^i(X,\mathcal{F})$, and since ρ' is affine, we know that the inverse images in X' can be used to compute $H^i(X',\rho'^*\mathcal{F})$. Then I will let you see that the complex we use to compute the cohomology of $\rho'^*\mathcal{F}$ is the complex we use to compute the cohomology of $\mathcal{F}$ tensored with $\otimes_C B$, then the HFFH theorem gives you the desired morphism.

Now since B is flat over C, then tensor $\otimes_C B$ is exact, then the HFFH map is going to be an isomorphism, we are done. $\square$

Remark 1.22. If we only want to prove $i = 0$, then we can just assume π is quasiseparated instead of separated since separated is only intended for the Cech cohomology to make sense. If we only care about the usual pushforward, we only need qcqs to make sure the pushforward is still quasicoherent and now we only care about the 0-th cohomology of the complex, so we don't care if the intersection is affine or not.

Remark 1.23. Recall that if $Y' \to Y$ is $Spec\mathcal{O}_{Y,q} \to Y$ where q is the generic point of a reduced component of Y (that is to make sure that $\mathcal{O}_{Y,q}$ is a field), then we know that since $Y' \to Y$ is flat, we can apply the above result and get an isomorphism between the stalk of the i-th higher pushforward and the i-th cohomology of the sheaf $\mathcal{F}$ on the fiber of the generic point.

This is the first example something that is important: understanding cohomology of quasicoherent sheaves on fibers through the stalk of the higher pushforward. This point of view provide the higher pushforward very important geometric meaning. In the next chapter, we will discuss when and how an when the cohomology a flat quasicoherent sheaf commutes with any base change.

Now we will study how closed subschemes and the corresponding ideal sheaf pullback under flat morphisms. Recall that if $i : X \hookrightarrow Y$ is a closed embedding and $\mu : Y' \to Y$ is any morphism, then we can form the fiber product $i' : X' \to Y'$ which is also a closed embedding:

$$\begin{tikzcd} X' \arrow[r] \arrow[d, hook, "i'"] & X \arrow[d, hook, "i"] \\ Y' \arrow[r, "\mu"] & Y \end{tikzcd}$$

We know that as closed embeddings, i, i' correspond to quasicoherent ideal sheaves $\mathcal{I}_{X/Y}$ and $\mathcal{I}_{X'/Y'}$. Consider the exact sequence of the closed embedding i:

$$0 \to \mathcal{I}_{X/Y} \to \mathcal{O}_Y \to i_*\mathcal{O}_X \to 0$$

We can apply the pullback functor μ^* to get a right exact sequence:

$$\mu^*\mathcal{I}_{X/Y} \to \mu^*\mathcal{O}_Y \to \mu^* i_*\mathcal{O}_X \to 0$$

Notice that we also have the exact sequence corresponding to the closed embedding i', and we will have:

$$\begin{tikzcd} & \mu^*\mathcal{I}_{X/Y} \arrow[r] & \mu^*\mathcal{O}_Y \arrow[r] \arrow[d, "\simeq"] & \mu^* i_*\mathcal{O}_X \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{I}_{X'/Y'} \arrow[r] & \mathcal{O}_{Y'} \arrow[r] & i'_*\mathcal{O}_{X'} \arrow[r] & 0 \end{tikzcd}$$

where the isomorphism between $\mu^*\mathcal{O}_Y \to \mathcal{O}_{Y'}$ is the canonical one. Then notice that if $Y = SpecA$ then we know that the exact sequence of i is:

$$0 \to I \to A \to A/I \to 0$$

Say we pick $SpecB$ in $\mu^{-1}SpecA$, then we know that the pullback morphism on $SpecB$ is:

$$I \otimes_A B \to B \to A/I \otimes_A B$$

Then you can check that $\mu^*\mathcal{I}_{X/Y} \to \mu^*\mathcal{O}_Y \to \mathcal{O}_{Y'} \to i'_*\mathcal{O}_{X'}$ on $SpecB$ is:

$$I \otimes_A B \to B \to B \to B/IB$$

Therefore $\mu^*\mathcal{I}_{X/Y} \to \mu^*\mathcal{O}_Y \to \mathcal{O}_{Y'} \to i'_*\mathcal{O}_{X'}$ is 0, then the universal property cokernel gives a canonical map $\mu^* i_*\mathcal{O}_X \to i'_*\mathcal{O}_{X'}$, and we know that affine locally it's just $A/I \otimes_A B \to B/IB$ and you can see that this an isomorphism, therefore we actually have the following diagram:

$$\begin{tikzcd} & \mu^*\mathcal{I}_{X/Y} \arrow[r] \arrow[d, dashed] & \mu^*\mathcal{O}_Y \arrow[r] \arrow[d, "\simeq"] & i_*\mathcal{O}_X \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & \mathcal{I}_{X'/Y'} \arrow[r] & \mathcal{O}_{Y'} \arrow[r] & i'_*\mathcal{O}_{X'} \arrow[r] & 0 \end{tikzcd}$$

where the dotted morphism is induced by the universal property of kernel which affine locally looks like $I \otimes_A B \to IB$. This maps is clearly surjective and if the row on the top is exact, then we can use five lemma to show that the induced morphism is an isomorphism. Recall that when μ is flat, the pullback is exact, then in this case the induced morphism is indeed an isomorphism. We will later using the ideal theoretic criterion to prove that not only flat morphsim implies this above isomorphism, the above isomorphism also characterizes flatness.

Remark 1.24. As an example of what happens when μ is not flat, consider the map $Speck \to Speck[x]$ which maps $Speck$ to the origin. We know that this is a closed embedding and the exact sequence is:

$$0 \to (t) \to k[t] \to k \to 0$$

However, if we again pullback this along $Speck \to Speck[t]$, we get:

$$(t)/(t^2) = (t) \otimes_{k[t]} k[t]/(t) \to k[t]/(t) \to k \to 0$$

Notice that $(t)/(t^2) \to k[t]/(t)$ is $at \mapsto 0$, therefore not injective.

Moreover we know that the induced map between the pullback ideal sheaf and the ideal sheaf of the pullback is:

$$(t) \otimes_{k[t]} k[t]/(t) \to 0$$

this is clearly not an isomorphism.

Therefore we know that the ideal sheaf of the pullback closed embedding is the pullback of the ideal sheaf of the original closed embedding in a canonical way. We not only just have this:

Proposition 1.25. Let D be an effective Cartier divisor on Y and $\pi : X \to Y$ is a flat morphism, then the pullback of D by π is also an effective Cartier divisor of X.

Proof. Let $\mathcal{I}_D$ be the ideal sheaf of D. We just prove that the ideal sheaf corresponding to the pullback of D is just $\pi^*\mathcal{I}_D$. We want to check that locally it's generated by a nonzero divisor, I will let you see that we can assume that Y is $SpecA$ and $\mathcal{I}_D$ is just (f) where f in A is a nonzero divisor, say X is $SpecB$ we know that the ideal sheaf is (f) in B. Notice that to check f is a nonzerodivisor is to check $B \xrightarrow{\times f} B$ is injective, but this is the tensor of $A \xrightarrow{\times f} A$ with B, which is injective since B is flat over A. ☐

Corollary 1.26. If $i : X \hookrightarrow Y$ is a regular embedding, then if $Y' \to Y$ is flat, then the pullback $i' : X' \hookrightarrow Y'$ is also a regular embedding.

Proof. Recall that to check that the pullback is still a regular embedding, it suffices to check affine locally, then again we can think that $X = SpecA/I, Y = SpecA, Y' = SpecB$ are affine and we can assume that I is generated by a regular sequence $f_1, \cdots, f_d$, then we know that we can factor this embedding as a sequence of effective Cartier divisors:

$$\begin{tikzcd} \cdots \arrow[r] & SpecA/(f_1, f_2) \arrow[r, hook] & SpecA/(f_1) \arrow[r, hook] & SpecA \\ \cdots \arrow[r] & SpecB/(f_1, f_2) \arrow[r, hook] \arrow[u] & SpecB/(f_1) \arrow[r, hook] \arrow[u] & SpecB \arrow[u] \end{tikzcd}$$

This implies that IB is also generated by a regular sequence since each embedding in the row at the bottom is also an effective Cartier divisor. $\square$

For later use, we here explore more on pullbacks of closed embeddings:

Proposition 1.27. Suppose that $\pi : X \to Y$ is a morphism and $Z \hookrightarrow Y$ is a closed embedding cut out by an ideal sheaf $\mathcal{I} \subset \mathcal{O}_Y$. Just for this exercise, we use $\pi^\flat \mathcal{I} \subset \mathcal{O}_X$ to denote the ideal sheaf of the pullback of $Z \hookrightarrow Y$ by π (so $\pi^*\mathcal{I} \to \pi^\flat\mathcal{I}$ is also surjective but not injective if π is not flat). Then $(\pi^\flat\mathcal{I})^n = \pi^\flat(\mathcal{I}^n)$.

Proof. Recall that the two quasicoherent sheaf are both ideal sheaves of $\mathcal{O}_X$. Then we only need to check that they are locally the same, therefore we can again assume $X = SpecA/I, Y = SpecA, Y' = SpecB$ are affine. Then we know that locally $\pi^\flat\mathcal{I}$ is just $\widetilde{IB} \hookrightarrow \widetilde{B}$ and $\pi^\flat(\mathcal{I}^n)$ is just $\widetilde{I^nB} \hookrightarrow \widetilde{B}$. By the definition of $(\pi^\flat\mathcal{I})^n$, locally on $SpecB$ it's $\widetilde{IB^n} = \widetilde{I^nB}$, therefore we are done. $\square$

Proposition 1.28. If we use the notation from above, and assume π is flat with $Y = \mathbb{A}^n_k$ and Z is the origin.

$$\begin{tikzcd} W \arrow[r] \arrow[d, hook] & Z = Speck \arrow[d, hook] \\ X \arrow[r, "\pi"'] & Y = \mathbb{A}^n_k \end{tikzcd}$$

Let $\mathcal{J} = \pi^*\mathcal{I}$ be the quasicoherent ideal sheaf of ideals on X cutting out W, the pullback of Z by X. Then the quasicoherent graded sheaf of $\mathcal{O}_X$ algebras $\oplus_{n\geqslant 0}\mathcal{J}^n/\mathcal{J}^{n+1}$ is isomorphic as $\mathcal{O}_X$-algebra to the pushforward of $\mathcal{O}_W[x_1, \cdots, x_n]$.

