Appendix 7 for Math 632: 2025 Winter Notes

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1 Line Bundles, Maps to Projective Spaces and Divisors

The goal of this chapter is to let you know the importance of line bundles, with many applications. The reasons behind are quite a lot: They are the simplest vector bundles we are often able to classify and we can always build and manipulate them via a lot constructions to get interesting new stuff. Finally, they help us to understand maps to projective spaces and divisors in general.

1.1 Some Line Bundles on Projective Spaces

We now describe an important family of invertible sheaves on projective spaces over a field k.

As a warm up, we begin with a line bundle called $\mathcal{O}_{\mathbb{P}^1_k}(1)$ on $\mathbb{P}^1_k = Projk[x_0, x_1]$. The subscript indicates the space on which the line bundle lives and is often omitted when it's clear from the context. We describe the invertible sheaves $\mathcal{O}(1)$ using transition functions:

Recall that we cover the space by $U_0 = D(x_0)$ and $U_1 = D(x_1)$ and they are just $U_0 = Speck[x_{1/0}]$ and $U_1 = Speck[x_{0/1}]$. We define the line bundle $\mathcal{O}(1)$ by being trivial on U_1 and U_2, we now need a transition function on the intersection. Notice that the global sections of the line bundle on U_0 are just polynomials in the variable $x_{1/0}$ and the global sections on U_1 are just polynomials in the variable $x_{0/1}$. Recall that the intersection is $D(x_0x_1)$ and we know that the structure sheaf restricts to the structure of $Speck[x_{1/0}, x_{0/1}]$ and we define the transition function from U_0 to U_1 by multiplying $x_{0/1}$ and from U_1 to u_0 by multiplying by $x_{1/0}$, then you see that the composition is the identity, thus indeed defined a transition function.

What you should notice here is that if we define the transition function on the intersection to be 1, then the stuff we get in the end is just the structure sheaf.

To test our understanding, let's compute the global section of this line bundle, we claim that the global section is isomorphic to the degree 1 piece of $k[x_0, x_1]$. Recall that the global section can be determined by the sections on U_0 and U_1 such that they agree on the intersection recall that for a polynomial $f(x_{1/0})$ in U_0 which is a section on U_1, it restrict to $f(x_{1/0})$ on the intersection then we multiply it by $x_{0/1}$ it should be equal to the restriction of $g(x_{0/1})$. Then we know that f can't have degree larger than 1, since we are just asking that for a one variable polynomial $f(x)$ when do we have $f(x)\frac{1}{x} = g(\frac{1}{x})$, thus the degree of f can only be 1, thus f is linear, that $f = a\frac{x_1}{x_0} + b$ and for each such f we get $g = a + b\frac{x_0}{x_1}$, thus we get the global section are just all linear functions in $\frac{x_1}{x_0}$, or equivalently linear combination of x_0, x_1.

Remark 1.1. Notice that this tells you that the dimension of $\Gamma(\mathbb{P}^1_k, \mathcal{O}(1)) = 2 \neq 1 = \Gamma(\mathbb{P}^1_k, \mathcal{O})$, thus this line bundle is not trivial!

Now we can define $\mathcal{O}(m)$ on $\mathbb{P}^1$ similarly, with the transition function on $Speck[x_{1/0}, x_{0/1}]$ being $\times x^m_{1/0}$ from U_0 to U_1 and $\times x^m_{0/1}$ from U_1 to U_0. Recall that the transition functions on $\mathcal{O}(1)^{\otimes m}$ is just the m power of the transition function on $\mathcal{O}(1)$, thus we see that $\mathcal{O}(m) \simeq \mathcal{O}(1)^{\otimes m}$.

This construction also makes sense for $m < 0$, since the transition function with m being negative also makes sense, thus you see that $\mathcal{O}(m) \simeq \mathcal{O}(1)^{\otimes m}$ makes sense even for $m < 0$ with the meaning of $\mathcal{O}(1)^{\otimes m}$ being $(\mathcal{O}(1)^{\otimes -m})^\vee$ since the transition of the dual is the inverse. Also, we see that $\mathcal{O}(m) \otimes \mathcal{O}(n) \simeq \mathcal{O}(m+n)$ still for the reason of they have the same transition functions. And this again comes to the fact that tensor product can be computed locally.

Proposition 1.2. The dimension of $\Gamma(\mathbb{P}^1, \mathcal{O}(m)) = m+1$ for $m \geqslant 1$ and 0 otherwise, moreover, for $m \geqslant 0$ you can identify the global section to be the space of homogeneous polynomials of degree m with variables x_0, x_1.

Proof. This is just a generalization of the computation we had earlier. Let's first consider $m \geqslant 2$, it just reduces to find a pair of polynomial f, g such that:

$$f(x)(\frac{1}{x})^m = g(\frac{1}{x})$$

Thus you see that the degree of f must be less than or equal to m, and for a $f = a_m x^m + \cdots + a_1 x + a_0$, we know that $g = a_0 x^m + a_1 x^{m-1} + \cdots + a_m$, thus we can identify the global section as span of degree m homogeneous polynomials in two variables thus $m+1$.

For $m < 0$, we see that we are looking for:

$$f(x)x^{-m} = g(\frac{1}{x})$$

and just for degree reasons, there is no such polynomials, we are done. □

Example 1.3. The line bundles $\mathcal{O}(m)$ on $\mathbb{P}^1$ gives a surprising example of the importance of sheafification when doing tensor product.

Recall that for $\mathcal{O}(1) \otimes \mathcal{O}(-1) \simeq \mathcal{O}$, the global secitions of $\mathcal{O}(-1)$ is just 0, thus if we don't do any sheafification and just simply tensor the global sections, we get 0, however, we know that the the global sections of the sturcture sheaf is not 0.

Proposition 1.4. If $m \neq n$, then $\mathcal{O}(m)$ is not isomorphic to $\mathcal{O}(n)$, thus we have an injective map:

$$\mathbb{Z} \hookrightarrow Pic(\mathbb{P}^1_k)$$

Proof. We only have to worry about when both m, n are negative since the discussion above have settled other cases.

Say $m > n$ Now if they are isomorphic, then tensor preserves being isomorphic, thus we have:

$$\mathcal{O}(m-m) = \mathcal{O} \simeq \mathcal{O}(n-m)$$

However, since $n-m$ is negative, we get a contradiction. □

Remark 1.5. As what we have seen in the proposition of determining global sections of $\mathcal{O}(m)$ for positive m, it's intuitive to identify the global sections of these invertible sheaves as homogeneous polynomials of degree d. This is not just for the sake of looking good. For example, consider where does the global section $x_0^3 - x_0 x_1^2$ of $\mathcal{O}(3)$ vanish. We claim that they vanish exactly at homogeneous primes that contain them. To see this, of course you pass to the two affine covers, for example in $D(x_0)$, the global section becomes $1 - (\frac{x_1}{x_0})^2$, then for a prime in the affine open containing this, the corresponding homogeneous polynomial contains $x_0^3 - x_0 x_1^2$ by just multiply by x_0^3 and conversely any homogeneous prime lying in $D(x_0)$ that contains this section will translate to a prime in the affine open that contains $1 - (\frac{x_1}{x_0})^2$.

There are other intuitive reasons for why the identification is useful: the section $x_0 + x_1$ of $\mathcal{O}(1)$ can be multiplied by a section x_0^2 of $\mathcal{O}(2)$ to get a section of $\mathcal{O}(3)$, and you intuitively think of the section as $x_0^3 + x_1 x_0^2$, and this is the correct answer if we use the rigorous way to find this section. To see how this is done, recall that when we mean multiply sections to get a section in the tensor product, we mean the image of the tensor of the two elements under the morphism from the presheaf to the tensor sheaf. This operation is well defined wrt restrictions, in the sense that we can use an affine cover and restrict to open covers to compute the global section. However, when you restrict to U_0, U_1, both the line bundles are trivial, hence here we don't have to do sheafification and the tensor product just correspond to product here and use this interpretation you will find the section I told you is correct one.

For one more example, consider $\mathcal{O}(-2)$ and the rational section $\frac{x_0^4(x_1+x_0)}{x_1^7}$ defined on $D(x_0)$ (it's indeed a rational section since the only associated point is the generic point). We will ask where does it have poles and zeros and what is the order. To rigorously answer the questions requires some work, but you can already from the global section itself where are the vanishing points intuitively.

More generally we can define $\mathcal{O}(m)$ on $\mathbb{P}^n_k$ by cover $\mathbb{P}^n_k$ with the $n+1$ standard affine cover and still defines it to be trivial on each $D(x_i)$, while the transition function between $D(x_i)$ and $D(x_j)$ is defined to be multiplication by $x^m_{i/j}$ and from $D(x_j)$ to $D(x_i)$ is multiplication by $x^m_{j/i}$. We need to check the cocycle condition, recall that the triple intersection of $D(x_i), D(x_j)$ and $D(x_k)$ is just $Spec(k[x_0, \cdots, x_n]_{x_i, x_j, x_k})_0$ and the restriction of $x^m_{j/i}$ is still itself, then we are just verifying:

$$x^m_{i/j} \cdot x^m_{j/k} \cdot x^m_{k/i} = 1$$

which is obvious since we the symbol $x_{i/j}$ just stands for $\frac{x_i}{x_j}$ and similarly for other variables.

The following proposition tells you the global sections of $\mathcal{O}(m)$ on $\mathcal{P}^n_k$, which is essential:

Proposition 1.6. We can identify the global sections of $\mathcal{O}(m)$ on $\mathbb{P}^n_k$ as the space of homogeneous polynomials of degree m in variables $x_0, \cdots, x_n$, thus the dimension of which will be $\binom{n+m}{n}$.

Proof. Similar to what we did for $\mathbb{P}^1$, this time we cover $\mathbb{P}^n_k$ with the standard $n+1$ affine opens, and we want to find a $n+1$ tuple of polynomials:

$$(f_0, \cdots, f_n)$$

with f_i in variables $\frac{x_0}{x_i}, \cdots, \frac{x_i}{x_i} = 1, \cdots, \frac{x_n}{x_j}$, such that they agree on intersections. For example we need:

$$f_0(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}) \cdot \frac{x_0^m}{x_1^m} = f_1(\frac{x_0}{x_1}, \frac{x_2}{x_1}, \cdots, \frac{x_n}{x_1})$$

in $(k[x_0, \cdots, x_n]_{x_0, x_1})_0$.

From this, we notice that each term in f won't have more than m variables, if so, $f_0(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}) \cdot \frac{x_0^m}{x_1^m}$ will have terms with x_0 on the denominator while the stuff on the right hand side doesn't have such terms, thus they can't be equal. This immediately tells you that when $m < 0$, there is no global section.

Therefore, f should be a polynomial in these variables with degree less than m. Conversely, for each such polynomial, we can produce other n polynomials uniquely and get a global section juts by homogenizing the polynomials. This also helps us to identify the global section as degree m homogeneous polynomials. □

Remark 1.7. Now from the above proposition, the function that maps $n \in \mathbb{Z} \mapsto dim_k \Gamma(\mathbb{P}^n_k, \mathcal{O}(m)) = \binom{n+m}{n}$ is a polynomial with coefficients in $\mathbb{Q}$ and with variable m for any $m \geqslant 0$. This suggests that this function want to be a polynomial in m, but won't succeed without help. To see which part falls apart, we need the higher cohomology and will also need to define Euler characteristic using higher cohomology. You will find that this the polynomial that the funcition in m tend to be is the Euler characteristic, and it will agree with the dimension of the global section when $m \geqslant 0$ precisely because all the other cohomology vanishes.

Remark 1.8. The above definition can be applied without change to any ring A and define $\mathcal{O}(m)$ on $\mathbb{P}^n_A$, you will find that the dimension of the global section is just a rank $\binom{n+m}{n}$ A-module still identified with homogeneous polynomials of degree m with coefficients in A.

The following example might be enlightenling:

Example 1.9. Recall that, for $n > 0$ we defined a morphism before:

$$\pi : A^{n+1} - \{0\} \to \mathbb{P}^n$$

You can define it by divide up into affine opens and define morphisms there and show that they glue together. Or if you recall, we know that the functions $x_0, \cdots, x_n$ have no common zeros on $\mathbb{A}^{n+1} - \{0\}$, thus they determine a unique morphism $A^{n+1} - \{0\} \to \mathbb{P}^n$, but the moral is still the same.

We claim that for every $m \in \mathbb{Z}$, we have an isomorphism:

$$\pi^* \mathcal{O}(m) \simeq \mathcal{O}_{A^{n+1}_A - \{0\}}$$

To see how this is isomorphism is defined, we still have to first work locally, notice that $D_+(x_i)$ is pulled back to $D(x_i)$. Then on $D(x_i)$, we define a rational polynomial f is mapped to $f \cdot x_i^m$, we claim that this glues well on the intersections. Recall that on $D(x_i)$, the pullback is identified with the structure sheaf by:

$$k[\frac{x_0}{x_i}, \cdots, \frac{x_n}{x_i}] \otimes_{k[\frac{x_0}{x_i}, \cdots, \frac{x_n}{x_i}]} k[x_0, \cdots, x_n, \frac{1}{x_i}]$$

Then you notice that the transition function on the intersection is still times $\frac{x_i^m}{x_j}$, then you can use this to show that this is a well defined morphism.