Proof. The multiplication in $\oplus_{n\geqslant 0}\mathcal{J}^n/\mathcal{J}^{n+1}$ can be defined by working affine locally and then check it is compatible with restricting to distinguished opens since $\oplus_{n\geqslant 0}\mathcal{J}^n/\mathcal{J}^{n+1}$ is quasicoherent. We assume that affine locally we have $\oplus_{n\geqslant 0}J^n/J^{n+1}$ and multiplication from J^r/J^{r+1} and J^m/J^{m+1} goes to J^{m+r}/J^{m+r+1}. You should check this is indeed well defined and compatible with localization to a distinguished open.

Then let's prove that $\mathcal{J}^r/\mathcal{J}^{r+1} \simeq \mathrm{Sym}^r(\mathcal{J}/\mathcal{J}^2)$. Note that Y is affine, therefore we know that each affine open $SpecB$ in X, we can assume it's in the inverse image of an affine open of Y, i.e. itself. We will use A to denote $k[x_1, \cdots, x_n]$. Let's still say that the image of x_i in B is x_i. Then we know that $\mathcal{J}$ locally looks like $J = (x_1, \cdots, x_n)$. Let $C = B/J$, we know that J^r/J^{r+1} are C-modules and I claim that we have a canonical C-module isomorphism:

$$J^r/J^{r+1} \simeq \oplus_{a_1+\cdots+a_n=r}C[x_1^{a_1}\cdots x_n^{a_n}]$$

where the right hand is the direct sum labeled by monomials of degree n in $x_1, \cdots, x_n$. Clearly we have a map from right to left defined by $x_{i_1}\cdots x_{i_n}$ to $\overline{x_{i_1}\cdots x_{i_n}}$. This is clearly surjective and we only need to show it's injective. To show this, we will show that if:

$$\sum_{a_1\cdots a_n=r} b_{a_1\cdots a_n}x_1^{a_1}\cdots x_n^{a_n} = 0$$

Then $b_{a_1\cdots a_r} \in J$. To see this, we will need the fact that $x_1, \cdots, x_n$ is a regular sequence. Moreover, this is not just any arbitrary regular sequence, since we know that in $k[x_1, \cdots, x_n]$, we can arbitrarily reorder $x_1, \cdots, x_n$ and it remains a regular sequence, therefore so is $(x_1, \cdots, x_n) \in B$. Let's first prove the case when $r = 2$, i.e. J/J^2 is $\oplus_{1\leqslant i\leqslant n}C[x_i]$. Then we first module by $(x_1, \cdots, x_{n+1})$, we get $b_n \in J$, then we proceed by induction. Then we know that $\mathrm{Sym}^r(J/J^2)$ is free over C and can be though of as the monomials of degree r with variables $x_1, \cdots, x_n$, and we have a canonical C-linear map:

$$\mathrm{Sym}^r(J/J^2) \to J^r/J^{r+1}$$

which is defined by the universal property of symmetric products and it's clearly surjective, therefore only injective is the issue. We know that under our identification of J/J^2 as $\oplus_{1\leqslant i\leqslant n}C[x_i]$, the above map is the obvious map:

$$\oplus_{a_1+\cdots+a_n=r}C[x_1^{a_1}\cdots x_n^{a_n}] \to J^r/J^{r+1}$$

We only need to show this is injective. Let's again prove this using induction on n. First, let's assume that there is only one variable, i.e. we only have x_1 and the result is trivial. Now suppose that this holds for $n-1$ and we want to prove for n, that is to say if:

$$\sum_{a_1+\cdots+a_n=r} b_{a_1\cdots a_n} x_1^{a_1}\cdots x_n^{a_n} \in J^{r+1}$$

we want $b_{a_1\cdots a_n} \in J$. Let's group these monimials in terms of exponents of x_n, that is:

$$\sum_{j=0}^{r} x_n^j \left(\sum_{a_1+\cdots+a_{n-1}=r-j} b_{a_1\cdots a_{n-1}j} x_1^{a_2}\cdots x_{n-1}^{a_{n-1}} \right) \in J^{r+1}$$

Let's modulo x_r, we know that $x_1, \cdots, x_{n-1}$ is still a regular sequence in $B/(x_r)$, and we know that:

$$\sum_{a_1+\cdots+a_{n-1}=n} b_{a_1 a_2\cdots a_{n-1}0} x_1^{a_1}\cdots x_{n-1}^{a_{n-1}} \in J^{r+1}/(x_r)$$

By our inductive hypothesis, we know that $b_{0a_2\cdots a_n} \in J/(x_r)$, therefore we know that $b_{a_1 a_2\cdots a_{n-1}0} \in J+(x_1) = J$. We can subtract these terms and the result are still in J^{n+1}, and now we have:

$$\sum_{j=1}^{r} x_n^j \left(\sum_{a_1+\cdots+a_{n-1}=r-j} b_{a_1\cdots a_{n-1}j} x_2^{a_1}\cdots x_{n-1}^{a_{n-1}} \right) \in J^{r+1}$$

Now we can factor out a x_n and we have $x_n \cdot z \in J^{n+1}$ where z is what is left after we factor out x_r. Notice that each term in z is in J^r therefore we know that $z \in J^r$, then we can use the same method to conclude that the coefficients of these terms with 0 exponent of x_n is in J. Then we can keep doing this and conclude all coefficients are in J. This concludes injectivity.

Finally you can check this is compatible with localizations to distinguished opens which finishes the isomorphism. Therefore:

$$\oplus_{n\geqslant 0}\mathcal{J}^n/\mathcal{J}^{n+1} \simeq \oplus_{n\geqslant 0}\mathrm{Sym}^n(\mathcal{J}/\mathcal{J}^2)$$

Therefore now we only need to show that:

$$\oplus_{n\geqslant 0}\mathrm{Sym}^n(\mathcal{J}/\mathcal{J}^2) \simeq \mathcal{O}_W[x_1,\cdots,x_n]$$

Notice that if we can prove $\mathcal{J}/\mathcal{J}^2$ is $\mathcal{O}_W[x_1]\oplus\cdots\mathcal{O}_W[x_n]$, then we are done. We already proved the local version above and notice that both are quasicoherent, and the isomorphism build affine locally obvious is compatible with restricting to a distinguished open, thus we are done. ◻

Proposition 1.29. Let $\pi : X \to Y$ be a flat morphism and $Z \hookrightarrow Y$ is a closed embedding. Then there is a canonical isomorphism:

$$(\mathrm{Bl}_Z Y) \times_Y X \simeq \mathrm{Bl}_{Z\times_Y X} X$$

Proof. Consider the following diagram:

$$\begin{tikzcd}
 & & E_Z Y \arrow[r, hook] \arrow[dd] & \mathrm{Bl}_Z Y \arrow[dd] \\
Z \times_Y (X \times_Y \mathrm{Bl}_Z Y) \arrow[r, hook] \arrow[dd] \arrow[urr, dashed] & X \times_Y \mathrm{Bl}_Z Y \arrow[urr] \arrow[dd] & & \\
 & & Z \arrow[r, hook] & Y \\
Z \times_Y X \arrow[r, hook] \arrow[urr] & X \arrow[urr, "\pi"'] & &
\end{tikzcd}$$

Here the dotted morphism is induced by the universal property, and you can see that all sides of the square are Cartesian. Then we know that since π is flat, so is $X \times_Y \mathrm{Bl}_Z Y \to \mathrm{Bl}_Z Y$. Then we know that since the pullback of an effective Cartier divisor by a flat morphism is also an effective Cartier divisor, then $Z \times_Y (X \times_Y \mathrm{Bl}_Z Y) \to X \times_Y \mathrm{Bl}_Z Y$ is also an effective Cartier divisor. Then we need to check the universal property. Consider the following diagram:

$$\begin{tikzcd}
D \arrow[r, hook] \arrow[drrr, dashed] \arrow[ddr, dashed] \arrow[ddddr] & W \arrow[drrr, dashed] \arrow[ddr, dashed] \arrow[ddddr] & & & \\
& & & E_Z Y \arrow[r, hook] \arrow[dd] & \mathrm{Bl}_Z Y \arrow[dd] \\
& Z \times_Y (X \times_Y \mathrm{Bl}_Z Y) \arrow[urr] \arrow[r, hook] \arrow[dd] & X \times_Y \mathrm{Bl}_Z Y \arrow[urr] \arrow[dd] & & \\
& & & Z \arrow[r, hook] & Y \\
& Z \times_Y X \arrow[urr] \arrow[r, hook] & X \arrow[urr, "\pi"'] & &
\end{tikzcd}$$

where $D \hookrightarrow W$ is an effective Cartier divisor and the square $D, W, X, Z \times_Y X$ is Cartesian. The dotted morphisms are induced by the obvious universal properties of Cartesian squares which are all uniquely given. This suffice to check the universal property of blow-ups. $\square$

Example 1.30. Here is an example that tells you that without flatness, blow-up does not necessarily commute with base change.

Let's consider the blow-up of the plane at the origin and the following diagram:

$$\begin{tikzcd}
& & E_{(0,0)}\mathbb{A}^2_k \arrow[r, hook] \arrow[dd] & \mathrm{Bl}_{(0,0)}\mathbb{A}^2_k \arrow[dd] \\
Projk[X,Y] \arrow[r] \arrow[urr, dashed] \arrow[dd] & Projk[X,Y] \arrow[urr] \arrow[dd] & & \\
& & Speck \arrow[r, hook] & Speck[x,y] \\
Speck \arrow[urr] \arrow[r, hook] & Speck \arrow[urr] & &
\end{tikzcd}$$

Clearly $Speck \to Speck[x, y]$ defined by $x, y \mapsto 0$ is not flat then the blow-up of $Speck$ at (0) should still be $Speck$, instead of $\mathbb{P}^1$.

2 Flatness through Tor

Recall that in the previous section, we defined the Tor functor and we prove that an A-module M is flat iff $\mathrm{Tor}_1^A(M, N) = 0$ for all A-modules N. In this section, we will exploit this easy fact and see that actually a lot of useful results comes out of this.

Sometimes, we can compute the Tor functor using definition. For example, we will need the following computation a lot:

Example 2.1. Let M be an A-module and $x \in A$, we define $(M : x)$ as the submodule of M killed by x. Then if x is not a zerodivisor of A (not of M), then we have the following computation:

$$\mathrm{Tor}_i^A(M, A/(x)) = \begin{cases} M/xM & \text{if } i = 0, \\ (M : x) & \text{if } i = 1, \\ 0 & \text{if } i > 1. \end{cases}$$

This is easy to compute, recall that to compute this Tor, we first need a free resolution of $A/(x)$, and since x is not a zerodivisor, we use:

$$\cdots \to 0 \to A \xrightarrow{\times x} A \to A/(x) \to 0$$

Then we know that after we tensor with M, we get:

$$\cdots \to 0 \to M \xrightarrow{\times x} M \to 0$$

Compute the homology and we are done.

Remark 2.2. As a corollary of the above example, we see that if A is an integral domain (which means that all nonzero elements are nonzerodivisors), then if M is flat, then since the first Tor vanishes, we know that M is in fact torsion free. Moreover, recall that we only talk about torsion of a module over nonzero divisors, therefore this above examples justifies the notation for Tor, which is just short for torsion.

Now if $A \to B$ is flat and M, N are two A-module. Given a free resolution of N:

$$\cdots \to A^{\oplus n_2} \to A^{\oplus n_1} \to A^{\oplus n_0} \to N \to 0$$

To compute $\mathrm{Tor}_i^A(M, N)$, we tensor M with the above resolution with N truncated and take the homology:

$$\cdots \to M^{\oplus n_2} \to M^{\oplus n_1} \to M^{\oplus n_0} \to 0$$

Since B is flat over A, we know that the FHHF theorem gives us an isomorphism between $B \otimes_A \mathrm{Tor}_i^A(M, N)$ and the i-th homology of:

$$\cdots \to (M \otimes_A B)^{\oplus n_2} \to (M \otimes_A B)^{\oplus n_1} \to (M \otimes_A B)^{\oplus n_0} \to 0$$

Notice that since B is flat over A, we have a free resolution of $N \otimes_A B$ by tensoring B with the free resolution of N:

$$\cdots \to B^{\oplus n_2} \to B^{\oplus n_1} \to B^{\oplus n_0} \to N \otimes_A B \to 0$$

We know that we can truncate $N \otimes_A B$ and then tensor with $\otimes_B (M \otimes_A B)$, and then we get the complex:

$$\cdots \to (M \otimes_A B)^{\oplus n_2} \to (M \otimes_A B)^{\oplus n_1} \to (M \otimes_A B)^{\oplus n_0} \to 0$$

Therefore the homology of the above complex is just $\mathrm{Tor}_i^B(M \otimes_A B, N \otimes_A B)$, therefore we have an isomorphism:

$$B \otimes_A \mathrm{Tor}_i^A(M, N) \simeq \mathrm{Tor}_i^B(M \otimes_A B, N \otimes_A B)$$

Recall that Tor is symmetric, i.e. we have isomorphisms $\mathrm{Tor}_i^A(M, N) \simeq \mathrm{Tor}_i^A(N, M)$, and this gives us a very important result:

Proposition 2.3. If $0 \to N' \to N \to N'' \to 0$ is a SES of A-modules with N'' flat, then for any A-module M, we have a SES:

$$0 \to N' \otimes_A M \to N \otimes_A M \to N'' \otimes_A M \to 0$$

Proof. Recall that for such a SES, we have a LES:

$$\cdots \to \mathrm{Tor}_1^A(M, N'') \to N' \otimes_A M \to N \otimes_A M \to N'' \otimes_A M \to 0$$

Note that using symmetry we have $\mathrm{Tor}_1^A(M, N'') \simeq \mathrm{Tor}_1^A(N'', M) = 0$, we are done. □

Corollary 2.4. Suppose that $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a SES of quasicoherent sheaves on a scheme Y with $\mathcal{F}''$ flat over Y, then if $\pi : X \to Y$ is any morphism of schemes, we have a SES:

$$0 \to \pi^*\mathcal{F}' \to \pi^*\mathcal{F} \to \pi^*\mathcal{F}'' \to 0$$

Proof. Recall that we proved before that if π is flat, then we have this SES, but now we are saying that no matter what morphism we use, this is still exact. We check it's exact on the level of stalks. Say $x \in X$, then the induced morphism on stalks is:

$$0 \to \mathcal{F}'_{\pi(x)} \otimes_{\mathcal{O}_{Y,\pi(x)}} \mathcal{O}_{X,x} \to \mathcal{F}_{\pi(x)} \otimes_{\mathcal{O}_{Y,\pi(x)}} \mathcal{O}_{X,x} \to \mathcal{F}''_{\pi(x)} \otimes_{\mathcal{O}_{Y,\pi(x)}} \mathcal{O}_{X,x} \to 0$$

Then we use the above proposition. $\square$

Proposition 2.5. Suppose that $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules. Then if M, M'' are both flat, so is M'.

Proof. Recall that to prove M' is flat, we need to show that for all N, we have $\operatorname{Tor}_1^A(M', N)$ is 0. For N, we consider the LES of $\operatorname{Tor}(N, -)$, which is:

$$\cdots \to \operatorname{Tor}_2^A(N, M'') \to \operatorname{Tor}_1^A(N, M') \to \operatorname{Tor}_A^1(N, M) \to \cdots$$

Notice that since M'', M are flat and Tor is symmetric, we know that $\operatorname{Tor}_2^A(N, M'') = 0$ and $\operatorname{Tor}_A^1(N, M) = 0$, therefore $\operatorname{Tor}_1^A(N, M') \simeq \operatorname{Tor}_1^A(M', N) = 0$. $\square$

Proposition 2.6. Using the same notation as above, then if M', M'' are flat, so is M.

Proof. Use the same idea as above, i.e. consider the LES. $\square$

Example 2.7. Here is an easy example that M', M are flat but M' is not flat: let A be an integral domain $x \in A$ is not zero, then consider:

$$0 \to A \xrightarrow{\times x} A \to A/(x) \to 0$$

We know that as an A-module, $A/(x)$ is not trosion free, thus not flat.

Remark 2.8. This is very much like the result we had for SES of vector bundles. If we have a SES sequence of quasicoherent sheaves, then if the first and third are vector bundles, then so is the one in the middle. If the one in the middle and the last one are vector bundles, then so is the first one. Similarly, we still can not conclude the last one is a vector bundle just from the fact that the first two are.

Proposition 2.9. If we have a exact sequence of flat A-modules:

$$\cdots M_{-2} \to M_{-1} \to M_0 \to 0$$

Then for any A-module N, tensor this sequence with N remains exact.

Proof. Recall that we can always always divide a LES into a sequence of SES. For example, we have:

$$\begin{tikzcd}
0 \arrow[dr] & & & & & & 0 \\
& \mathrm{im} = \ker \arrow[dr] & & & & \mathrm{im} = \ker \arrow[ur] \arrow[dr] & \\
M_{-2} \arrow[ur] \arrow[rr] & & M_{-1} \arrow[rr] \arrow[dr] & & M_0 \arrow[ur] \arrow[rr] & & 0 \\
& & & \mathrm{im} = \ker \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Notice that in $M_0 \to \mathrm{im} = \ker = 0$, 0 is clearly flat, therefore we know that the $\mathrm{im} = \ker$ in $\mathrm{im} = \ker \to M_0$ is still flat, then we keep going and we know that each term in these SES are flat, then tensor with any module preserves exactness. Therefore tensor with any N preserves kernels and images, then we are done. $\square$

Proposition 2.10. If $0 \to M_0 \to M_1 \to \cdots \to M_n \to 0$ is an exact sequence and M_i is flat for $i > 0$, then M_0 is also flat.

Proof. Let's break this LES into SESs, the first of which is:

$$0 \to M_0 \to M_1 \to \mathrm{im} \to 0$$

First, we know that M_1 is flat, then if Im is flat, we can use the previous proposition to conclude M_0 is flat.

To show im is flat, we use the next SES and we keep proceeding, until we need the fact that M_n is flat, which is given, then we are done. □

Suppose that the following is a fibered product diagram:

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[d] & X \arrow[d] \\ Y \arrow[r, "\beta"'] & Z \end{tikzcd}$$

Let $V \hookrightarrow X$ is a closed subscheme. Then $Y \times_Z V$ is a closed subscheme of W. Again, there are two possible senses in which we can "pullback" $\mathcal{I}_{V/X}$ to W. First, we can think of it as the quasicoherent sheaf $\alpha^*\mathcal{I}_{V/X}$ and then secondly we can think of the ideal sheaf corresponding to $Y \times_Z V$, i.e. the quasicoherent ideal sheaf $\mathcal{I}_{Y\times_Z V/W}$. As what we saw before, there is a canonical surjective morphism $\alpha^*\mathcal{I}_{V/X} \to \mathcal{I}_{Y\times_Z V/W}$, but this is not necessarily an isomorphism. We also saw that if β is flat, then the above morphism is an isomorphism. Now we give another criterion when this morphism is an isomorphism.

Proposition 2.11. Let's use the above notation, but with one more hypothesis that V is flat over Z (with no hypothesis on β), then the canonical morphism $\alpha^*\mathcal{I}_{V/X} \to \mathcal{I}_{Y\times_Z V/W}$ is an isomorphism.

Proof. Let's call the closed embedding $V \hookrightarrow X$ i. Consider the canonical SES of closed embedding:

$$0 \to \mathcal{I}_{V/X} \to \mathcal{O}_X \to i_*\mathcal{O}_V \to 0$$

Recall that if $i_*\mathcal{O}_V$ is flat over X, then we can apply the pullback functor α^* and this sequence will remain exact, then we know from the five lemma that the canonical morphism is an isomorphism. However, it almost never happens that $i_*\mathcal{O}_V$ is flat over X for the same reason that A/I is almost never flat over A, as what we will see in the next section. Therefore we can not direct use the fact we proved before that if we have a SES of quasicoherent sheaves with the last one being flat, then pullback functor is exact on this SES.

If we still want to prove the canonical morphism $\alpha^*\mathcal{I}_{V/X} \to \mathcal{I}_{W\times_X V/W}$ is an isomorphism, let's check locally. Recall that W can be covered by $SpecA \otimes_C B$ where $SpecA, SpecB$ are affine opnes in Y, X that maps affine open $SpecC$ on Z. Then we know that locally this Cartesian diagram looks like:

$$\begin{tikzcd} \cdots \arrow[r] \arrow[d] & SpecB/I \arrow[d] \\ SpecB \otimes_C A \arrow[r] \arrow[d] & SpecB \arrow[d] \\ SpecA \arrow[r] & SpecC \end{tikzcd}$$

We know that by our assumption B/I is flat over C. Then we know that in this local case, the canonical morphism is defined by:

$$\begin{tikzcd} & I \otimes_B (B \otimes_C A) \arrow[r] \arrow[d, dashed] & B \otimes_C A \arrow[r] \arrow[d, "\mathrm{id}"] & (B/I) \otimes_B (B \otimes_C A) \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & I(B \otimes_C A) \arrow[r] & B \otimes_C A \arrow[r] & B \otimes_C A/I(B \otimes_C A) \arrow[r] & 0 \end{tikzcd}$$

We know from five lemma that if the top row is exact, then we can conclude the dotted morphism is an isomorphism. Notice that the top row can be identified as tensor by $\otimes_C A$, however, recall that B/I is flat over C, therefore we know that the tensor is exact, we are done. □

Before we end this section, let's spend sometime discussing the Kunneth formula for quasicoherent sheaves. Since we will not use it later, we will only prove a special case of it. The way we prove it can be generalized considerably to cover a wide variety of situations.