It's also an isomorphism since it's an isomorphism of $k[x_0, \cdots, x_n, \frac{1}{x_i}]$-modules on each $D(x_i)$. It's obviously injective and it's also surjective since we know x_i is invertible on each $D(x_i)$.

The above construction depends on you understanding of how the pullback controls the transition functions on affine schemes. For example, we have a morphism of affine spaces:

$$SpecA \to SpecB$$

and we have an endomorphism for quasicoherent sheaf $\widetilde{M}$ on $SpecB$, which is induced by the B-module morphism $M \to M$, then we claim that if you consider the endomorphism induced on the pullback, it's still induced induced by $M \to M$ on:

$$M \otimes_B A \to M \otimes_B A$$

which is something we discussed before.

This isomorphism π^* induces a morphism $\pi^* : \Gamma(\mathbb{P}_A^n, \mathcal{O}(m)) \to \Gamma(\mathbb{A}^{n+1} - \{0\}, \mathcal{O}_{\mathbb{A}^{n+1}-\{0\}})$ by first maps it to the tensor product and then to the sheafification, you can check that this correspond to the map that maps the global section correspond to a homogeneous polynomial of degree d to the global section of $\mathcal{A}_A^{n+1} - \{0\}$ which correspond to the homogeneous polynomial, thus obviously injective.

Moreover, recall we know that we have an isomorphism on global sections $\alpha : \Gamma(\mathbb{A}^{n+1}, \mathcal{O}_{\mathbb{A}^{n+1}}) \to \Gamma(\mathbb{A}^{n+1} - \{0\}, \mathcal{O}_{\mathbb{A}^{n+1}-\{0\}})$ induced by the inclusion morphism:

$$\mathbb{A}^{n+1} - \{0\} \hookrightarrow \mathbb{A}^{n+1}$$

Then the composition gives you an morphism:

$$\alpha^{-1} \circ \pi^* : \Gamma(\mathbb{P}_A^n, \mathcal{O}(m)) \to A[x_0, \cdots, x_n]$$

You should check that it's injective and maps the section that correspond to the homogeneous polynomial of degree m to itself in $A[x_0, \cdots, x_n]$.

You will see later that the only line bundles on $\mathbb{P}_k^n$ are these $\mathcal{O}(m)$, which uses more tools than we have now, but for $\mathbb{P}_k^1$, we can prove it right now.

Proposition 1.10. The only line bundles on $\mathbb{P}_k^1$ up to isomorphism are $\mathcal{O}(m)$ we defined before.

Proof. Recall that we prove that all locally free sheaf of finite rank on $\mathbb{A}_k^1$ is free of rank n, thus every invertible sheaf on $\mathbb{A}^1$ is trivial.

Now suppose that $\mathbb{P}_k^1$ has a line bundle $\mathcal{L}$, then from the discussion above, we know that it's trivial on $D(x_0)$ and $D(x_1)$, thus we know that on the intersection, $Speck[x, \frac{1}{x}]$ if $\mathcal{L}'$ is another line bundle that has the same transition function on the intersection, then we know that $\mathcal{L}'$ is isomorphic to $\mathcal{L}$, thus the question just becomes how to find all the invertible functions on $k[x, \frac{1}{x}]$, now suppose that $\frac{f}{x^n} \cdot \frac{g}{x^m} = 1$ we know that $fg = x^{m+n}$, therefore we know that f and g are all powers of x, thus the only invertible functions in this ring are x^m for all $m \in \mathbb{Z}$, we are done. $\square$

Remark 1.11. One thing you should pay attention to is that if we are not working with k, but an arbitrary ring A, $\mathbb{P}_A^n$ may have more line bundles than $\mathcal{O}(m)$, and you will see an example later.

The last part of this section will generalize $\mathcal{O}(m)$ construction to $ProjS_\bullet$, where $S_\bullet$ is a finitely generated graded ring over A, or in other words we are generalizing the construction to projective A-schemes, let's first focus on the case where $S_\bullet$ is generated in degree 1.

If you recall that for any graded $S_\bullet$-module $M_\bullet$, we have a quasicoherent sheaf on $ProjS_\bullet$. Now there is an operation called twisting defined for all such graded modules. We define $M(m)_\bullet$ to be the graded module where each graded piece:

$$M(m)_d = M(m + d)$$

It's still the module $M_\bullet$, just with a different grading. It's easy to check that it's indeed still a graded $S_\bullet$-module, that we can apply the associated sheaf operation and get a quasicoherent sheaf on $ProjS_\bullet$. If you do the computation you will find that on each $D(f)$, we have:

$$\Gamma(D(f), \widetilde{M(m)_\bullet}) = ((M_\bullet)_f)_m$$

where the subscript $_m$ means that you take the $_m$'s graded piece.

Proposition 1.12. Recall that when $S_\bullet = A[x_0, \cdots, x_n]$ with the usual grading, we have $ProjS_\bullet = \mathbb{P}^n_A$. Then $\widetilde{S(m)_\bullet} \simeq \mathcal{O}(m)$ on $\mathbb{P}^n_A$.

Proof. Let's use transition functions to determine the isomorphism, which means that we need to show they have the same transition function.

Since we already know the transition function for $\mathcal{O}(m)$, we only have to deal with $\underline{S(m)}$. To see this, we first need to show that $\widetilde{M_\bullet(m)}$ is indeed trivial on $D_+(x_i)$. Notice that on $D_+(x_i)$, $\widetilde{M_\bullet(m)}$ is just $(\widetilde{(S_\bullet)_{x_i}})_1$, and there is an $((S_\bullet)_{x_i})_0$-module isomorphism:

$$((S_\bullet)_{x_i})_0 \to ((S_\bullet)_{x_i})_m$$

defined by $\frac{f}{x_i^n} \mapsto \frac{f x_i^m}{x_i^n}$, this is clearly injective and surjective thus an isomorphism. Now we need to figure out the transition function. Notice that the isomorphism is compatible with restriction to the intersection in the sense that we have the following diagram:

$$\begin{tikzcd}
(S_{x_i})_m \arrow[r, "res"] & (S_{x_i x_j})_m \arrow[r, "id"'] & (S_{x_i x_j})_m & (S_{x_j})_m \arrow[l, "res"'] \\
(S_{x_i})_0 \arrow[r, "res"'] \arrow[u, "\times x_i^m"] & (S_{x_i x_j})_0 \arrow[r, dashed, "?"'] \arrow[u, "\times x_i^m"] & (S_{x_i x_j})_0 \arrow[u, dashed, "\times x_j^m"] & (S_{x_j})_0 \arrow[l, "res"] \arrow[u, dashed, "\times x_j^m"]
\end{tikzcd}$$

thus we are only have to consider that is the invertible map, indicated by the dotted arrow such that the diagram commutes, and you can check it's exactly the transition function of $\mathcal{O}(m)$. $\square$

Motivated by the above computation, we define:

Definition 1.13. Suppose that $S_\bullet$ is a finitely generated A-graded ring such that it's generated in degree 1, then we define $\mathcal{O}(m)$ on $ProjS_\bullet$ to be $\widetilde{S(m)_\bullet}$.

Proposition 1.14. If $S_\bullet$ is generated in degree 1, then $\mathcal{O}(m)$ is an invertible sheaf on $ProjS_\bullet$.

Proof. The main reason we want $S_\bullet$ to be generated in degree one is that we want $(S_f)_m \simeq (S_f)_0$. Since f is of degree 1, we can define a morphism by:

$$(S_f)_0 \to (S_f)_m$$

$\frac{x}{f^n} \mapsto \frac{x f^m}{f^n}$, and if f is not of degree 1, we know that this may not be defined just for degree reasons.

And we know that $ProjS_\bullet$ is covered by $D_+(f)$ for finitely many degree 1 homogeneous elements, then we are done. $\square$

Definition 1.15. If $S_\bullet$ is generated in degree 1 and $\mathcal{F}$ is a quasicoherent sheaf on $ProjS_\bullet$, we define $\mathcal{F}(m) = \mathcal{F} \otimes \mathcal{O}(m)$, this is called twisting $\mathcal{F}$ by $\mathcal{O}(m)$ or twisting $\mathcal{F}$ by m.

More generally, for an invertible sheaf $\mathcal{L}$ on $ProjS_\bullet$ we define $\mathcal{F} \otimes \mathcal{L}$ to be twisting $\mathcal{F}$ by $\mathcal{L}$.

Proposition 1.16. If $S_\bullet$ is generated in degree 1, then $\widetilde{M_\bullet}(m) \simeq \widetilde{M(m)_\bullet}$, thus the placement of the dot is not important.

Proof. We can define the morphism locally and show that they patch together. Again, suppose that $ProjS_\bullet$ is covered by finitely many $D_+(f)$ where f is homogeneous of positive degree 1. Let's first define the morphism on $D_+(f)$.

Since $\widetilde{M_\bullet}$ and $\widetilde{S(m)_\bullet}$ are quasicoherent, we know that the result is quasicoherent and is described on $D_+(f)$ by $(M_f)_0 \otimes (S_f)_m$, then since $\widetilde{M(m)}$ on $D_+(f)$ is $(M_f)_m$, we can just define the morphism by the module morphism corresponding to the multiplication. Then I claim that the following morphism is commutative for another g, homogeneous of positive degree 1:

$$\begin{tikzcd}
(M_f)_0 \otimes (S_f)_m \arrow[r] \arrow[d] & (M_f)_m \arrow[d] \\
(M_{fg})_0 \otimes (S_{fg})_m \arrow[r] & (M_{fg})_m \\
(M_g)_0 \otimes (S_g)_m \arrow[r] \arrow[u] & (M_g)_m \arrow[u]
\end{tikzcd}$$

this tells you that they agree on intersections, which I will let you check since there are indeed some identifications you need to notice, but after you check it, it defines a morphism.

Now we have to prove it's an isomorphism, we can do this by defining a inverse $(M_f)_m \to (M_f)_0 \otimes (S_f)_m$, and I will let you figure out what is it. $\square$

Using the above exercise, we are able to show that:

Proposition 1.17. If $S_\bullet$ is generated in degree 1, then $\mathcal{O}(m_1 + m_2) \simeq \mathcal{O}(m_1) \otimes \mathcal{O}(m_2)$.

Proof. Notice that $\mathcal{O}(m_1)$ is just $\widetilde{S(m_1)}$, then twist it by m_2, from the previous proposition, you get it's isomorphic to $S\widetilde{(m_1)(m_2)} = S\widetilde{(m_1 + m_2)} = \mathcal{O}(m_1 + m_2)$. $\square$

1.2 Line Bundles and Maps to Projective Spaces

Before we discuss what we will do, let's tell you our goal of this section and tells you a non-formal example to give you some flavor on what we will do later.

First, recall that a morphism from any scheme X to $\mathbb{A}^n$ corresponds to choose n functions on X. And from the Yoneda representable functor point of view, we can define $\mathbb{A}^n$ as the object such that morphims to it correspond to n functions. In this section we will reinterpret $\mathbb{P}^n$ using this idea. To do this, we need to understand how to use line bundles to give morphisms to $\mathbb{P}^n$, which is the core of this section. To give you a feeling of what happens here, I claim that:

$$[x_0^3, x_0^2 x_2, x_0 x_1 x_2 + x_1^3, x_2^3]$$

give a morphism from $\mathbb{P}^2$ to $\mathbb{P}^3$. Clearly, none of the stuff in the bracket is a function on $\mathbb{P}^2$, thus the strategy we used before of giving functions that has no common vanishing points doesn't work here. However, say for a k-valued point $[1 : 1 : 1]$, it's sent to $[1 : 1 : 2 : 1]$, and the same point is sent to $[8 : 8 : 16 : 8]$, thus their rations still makes sense. Therefore, you know where each point should go.

However, the following bracket doesn't give a morphism from $\mathbb{P}^2$ to $\mathbb{P}^3$:

$$[x_0^3, x_0^2 x_2, x_0 x_1 x_2 + x_1^3, x_1^3]$$

This is because the point $[0 : 0 : 1]$ will be mapped to $[0 : 0 : 0 : 0]$, which doesn't make sense. These stuff in the bracket, as what we have mentioned before, correspond to global sections of the line bundle $\mathcal{O}_3$ on $\mathbb{P}^2$, the difference between the two collections is that the first collection has no common zero locus while the second collection has such a common zero. This observation is key for the following vital construction:

Proposition 1.18. Suppose that X is any A-scheme and there is a line bundle $\mathcal{L}$ on X with $n+1$ global sections of $\mathcal{L}$ with no common zeros, then there is a corresponding morphism to $\mathbb{P}^n$:

$$X \overset{[s_0:\cdots:s_n]}{\longrightarrow} \mathbb{P}^n$$

This works over any base ring.

Proof. Here the meaning of no common zero means that for every point p, there is at least one of the s_i's such that it's image in the fiber $\mathcal{L}|_p$ is not 0.

To see how this works, let's pick a collection of trivialization opens of $\mathcal{L}$, say one of them is U, then we know that $s_0, \cdots, s_n$ will correspond to n functions on U, and they have no common zero, then the previous idea works which allows up to give a morphism from U to $\mathbb{P}^n$. We first need to show that these morphisms agree. Notice that if V is another trivialization neighborhood, using the transition function on the intersection, we know that the functions f_i, g_i differs by an invertible function on the intersection, thus they define the same morphism on the intersection, which is the restriction of the morphism from U, V to $U \cap V$, thus the map is well defined.