Proposition 2.12. If X is a separated scheme with $j : SpecA \hookrightarrow X$ an open embedding. Then $j_*\mathcal{O}_{SpecA}$ is quasicoherent on X which is flat over X.

Proof. First I claim that this open embedding $SpecA \hookrightarrow X$ is an affine morphism. To see this, recall that in a separated scheme, intersection of two affine opens is still affine. Or you can consider the following diagram:

$$\begin{tikzcd} SpecA \arrow[rr, hook] \arrow[dr] & & X \arrow[dl] \\ & Spec\mathbb{Z} & \end{tikzcd}$$

Since $X \to Spec\mathbb{Z}$ is separated, we know that the diagonal is a closed embedding, in particular affine, therefore we know from the cancellation theorem that $SpecA \hookrightarrow X$ is also affine.

Recall that affine morphisms are all qcqs, therefore the pushforwad of quasicoherent sheaves is quasicoherent. Then we know that $j_*\mathcal{O}_{SpecA}$ is quasicoherent. To show it's flat, for $x \in SpecA$, we know that the ring map is just the identity $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$, flat. If $x \notin SpecA$, you might be tempted to try to show that there is always some open of x that doesn't intersect $SpecA$ (since this is what Hausdorffness implies in topology). But this is not true here as one easy example is to consider:

$$Speck[t, t^{-1}] \hookrightarrow Speck[t]$$

So we need to use another way to prove this. Recall that we prove that for an affine morphism $\pi : X \to Y$ and a quasicoherent sheaf $\mathcal{F}$ on X, we know that $\mathcal{F}$ is flat over Y iff $\pi_*\mathcal{F}$ is flat over Y. Therefore here we only need to prove that $\mathcal{O}_{SpecA}$ is flat over X, that if we need to show that for each $x \in SpecA$, $\mathcal{O}_{SpecA,x}$ is flat over $\mathcal{O}_{X,x}$, but the corresponding morphism is just the identity, thus flat. $\square$

Now let X be a quasicompact separated scheme and we have a finite affine cover $X = \cup_{i=1}^n U_i$. We will use $\mathfrak{U} = \{U_i\}$ to denote this cover. Using the usual notation from the Cech complex, for $I \subset \{1, 2, \cdots, n\}$, we use U_I to denote $\cap_{i\in I} U_i$ and let $j^I : U_I \hookrightarrow X$ be the corresponding open cover. Recall that since X is separate, then any U_I is still going to be affine.

For the purpose of proving the Kunneth formula, we define the **Cech complex of sheaves for the cover** $\mathfrak{U}$, denoted by $C^\bullet_X(\mathfrak{U})$:

$$0 \to \oplus_{|I|=1} j^I_* \mathcal{O}_{U_I} \to \oplus_{|I|=2} j^I_* \mathcal{O}_{U_I} \to \cdots \to \oplus_{|I|=n} j^I_* \mathcal{O}_{U_I} \to 0$$

The differential in this complex is again defined like what we have seen in the usual Cech complex. For $V \subset X$, we define for example:

$$\oplus_{|I|=i} j^I_* \mathcal{O}_{U_I}(V) \to \oplus_{|I|=i+1} j^I_* \mathcal{O}_{U_I}(V)$$

which is:

$$\oplus_{|I|=i} \mathcal{O}_X(U_I \cap V) \to \oplus_{|I|=i+1} \mathcal{O}_X(U_I \cap V)$$

This is defined to be the Cech complex differential for the quasicoherent sheaf $\mathcal{O}_V$ on V, with respect to the cover $U_1 \cap V, \cdots, U_n \cap V$. You can easily check this is indeed a morphism of $\mathcal{O}_X$-modules and the composition of two differentials is indeed 0.

We also define the **augmented Cech complex of sheaves for** the cover $\mathfrak{U} = \{U_i\}$ denoted by $C^\bullet_{X,\mathrm{aug}}(\mathfrak{U})$ by appending $\mathcal{O}_X$, i.e.:

$$0 \to \mathcal{O}_X \to \oplus_{|I|=1} j^I_* \mathcal{O}_{U_I} \to \oplus_{|I|=2} j^I_* \mathcal{O}_{U_I} \to \cdots \to \oplus_{|I|=n} j^I_* \mathcal{O}_{U_I} \to 0$$

The first morphism $\mathcal{O}_X \to \oplus_{|I|=1} j^I_* \mathcal{O}_{U_I}$ is defined by restriction and you can check the composition is still 0.

Proposition 2.13. The augmented Cech complex of sheaves $C^\bullet_{X,\mathrm{aug}}(\mathfrak{U})$ is an exact complex of flat sheaves over X.

Proof. Flatness is easy by the previous proposition. To prove exactness, let's check this locally. Suppose that SV is an affine open of X, since each of these sheaves are quasicoherent, we know that this sequence is determined by the morphism on the global sections. Recall that the first few maps are:

$$0 \to \Gamma(V, \mathcal{O}_X) \to \oplus_i \Gamma(V \cap U_i, \mathcal{O}_X) \to \oplus_{i,j} \Gamma(V \cap U_i \cap U_j, \mathcal{O}_X)$$

We know from the sheaf axioms that this sequence is exact at $\Gamma(V, \mathcal{O}_X)$ and at $\oplus_i \Gamma(V \cap U_i, \mathcal{O}_X)$. Exactness later comes from the fact that quasicoherent sheaves on an affine scheme has no higher Cech cohomology. ☐

Before proceeding further, let's prove a useful homological statement. Recall that we proved before that a finite exact sequence of flat modules remain exact upon tensoring with any module and tensoring any exact sequence by a flat module preserves exactness. The following result generalize these two results:

Proposition 2.14. Let $C^\bullet$ be a finite exact sequence of A-modules and $F^\bullet$ is a finite exact sequence of flat modules. Then the total complex of the double complex $C^\bullet \otimes_A F^\bullet$ is also exact.

Proof. Recall that to prove a complex is exact is to prove that all cohomology groups are 0. The following is the double complex:

$$\begin{tikzcd}
 & \cdots & \cdots & \cdots & \\
0 \arrow[r] & C_0 \otimes_A F_2 \arrow[r] \arrow[u] & C_1 \otimes_A F_2 \arrow[r] \arrow[u] & C_2 \otimes_A F_2 \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & C_0 \otimes_A F_1 \arrow[r] \arrow[u] & C_1 \otimes_A F_1 \arrow[r] \arrow[u] & C_2 \otimes_A F_1 \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & C_0 \otimes_A F_0 \arrow[r] \arrow[u] & C_1 \otimes_A F_0 \arrow[r] \arrow[u] & C_2 \otimes_A F_0 \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

We can first compute the spectral sequence in the rightward orientation. Notice that since F_i are all flat, then we know that tensoring with F_i remains exact, therefore we know that the first page are all 0. Therefore we know that the spectral sequence stabilizes at the first page and are all 0, therefore we know that the cohomology of the total complex are all 0. This agrees with the result if we compute the spectral sequence in the upward direction since we know that a finite sequence of flat modules remains exact upon tensoring with any modules. ☐

Remark 2.15. First notice that the above result indeed generalize the two result we stated before the proposition. For example, if we want to prove that a finite exact sequence of flat modules remains exact upon tensoring with any module, say C. We can take the finite exact sequence to be $0 \to C \to C \to 0$. I will let you see why this suffices. Similarly, we can take the finite sequence of flat modules to be $0 \to F \to F \to 0$.

Moreover, we notice that the spectral sequence argument we use doesn't need $C^\bullet$ to be finite. Moreover, we don't even need to require $C^\bullet$ to be exact. The spectral argument can still be ran and we still use the rightward orientation. This time the first page looks like:

$$\begin{tikzcd}
\cdots & \cdots & \cdots \\
H^0(C^\bullet) \otimes F_2 \arrow[u] & H^1(C^\bullet) \otimes F_2 \arrow[u] & H^2(C^\bullet) \otimes F_2 \arrow[u] \\
H^0(C^\bullet) \otimes F_1 \arrow[u] & H^1(C^\bullet) \otimes F_1 \arrow[u] & H^2(C^\bullet) \otimes F_1 \arrow[u] \\
H^0(C^\bullet) \otimes F_0 \arrow[u] & H^1(C^\bullet) \otimes F_0 \arrow[u] & H^2(C^\bullet) \otimes F_0 \arrow[u] \\
0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

Note that $F^\bullet$ is still finite, therefore it remains exact upon tensoring any module, therefore the 2-th page vanishes and therefore this spectral sequence stabilizes at 2-th page and the cohomology of the total complex is still 0.

Remark 2.16. One case of a nonexact complex of flat $F^\bullet$ is likely to be useful in general (though we will not use it). Say that $F^\bullet$ is a flat resolution of A. Then we can run the same rightward orientation spectral sequence argument it's just this time on the second page, we will have on the first row the cohomology of $C^\bullet$ instead of all being 0. But it still stabilizes on 2-th page, therefore in this case we have the conclusion that the cohomology of the total complex is the same as the cohomology of $C^\bullet$.

Now we go back to the case of sheaves:

Proposition 2.17. Again, let X be a quasicompact and separated scheme, we use the notation from the Cech complex of sheaves of the cover $\mathfrak{U}$ as above. Let $\mathcal{F}$ be any quasicoherent sheaves on X, then I claim that $\mathcal{F} \otimes C^\bullet_{X,\mathrm{aug}}(\mathfrak{U})$ is still exact. Similarly $\mathcal{F} \otimes C^\bullet_X(\mathfrak{U})$ is exact except at the first slot, with cohomology sheaf canonically isomorphic to $\mathcal{F}$.

Proof. Let's first check $\mathcal{F} \otimes C^\bullet_{X,\mathrm{aug}}(\mathfrak{U})$ is still exact. Again, we can check it locally on affine opens. Say V is an affine open of X, then if we use F to denote $\mathcal{F}(V)$ we know that the complex of sheaves corresponds to:

$$0 \to \Gamma(V, \mathcal{O}_X) \otimes F \to \oplus_i \Gamma(V \cap U_i, \mathcal{O}_X) \otimes F \to \oplus_{i,j} \Gamma(V \cap U_i \cap U_j, \mathcal{O}_X) \otimes F \to \cdots$$

Recall that each term in the complex before we tensor with F is exact over $\Gamma(V, \mathcal{O}_X)$, therefore it remains exact even after we tensor F. This proves exactness.

Now for the nonaugmented complex, the way we prove it's exact beyong the first term is exactly the same as the above proof. To show that the kernel of the first morphism is canonically isomorphisms to $\mathcal{F}$. Notice that we have a morphism:

$$0 \to \mathcal{F} \to \oplus_i j^i_* \mathcal{O}_{U_i} \otimes \mathcal{F} \to \oplus_{i,j} j^{i,j}_* \mathcal{O}_{U_i \cap U_j} \otimes \mathcal{F}$$

which you can think of as tensor the agumented complex by $\mathcal{F}$, which is exact, therefore we know that $\mathcal{F}$ is the kernel. $\square$

Using these pleasant results from homological algebra, we can prove a version of **Kunneth formula**. Say X, Y are both quasicompact separated k-schemes (for example both are k-varieties), we consider the fibered product $X \times_k Y$ and we will call the two projections $\pi_X : X \times_k Y \to X$ and $\pi_Y : X \times_k Y \to Y$. Let $\mathfrak{U}_X, \mathfrak{U}_Y$ be finite affine open covers of X, Y.

Proposition 2.18. $\pi_X^* C^\bullet_{X,\mathrm{aug}}(\mathfrak{U}_X)$ is an exact complex of flat quasicoherent sheaves on $X \times_k Y$.

Proof. Recall that $C^\bullet_{X,\mathrm{aug}}(\mathfrak{U}_X)$ is a sequence of flat quasicoherent sheaves on X, therefore after pulling back to $X \times_k Y$, it remains exact. Recall that flatness is preserved by base change, then we can consider the following Cartesian diagram:

$$\begin{array}{ccc} X \times_k Y & \xrightarrow{\pi_X} & X \\ \downarrow & & \downarrow \\ X \times_k Y & \xrightarrow{\pi_X} & X \end{array}$$

Then we can prove that π_X^* takes quasicoherent sheaves that is flat over X to quasicoherent sheaves that is flat over $X \times_k Y$. $\square$

Similarly, we have $\pi_Y^* C^\bullet_{Y,\mathrm{aug}}(\mathfrak{U}_Y)$ is an exact complex of quasicoherent sheaf that is flat over $X \times_k Y$.

We will use the notation $C^\bullet_X \boxtimes C^\bullet_Y$ to denote the total complex of the double complex $\pi_X^* C^\bullet_X(\mathfrak{U}_X) \otimes \pi_Y^* C^\bullet_Y(\mathfrak{U}_Y)$. Similarly we can define $C^\bullet_{X,\mathrm{aug}} \boxtimes C^\bullet_{X,\mathrm{aug}}$ to be the total complex of the double complex $\pi_X^* C^\bullet_{X,\mathrm{aug}}(\mathfrak{U}_X) \otimes \pi_Y^* C^\bullet_{Y,\mathrm{aug}}(\mathfrak{U}_Y)$.

Proposition 2.19. This complex $C^\bullet_X \boxtimes C^\bullet_Y$ is a complex of flat quasicoherent sheaves on $X \times_k Y$ that is exact except at the first step where the kernel is canonically identified with $\mathcal{O}_{X \times_k Y}$.

Proof. To prove this result, again we use spectral sequence argument: we know that the double complex looks like:

$$\begin{tikzcd}
\pi_X^*(\oplus_{|I|=1}j_*^I\mathcal{O}_{U_I})\otimes\pi_Y^*(\oplus_{|J|=3}j_*^J\mathcal{O}_{U_J}) \arrow[r] & \pi_X^*(\oplus_{|I|=2}j_*^I\mathcal{O}_{U_I})\otimes\pi_Y^*(\oplus_{|J|=3}j_*^J\mathcal{O}_{U_J}) \arrow[r] & \pi_X^*(\oplus_{|I|=3}j_*^I\mathcal{O}_{U_I})\otimes\pi_Y^*(\oplus_{|J|=3}j_*^J\mathcal{O}_{U_J}) \\
\pi_X^*(\oplus_{|I|=1}j_*^I\mathcal{O}_{U_I})\otimes\pi_Y^*(\oplus_{|J|=2}j_*^J\mathcal{O}_{U_J}) \arrow[r] \arrow[u] & \pi_X^*(\oplus_{|I|=2}j_*^I\mathcal{O}_{U_I})\otimes\pi_Y^*(\oplus_{|J|=2}j_*^J\mathcal{O}_{U_J}) \arrow[r] \arrow[u] & \pi_X^*(\oplus_{|I|=3}j_*^I\mathcal{O}_{U_I})\otimes\pi_Y^*(\oplus_{|J|=2}j_*^J\mathcal{O}_{U_J}) \arrow[u] \\
\pi_X^*(\oplus_{|I|=1}j_*^I\mathcal{O}_{U_I})\otimes\pi_Y^*(\oplus_{|J|=1}j_*^J\mathcal{O}_{U_J}) \arrow[r] \arrow[u] & \pi_X^*(\oplus_{|I|=2}j_*^I\mathcal{O}_{U_I})\otimes\pi_Y^*(\oplus_{|J|=1}j_*^J\mathcal{O}_{U_J}) \arrow[r] \arrow[u] & \pi_X^*(\oplus_{|I|=3}j_*^I\mathcal{O}_{U_I})\otimes\pi_Y^*(\oplus_{|J|=1}j_*^J\mathcal{O}_{U_J}) \arrow[u]
\end{tikzcd}$$

Let's first compute the spectral sequence in the rightward direction. The first page looks like:

$$\begin{tikzcd}
\pi_X^*\mathcal{O}_X\otimes\pi_Y^*(\oplus_{|J|=3}j_*^J\mathcal{O}_{U_J}) & 0 & 0 \\
\pi_X^*\mathcal{O}_X\otimes\pi_Y^*(\oplus_{|J|=2}j_*^J\mathcal{O}_{U_J}) \arrow[u] & 0 \arrow[u] & 0 \arrow[u] \\
\pi_X^*\mathcal{O}_X\otimes\pi_Y^*(\oplus_{|J|=1}j_*^J\mathcal{O}_{U_J}) \arrow[u] & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

we know that only the first column is nonzero because we know that the complex $\pi_X^*C_X^\bullet(\mathfrak{U}_X)$ is exact only except at the first step and elements in $\pi_Y^*C_Y^\bullet(\mathfrak{U}_Y)$ are all flat. Then we compute the second page, we know that since $\pi_Y^*C_Y^\bullet(\mathfrak{U}_Y)$ is exact except at the first step and objects in $\pi_Y^*C_Y^\bullet(\mathfrak{U}_Y)$ are all flat, we know that the second page looks like:

$$\begin{tikzcd}
0 & & \\
0 & 0 & \\
 & \pi_X^*\mathcal{O}_X\otimes\pi_Y^*\mathcal{O}_Y \arrow[uul] & \\
 & 0 \arrow[uul] & 0 \arrow[uul] \\
 & & 0 \arrow[uul]
\end{tikzcd}$$

Therefore the spectral sequence stabilize on the second page and the cohomology of the double complex is isomorphic to $\pi_X^*\mathcal{O}_X\otimes\pi_Y^*\mathcal{O}_Y\simeq\mathcal{O}_{X\times_k Y}$. To show that this complex consists of flat quasicoherent sheaves over $X\times_k Y$, recall that each object in $\pi_X^*C_X^\bullet(\mathfrak{U}_X)$ and $\pi_Y^*C_Y^\bullet(\mathfrak{U}_Y)$ are flat. Then we know that the tensor product is also flat since we know that if M, N are flat over A, then $M\otimes_A N$ is also flat over A.

If you want you can also prove a similar result for $C_{X,\mathrm{aug}}^\bullet\boxtimes C_{X,\mathrm{aug}}^\bullet$, notice that part of the double complex is:

$$\begin{tikzcd}
\pi_X^*\mathcal{O}_X\otimes\pi_Y^*(\oplus_{|J|=3}j_*^J\mathcal{O}_{U_J}) \arrow[r] & \pi_X^*(\oplus_{|I|=1}j_*^I\mathcal{O}_{U_I})\otimes\pi_Y^*(\oplus_{|J|=3}j_*^J\mathcal{O}_{U_J}) \arrow[r] & \pi_X^*(\oplus_{|I|=2}j_*^I\mathcal{O}_{U_I})\otimes\pi_Y^*(\oplus_{|J|=3}j_*^J\mathcal{O}_{U_J}) \\
\pi_X^*\mathcal{O}_X\otimes\pi_Y^*(\oplus_{|J|=2}j_*^J\mathcal{O}_{U_J}) \arrow[r] \arrow[u] & \pi_X^*(\oplus_{|I|=1}j_*^I\mathcal{O}_{U_I})\otimes\pi_Y^*(\oplus_{|J|=2}j_*^J\mathcal{O}_{U_J}) \arrow[r] \arrow[u] & \pi_X^*(\oplus_{|I|=2}j_*^I\mathcal{O}_{U_I})\otimes\pi_Y^*(\oplus_{|J|=2}j_*^J\mathcal{O}_{U_J}) \arrow[u] \\
\pi_X^*\mathcal{O}_X\otimes\pi_Y^*(\oplus_{|J|=1}j_*^J\mathcal{O}_{U_J}) \arrow[r] \arrow[u] & \pi_X^*(\oplus_{|I|=1}j_*^I\mathcal{O}_{U_I})\otimes\pi_Y^*(\oplus_{|J|=1}j_*^J\mathcal{O}_{U_J}) \arrow[r] \arrow[u] & \pi_X^*(\oplus_{|I|=2}j_*^I\mathcal{O}_{U_I})\otimes\pi_Y^*(\oplus_{|J|=1}j_*^J\mathcal{O}_{U_J}) \arrow[u] \\
\pi_X^*\mathcal{O}_X\otimes\pi_Y^*\mathcal{O}_Y \arrow[r] \arrow[u] & \pi_X^*(\oplus_{|I|=1}j_*^I\mathcal{O}_{U_I})\otimes\pi_Y^*\mathcal{O}_Y \arrow[r] \arrow[u] & \pi_X^*(\oplus_{|I|=2}j_*^I\mathcal{O}_{U_I})\otimes\pi_Y^*\mathcal{O}_Y \arrow[u]
\end{tikzcd}$$

This times you will see that the sequence still stabilizes on the second page but all terms are 0, therefore the cohomology of the double complex is 0. $\square$

Recall that if $\mathcal{F}, \mathcal{G}$ are quasicoherent sheaves on X, Y, we use the notation $\mathcal{F} \boxtimes \mathcal{G}$ to denote the quasicoherent sheaf $\pi_X^* \mathcal{F} \otimes \pi_Y^* \mathcal{G}$.