We also have to show that this map doesn't depend on the trivialization we pick. Say we pick another trivialization $\{V_j\}$ and we want to show the morphism given by this trivilization agrees with the morphism given by $\{U_i\}$. Notice that we can combine the two trivializations and get a larger one $\{U_i, V_j\}$, then you can show that the morphisms given by $\{V_j\}$ and $\{U_i\}$ agrees with the one given by $\{U_i, V_j\}$. □

We will see later that these are all the morphisms from X to $\mathbb{P}^n$, but before that let's give some useful definitions:

Definition 1.19. If $\mathcal{L}$ is an invertible sheaf on X, then those points where all global sections of $\mathcal{L}$ vanish are called the **base points** of $\mathcal{L}$, thus if $\mathcal{L}$ has no nonzero sections, every point is a base point of $\mathcal{L}$.

The set of base points are called the **base locus** of $\mathcal{L}$ and it will be closed in X since it's the intersection of the vanishing points of all global sections, and the vanishing points for a global section is a closed subset of X, you can prove that pretty easily by the fact that vanishing points of a function is closed and you can choose trivializations. You can also refine this to the scheme theoretic base locus by taking the scheme-theoretic intersection of all the vanishing loci of global sections of $\mathcal{L}$, each equipped with the reduced closed subscheme structure. It's called the **scheme-theoretic base locus**.

The complement of the base locus is called the **base-point-free locus**. If $\mathcal{L}$ has no base point, we will say that it is **base-point-free**. As the complement of a closed subset, the base-point-free locus is an open subset of X.

Definition 1.20. Similar to the above definition, if we only have a collection of global sections, perhaps not all of the global sections. We can still define the base point of this collection to be the point where all of the sections in the collection vanish. You can also define the scheme-theoretic version.

The complement is call the base-point-free locus of the collection, and this collection is called base-point-free if it has no common base point.

Based on this definition, the previous proposition just tells you that a base-point-free collection of $n+1$ global sections of $\mathcal{L}$ gives you a morphism to $\mathbb{P}^n$.

Definition 1.21. A **linear series** or a **linear system** on a k-scheme X is a k-vector space V usually finite dimensional, an invertible sheaf $\mathcal{L}$ on X and a k linear map from V to the global sections of $\mathcal{L}$: $\lambda : V \to \Gamma(X, \mathcal{L})$.

Such a linear series is always only written as V, with other data being implicit. If the map λ is an isomorphism, it's called a **complete linear series** and is often written as $|\mathcal{L}|$.

You can also define base points of V, base locus of V, base-point-free locus of V and when V is called base-point-free. These all refer to the image of V in $\Gamma(X, \mathcal{L})$ via the map λ.

As a reality check, the previous proposition tells you that a $n+1$-dimensional linear series on a k-scheme X, with a choice of basis and with base point free locus U, gives a morphism from U to $\mathbb{P}^n$.

Proposition 1.22. If $\mathcal{L}$ and $\mathcal{M}$ are base point free on X, then so is their tensor product $\mathcal{L} \otimes \mathcal{M}$.

Proof. Since $\mathcal{L}$ and $\mathcal{M}$ are all invertible sheaves on X, the stalk at p is identified with $\mathcal{O}_{X,p}$ via:

$$\mathcal{O}_{X,p} \otimes \mathcal{O}_{X,p} \simeq \mathcal{O}_{X,p}$$

and the map is just multiplication. Then notice that we have the following commutative diagram:

$$\begin{tikzcd} \Gamma(X, \mathcal{M}) \otimes_k \Gamma(X, \mathcal{L}) \arrow[r] \arrow[d] & \Gamma(X, \mathcal{M} \otimes \mathcal{L}) \arrow[d] \\ \mathcal{O}_{X,p} \arrow[r] & \mathcal{O}_{X,p} \end{tikzcd}$$

where the left vertical map is $x \times y \mapsto x_p \cdot y_p$ the top horizontal map is $x, y \mapsto (x_p \otimes y_p = x_p \cdot y_p)_{p \in X}$ and the left vertical map is the projection (thus the stalk map) and the bottom horizontal morphism is just the identity. In other words, this is the diagram that tells you that the stalk of the presheaf is the stalk of the sheafified sheaf.

This tells you that if $\mathcal{M} \otimes \mathcal{L}$ has a base point, then in particular $x_p \cdot y_p$ vanishes at p for all x, y global sections of $\mathcal{L}$ an $\mathcal{M}$, but since they are base point free, there is at least some x, y such that their image in $\mathcal{O}_{X,p}$ is not in the maximal ideal, thus invertible, thus the product is invertible, not in the maximal ideal, thus a contradiction to $x_p \cdot y_p$ vanish. $\square$

In the next exercise we introduce the important notion of degeneracy. Suppose that we have a morphism $X \to \mathbb{P}^n_k$, we say this morphism is **degenerate** or X is degenerate if the scheme-theoretic image of X is in a hyperplane of $\mathbb{P}^n$. If this morphism corresponds to a base-point-free linear series $V \to \Gamma(X, \mathcal{L})$, we say the linear series is **degenerate** if the scheme theoretic image of X is contained in a hyperplane. If otherwise, we say it's **nondegenerate**.

Proposition 1.23. A base-point-free linear series $V \to \Gamma(X, \mathcal{L})$ is nondegenerate iff it is injective.

In particular, a complete linear series is always nondegenerate.

Proof. We will prove that if it's not injective, then there is a hyperplane in $\mathbb{P}^n$ that contains the scheme theoretic image. The main tool we will use is that if $X \to Y$ always factors through it's scheme theoretic image and if $X \to Y$ factor through a closed subscheme Z of Y, then it's scheme theoretic image factors through Z, thus in particular the scheme theoretic image is contained in Z, which we proved in a previous appendix.

Suppose that the basis of V is $e_0, \cdots, e_n$ whose image in $\Gamma(X, \mathcal{L})$ is base point free, but it's not linearly independent. If we still call the image e_i, then there is a k-linear equation $\sum r_i e_i = 0$ with not all of r_i zero. Then recall how we defined the morphism from X to $\mathbb{P}^n$ using this series. Suppose that U is a trivializing neighborhood of $\mathcal{L}$, then we claim that the scheme theoretic image of $U \to \mathbb{P}^n$ is in a hyperplane. We claim that this hyperplane is $\sum r_i x_i = 0$, this is because that if you consider this hyperplane locally in each chart $D_+(x_i)$, you find that the morphism from U to $\mathbb{P}^n$ is just:

$$U_{e_i} \to Speck[\frac{x_0}{x_i}, \cdots, \frac{x_n}{x_i}]$$

defined by $\frac{x_j}{x_i} \mapsto \frac{e_j}{e_i}$ where U_{e_i} stands for the locus where e_i doesn't vanish, however, notice that $(\sum r_j \frac{e_j}{e_i}) = 0$, thus it factors through the hyperplane locally in $D_+(x_i)$. This is summarized in the following commutative:

$$\begin{tikzcd}
U \arrow[r] & \mathbb{P}^n & H \arrow[l, hook'] \\
U_{e_i} \arrow[u, hook] \arrow[r] \arrow[rr, bend right] & D_+(x_i) \arrow[u, hook] & D_+(x_i) \cap H \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd}$$

Then if you consider other e_j, you will show that $U \to \mathbb{P}^n$ factors through H. Then this tells you that $X \to \mathbb{P}^n$ factors through H.

Conversely, if the scheme-theoretic image is contained in the hyperplane $\sum r_i x_i$, and the morphism corresponds to the basis $e_0, \cdots, e_n$ in V, we claim that the image satisfies $\sum r_i e_i = 0$. To see this, we still choose a trivializing cover of X and on each U we use the same argument as above to show that since the scheme-theoretic image is contained in the hyperplane the morphism factors through the hyperplane, thus locally it factors through the hyperplane, therefore we know from the universal property of the quotient, we know that locally on U, $\sum r_i e_i = 0$ is satisfied. Since U covers X, we know that it's satisfied globally by the sheaf axioms. $\square$

The following theorem provides the converse to the statement that base-point-free linear series give morphisms into projective spaces. It tells you that they are all the possible morphisms you can ever get. It tells you why line bundles are of crucial importance.

Theorem 1.24. For a fixed scheme X over any base ring, morphisms $X \to \mathbb{P}^n$ are in bijection with the data $(\mathcal{L}, s_0, \cdots, s_n)$ where $\mathcal{L}$ is an invertible sheaf and $s_0, \cdots, s_n$ are global sections of $\mathcal{L}$ with no common zero, up to isomorphism of these data.

Before prove it, let's make some useful remark which helps proving the theorem:

Remark 1.25. First, we say that an isomorphism of the data $(\mathcal{L}, s_0, \cdots, s_n)$ and $(\mathcal{M}, t_0, \cdots, t_n)$ is an isomorphism of line bundles $\mathcal{L} \simeq \mathcal{M}$ such that under this isomorphism s_i is mapped to t_i.

Then, we give another way of think the morphism $X \to \mathbb{P}^n$ given by $s_0, \cdots, s_n$. First, recall that since the locus where a section of a line bundle vanishes is closed, the locus where it doesn't vanish is open. Then we can consider X_{s_i}, which is open. Now we claim that for a line bundle $\mathcal{L}$ on X_{s_i}, $\mathcal{L}$ is trivial, and the isomorphism is given by:

$$\mathcal{O}_{X_{s_i}} \to \mathcal{L}|_{X_{s_i}}$$

where 1 is mapped to s_i. To see it's indeed an isomorphism, let's consider it's action on the stalk. It induces a morphism $\mathcal{O}_{X,p} \to \mathcal{O}_{X,p}$ where 1 is mapped to some invertible function in $\mathcal{O}_{X,p}$ for any $p \in X_{s_i}$, and I will let you see why. This tells you that this is indeed an isomorphism.

Then, on X_{s_i}, I can define a morphism $X_{s_i} \to D_+(x_i)$ by the same method we define a morphism from any trivilizing neighborhoods of X, this is the same as any other morphism defined by find trivializations not X_{s_i}.

Moreover, under this trivilization, you can check the the transition function on $U_i \cap U_i$ is given by multiply by $\frac{s_i}{s_j}$. And the inverse image of $D_+(x_i)$ is just X_{s_i}.

Remark 1.26. To show we have a bijection, we need to take a look at what a morphism $\pi : X \to \mathbb{P}^n$ gives you.

First, you know that there is a line bundle on the projective space $\mathcal{O}(1)$, with global sections $x_0, \cdots, x_n$, no common zero. Surprisingly, the locus where x_i is not zero is exactly $D_+(x_i)$ which you can check quite easily and the transition function of this line bundle on intersections of $D_+(x_i) \cap D_+(x_j)$ is given by multiplication by $\frac{x_i}{x_j}$, which is quite similar to the transition function we see up there. Then if you consider the pullback $\pi^*\mathcal{O}$, with global sections $\pi^*x_0, \cdots, \pi^*x_n$. Then we know that the locus where π^*x_0 is not zero is just $\pi^{-1}(D_+(x_i))$ since we already prove that the fiber of $\pi^*\mathcal{O}(1)$ at p is $k(q) \otimes_{k(q)} k(p)$, and the image of π^*x_i in the fiber is $x_i(q) \otimes 1$, then it's not zero iff $x_i(q)$ is not 0 where $\pi(p) = q$, thus we know that only those p in the inverse image of the nonvanishing locus of x_i gives you nonvanishing locus of π^*x_i.

Again, here are some identifications you need to check, for example we know that the global sections x_i is given by the morphism that maps 1 to s:

$$\mathcal{O}_Y \to \mathcal{O}(1)$$

where $Y = \mathbb{P}^n$, then the global sections π^*x_i is actually the section determined by:

$$f^{-1}\mathcal{O}_Y \otimes \mathcal{O}_X \to f^{-1}\mathcal{O}(1) \otimes \mathcal{O}_X$$

we want to see what is the stalk of this new global section, it's the same to watch the morphism induced on stalks by $f^{-1}\mathcal{O}_Y \otimes \mathcal{O}_X \to f^{-1}\mathcal{O}(1) \otimes \mathcal{O}_X$ and see where does 1 go. To figure where does 1 go, we need to first focus on:

$$f^{-1}\mathcal{O}_Y \to f^{-1}\mathcal{O}(1)$$

Then use the colimit description of the inverse image, you see that 1 in the stalk at p is mapped to the element corresponding to the image of s at the stalk of of q where $\pi(p) = q$. I will let you do the detailed work.

But this tells you that if we take the trivializing neighborhoods of $\pi^*\mathcal{O}(1)$ on X determined by the nonvanishing locus of π^*x_i. The transition functions will be multiplication by $\frac{\pi^*x_i}{\pi^*x_j}$. This comes from the fact that if we have an endomorphism of $\mathcal{O}_Y$, defined by multiplication by some global section of $\mathcal{O}_Y$, then after you apply the pullback functor, the endomorphism induced on the pullback is the one by multiplication by the pullback global section, which is summarized in the following diagram:

$$\mathcal{O}_Y \xrightarrow{s} \mathcal{O}_Y \qquad f^{-1}\mathcal{O}_Y \otimes \mathcal{O}_X \xrightarrow{f^*s} f^{-1}\mathcal{O}_Y \otimes \mathcal{O}_X$$

then since pullback functor commutes with restriction to open subschemes, thus we have the following diagram:

$$\begin{tikzcd}
\pi^*(\mathcal{O}_Y|_U) \arrow[r, "\pi^*(s|_U)"] \arrow[d, "\simeq"'] & \pi^*(\mathcal{O}(1)|_U) \arrow[d, "\simeq"] \\
(\pi^*\mathcal{O}_Y)|_{\pi^{-1}U} \arrow[r, "(\pi^*s)|_{\pi^{-1}U}"'] & (\pi^*\mathcal{O}(1))|_{\pi^{-1}U}
\end{tikzcd}$$

this tells you the transition function of the pullback.