Proposition 2.20. $(\mathcal{F} \boxtimes \mathcal{G}) \otimes (C_X^\bullet \boxtimes C_Y^\bullet)$ is a complex of quasicoherent sheaves on $X \times_k Y$ that is exact except at the first step, with the kernel canonically identified with $\mathcal{F} \boxtimes \mathcal{G}$.

Proof. Let's append the kernel of the first morphism in the complex $C_X^\bullet \boxtimes C_Y^\bullet$, which by our previous proposition is canonically identified with $\mathcal{O}_{X \times_k Y}$. Then we obtained an finite exact complex of flat quasicoherent sheaves on $X \times_k Y$, therefore after you tensor with $\mathcal{F} \boxtimes \mathcal{G}$, it still remains exact. Therefore the kernel of the first map is identified with:

$$\mathcal{O}_{X \times_k Y} \otimes (\mathcal{F} \boxtimes \mathcal{G}) \simeq \mathcal{F} \boxtimes \mathcal{G}$$

$\square$

Now we are able to prove a version of Kunneth formula:

Theorem 2.21. (Kunneth Formula) We use the above notation, then by identifying $(\mathcal{F} \boxtimes \mathcal{G}) \otimes (C_X^\bullet \boxtimes C_Y^\bullet)$ with $(\mathcal{F} \otimes C_X^\bullet) \boxtimes (\mathcal{G} \otimes C_Y^\bullet)$, there is a canoncial isomorphism:

$$H^n(X \times_k Y, \mathcal{F} \boxtimes \mathcal{G}) \simeq \oplus_{i+j=n} H^i(X, \mathcal{F}) \otimes_k H^j(Y, \mathcal{G})$$

Proof. Let's first consider the complex $\mathcal{F} \otimes C_X^\bullet$. Suppose that $j : U \hookrightarrow X$ is any affine open embedding, I claim that we have a canonical isomorphism:

$$\mathcal{F} \otimes j_* \mathcal{O}_U \simeq j_*(\mathcal{F}|_U)$$

Notice that there is a canonical morphism of presheafs on X:

$$\mathcal{F}(V) \otimes \mathcal{O}_X(U \cap V) \to \mathcal{F}(U \cap V)$$

which is defined by restriction, this gives a morphism from the tensor product to $j_*(\mathcal{F}|_U)$ and we need to show it's an isomorphism.

Recall that these two sheaves are all quasicoherent, therefore we only need to check it's an isomorphism on each affine. Say V is an affine, then we know that since X is separated, $V \cap U$ is an affine open of V, then I claim that since $\mathcal{F}$ is a quasicoherent sheaf, then the above morphism $\mathcal{F}(V) \otimes \mathcal{O}_X(U \cap V) \to \mathcal{F}(U \cap V)$ is an isomorphism. To see why, suppose that $\mathcal{F}$ is a quasicoherent $SpecA$ morphism and $SpecB \hookrightarrow SpecA$ is an open embedding of affine schemes, then I claim that:

$$\mathcal{F}(SpecA) \otimes_A B \to \mathcal{F}(SpecB)$$

is an isomorphism of B-modules. To check that this is indeed an isomorphism, we only need to show that for all $f \in B$, after we localize at f, we get an isomorphism. Since $SpecB \hookrightarrow SpecA$ is an open embedding, we can assume that f comes from some $f \in A$ and the maps to B by $A \to B$. Then the open embedding gives us an A-algebra isomorphism:

$$A_f \simeq B_f$$

Therefore we are showing the following isomorphism:

$$\mathcal{F}(SpecA) \otimes_A B_f \simeq \mathcal{F}(SpecB) \otimes_B B_f$$

Notice that since $\mathcal{F}$ is quasicoherent, suppose that the open of $SpecB$ where f doesn't vanish is $D(f) \subset SpecB \subset SpecA$, the left hand side is $\mathcal{F}(D(f))$ so is the right hand side and the map between then can be identified the horizontal map in the following diagram:

$$\begin{tikzcd} & \mathcal{F}(SpecA) \arrow[dl, "\mathrm{Res}"'] \arrow[dr, "\mathrm{Res}"] & \\ \mathcal{F}(D(f)) \arrow[rr] & & \mathcal{F}(D(f)) \end{tikzcd}$$

which is an isomorphism.

This implies that the complex $\mathcal{F}\otimes C_X^\bullet$ is identified with:

$$\oplus_{|I|=1}j_*^I(\mathcal{F}|_{U_I}) \to \oplus_{|I|=2}j_*^I(\mathcal{F}|_{U_I}) \to \cdots \to \oplus_{|I|=n}j_*^I(\mathcal{F}|_{U_I})$$

with the differentials identified as the Cech complex differential. Then we know that if we apply the functor $\Gamma(X,-)$, we get the Cech complex to compute the Cech cohomology of $\mathcal{F}$ on X. Similarly, we know that the complex $\mathcal{G}\otimes C_Y^\bullet$ is identified with:

$$\oplus_{|J|=1}j_*^J(\mathcal{F}|_{U_J}) \to \oplus_{|J|=2}j_*^J(\mathcal{F}|_{U_J}) \to \cdots \to \oplus_{|J|=m}j_*^J(\mathcal{F}|_{U_J})$$

Similarly, after we apply $\Gamma(Y,-)$, we get the Cech complex to compute the Cech cohomology of $\mathcal{G}$.

Then let's consider the total complex $(\mathcal{F}\otimes C_X^\bullet)\boxtimes(\mathcal{G}\otimes C_Y^\bullet)$. This is the total complex of a double complex and in the double complex, each term looks like:

$$\pi_X^*\mathcal{F}\otimes\pi_X^*j_*^I\mathcal{O}_{U_I}\otimes\pi_X^*\mathcal{G}\otimes\pi_Y^*j_*^J\mathcal{O}_{U_J}$$

I claim that this is canonically isomorphic to $j_*^{IJ}\mathcal{O}_{U_I\times U_J}\otimes(\mathcal{F}\boxtimes\mathcal{G})$. To prove this, we need to show a canonical isomorphism

$$\pi_X^*j_*^I\mathcal{O}_{U_I}\otimes\pi_Y^*j_*^J\mathcal{O}_{U_J}\simeq j_*^{IJ}\mathcal{O}_{U_I\times U_J}$$

Notice that these are all $\mathcal{O}_{X\times_k Y}$-algebras, therefore we know that to define such a morphism, we only need to define $\mathcal{O}_{X\times_k Y}$-algebra morphisms:

$$\pi_X^*j_*^I\mathcal{O}_{U_I}\to j_*^{IJ}\mathcal{O}_{U_I\times U_J},\quad \pi_Y^*j_*^J\mathcal{O}_{U_J}\to j_*^{IJ}\mathcal{O}_{U_I\times U_J}$$

Using the adjoint property between $(\pi_X^*,(\pi_X)_*)$ and $(\pi_Y^*,(\pi_Y)_*)$, we know that we need to define $\mathcal{O}_X$-algebra and $\mathcal{O}_Y$-algebra morphisms:

$$j_*^I\mathcal{O}_{U_I}\to(\pi_X)_*j_*^{IJ}\mathcal{O}_{U_I\times U_J},\quad j_*^J\mathcal{O}_{U_J}\to(\pi_Y)_*j_*^{IJ}\mathcal{O}_{U_I\times U_J}$$

For example let's define $j_*^I\mathcal{O}_{U_I}\to(\pi_X)_*j_*^{IJ}\mathcal{O}_{U_I\times U_J}$. For each open V, we need to define:

$$\mathcal{O}_X(U_I\cap V)\to\mathcal{O}_{X\times_k Y}(U_I\cap V\times U_J)$$

This can be defined by:

$$\mathcal{O}_X(U_I\cap V)\stackrel{\pi_X^\sharp}{\to}\mathcal{O}_{X\times_k Y}(U_I\cap V\times Y)\stackrel{\text{Res}}{\to}\mathcal{O}_{X\times_k Y}(U_I\cap V\times U_J)$$

This obvious is a morphism of $\mathcal{O}_X$-algebras. Similarly, we can define $j_*^J\mathcal{O}_{U_J}\to(\pi_Y)_*j_*^{IJ}\mathcal{O}_{U_I\times U_J}$, which gives us $\pi_X^*j_*^I\mathcal{O}_{U_I}\to j_*^{IJ}\mathcal{O}_{U_I\times U_J},\pi_Y^*j_*^J\mathcal{O}_{U_J}\to j_*^{IJ}\mathcal{O}_{U_I\times U_J}$ and further:

$$\pi_X^*j_*^I\mathcal{O}_{U_I}\otimes\pi_Y^*j_*^J\mathcal{O}_{U_J}\to j_*^{IJ}\mathcal{O}_{U_I\times U_J}$$

We want to check this is an isomorphism. Notice that both sides are quasicoherent sheaves, then we can check on any affine cover of $X\times_k Y$ (which waives us of the problem of having to care about the sheafifying problem), recall that if we cover X,Y with affine opens, then their fibered product covers $X\times_k Y$ say $SpecA, SpecB$ are two such opens of X,Y, then we need to show that on $SpecA\times_k SpecB$, the above defined morphism is an isomorphism. Recall that the adjoint property commute with restrictions, therefore we know that the following morphism on $SpecA\times_k SpecB$:

$$\pi_X^*j_*^I\mathcal{O}_{U_I}\to j_*^{IJ}\mathcal{O}_{U_I\times U_J}$$

is defined by:

$$\mathcal{O}_X(U_I\cap SpecA)\otimes_A(A\otimes_k B)\to\mathcal{O}_{X\times_k Y}(U_I\cap SpecA\times U_J\cap SpecB)\simeq\mathcal{O}(U_I\cap SpecA)\otimes_k\mathcal{O}_Y(U_J\cap SpecB)$$

This is identified with:

$$\mathcal{O}_X(U_I\cap SpecA)\otimes_k B\to\mathcal{O}(U_I\cap SpecA)\otimes_k\mathcal{O}_Y(U_J\cap SpecB)$$

Similarly we can figure out $\pi_Y^* j_*^J \mathcal{O}_{U_J} \to j_*^{IJ} \mathcal{O}_{U_I \times U_J}$ on $SpecA \times_k SpecB$ is determined by:

$$A \otimes_k \mathcal{O}_Y(U_J \cap SpecB) \to \mathcal{O}(U_I \cap SpecA) \otimes_k \mathcal{O}_Y(U_J \cap SpecB)$$