Remark 1.27. The last remark we would like to give is still about the pullback sections. Suppose that $f : X \to Y$ is a morphism of scheme, and a global section y of $\mathcal{O}_Y$ is mapped to the global section x of $\mathcal{O}_X$, then we claim that the pullback section $f^*y = x$.

Here is why, recall that the image of the pullback section f^*y in the stalk of p where $q = f(p)$ is just:

$$y' \otimes 1 \in \mathcal{O}_{Y,q} \otimes_{\mathcal{O}_{Y,q}} \mathcal{O}_{X,p}$$

where y' is the image of y in the stalk of $f^{-1}\mathcal{O}_Y$, I will let you show that in the induced morphism:

$$\mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$$

y' is mapped to x' where x' is the image of x in $\mathcal{O}_{X,p}$, thus the stalk of f^*y at each point is the same as the stalk of x at each point, thus they are the same since sections are determined by stalks.

Now we are ready to prove the theorem and it's surprisingly easy if you figure out the right notation to use:

Proof. Suppose that we have $s_0, \cdots, s_n$, base point free sections of $\mathcal{L}$ on X, which determines the morphism $\pi : X \to \mathbb{P}^n$, we claim the inverse is given by $\pi^*\mathcal{O}(1)$, i.e. there is an isomorphism of line bundles:

$$\mathcal{L} \simeq \pi^*\mathcal{O}(1)$$

and such an isomorphism can be chosen such that the image of s_i is π^*x_i for global sections $x_0, \cdots, x_n$ of $\mathcal{O}(1)$.

To see how this works, we already see that the inverse image of $D_+(x_i)$ is X_{s_i} and the transition function of $\mathcal{L}$ associated to this trivilization is $\frac{s_i}{s_j}$(of course the quotient of two sections of line bundle doesn't make sense therefore this notation stands for the element r in the ring such that $rs_j = s_i$ can you see why there is indeed a unique r). On the other hand, we also have seen that $\pi^*\mathcal{O}(1)$ is also a line bundle on X with same trivializing opens X_{s_i} with transition functions $\pi^*\frac{x_i}{x_j}$(Here this fraction $\frac{x_i}{x_j}$ is still interpreted to be the ring element such that $x_i \cdot \frac{x_j}{x_i} = x_j$). However, from the previous remark, since $\frac{x_i}{x_j}$ (now considered an element in the ring $(k[x_0, \cdots, x_n]_{x_i,x_j})_0$) is mapped to s_i/s_j then the pullback $\pi^*(\frac{x_i}{x_j}) = \frac{s_i}{s_j}$, thus the transition functions of $\mathcal{L}$ and $\pi^*\mathcal{O}(1)$ are the same, thus they are isomorphic.

Finally, we only have to show the isomorphism can be chosen so that the image of s_i's are π^*x_i's or vice versa. To show this, notice that on each intersection we have: $s_i/s_j = \pi^*(\frac{x_i}{x_j}) = k$ Recall that the transition function $\frac{x_i}{x_j}$ is computed under the trivialization that on each $D_+(x_i)$ we identify the restriction of $\mathcal{O}(1)$ with the structure by mapping 1 to $x_i|_{D_+(x_i)}$, thus we know that the way we trivialize $\pi^*\mathcal{O}(1)$ is via:

$$\mathcal{O}_{X_{s_i}} \simeq \pi^*\mathcal{O}(1)|_{X_{s_i}} \qquad 1 \mapsto \pi^*x_i|_{X_{s_i}}$$

Recall that we can conclude $\pi^*\mathcal{O}(1)$ and $\mathcal{L}$ by the fact that they have the same transition function is because that we can use the function to glue a new invertible sheaf and show that it's isomorphic to both $\mathcal{L}$ and $\pi^*\mathcal{O}(1)$ where on each X_{s_i} the isomorphism locally looks like:

$$\begin{tikzcd}
\pi^*\mathcal{O}(1)|_{X_{s_i}} \arrow[rr, "\simeq"] & & \mathcal{L}|_{X_{s_i}} \\
& \mathcal{O}_{X_{s_i}} \arrow[ul, "1\mapsto\pi^*x_i"] \arrow[ur, "1\mapsto s_i"'] &
\end{tikzcd}$$

Now to show that the image of s_i under the global isomorphism is indeed π^*x_i, we only have to check this on an open cover, and we will take the open cover to be $X_{s_i}, X_{s_i} \cap X_{s_j}$ for all $j \neq i$. The checking in X_{s_i} is trivial. For simplicity, we will call the global isomorphism $\Phi : \mathcal{L} \to \pi^*\mathcal{O}(1)$, then $\Phi(s_i|_{X_{s_i} \cap X_{s_j}}) = \Phi(s_j|_{X_{s_i} \cap X_{s_j}} \cdot \frac{s_i}{s_j}) = \frac{s_i}{s_j}\Phi(s_j|_{X_{s_i} \cap X_{s_j}}) = \frac{s_i}{s_j} \cdot (\pi^*x_j)|_{X_{s_i} \cap X_{s_j}} = \pi^*\frac{x_i}{x_j}(\pi^*x_j)_{X_{s_i} \cap X_{s_j}} = (\pi^*x_i)|_{(\pi^*x_j)_{X_{s_i} \cap X_{s_j}}}$, thus we are done.

For the other direction, suppose that we are given a map $\pi : X \to \mathbb{P}^n$, let's first produce a line bundle $\mathcal{L}$ on X with sections $s_0, \cdots, s_n$ no common zeros. A natural thought is to take $\mathcal{L} = \pi^*\mathcal{O}(1)$ and take $s_i = \pi^*x_i$. Since we know that x_i's have no common zeros on $\mathbb{P}^n$, we have proved that their pullbacks have no common zeros. Then we know that $\mathcal{L}$ and s_i gives a morphism to $\mathbb{P}^n$, we need to prove that the two morphisms agree. Recall that the inverse image $\pi^{-1}D_+(x_i)$ is $D(\pi^*x_i) = D(s_i)$. Thus we only have to show that the two morphisms $D(s_i) \to D_+(x_i)$ agree.

Recall that $\pi : D(s_i) \to D_+(x_i)$ is determined by where each $\frac{x_j}{x_i}$ goes, we know that the pullback section of $\frac{x_j}{x_i}$ is equal to $\pi^*\frac{x_j}{x_i} = s_j|_{X_{s_i}}$. However, notice that $s_i|_{X_{s_i}}$ is identified with 1, thus $s_j|_{X_{s_i}} = s_j|_{X_{s_i}}/s_i|_{X_{s_i}}$, thus the two morphisms agree. □

Remark 1.28. The above proof can be carried out with no change for a collection of global sections $s_0, \cdots, s_n$ on X, but with vanishing locus $V(s_0, \cdots, s_n)$ defined to be the intersection of the vanishing locus on each s_0, which is closed, then you can define a morphism:

$$X - V(s_0, \cdots, s_n) \to \mathbb{P}^n$$

In particular, if X is integral, then this gives you a rational morphism.

Now it's time to see a few examples and applications of this idea. One important thing to keep in mind is that whenever you see a morphism into projective space, you should think what line bundle and global sections this morphism corresponds to.

Example 1.29. In this example, you will see that Veronese embedding corresponds to the complete linear series $|\mathcal{O}_{\mathbb{P}^n_k}(d)|$ with the choice of the standard basis corresponding to the degree d homogeneous polynomials. They have no common zeros since you only have to look at $x_0^d, \cdots, x_n^d$. Therefore this complete linear series gives you a morphism:

$$\mathbb{P}^n_k \to \mathbb{P}^N_k$$

where $N = \binom{n+d}{d} - 1$. I claim that this is the Veronese embedding.

Consider the d-th Veronese embedding $v_d : ProjS_{d\bullet} \to \mathbb{P}^N_k$, where $S_\bullet = k[x_0, \cdots, x_n]$ this morphism correspond to the morphism induced by the pullback of $\mathcal{O}_{\mathbb{P}^N_k}(1)$, we also have an isomorphism between $\mathbb{P}^n_k \to ProjS_{d\bullet}$ induced by the natural inclusion $S_{d\bullet} \hookrightarrow S_\bullet$. Together, they give us a closed embedding:

$$\mathbb{P}^n_k \to \mathbb{P}^N_k$$

We only have to check they agree locally, I claim that the morphism given by the complete linear series is locally given, for example on the locus where x_0^d doesn't vanish, by the obvious ring morphism:

$$(k[y_0, \ldots, y_N]_{y_0})_0 \to ((k[x_0, \cdots, x_n])_{x_0^d})_0 \simeq (k[x_0, \cdots, x_n]_{x_0})_0$$

where each y_i is mapped to a homogeneous monomial of degree d. To see this, you only have to worry about what is the trivialization on $\mathcal{O}(d)$ on the locus where x_0^d doesn't vanish, we first have the trivialization on $D(x_0)$, the use the isomorphism $((k[x_0, \cdots, x_n])_{x_0^d})_0 \simeq (k[x_0, \cdots, x_n]_{x_0})_0$, you get a trivialization on $((k[x_0, \cdots, x_n])_{x_0^d})_0$, then you will see what I promised you is correct. You can also do this for all the other x_i^d's.

Now for the Veronese embedding, we know it's induced by the ring morphism:

$$k[y_0, \cdots, y_N] \to S_{d\bullet} \to S_\bullet$$

Then recall how do we define a morphism out of a graded ring morphism, then you will find the two morphisms locally are the same.

Remark 1.30. When we defined $\mathcal{O}(d)$, we only consider the standard affine open $D_+(x_i)$ and the transition functions between them. Actually, we can consider this construction for all f with f homogeneous of positive degree. One reason for this construction is obviously that it doesn't depend on the choice of coordinates, another reason for doing this is that we can get a trivialization on each $D_+(f)$, which can be convenient in situation like above where we want to know the trivialization on $D_+(x_0^d)$ or other distinguished opens.

The following is a more general method of checking morphisms to projective spaces are a closed embedding:

Proposition 1.31. Suppose $\pi : X \to \mathbb{P}^n_A$ is a morphism corresponding to an invertible sheaf $\mathcal{L}$ on X and sections $s_0, \cdots, s_n$, then it's a closed embedding if and only if:

1. Each X_{s_i} is an affine open.
2. For each i, the map of rings $A[y_0, \cdots, y_n] \to \Gamma(X_{s_i}, \mathcal{O})$ given by $y_j \mapsto \frac{s_j}{s_i}$ is surjective.

Proof. Suppose that π is a closed embedding, notice that we already proved that the inverse image of $D_+(x_i)$ is X_{s_i}, and since a closed embedding is affine, we know that X_{s_i} is affine and surjective is also from definition.

The converse is also easy to check. $\square$

Example 1.32. Recall that the image of the Veronese embedding when n is 1 is called a rational normal curve of degree d. Then it's given by:

$$\mathbb{P}^1_k \to \mathbb{P}^d_k$$

where $[x, y] \mapsto [x^d, x^d y, \cdots, xy^{d-1}, y^d]$ and $d = \binom{d+1}{1} - 1$. This notation stands for the morphism induced by $|\mathcal{O}(d)|$ on $\mathbb{P}^1_k$.

Proposition 1.33. Suppose that $\pi : \mathbb{P}^1 \to \mathbb{P}^n_k$ which correspond to the invertible sheaf $\mathcal{O}(d)$(this can be called a degree d morphism), then if $d < n$, the image is degenerate and if $d = n$ and the image is not degenerate, it's isomorphic to a rational normal curve via an automorphism of the projective space.

Proof. Say $d < n$, we know that we picked $n + 1$ base point free elements in $\Gamma(\mathbb{P}^1, \mathcal{O}(d))$, which are linearly dependent, since the dimension of this vector space is only of dimension $d + 1$, thus what we proved earlier tells you that this is degenerate since the corresponding linear series is not an inclusion.

Suppose that $d = n$ and the morphism is not degenerate, we know that the $n + 1$ elements should be linear independent (thus a basis). Now since there is an invertible transformation taking the standard basis of $\Gamma(\mathbb{P}^1, \mathcal{O}(d))$ to the basis we just picked.

Recall that such a linear transformation can also be applied to $k[x_0, \cdots, x_n]$ and induces an automorphism on $\mathbb{P}^n$, now consider the following diagram:

$$\begin{tikzcd} & \mathbb{P}^n \arrow[dd, "\simeq"] \\ \mathbb{P}^1 \arrow[ur, "\pi"] \arrow[dr, "standard"'] & \\ & \mathbb{P}^n \end{tikzcd}$$

where the isomorphism is induced by the inverse of the invertible linear transformation and the lower arrow is induced by the standard basis of $\Gamma(\mathbb{P}^1, \mathcal{O}(n))$. We claim that it's commutative, which will be proved by showing that the data corresponding to the morphism are isomorphic.

Consider the morphism $\mathbb{P}^1 \to \mathbb{P}^n \to \mathbb{P}^1$, notice that $\mathbb{P}^n \to \mathbb{P}^1$ is induced by $\mathcal{O}(1)$ and the global sections are the sections you get after you apply the inverse of the linear transformation to the standard basis. Then you pullback these sections along π, which commutes with the linear transformation, then you get they pullback to the standard basis. But this is just the data induce the lower arrow, we are done. $\square$

Below is a base point free description of the main theorem of this section:

Proposition 1.34. Suppose that X is a k-scheme then a finite dimensional base-point-free linear series V on X corresponding to $\mathcal{L}$ induces a morphism to projective space:

$$\Phi_V : X \to ProjV^\vee$$

Proof. **** later* ☐

Proposition 1.35. Recall the graded ring $R_\bullet = k[u,v,w]/(wu - v^2)$ and $S_\bullet = k[x,y]$, we know $ProjS_\bullet \simeq ProjR_\bullet$ when we discussed Veronese embedding. This isomorphism doesn't come from any map of graded rings $S_\bullet \to R_\bullet$

Proof. **** later* ☐

Below, we take a look at some intersection theory related to Bezout's theorem. Let's first define some notions. We will focus on the degree d morphism $\pi : \mathbb{P}^1_k \to \mathbb{P}^n_k$ defined by the standard basis of $\mathcal{O}(d)$ on $\mathbb{P}^1$. We will interpret the classical degree of a curve in the projective space in this following proposition. To make sense of this, first I claim that there is a hyperplane H in $\mathbb{P}^n$ such that the it doesn't contain the image of π. Suppose that H is defined by $\sum a_i x_i = 0$, then this is equivalent to find a $H = a_i x_i \in \mathcal{O}_{\mathbb{P}^n_k}(1)$ such that the pullback of H, $\pi^* H$ is not 0 in $\Gamma(\mathbb{P}^1, \mathcal{O}(1))$. Now we will define the degree of π to be the zeros of $\pi^* H$ with the appropriate multiplicity counted. To be precise, the multiplicity **** to be completed later*

Proposition 1.36.

Example 1.37. The Segre embedding can also be interpreted in this way and it gives us some excuse to intorduce a notation. Suppose that $\mathcal{F}$ is a quasicoherent sheaf on a Z-scheme X and $\mathcal{G}$ is a quasicoherent sheaf on a Z-scheme Y, let π_X, π_Y be the two canonical projections from $X \times_Z Y$ to X, Y, we define $\mathcal{F} \boxtimes \mathcal{G}$ to be $\pi_X^* \mathcal{F} \otimes \pi_Y^* \mathcal{G}$. In particular, we define $\mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}(a,b)$ to be $\mathcal{O}_{\mathbb{P}^m}(a) \boxtimes \mathcal{O}_{\mathbb{P}^n}(b)$, over any base Z.

Then we claim that the Segre embedding $\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}m + n + mn$ is determined by the complete linear series $\mathcal{O}(1,1)$. **** to be completed later*

This is also a good place to review Segre embedding since we haven't done it really seriously before. To get the motivation of what happens in the Segre embedding, first we can informally consider the following map:

$$\mathbb{P}^2 \to \mathbb{P}^1 \to \mathbb{P}^5$$

define to be:

$$[x_0, x_1, x_0] \times [y_0, y_1] \to \begin{bmatrix} x_0 y_0 & x_1 y_0 & x_2 y_0 \\ x_0 y_1 & x_1 y_1 & x_2 y_1 \end{bmatrix}$$

You can see that this is well defined up to nonzero scalar multiplications and won't be all 0 if x_i and y_j's are not all 0. Again, from the result, we can recover x_i and y_j up to scalar multiplication. For example, we can first look at the first row, if they are not all 0, we recover $[x_0, x_1, x_2]$, it's zero we go to the second row, and it must not be all 0. This suggest that if we define a morphism like this, it should be some kind of embedding, and indeed it's true, which is called the Segre embedding defined below:

Suppose that A is any ring, then we will define a morphism: $\mathbb{P}^m \to \mathbb{P}^n \to \mathbb{P}^{mn+m+n}$ that looks like the morphism above. Locally on $U_i \times V_j$ where $U_i = D_+(x_i)$ and $V_j = D_+(y_j)$, this morphism will map to $W_{ij} = D_+(z_{ij})$ where the coordinate on $\mathbb{P}^{mn+m+n}$ is given by $z_{00}, z_{0,1}, \cdots, z_{ij}, \cdots, z_{mn}$. And it's defined by the ring morphism:

$$A[\frac{z_{st}}{z_{ij}}] \to k[\frac{x_0}{x_i}, \cdots, \frac{x_m}{x_i}, \frac{y_0}{y_j}, \cdots, \frac{y_n}{y_j}]$$

where Z_{st}/z_{ij} is mapped to $\frac{x_s}{x_i} \cdot \frac{y_t}{y_j}$. Notice that this is a surjective morphism, thus if it's well defined globally, we get a closed embedding.

Now we only have to check that they glue on intersections. The only point you need to figure out is what is the restriction to $U_i \times V_j \cap U_a \times V_b = (U_i \cap U_j) \times (V_j \cap V_b)$ and the answer is just, for example:

$$k[\frac{x_0}{x_i}, \cdots, \frac{x_m}{x_i}, \frac{y_0}{y_j}, \cdots, \frac{y_n}{y_j}] \to k[\frac{x_0}{x_i}, \cdots, \frac{x_m}{x_i}, \frac{y_0}{y_j}, \cdots, \frac{y_n}{y_j}, \frac{x_0}{x_1}, \cdots, \frac{x_m}{x_a}, \frac{y_0}{y_b}, \cdots, \frac{y_n}{y_b}]$$

Then the glue should be obvious. The morphism we get is always called Segre morphism or Segre embedding, and if k is a field, the image will be called the Segre variety.

The image of the embedding, as a closed subvariety can be described by the closed subscheme corresponding to the equations that makes the rank of the following matrix 1:

$$\begin{bmatrix} z_{00} & \cdots & z_{0n} \\ \cdots & \cdots & \cdots \\ z_{m0} & \cdots & z_{mn} \end{bmatrix}$$

To see this, notice that the Segre closed subscheme is lcoally defined very clearly by the ideal generated by the polynomials f in variables $\frac{z_{st}}{z_{ij}}$ such that if we make the substitution $z_{st} = x_s y_t$, it becomes 0, notice that if this conditions holds, then after we homogenize this polynomial by multiply a suitable power of z_{ij}, we get a polynomial in z_{st} such that we substitue $z_{st} = x_s y_t$, we get zero. To proceed further, we need some combinitoric on polynomial in several variables index by s, t. For simplicity, we will order all the z_{st} lexicographically, a monomial $z_{s_1 t_1} \cdots z_{s_n t_n}$ is called standard if the subscript is lexicographic in the sense that $s_1 \leqslant s_2 \cdots \leqslant s_n$ and similarly for t, then there is an algorithm, which is not complicated, which allows you to convert any monimial to a standard monomial plus some terms that has a factor in the 2×2 minor. Then suppose that f is any homogeneous polynomial, we convert each monomial in the term using the above algorithm. Notice that if f vanishes after we substitute $z_{st} = x_s y_t$, then the terms which contains terms coming from the 2×2 minors vanishes, the remain terms are linearly independent since they are of standard form, this tells you that f is in the ideal generated by the 2×2 minors.

This algorithm tells you that locally the Segre subscheme is the same with the closed subscheme defined by the 2×2 minors and it's easy to see that this local isomorphism is compatible on the intersections, thus they are indeed isomorphic.

As an important example, we consider $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$, which is the closed subscheme defined by:

$$rank \begin{bmatrix} z_{00} & z_{01} \\ z_{10} & z_{11} \end{bmatrix} = 1$$

You should recall that this is just our old friend, the quadric surface. Therefore, we know the smooth quadric $wz - xy = 0$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

1.3 Curve to Projective Extension Theorem

Recall that in the previous section, we had a failed attempt in defining a morphism $\mathbb{P}^2 \to \mathbb{P}^3$ where one point of the domain doesn't know where to go. However, in this section you will see that this kind of failure doesn't happen when we are dealing with morphisms with domains $\mathbb{P}^1$. For example it appears that the rational morphism $\mathbb{P}^1 \to \mathbb{P}^3$ given by:

$$[x^7, x^3y^4, x^4y^3 + 4x^5y^2, x^5y^2 + x^7]$$

fails to give you where does $[0, 1]$ go, however, if you divide each of the entry by x^3, it becomes $[x^4, y^4, xy^3 + 4x^2y^2, x^2y^2 + x^4]$ whihc gives you a well defined morphism. We will generalize this kind of behavior to the following theorem:

Theorem 1.38. Suppose that C is a pure dimension 1 Noetherian scheme over an affine base $S = SpecA$, and $p \in C$ is a regular closed point, suppose that Y is a projective A-scheme then any morphism $C - \{p\} \to Y$ extends to all of C.

Remark 1.39. To avoid any confusion, pure dimension 1 here means that it's topologically dimension 1, not relative of pure dimension 1 in any sense.

In practice, we will most often use $S = Speck$ and C is a k-variety. The only reason we are assuming S is affine is because we still don't know the meaning of a projective S-scheme until we know the definition of a projective morphism. But after we introduce that definition, you will find the proof extends to that situation as well.

Remark 1.40. Another point you need to notice is that if the extension exists then it must be unique. Recall that a regular point of C is reduced, and we know that the nonreduced points are closed in C, therefore there is a small neighborhood of p which is reduced, then since projective A-schemes are separated, you can use the reduced to separated theorem to guarantee the uniqueness.

The following examples tell you that our hypothesis are necessary:

Proposition 1.41. Let $\mathbb{A}^1_k = C = Y$ and $p = (x)$. Consider the morphism from $C - \{p\}$ to Y given by $t \mapsto \frac{1}{t}$. Or in other words, it's given by the ring morphism:

$$k[t] \to k[t, t^{-1}] \qquad t \mapsto \frac{1}{t}$$

Then this morphism can not extend to C and the reason is just there is no k-algebra homomorphism such that he following diagram is commutative:

$$\begin{tikzcd} k[t] \arrow[rr] \arrow[dr] & & k[t, \frac{1}{t}] \\ & k[t] \arrow[ur] & \end{tikzcd}$$

This tells you that Projectivity of Y is necessary.

The second example tells you that if C is not of dimension 1, extension may not hold. Let $C = \mathbb{A}^2_k$ and $p = (x, y)$. Let $\mathbb{P}^1_k = Y$, the morphism $C - \{p\} \to \mathbb{P}^1$ is given by $(x, y) \mapsto [x, y]$ or in other words, given by the two sections x, y of the trivial line bundle. We have seen that this morphism doesn't extend to the origin. For the sake of contradiction, if it extends, we know that we have the following diagram:

$$\begin{tikzcd} C - \{p\} \arrow[r] \arrow[d, hook] & \mathbb{P}^1 \\ \mathbb{A}^2 \arrow[ur] & \end{tikzcd}$$

where the pullback of x_0, x_1 of $\mathcal{O}(1)$ on $C - \{p\}$ should be x, y, then we know the pullback along another path is also x, y, since we know that there is only thing on $\mathbb{A}^2$ that pulls to x, y are x, y, thus we know that x_0, x_1 pulls back to x, y on $\mathbb{A}^2$, but this doesn't give you a morphism from all of $\mathbb{A}^2$ since x, y vanishes at the origin, thus a contradiction to the theorem.

The third example shows that the regularity of p is also necessary: Let $C = k[x, y]/(y^2 - x^3)$ and p is still the origin. Let $Y = \mathbb{P}^1$. Consider the morphism from $C - \{p\} \to Y$ still defined by $(x, y) \mapsto [x, y]$. This is the cusp and we know that the origin is not regular

Now, let's say a little bit about the central ideal of the proof. The main idea to to clear powers of the uniformizer. To see what this means, suppose that you have a morphism $\mathbb{A}^1 - 0 \to \mathbb{P}^1$ defined by $t \mapsto [t^4 + t^{-3}, t^{-2} + 4t]$, then if you want to extend to the origin, you would definitely multiply by t^3 and get $[t^7 + 1, t + t^4]$. Or if you have something like $[t^2 + t^3, t^2 + t^4]$, you will divide by t^2. And as what you will see in the proof, we will maneuver ourselves into this situation:

Proof. Let's begin with two reductions, we first know that p is a reduced point and nonreduced points are closed, thus we can assume C is affine and reduced.

Next, we claim that we can require $Y = \mathbb{P}^n_A$ since we can pick a closed embedding $X \hookrightarrow \mathbb{P}^n_A$ and if we can prove $C - \{p\}$ is mapped to Y then we claim that if the map extends, then p is also in X. The reason is not complicated, suppose that p is mapped to some distinguished open, we can take the inverse image and assume that we are working with an affine source and the target is $\mathbb{A}^n_A$. Consider $Y \cap \mathbb{P}^n$, if we want to prove C maps to Y, we only need to prove that pullback of those functions of $\mathbb{A}^n$ that lies in I, the ideal corresponding to the closed subscheme $Y \cap \mathbb{A}^n$ vanishes on C (can you see why? Hint use the reducedness of C thus if a function vanishes every where, then it's 0). However, we already know that the functions vanishing on $Y \cap \mathbb{A}^n$ vanishes on points not at p, which contains all the generic points (finitely many) of irreducible components of C, thus they vanish on all of C and since C is reduced, the function is pulled back to 0, thus we are done.