Therefore the induced morphism on the tensor product is:

$$(\mathcal{O}_X(U_I \cap SpecA) \otimes_k B) \otimes_{A \otimes_k B} (A \otimes_k \mathcal{O}_Y(U_J \cap SpecB)) \to \mathcal{O}(U_I \cap SpecA) \otimes_k \mathcal{O}_Y(U_J \cap SpecB)$$

This is the same as the identity map:

$$\mathcal{O}(U_I \cap SpecA) \otimes_k \mathcal{O}_Y(U_J \cap SpecB) \to \mathcal{O}(U_I \cap SpecA) \otimes_k \mathcal{O}_Y(U_J \cap SpecB)$$

Therefore we are done and we know that each term in the double complex indeed is:

$$(\mathcal{F} \boxtimes \mathcal{G}) \otimes j_*^{IJ} \mathcal{O}_{U_I \times U_J} \simeq j_*^{IJ}(\mathcal{F} \boxtimes \mathcal{G}|_{U_I \times U_J})$$

The isomorphism holds because $\mathcal{F} \boxtimes \mathcal{G}$ is still quasicoherent, then we can use the same argument as $j_*^I \mathcal{O}_{U_I} \otimes \mathcal{F} \simeq j_*^I(\mathcal{F}|_{U_I})$ that we gave before. Notice that $U_i \times_k U_j$ where $i \in I, j \in J$ forms an affine open cover of $X \times_k Y$. If we want to consider the Cech complex of $\mathcal{F} \boxtimes \mathcal{G}$ with respect to this cover, we need to give this finite set consisting of all the possible subsets of $I \times J$ an order so that we can define the differentials. We notice that the double complex of $(\mathcal{F} \boxtimes \mathcal{G})$ can be schematically written as:

$$\begin{array}{ccccc}
(I=1, J=3) & \longrightarrow & (I=2, J=3) & \longrightarrow & (I=3, J=3) \\
\uparrow & & \uparrow & & \uparrow \\
(I=1, J=2) & \longrightarrow & (I=2, J=2) & \longrightarrow & (I=3, J=2) \\
\uparrow & & \uparrow & & \uparrow \\
(I=1, J=1) & \longrightarrow & (I=2, J=1) & \longrightarrow & (I=3, J=1)
\end{array}$$

Then we know that if we order I, J using lexicographic order where I is the first index and J is the second order, I will let you that after we apply global section functor $\Gamma(X \times_k)$ to the total complex, we obtain the complex to compute the cohomology of $\mathcal{F} \boxtimes \mathcal{G}$ over $Speck$. Now it's not hard to check that after we apply $\Gamma(X \times_k Y, -)$ to the total complex $(\mathcal{F} \otimes C_X^\bullet) \boxtimes (\mathcal{G} \otimes C_Y^\bullet)$, what we get is the total complex we get from the double complex that tensors up $\Gamma(X, \mathcal{F} \otimes C_X^\bullet)$ and $\Gamma(Y, \mathcal{G} \otimes C_Y^\bullet)$ over k. (Here we again use something that we used above, suppose that M is an A-module N is a B-module and A, B are all k-algebra, then there is a canonical isomorphism $(M \otimes_k B) \otimes_{A \otimes_k B} (N \otimes_k A) \simeq M \otimes_k N$). Therefore we only need to show the following last step: $\square$

Lemma 2.22. Suppose that we have two complexs of k-vector spaces:

$$A^\bullet = 0 \to A_0 \to A_1 \to \cdots A_n \to \cdots$$

$$B^\bullet = 0 \to B_0 \to B_1 \to \cdots B_n \to \cdots$$

Then there is a canonical isomorphism between:

$$H^n(A^\bullet \otimes_k B^\bullet) \simeq \oplus_{i+j=n} H^i(A^\bullet) \otimes_k H^j(B^\bullet)$$

Proof. We could directly use spectral sequences since we know that all k-modules are flat, but here let's compute it by hand so it's more clear.

Since we work with a field k, consider the complex $A^\bullet$, notice that for each A^i, we can first decompose it to:

$$A^i \simeq Z^i \oplus L^{i+1}, \quad Z^i = \ker(A^i \to A^{i+1}), \quad L^{i+1} = \operatorname{im}(A^i \to A^{i+1})$$

Further we can decompose Z^i to:

$$Z^i = H^i(A^\bullet) \oplus L^i$$

Then we know that for each i, we have an isomorphism:

$$A^i = L^{i+1} \oplus H^i(A^\bullet) \oplus L^i$$

The differential is defined by:

$$L^{i+1} \oplus H^i(A^\bullet) \oplus L^i \to L^{i+2} \oplus H^{i+1}(A^\bullet) \oplus L^{i+1}$$

where L^{i+1} is mapped by identity to L^{i+1} and the other two are mapped to 0. To differential A and B, we will use subscript A, B. Then to see out assertion, you can write every term in the complex in terms of tensors of:

$$L_A^\bullet, H_A^\bullet, L_B^\bullet, H^B\bullet$$

Then you will find the conclusion not hard to see. □

Remark 2.23. It's not hard to see why we assumed quasicoherence of $\mathcal{F}$ and $\mathcal{G}$ in the proof as what we have pointed put in the proof. Notice that he only place where we used the fact that X, Y are over k is in the last lemma where you either need flatness to use the spectral sequence or you need to use decomposition properties of vector spaces to do the above explicit computation.

Then one natural question to ask after the above proof is: instead of working with $Speck$, which is a point, can we work with quasicompact and separated morphisms $\rho^X : X \to Z, \rho^Y : Y \to Z$ over a general base Z? In this case if we use $\rho^{X\times Y}$ to denoted morphism from $X \times_Z Y$ to Z, of course it doesn't make sense to take about Cech cohomology, but rather we need to ask what is he relation between $\mathrm{R}^n\rho_*^{X\times Y}\mathcal{F}\boxtimes_Z\mathcal{G}$ $\mathrm{R}^i\rho_*^X\mathcal{F}$ and $\mathrm{R}^j\rho_*^Y\mathcal{F}$. You guess that you will have a similar Kunneth formula (but of course with more conditions imposed on ρ^X, ρ^Y and we may loose the claim that the formula is an isomorphism for nonaffine base Z) however, a clear statement and proof will require derived categories therefore we don't put it here.

Remark 2.24. Here is a proof which gives you a high point of view proof of why we can to compute the Cech cohomology of $\mathcal{F}, \mathcal{G}, \mathcal{F}\boxtimes\mathcal{G}$ by apply the global function to the complexes $\mathcal{F}\otimes C_X^\bullet, \mathcal{G}\otimes C_Y^\bullet$ and $(\mathcal{F}\boxtimes\mathcal{G})\otimes(C_X^\bullet\boxtimes C_Y^\bullet)$ without show that after apply the global section functor we indeed get the Cech complex.

Recall that affine morphisms gives an isomorphism between the cohomology of the pushforward and the original cohomology. Also recall that quasicoherent sheaves on an affine scheme has no higher degree cohomology. Finally recall that when the scheme is qusicompact and separated, the Cech cohomology of quasicoherent sheaves is isomorphic to the derived functor cohomology and we can compute derived functor using acyclic resolutions. Then you only need to show that $\mathcal{F}\otimes C_X^\bullet, \mathcal{G}\otimes C_Y^\bullet$ and $(\mathcal{F}\boxtimes\mathcal{G})\otimes(C_X^\bullet\boxtimes C_Y^\bullet)$ are acyclic resolutions of $\mathcal{F}, \mathcal{G}, \mathcal{F}\boxtimes\mathcal{G}$, which are what we proved above.

3 Ideal-theoretic criteria for flatness

The following theorem which characterizes flatness using ideal theoretic methods will allow us to characterize flatness over a number of rings:

Theorem 3.1. Let M be an A-module, then M is flat over A iff for any ideal $I \subset A$, $\mathrm{Tor}_1^A(M, A/I) = 0$.

Remark 3.2. Before we prove the above theorem, let's make a side remark about how you can think of this theorem and why it helps you understand flatness.

Notice that the statement of the above theorem is equivalent to the following: for each ideal $I \subset A$, the map $I \otimes_A M \to M$ is an injection, which is further equivalent to $I \otimes_A M \to IM$ is an isomorphism. Flatness is often informally described as continuously varying fibers and this can be made precise as follows. An A-module M is flat iff it restricts nicely to closed subschemes of $SpecA$, i.e. if we consider $SpecA/I \hookrightarrow M$ and restrict M to $SpecA/I$, we know that in the process, the information we loose in M is the submodule IM, that is the elements that vanishes on $SpecA/I$. Then we can state that M is flat iff IM is easy to understand in the sense that elements in IM all comes from linear combinations of $i \otimes m$ with no surprising relations.

Let's also take a moment to explain what this no surprising relations mean. Recall that the tensors are defined as the quotient of a free abelian group and the subgroup we are quotienting out is the subgroup generated by the A-bilinear relations. Therefore we can think of $I \otimes M$ as formal linear combinations of $i \otimes m$, only subject to A-bilinear relations. Notice that in IM, we know that the elements can also be written as linear combinations im, but here is may be subject to more relations in M. Notice that in M, these linear combinations im clearly subject to the A-bilinear relation. Then the fact that $I \otimes M \to IM$ being an isomorphism means that in M there are no other extra relations other than the bilinear relations. This is the content of the following result:

Proposition 3.3. (Equation criterion for flatness) An A-module M is flat iff for any relation $\sum_i a_i m_i = 0$ with $a_i \in A$ and $m_i \in M$, there exists $m'_j \in M$ independent of i and $a_{ij} \in A$ such that $\sum_j a_{ij} m'_j = m_i$ for all i and $\sum_i a_i a_{ij} = 0$ for all j.

Proof. Let's first take a look at what does this relation mean:

$$\sum_i a_i m_i = \sum_j \sum_i a_i a_{ij} m'_j$$

We know that the right hand side comes from the tensor:

$$\sum_j \sum_i a_i a_{ij} \otimes m'_j$$

But we know that by assumption that:

$$\sum_i a_i a_{ij} \otimes m'_j = (\sum_i a_i a_{ij}) \otimes m'_j = 0$$

Therefore if we assume that $a_i \in I$ for some ideal I, we know that some linear combination in IM is 0 because it comes from 0 in the tensor product $I \otimes M$. Moreover, if you compare:

$$\sum_i a_i \otimes m_i, \quad \sum_j \sum_i a_i a_{ij} \otimes m_j$$

we know that the second term is $\sum_i \sum_j a_i a_{ij} \otimes m_j = \sum_i \sum_j a_i \otimes a_{ij} m_i = \sum_i a_i \otimes m_i$, therefore $\sum_i a_i \otimes m_i = 0$, therefore $I \otimes M \to IM$ is an isomorphism.