Now, since p is a regular closed point, it's of codimension 1, thus the local ring at p is a regular local ring of dimension 1, thus a DVR, then we can choose a uniformizer $t \in m - m^2$ at p. As t comes from some functions on a neighborhood of p we can assume that t is a global function by restricting to that neighborhood of t. Now $V(t)$ contains p, but it doesn't contain the generic point of the component that contains p. This is because if so, suppose that the generic point is q, then q correspond to the zero ideal 0 after we loclize at m since the local ring is a integral domain of dimension 1, but this implies that $t = 0$, contradiction that t is a uniformizer.

We claim that $C - V(t) \cup \{p\}$ is open in C. To see this, in a Noetherian scheme of pure dimension 1 every proper closed subset is a finite union of closed points, thus since $V(t)$ is not all of C its just a finite union of closed points, then if it only has p, we are good, if it has more than just p, we take out those points and get the desired result, thus we can assume that on C, p is the only zero of t.

Now suppose that we have a morphism $C - \{p\} \to \mathbb{P}^n_k$ corresponding to a line bundle $\mathcal{L}$ on $C - \{p\}$ with $n+1$ base point free sections. Let $U \subset C - \{p\}$ be a trivializing neighborhood of $\mathcal{L}$, then again by replacing C by $U \cup \{p\}$ (it's open in C again because we know the topology of C quite well), we can assume that the morphism on $C - \{p\}$ is defined by $f_0, \cdots, f_n$, $n+1$ rational functions with no common zero away from p. Now, let $N = min_i(val_p(f_i))$, we claim that $t^{-N} f_i$ now has no zero at p(you should think a little bit on why after we multiply t^{-N}, these rational functions extend to a regular section or in other words, extend to a neighborhood of p. Hint: You can take the trivializing neighborhood to be a distinguished open say $D(f)$, then we know that we can assume that $f_0, \cdots, f_n$ are restrictions of functions on the larger affine space since multiplication by powers of an invertible f doesn't affect the morphism we define), thus regular functions defined on C. To see this, notice that f has a zero at p iff $val(f) > 0$, then after we multiply by t^{-N}, the valuation at p is at most 0, thus there is no 0 at p, and since t doesn't vanish anywhere away from p, we know that $t^{-N} f_i$ indeed are regular functions with no base point, thus induces a morphism from C to $\mathbb{P}^n$ and since on $C - \{p\}$, t is invertible thus the morphism restricted on $C - \{p\}$ is the same as the original morphism since the sections only differ by an invertible function. ☐

The following proposition is a good practice of the above idea:

Proposition 1.42. Suppose that X is a Noetherian k-scheme and Z is an irreducible codimension 1 subvariety whose generic point p is a regular point of X. If $X \dashrightarrow Y$ is a rational map to a projective k-scheme, then the domain of definition of the ratiinal map includes a dense open subset of Z.

Proof. First, let's prove a topology statement, in a Noetherian space, a dense open necessarily intersects each component nontrivially. This is because a Noetherian space only has finitely many components and suppose that U is a dense open such that it doesn't intersect with the component X_1, then we know that U is contained in $X_2 \cup \cdots \cup X_n$, then the closure of U is contained in $X_2 \cup \cdots \cup X_n$ since a finite union of closed set is still closed. This is a contradiction to U is dense since $X_2 \cup \cdots \cup X_n \neq X$.

It suffice to prove that the domain of definition include the generic point of Z. First, we can choose a reduced affine open U of the generic point, we know that the rational map is defined on a dense open of this affine scheme to Y, then by choosing a embedding of Y into $\mathbb{P}^n$ we can assume the target is $\mathbb{P}^n$ since if the rational map extends to p it will lie in Y since the affine open is reduced and all the generic points of the irreducible components of the affine open is in the dense open.

Now choose a uniformizer t of m/m^2 where m is the maximal ideal of the local ring at p. Again by further shrink U we can assume t is a function on U. Notice that $V(t)$ again contains p but not the generic point of the component that contains p since that will contradict to the fact that t is a uniformizer just like what happened above. Now since t is not 0 on U, $V(t)$ is a closed subset of codimension 1, we discard the components that doesn't contain p, we can assume that t can only possibly be 0 on the closure of p.

Now choose a a trivializing neighborhood of U minus the zero locus of t, we can assume that it's still a distinguished open contained in the domain of the rational map, then we can assume that the rational map is defined by $f_0, \cdots, f_n$, n rational functions defined as the restriction of global functions and with no common zeros on the trivializing neighborhood. Then we still let $N = min_i(val_p(f_i))$ and consider the global functions $t^{-N} f_i$, at least one of them is not 0 at p, thus now the morphism defined by $t^{-N} f_i$'s has the domain that contains an open neighborhood of p, and on the original domain the definition agrees with the rational function we start with since the data that defines the two morphisms only differ by an invertible element t on the intersection. $\square$

1.4 Line Bundles and Weil Divisors

This section is the another core sections of line bundles, since using the notion of Weil divisors we can understand and classify line bundles very well at least on Noetherian normal schemes thanks to the Algebraic Hartog's lemma we introduced before. Some of the discussions can be applied to more general setting but let's stick to the basics here and try to generalize later.

For the rest of the section, we will consider only Noetherian normal schemes.

Definition 1.43. A Weil divisor is a formal $\mathbb{Z}$-linear combination of codimension 1 irreducible closed subsets of X, in other words, a Weil divisor is defined to be an object of the form:

$$\sum_{Y\subset X, codim_X Y=1} n_Y[Y]$$

where n_Y's are integers, all but a finite number of which are 0. Weil divisors obviously form an abelian group, denoted by Weil$X$. For example if X is a curve, then Weil$X$ is just a formal linear combination of closed points. First, you can easily show that closed points on a curve has codimension 1. Then say Y is an irreducible closed subset of codimension 1, then we know the dimension of Y is 0 using the dimension codimension inequality, then a 0-dim irreducible closed subset of a Noetherian space is just a closed point.

Definition 1.44. We say $[Y]$, the class of Y is an irreducible Weil divisor. A Weil divisor $D = \sum n_Y[Y]$ is said to be effective if $n_Y \geqslant 0$ for all Y. In this case, we will write $D \geqslant 0$, and by $D_1 \geqslant D_2$, we just mean $D_1 - D_2 \geqslant 0$. The support of the Weil divisor is the subset $\bigcup_{n_Y\neq 0} Y$. Recall that intersecting an irreducible closed subset with an open subset doesn't change the dimension and codimension of the irreducible closed subset, then we can define the restriction map:

$$WeilX \to WeilU \qquad [Y] \mapsto [Y \cap U]$$

and extend linearly for an open subset U of X.

To practice a little bit, we can show:

Proposition 1.45. Irreducible divisors on $\mathbb{P}^n_k$ correspond to irreducible homogeneous polynomials in $k[x_0, \cdots, x_n]$, up to multiplication by nonzero scalars in k^*.

Proof. Recall that for any irreducible homogeneous polynomial f, We already used Krull's Theorem to show that $V(f)$, the locus where f vanish is irreducible closed of codimension 1 and you can easily see that a multiplication by an invertible element in k defines the same closed subset.

Conversely, suppose that Y is an irreducible closed subset of $\mathbb{P}^n_k$ with codimension 1, then consider $I(Y)$, we claim that it's generated by an irreducible homogeneous polynomial. First, we can assume that it's $V(p)$ for some codimension 1 homogeneous prime ideal p, then since S is a UFD, then any codimension 1 prime is principal, thus it's generated by f, a homogeneous irreducible polynomial, up to multiplications by k^*.

The fact that these are inverses is left for you to check. $\square$

Suppose that from now on, X is regular in codimension 1 since we would like to use the properties of discrete valuation rings. We also want to assume that X is reduced since in this case rational functions will not be defined at embedded points.

Suppose that $\mathcal{L}$ is an invertible sheaf on X and s is a rational section not vanishing everywhere on any irreducible component of X. (This is equivalent to he fact that it doesn't vanish at the generic point.) Recall that rational sections are sections on a dense open subset with obvious restriction equivalences. (A rational function should be defined on the open subset that contains all the generic points of irreducible components, since the open subset must contain all the associated points, and generic points are indeed associated points thus dense.)

We can now make sense of the valuation of s along an irreducible Weil divisor Y. Recall that if X is an Noetherian integral scheme, and f is a nonzero element of it's function field. Then we claim that f only has a finite number of zeros and poles. To see this, we can cover X by finitely many $SpecA$ where A is an integral domain. Then we can assume that f is just f_1/f_2 where $f_i \in A$, then we can show each f_i has only finitely many zeros and poles since at a point p, zeros and poles are additive on multiplication. Then consider $f_1 \neq 0$ for example, first, we know that by Krull's principal ideal theorem, since f_1 is not zero, $V(f)$ is codimension 1, then has finitely many irreducible components each of codimension 1 with a unique codimension 1 point. Then since if p is a regular codimension 1 point, and f has a zero at p, we know $p \in V(f)$, since such p can only be finitely many by the discussion above, we are done.

Then on a Noetherian normal scheme, we can cover it by finitely many $SpecA$ each of A is an integral domain. And if f is a rational function on X defined on a dense open, then each of the $SpecA$ meets U and we can restrict f to the intersection to compute the zeros and poles. Now suppose that Y is an irreducible Weil divisor, we can make sense of the **valuation of s along Y**. Now take any open subset U containing the unique generic point of Y on which $\mathcal{L}$ is trivial with any trivialization, then we know that under this trivialization, s is a rational function defined on a dense open of that trivializing neighborhood. This allows us to talk about the valuation of s at p. Notice that any two such trivializatons differ by an invertible function on the intersection, and since computing valuations at p is compatible with doing restrictions, we know any two such trivializations give you the same valuation, thus the valuation of s at p makes sense. And we will define this number to be $val_Y(s)$, which is just the valuation of s at the generic point of Y.

Now I claim that for a rational section s, the valuation along Y, is 0 for all but finitely many Y. This is due to the discussion we had above, a rational function on a Noetherian integral scheme only has finitely many zeros and poles, this simply implies that for a Noetherian normal scheme, rational functions only have finitely many zeros and poles since we can always cover X by finitely many $SpecA$ where A is an integral normal domain and computing valuations at a regular codimension 1 point is compatible with restriction. The exact same argument tells you the assertion about $val_Y(s)$.

Thus, s determine a unique Weil divisor, which will be denoted by:

$$div(s) = \sum_Y val_Y(s)[Y]$$

The summation runs over all the irreducible Weil divisors of X we call $div(s)$ the divisor of zeor and poles of the rational section s.

Now we introduce the important notion of the **group of line bundles with rational sections**.

Now consider the set of pairs $\{(\mathcal{L}, s)\}$ where $\mathcal{L}$ is a line bundle and s is a rational section, not vanishing everywhere on any irreducible component of X, up to isomorphism $(\mathcal{L}, s) \simeq (\mathcal{L}', s')$. Here the isomorphism is meant to be an isomorphism of the line bundles $\mathcal{L} \simeq \mathcal{L}'$ such that s is sent to s'. You should notice since that s in the pair $(\mathcal{L}, s)$ is defined to be a rational section, there is always a question of restricting to another dense opens, here the meaning of s is mapped to s' can be taken on any dense open subset where s, s' are defined. The you can check that this is indeed an equivalence relation, thus quotienting out by this relation makes sense.

I claim that this set, after taking quotient, forms an abelian group under taking tensor product $\otimes$. The new element $(\mathcal{L}, s) \otimes (\mathcal{L}', s')$ is define to be $(\mathcal{L} \otimes \mathcal{L}', ss')$ where we first can assume s, s' lives on the same dense open by taking intersection and ss' is the tensor element of s, s' on that dense open. We need to show this is well defined, which I will let you do.

One trickier question is: what is the inverse element of $(\mathcal{L}, s)$. To answer this, we first notice that the identity element is $(\mathcal{O}_X, 1)$, then I claim the inverse element is $(\mathcal{L}^\vee, s^\vee)$, to see how $s^\vee$ is defined. We first notice that since s doesn't vanish at any irreducible components of X, it doesn't vanish on a dense open of X. On that dense open, we have an isomorphism:

$$\mathcal{O}_U \to \mathcal{L}|_U$$

where 1 is mapped to s, taking the dual of this morphism, we get the isomorphism:

$$\mathcal{L}|U^\vee \to \mathcal{O}_U$$

there there is an morphism $s^\vee$ from $\mathcal{L}_U \to \mathcal{O}_U$ such that it maps s to 1, then I claim that $s \otimes s^\vee$ is also mapped to 1 in the dual isomorphism:

$$\mathcal{L} \otimes \mathcal{L}^\vee \simeq \mathcal{O}_X$$

We can check this on the presheaf of the tensors, you find that in the stalk, this can be checked, then you pass to the sheafification to conclude the result.

It's also important to notice that if t in an invertible function of X, then multiplication by t gives you an iosmorphism:

$$(\mathcal{L}, s) \simeq (\mathcal{L}, ts)$$

Similarly, suppose that $(\mathcal{L}, s)$ and $(\mathcal{L}, u)$ are two sections, s, u both don't vanish on a dense open, then this dense open is a trivializing neighborhood of $\mathcal{L}$, and the transition function $\frac{s}{u}$ in an invertible function and takes s to u, thus we have an isomorphism:

$$\mathcal{O}_X \otimes \mathcal{L} \simeq \mathcal{L}$$

defined by multiplication, and via this isomorphism, we see that $\frac{s}{u} \otimes u$ is mapped to s, thus we have an equation:

$$(\mathcal{L}, s) \simeq (\mathcal{L}, s)(\mathcal{O}_X, \frac{s}{u})$$

which is equivalent to $\frac{(\mathcal{L},s)}{(\mathcal{L},u)} = (\mathcal{O}_X, \frac{s}{u})$

I claim that the above map div induces a group homomorphism:

$$div : \{(\mathcal{L}, s)\}/\text{isomorphism} \to WeilX \qquad (\mathcal{L}, s) \mapsto div(s)$$

To see this you need to first show this is well defined. The reason is just they differ by an invertible element on the intersection of the trivializing neighborhoods and this doesn't affect the valuation. We also have to show it's additive, but this just comes from the fact that valuation is additive upon multiplication and tensor product of line bundles induced multiplications on the tirvialization. I will let you fill the details.

As some practice, let's consider the following computations of Weil divisors:

Example 1.46. First, consider the easy example that on $\mathbb{A}^1_k$, we have a rational function $\frac{x^3}{x+1}$, we can consider the divisor determined by this rational function.

By definition, we need to consider all the closed points since we know that $\mathbb{A}^1_k$ is regular, thus all the codimension 1 points are regular. Recall that $k[x]$ is a PID, thus a prime ideal is generated by an irreducible polynomial. Thus the maximal ideal of the local ring is generated by that polynomial, then the valuation is just to determine the power of that polynomial. Then consider $x^3/(x+1)$, we can easily see that it has valuation 3 at (x) and -1 at $(x+1)$ and 0 everywhere else.

Now consider the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^1_k$, we have a rational section that informally corresponds to $\frac{x^2}{x+y}$. I claim that this is a section that only defined on $D_+(x+y)$ and is determined by specifying it on $D_+(x) \cap D_+(x+y)$ and $D_+(y) \cap D_+(x, y)$. On $D_+(x) \cap D_+(x+y)$ it correspond to the function $\frac{x^2}{x(x+y)}$ and on $D_+(y) \cap D_+(x+y)$, it correspond to the function $\frac{x^2}{y(x+y)}$. We need to make sure it agrees on the intersection, and you see that:

$$\frac{x^2 y}{x(x+y)y} \cdot \frac{x^2(x+y)}{yx(x+y)} = \frac{x^3}{xy(x+y)}$$

Thus this section is well defined.

Then we can talk about the divisor it defines. Then recall that irreducible codimension 1 closed subset in $\mathbb{P}^1_k$ correspond to irreducible homogeneous polynomials in x, y and to determine the valuation we can pass to the affine open in $D_+(x)$ or $D_+(y)$, then you will find that the divisor $\frac{x^2}{x+y}$ it defines is just $2[x] - [x+y]$ as you can check since after you reduce to the affine case it's essential the same as the example above in $\mathbb{A}^1_k$.

Remark 1.47. Suppose that $\mathcal{L}$ is an invertible sheaf on X where X is Noetherian, we claim that there is a rational section s such that it doesn't vanish at any generic points of irreducible components of X.

Suppose that p is such a generic point, we pick a trivializing neighborhood of p such that it doesn't intersect with any other irreducible component (this is always doable since we can take the union of other components, which is still closed) and pick 1 as the rational function. We do this for all the generic points, and take the disjoint union of the trivializing neighborhoods to get a new trivializing neighborhood and take the section that correspond to 1, we are done.

We return to study the group homomorphism div, which is going to be essential in determining line bundles on X. We will see that div will be injective under reasonable assumptions and often an isomorphism. Thus, line bundles with rational sections are determined by Weil divisors. If we consider an appropraite quotient to quotient out the rational sections, we can say that Weil divisors determine the line bundles in that case. To head to this direction, we can first prove:

Proposition 1.48. If X is normal and Noetherian, then the map div is injective.

Proof. Suppose that $div(\mathcal{L}, s) = 0$, this implies that s has no poles, thus we have already proved that s extends to a regular section, still denoted by s.

We will now show that the morphism:

$$\times s : \mathcal{O}_X \to \mathcal{L}$$

is an isomorphism, which gives you $(\mathcal{O}_X, 1) = (\mathcal{L}, s)$, thus $(\mathcal{L}, s) = 0$.

To show this is an isomorphism, we still check that locally, and it will suffice to check that this is an isomorphism on any open neighborhood where $\mathbb{L}$ is trivial since these neighborhoods cover X. Suppose that U is such a trivialization such that $U = SpecA$ for an integral domain A and the isomorphism is given by:

$$i : \mathcal{L}|_U \to \mathcal{O}|_U$$

since $div(s) = 0$, we know that under this trivilization, the rational function on U has no zeros and no poles, the combining this with $\times s$, we get a morphism:

$$\times s' \mathcal{O}_U \to \mathcal{O}_U$$

defined by multiplication by $i(s) = s'$. Which is a function that has no zeros and no poles, then consider the element $\frac{1}{s'}$ in the fraction field of A, which has no poles, thus it also in A, then s' is invertible on U, thus $\times s'$ is an isomorphism, thus $\times s$ is an isomorphism. □

Motivated by this, we want to find the inverse of div, or at least, even if it's not surjective, find the image of div. We will still assume that X is Noetherian and normal. Moreover, we will also assume that X is irreducible, thus since X is also reduced, we are assume that X is an integral Noetherian normal scheme. To justify the assumption, let's consider the following propositions:

Proposition 1.49. A Noetherian scheme is integral iff X is nonempty, connected and all stalks are are integral domains.

Proof. First assume it's integral, then it's irreducible, thus connected, and since it has a cover by specs of integral domains, the stalks are indeed integral domains.

Conversely, since X is Noetherian, it's a finite union of irreducible components. Since X is connected, if X has more than 1 components, there is at least a point p such that it lies in two components. Then consider an affine open $SpecA$ of p, we claim that in this case $\mathcal{O}_{X,p}$ is not an integral domain. This is because we know that the generic points of the two irreducible components correspond to two distinct minimal primes in A, lying below p, then A_p has two distinct minimal primes, thus it can't be an integral domain. □

Remark 1.50. This tells you that at least in the Noetherian case, being integral is the union of the local algebra condition (stalks are domains) and the global topology condition (X is connected).

Proposition 1.51. A Noetherian scheme is normal iff it's a finite disjoint union of integral Noetherian normal schemes each of them both open and closed.

Proof. Suppose that X has irreducible component $X_1, \cdots, X_n$, then we claim that X_1 doesn't meet any other components, otherwise, it will ruin the property that the stalk is an integral domain. Thus X_1 both open and closed. This discussion holds for any other components.

Conversely, if X is a disjoint union of normal schemes, it's normal since being normal is a stalk local property and taking disjoint union doesn't affect the computation of stalks, and finite disjoint union of Noetherian space is still Noetherian. □

Remark 1.52. If X is $SpecR$ where R is a Noetherian normal ring, we can even prove something stronger, in the sense that we can prove that $R \to \prod_i R/p_i$ where p_i's are the minimal primes of R is an isomorphism, thus $SpecR \simeq \coprod SpecR/p_i$. I will let you to prove the natural morphism $R \to \prod_i R/p_i$ is indeed an isomorphism since injectivity is checked using R is reduced and surjectivity is checked using localization.

This gives the explanation why we we deal with locally Noetherian normal schemes we have a cover by Specs of Noetherian integral normal domains.

Thus, by the above discussion, any Noetherian normal scheme is a finite disjoint union of integral Noetherian normal schemes, and you will see that reduce to this case doesn't do any harm:

Definition 1.53. Assume that X is an integral Noetherian normal scheme, suppose that D is a Weil divisor, we define the sheaf $\mathcal{O}_X(D)$ to be:

$$\Gamma(U, \mathcal{O}_X(D)) = \{t \in K(X)^* : div|_U t + D|_U \geqslant 0\} \cup \{0\}$$

Here $D|_U$ is the restriction of D to U we already talked about. And $div|_U t$ here means that we first consider the restriction of t to U, which is still defined on a dense open of U, thus a rational function on U and we compute the divisor of t restricted to U as usual since U is still normal Noetherian.

We will often omit the subscript X if it's clear from the context.

Remark 1.54. One thing you should notice it that the codimension 1 points of X that are also in U are still of codimension 1 in U. Then recall that we already mentioned that computation of valuation at a regular codimension 1 point is compatible with restriction to an open subset if that open subset contains that point, then we know that $div|_U(t)$ can be thought as $div(t)$ but we taken out those codimension 1 irreducible closed subsets that doesn't intersect with U and for the rest remaining in $div(t)$, we don't change the coefficient, turn the class Y into $Y \cap U$.

Therefore, you can think the sections of $\mathcal{O}_X(D)$ as rational functions of X that have poles and zeros constrained by the coefficients of D: A positive coefficient of D allows a pole of the rational function up to that order and a negative coefficient of D demands a zero of the rational function at least that order.

Remark 1.55. For the restriction map, notice that if t satisfies $div|_U t + D|_U \geqslant 0$ on U, then for any $V \subset U$, we claim that $div|_V t + D|_V \geqslant 0$. I will let you figure this out based on the previous remark. But this tells you that the restriction is just the inclusion map.

Then we need to show that this is indeed a sheaf. The identity axiom is trivial to check since the restriction is just inclusion. Gluability is also trivial to check, suppose that U is covered by U_i, since agreeing on intersection just means that they are the same rational function, then we only have to show that this rational function t is ok. But this is also trivial since if p is a regular codimension 1 point in U, then it's in some U_i, then t in $\Gamma(U_i, \mathcal{O}(D))$ means that the valuation of t at p is larger than the cofficient of the closure of p in U_i, this suffice to show that t is in $\Gamma(U, \mathcal{O}(D))$.

Remark 1.56. Recall the definition of the support of a Weil divisor D is just the union of those terms with nonzero coefficients. Then since D only have a finite number of nonzero coefficients, the support of D is closed, thus the complement is open. We claim that away from $SuppD$, $\mathcal{O}(D)$ is isomorphic to the structure sheaf.

To see this,say the complement of the support is U, we need to establish an isomorphism:

$$\mathcal{O}(D)|_U \to \mathcal{O}|_U$$

For any $V \subset U$, if $t \in \Gamma(V, \mathcal{O}(D))$, since $D|_V = 0$, this is equivalent to $div|_V(t) \geqslant 0$, i.e. t is a rational function that has no poles on V, then since we work with Noetherian normal schemes, we proved before that algebraic Hartog's lemma tells us t is a function defined on all of V, i.e. $t \in \mathcal{O}(V)$, this gives you a natural morphism which is just the inclusion on each V and is obviously compatible with restrictions since you can view everything running inside $K(X)$. Moreover, on each V, the inclusion is also surjective, since for each $t \in \mathcal{O}(V)$, it won't have any pole on V, thus the divisor determined by t is effective, thus t is in $\mathcal{O}(D)(V)$.

Remark 1.57. Another important remark about $\mathcal{O}(D)$ is that it has a canonical ration section that correspond to $1 \in K(X)^*$, i.e. the section $1 \in K(X)^*$ on the complement of $SuppD$.

Remark 1.58. You can try to generalize the definition of $\mathcal{O}_X(D)$ to a Noetherian normal scheme that is not necessarily irreducible. To see what happens there, recall that X is a finite disjoint union of integral Noetherian normal schemes, then for any Y codimension 1 irreducible closed, it's contained in a unique component, that a Weil divisor D is just the summation of Weil divisors in each component.

On the other hand, notice that a rational function t that doesn't vanish on any component of X correspond to a pair $\prod_i K(X_i)^*$ where $K(X_i)$ is the function field of the component X_i, and you can see that computing the divisor corresponding to $(t_1, \cdots, t_n)$ is just computing the divisor of each t_i and then sum them up. Then you can just define $\mathcal{O}_X(D)$ to be:

$$\Gamma(U, \mathcal{O}_X(D)) = \{(t_1, \cdots, t_n) \in \prod_i K(X_i)^* : div|_U(t) + D|_U \geqslant 0\}$$

You can even further simplify this, but we don't use this later, thus we won't do it here.

Proposition 1.59. $\mathcal{O}_X(D)$ is a quasicoherent sheaf.

Proof. We still use the quasocoherent criterion related to localization.