Conversely, suppose that M is flat over A. Consider the following A-module map:

$$0 \to \ker\varphi \xrightarrow{\varphi} A^n \to A$$

where φ is defined by $(x_1, \cdots, x_n) \mapsto \sum_i a_i x_i$. Then we know that since M is flat over A, we can tensor with M and get:

$$0 \to \ker\varphi \otimes M \to M^n \to M$$

We know that by assumption that $(m_1, \cdots, m_n)$ is mapped to 0. Therefore there will be something in $\ker\varphi \otimes M$ that maps to $(m_1, \cdots, m_n)$.

Notice that we know that elements in $\ker\varphi \otimes M$ are of the form:

$$\sum_j k_j \otimes m'_j$$

suppose that the i-th component of k_j is a_{ij}, then we find a_{ij} and m_j such that $\sum_j a_{ij} m'_j = m_i$. Moreover, by assumption $k_j \in \ker\varphi$, therefore we know that:

$$\sum_i a_{ij} a_i = 0$$

We are done. ∎

Example 3.4. To give you a sense of when in IM, there are additional relations, think of the following example which we have seen before. Let $A = k[t]/(t^2)$, $I = (t)/(t^2)$ and $M = (k[t]/(t^2))/((t)/(t^2)) = k$. consider the map:

$$I \otimes_A M \to IM$$

Notice that M is annihilated by I, therefore $IM = 0$. However, we know that $I \otimes_A M = I \otimes A/I = I/I^2 = I \simeq k$, here we have:

$$\bar{t} \otimes \bar{1} \mapsto \bar{t} \cdot \bar{1} = 0$$

Notice that since $\bar{t} \otimes \bar{1}$ is mapped to 1 in k, we know that this is not 0 in the tensor product. Therefore this is not subject to bilinear conditions, but it's image in IM is 0, which means that the linear combinations in IM has more relations than just the bilinear relation.

Remark 3.5. Another remark before we prove the theorem is that: actually, in the statement, we can require I to be finitely generated instead of all ideals of A. To see why, we want to prove that if $I \otimes_A M \to M$ is injective for all finitely generated ideal I, then $I \otimes M \to M$ is injective for all ideal $I \subset A$. Let's prove the contrapositive, i.e. if there is an ideal I such that $I \otimes M \to M$ is not injective for some ideal I, then we will find a finitely generated ideal J such that $J \otimes M \to M$ is not injective.

Since $I \otimes M \to M$ is not injective, we know that there will be:

$$\sum a_i \otimes m_i, \quad \sum_j b_j \otimes m_j$$

which are mapped to the same element in M. Consider the ideal generated by a_i, b_j, say it's J, then we have:

$$J \otimes M \to I \otimes M \to M$$

we know that if we consider $\sum a_i \otimes m_i, \quad \sum_j b_j \otimes m_j$ in $J \otimes M$, they are different since their images in $I \otimes M$ are different, but the images in M are the same, we are done.

Now let's prove the theorem:

Proof. We only need to show that $\mathrm{Tor}_1^A(M, A/I) = 0$ implies $\mathrm{Tor}_1^A(M, N) = 0$ for all A-modules N.

Let's prove first for the case when N is finitely generated. Let's do induction on the number of generators, say first N is generated by one element n, then we know that $N \simeq A/\mathrm{Ann}(n)$, then we are done. If this holds for N with n generators, we want to prove it for $n+1$ generators. Say these generators are $a_1, \cdots, a_{n+1}$, we consider the following SES:

$$0 \to Aa_{n+1} \to N \to N/Aa_{n+1} \to 0$$

We consider the LES of $\mathrm{Tor}^\bullet(M, -)$, since N/Aa_{n+1} is generated by n elements, we know that we can apply the inductive hypothesis for Aa_{n+1} and N/Aa_{n+1}, and conclude $\mathrm{Tor}_1^A(M, N) = 0$.

Then for a general N, we know that N is the union of all it's finitely generated submodules N_α, which is a filtered colimit and we know that:

$$\mathrm{Tor}_1^A(M, N) \simeq \mathrm{Tor}_1^A(N, M)$$

Then if we can show that:

$$\mathrm{Tor}_1^A(N, M) \simeq \varinjlim_\alpha \mathrm{Tor}_1^A(N_\alpha, M)$$

Then we know that since $\mathrm{Tor}_1^A(N_\alpha, M) = 0$, $\mathrm{Tor}_1^A(N, M) = 0 = \mathrm{Tor}_1^A(M, N)$. Notice that if we have a projective resolution of M, we have a filtered colimit of complex after we tensor with N_α, then since we know that filtered colimit is exact, then it commutes with taking cohomology, therefore we are done. ☐

We will not use this to give explicit characterizations of flat modules over PIDs, dual numbers and several local rings. Recall that we prove before that over a domain, a module is torsion-free if it's flat, now if we assume further that the ring is a PID, the converse also holds:

Proposition 3.6. If A is a PID, then a module M is flat iff it's torsion free.

Proof. Recall that since A is a PID, then any ideal I is generated by one element, say (x), then $\mathrm{Tor}_1^A(M, A/I) = \mathrm{Tor}_1^A(M, A/(x)) = (M : x)$, therefore we know that if M is trosion free, then $\mathrm{Tor}_1^A(M, A/I) = 0$ for all I, thus M is flat. ☐

Then let's consider the condition for a module to be flat over the dual numbers $k[t]/(t^2)$. This fact will be important in deformation theory:

Proposition 3.7. *M is flat over $k[t]/(t^2)$ iff $M/tM \xrightarrow{\times t} tM$ is an isomorphism.*

Proof. One important observation is that $k[t]/(t^2)$ only has three ideals. Besides the 0 ideal and the whole ring, I claim that (t) is the only nontrivial ideal. To see why, notice that if we consider $a + bt$, we know that it's not invertible only when $a = 0$ and $b \neq 0$ since if not we know that:

$$(a + bt)(\frac{1}{a} - \frac{b}{a^2}t) = 1$$

Therefore we know that an nontrivial ideal can only contain elements of the form bt, that is the ideal (t).

Then for M to be flat, we only have to require $(t) \otimes M \to tM$ to be an isomorphism. Notice that (t) as an A-module is isomorphic to $A/(t)$ since we know that t acts on $A/(t)$ and (t) trivially, therefore we know that we have a diagram:

$$\begin{tikzcd}
(t) \otimes M \arrow[rr, "at\otimes m\mapsto atm"] \arrow[dr, "\simeq"'] & & tM \\
& M/tM \arrow[ur, "m\mapsto tm"'] &
\end{tikzcd}$$

Therefore we are done. ☐

Let's also see how flatness is characterized over a DVR:

Proposition 3.8. *Let M be a module over a DVR A with uniformizer t, then M is flat over A iff t is not a zerodivisor of M.*

Proof. Recall that any nonzero ideal of A is of the form (t^n), then we know that M is flat iff $\mathrm{Tor}_1^A(M, A/(t^n)) = 0$, that is to say that $(M : t^n) = 0$. This is equivalent to $(M : t) = 0$, that is t is not a zerodivisor. ☐

You will see later that the above result gives a useful geometric characterization of flat morphisms over a regular curve.

The following theorem will be absolutely important:

Theorem 3.9. *Suppose that (A, m) is a local ring and M is a finitely presented A-module, then the following are equivalent:*

1. *M is free.*
2. *M is projective.*
3. *M is flat.*
4. *$\mathrm{Tor}_1^A(M, A/m) = 0$.*

Remark 3.10. Here this finitely presented condition can not be dropped as there are flat modules over local ring that is not free. Consider the $\mathbb{Z}_{(p)}$-algebra (here this notation is localize $\mathbb{Z}$ at the prime (p)) $\mathbb{Q}$, notice that since $\mathbb{Q}$ is just a further localization, therefore is flat over $\mathbb{Z}_{(p)}$. However, I claim that it's not free. The reason is also easy, we know that elements in $\mathbb{Z}_{(p)}$ all have denominator that is prime to p, then we know that if we want to write elements in $\mathbb{Q}$ whose denominator is a power of p is as a $\mathbb{Z}_{(p)}$ linear combination, we will need some elements of the form $\frac{1}{p^n}$ for n that runs over all positive integers. At the same time, only finitely many such elements are not enough as we know that $\frac{1}{p^n}$ only suffices for elements in $\mathbb{Q}$ with denominator whose exponent of p is at most n. However, any two $\frac{1}{p^n}, \frac{1}{p^m}$ are not linearly independent.

The above example doesn't contradict the above result since our discussion already shows that $\mathbb{Q}$ is not finitely presented over $\mathbb{Z}_p$.

Then let's prove the theorem:

Proof. Note that the implications $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ are all trivial. Therefore the only thing we need to show is $4 \Rightarrow 1$.

Recall that by our assumption, M is firstly finitely generated, then we know that M/mM is a finite dimensional vector space over A/m, thus there is a basis, which we lift back arbitrarily to M. Say they are $m_1, \cdots, m_n$. Notice that by Nakayama's lemma the map:

$$A^{\oplus n} \to M, \qquad e_i \mapsto m_i$$

is surjective. We will prove it's an isomorphism. Notice that by our assumption that M is finitely presented, we know that the kernel K is also finitely generated. Consider the exact sequence:

$$0 \to K \to A^{\oplus n} \to M \to 0$$

Since we assumed that $\mathrm{Tor}_1^A(M, A/m) = 0$, then we know that after we tensor with A/m, the sequence remains exact:

$$0 \to K/mK \to (A/m)^{\oplus n} \to M/mM \to 0$$

by our assumption $(A/m)^{\oplus n} \to M/mM$ is an isomorphism, therefore $K/mK = 0$. Since K is finitely generated, $K/mK = 0$ implies $K = 0$. □

The geometric interpretation of this result is:

Corollary 3.11. A finitely presented sheaf $\mathcal{F}$ over any scheme is flat iff it's locally free.

Proof. Recall that for any finitely presented sheaf $\mathcal{F}$, we can check locally freeness on stalk level, i.e. it's locally free iff each stalk is free. (Recall that we prove this using the fact that finitely presented implies always finitely presented, thus the kernel is finitely generated, therefore the support of the kernel is closed). Then since $\mathcal{F}_p$ is finitely presented over $\mathcal{O}_{X,p}$ and is flat, we know that it's free. □

References