For any affine open $SpecA$ of X, we claim that the restriction:

$$\Gamma(SpecA, \mathcal{O}_X(D)) \to \Gamma(SpecA_f, \mathcal{O}_X(D))$$

is the localization at f. To see what happens, suppose that $x \in \Gamma(SpecA_f, \mathcal{O}_X(D))$, we know that it's just some nonzero element in $K(A)$ such that it's prder and pole satisfies some constraints at the codimension 1 primes which f doesn't belong to. Since f is not in these primes, we know that the valuation of f at all the primes are 0, then we know that xf^n for any $n \in \mathbb{Z}$ is in $\Gamma(SpecA_f, \mathcal{O}_X)$, on the other hand, since f has positive valuation at all the other codimension 1 points, we know that xf^n for a large enough n will belong $\Gamma(SpecA, \mathcal{O}_X(D))$, this implies that $x = y/f^m$ for some $y \in \Gamma(SpecA, \mathcal{O}_X(D))$. Then consider the natural diagram:

$$\begin{tikzcd}
\Gamma(SpecA, \mathcal{O}_X(D)) \arrow[rr] \arrow[dr, "loc"'] & & \Gamma(SpecA_f, \mathcal{O}_X(D)) \\
& \Gamma(SpecA, \mathcal{O}_X(D))_f \arrow[ur, dashed] &
\end{tikzcd}$$

the above discussion tells you that the induced morphism is an isomorphism, we are done. $\square$

Our next goal is to show that in good circumstances, $\mathcal{O}_X(D)$ is an invertivle sheaf. For example, let $X = \mathbb{A}^1_k$, consider the divisor $D = -2[x] + [x-1] + [x-2]$, often simply denoted as $D = -2[0] + [1] + [2]$ for convenience. We can compute $\mathcal{O}_X(-2[0] + [1] + [2])$, and I claim that it's invertible. To do this, for example, I claim that there is a global section $\frac{3x^2}{x-1}$, you can easily check that it has the desired zeros and poles, thus indeed a global section. Furthermore, we can show that any global section is a $k[x]$ multiple of $\frac{x^2}{(x-1)(x-2)}$.

This is because for any $f \in k(x)^*$, if it is in the global section, it must have at least a zero of order 2 at x and it can at most have a pole of order 1 at $x-1$ and $x-2$, thus there is a x^2 factor in the numerator, and it has a $\frac{1}{(x-1)(x-2)}$ factor, and if you don't want a pole, you can use $k[x]$ multiple to eliminate them.

This can be extended to an isomorphism $\mathcal{O}_X \to \mathcal{O}_X(D)$, and you can easily guess what is the isomorphism here, this is just induced by $k[x] \to k[x] \cdot \frac{x^2}{(x-1)(x-2)}$ where 1 is mapped to $\frac{x^2}{(x-1)(x-2)}$. Since $\mathcal{O}_X(D)$ is quasicoherent and we are over an affine space, the isomorphism on the global section is an isomorphism globally. For concreteness, let $\mathbb{C} = k$ in the following example:

Example 1.60. We still work with the above example just with k replaced $\mathbb{C}$.

First consider the global section $\frac{x^2(x-3)}{(x-1)(x-2)}$, I claim that this section vanishes at just one point, which is $(x-3)$, since it correspond to the global function $(x-3)$ of the structure sheaf. It doesn't vanish at x in particular, which is a littlbe bit wierd since it has x^2 on the numerator.

As another example, consider the global section $x^2(x-1)$, I claim that it only vanishes at two points, which are $x-1$ and $x-2$, since this correspond to the global function $(x-1)^2(x-2)$.

Finally consider the canonical rational section corresponding to $1 \in K(X)$. I claim that it has a zero of order 1 at $(x-2)$ and $(x-1)$, it also have a pole at x of order 2. To see this, notice that this section is obtained by first restrict $\frac{x^2}{(x-1)(x-2)}$ to the dense open where $x, x-1, x-2$ is not in it. Then we multiply it by $(x-1)(x-2)/x^2$, this tells you that this correspond to the rational function $(x-1)(x-2)/x^2$ on the dense open, and you can see what are the poles and zeros.

Common wrong answers to this question is obtained by forgetting the identification:

$$\mathcal{O}_X \to \mathcal{O}_X(D)$$

and directly think of elements in $\mathcal{O}_X(D)$ as rational functions living on X, though this is in some sense correct, but notice that in the above example, $\frac{x^2}{(x-1)(x-2)}$ is never a rational function defined on all of X, thus there must be some identification, never forget it.

We next can show that in good circumstances $\mathcal{O}_X(D)$ is invertible. We still assume X is Notherian normal and irreducible:

Proposition 1.61. Suppose that $\mathcal{L}$ is an invertible sheaf on X and s is a nonzero rational section of $\mathcal{L}$. (This already tells you that it doesn't vanish at the generic point.)

Then there is an isomorphism $\mathcal{O}(div(s)) \simeq \mathcal{L}$.

Proof. The first step is to show that there is an open cover $\{U\}$ of X such that on each open $\mathcal{O}(div(s))|_U \simeq \mathcal{O}|_U$ and these U forms a base for the Zariski topology.

To see this, let's first cover X by $SpecA$ where A are Noetherian normal integral domains such that $\mathcal{L}$ is trivial on $SpecA$. Then via the trivialization, $div|_{SpecA}(s)$ is just the divisor determined by a rational function s' on $SpecA$. Then we claim that we have a local isomorphism:

$$\mathcal{O}|_{SpecA} \to \mathcal{O}(div(s))|_{SpecA}$$

defined by mapping 1 to $\frac{1}{s'}$. First, recall how $\mathcal{O}(div(s))(SpecA)$ is defined, it's just elements t in $K(X)^*$ such that $div|_{SpecA}(t) + div|_{SpecA}(s') \geqslant 0$, then we know that if we consider the inverse of s, which is $\frac{1}{s'}$ viewed as an element in $K(a)$, since the divisor function is additive upon multiplication, we know that $div|_{SpecA}(\frac{1}{s'}s') = div|_{SpecA}(\frac{1}{s}) + div|_{SpecA}(s) = div|_{SpecA}(1) = 0$, thus $\frac{1}{s}$ is indeed in $\mathcal{O}(div(s))(SpecA)$, and moreover, you can see that any other elements in $\mathcal{O}(div(s))(SpecA)$ is an A-multiple of $\frac{1}{s'}$ since if $div(ts') \geqslant 0$ in $SpecA$, since it has no poles, algebraic Hartog's lemma tells you that $ts' = a$ for some $a \in A$, then $t = \frac{a}{s'}$, thus the morphism:

$$\mathcal{O}|_{SpecA} \to \mathcal{O}(div(s))|_{SpecA} \qquad 1 \mapsto \frac{1}{s}$$

is an isomorphism since it's a morphism of quasicoherent sheaves on an affine open.

Then we do the second step, for each of the U above, we can define an isomorpshim:

$$\phi|_U : \mathcal{O}(div)(U) \to \mathcal{L}(U)$$

sending a rational function t with zeros and poles constrainted by s to st. Here we know that $div(st) \geqslant 0$, thus this rational section on U extend to a regular section on U. We need to show this is indeed an isomorphism, this is because we have the following diagram:

$$\begin{tikzcd}
& \mathcal{O}(div(s))(U) \arrow[dl] & \\
\mathcal{L}(U) & & \mathcal{O}(U) \arrow[ul, "1 \mapsto \frac{1}{s'}"'] \arrow[ll, "\simeq"]
\end{tikzcd}$$

where the bottom isomorphism is the trivialization. I claim that the composition: $\mathcal{O}(div(s))(U) \to \mathcal{L}(U) \to \mathcal{O}(U) \to \mathcal{O}(div(s))(U)$ and $\mathcal{L}(U) \to \mathcal{O}(U) \to \mathcal{O}(divs)(U) \to \mathcal{L}(U)$ are identities. I will let you to check this.

Now you can easily see that this definition is compatible with restrictions for any open V in the base of the Zariski topology in the sense that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{O}(div(s))(U) \arrow[r, "\simeq"] \arrow[d, hook, "res"'] & \mathcal{L}(U) \arrow[d, hook, "res"'] \\
\mathcal{O}(div(s))(V) \arrow[r, "\simeq"] & \mathcal{L}(V)
\end{tikzcd}$$

which I will let you check, but this tells you that this gives an isomorphism on the sheaf of bases, which induces an isomorphism of the sheaves, we are done. ☐

Proposition 1.62. Let σ be the map from $K(X)$ to the rational sections of $\mathcal{L}$ where $\sigma(t)$ is the rational section under the isomorphism:

$$\mathcal{O}(div(s)) \simeq \mathcal{L}$$

then we claim that the canonical section $1 \in K(X)$ is mapped to the rational section s.

Proof. First, not every element in $K(X)$ is in the domain of σ, this σ is just a temporary notation and we will not use it later.

Consider the canonical section 1, we know that the base we defined in the previous proposition covers the complement of the support and for each of them in the complement, we know that via:

$$\phi_U : \mathcal{O}(div(s))(U) \to \mathcal{L}(U)$$

1 is mapped to s, thus we know that on the whole complement 1 is mapped to s. ☐

Remark 1.63. The above two propositions show that in the line bundle with rational sections group, $(\mathcal{O}(div(s)), 1) = (\mathcal{L}, s)$.

Therefore, the following composition is the identity:

$$\{(\mathcal{L}, s)\}/iso \xrightarrow{div} Weil(X) \xrightarrow{\mathcal{O}_X(-)} \{(\mathcal{L}, s)\}/iso$$

We want to conclude from this that Weil divisor D in the image of div: are those D such that $\mathcal{O}_X(D)$ is a line bundle. One direction is easy: if $D = divs$ is in the image of div, then $\mathcal{O}(D)$ is a line bundle. Conversely, suppose that D is a divisor such that $\mathcal{O}(D)$ is a line bundle, then I claim if you consider $(\mathcal{O}(D), 1)$ where 1 is the canonical section, then $div(1) = D$.

This relies on the following notion of principal and locally principal divisors, so let's first discuss them.

Definition 1.64. If D is a Weil divisor such that $D = div(f)$ for some rational function f, we say that D is **principal**. Clearly, principal divisors form a subgroup of Weil$X$ since if $div(f) = D$ and $div(g) = D'$, then you can show $D + D' = div(fg)$. We will denote this subgroup by Prin$X$. And the function div induces a group homomorphism:

$$div : K(X)^* \to PrinX$$

A similar but weaker notion is called locally principal. If X is covered by U_i's such that on each U_i, $D|_U$ is principal on U_i, we say that D is **locally principal**. Clearly locally principal divisors form a subgroup of Weil$X$ since if D and D' are locally principal with open cover U_i and V_i, then $D + D'$ is locally principal on the open cover $U_i \cap V_j$. We denote this subgroup by LocPrin$X$.

Here is an important observation, though not difficult. First, from the above two propositions, if $D = div(f)$, then we have an isomorphism $(\mathcal{O}_X, f) \simeq (\mathcal{O}_X(D), 1)$ and in particular $\mathcal{O}_X \simeq \mathcal{O}_X(D)$. Later we prove the converse also holds, for a divisor such that $\mathcal{O}_X(D) \simeq \mathcal{O}$, then D is principal. Similarly, if D is locally principal, the exact same argument shows that $\mathcal{O}_X(D)$ is locally isomorphic to $\mathcal{O}_X$, thus it's a line bundle. Let's first prove the converse holds:

Proposition 1.65. If $\mathcal{O}_X(D)$ is a line bundle, then D is locally principal.

Proof. Let U_i be an open cover of X by specs of Noetherian normal integral domains, where $\mathcal{O}_X(D)$ is trivial. Pick one of the $U_i = SpecA$, we have a local isomorphism:

$$\mathcal{O}_U \simeq \mathcal{O}_X(D)|_U$$

which is defined to be $1 \mapsto t$ for a rational function t. In particular we know that $div(t) + D|_U \geqslant 0$. I claim that this is strictly 0, otherwise, there will be a coefficient of some regular codimension 1 point in $SpecA$ that is at least 1, then pick a uniformizer η_i of that point, restrict to a smaller affine open of Noetherian integral domain we can assume the uniformizer η_i is in A. Then t/η_i is still in $\Gamma(SpecA, \mathcal{O}_X(D))$, however, this time the morphism:

$$A \to \Gamma(SpecA, \mathcal{O}_X(D))$$

defined by $1 \mapsto t$ is not an isomorphism since t/η^i is not in the image since we are working with an integral domain!

Thus we know that $D|_U = div(t)$. $\square$

Remark 1.66. The idea of the proposition above gives the converse of the statement we had before: If D is a divisor such that $\mathcal{O}(D)$ is a line bundle, then $div(\mathcal{O}(D), 1) = D$.

To see why this is true, since $\mathcal{O}_X(D)$ is a line bundle, the local isomorphism in the above proposition still holds and maps $1 \in A$ to t where $div(t) + D|_{SpecA} = 0$. This tells you that $div(t^{-1}) = D|_{SpecA}$.

Then here comes the confusing part, recall that if we view $\mathcal{O}(D)$ as a line bundle, then for the rational section 1, when we compute $div(1)$, we are think of it not as a rational function, but a rational section of a line bundle, thus we will use the subscript l to indicate this, i.e $div_l(1) = div(a)$ for some a under the local trivialization where a is mapped to 1 and div without the subscript denotes the divisor function on rational functions. The same notation applies to the preceding paragraph.

Notice that identifications is given by $A \to \Gamma(SpecA, \mathcal{O}(D))$ where $1 \mapsto t$, then on the smaller open V where $\frac{1}{t}$ is defined, we can compute that $\frac{1}{t}$ is mapped to 1, thus $div_l(1) = div(\frac{1}{t}) = D|_V$. You can do this for all the U's in the cover and conclude $div_l(1) = D$ globally.

Now, let's look at an example that tells you the Picard group of $\mathbb{P}^n_k = \mathbb{Z}$, but before that we can show the following:

Proposition 1.67. Suppose that $X = \mathbb{P}^n_k$ and $\mathcal{L} = \mathcal{O}(1)$, let s be the global section of $\mathcal{L}$ corresponding to x_0 and let $D = div(s)$. Then $\mathcal{O}(mD) \simeq \mathcal{O}(m)$ and the canonical section of $\mathcal{O}(mD)$ is precisely s^m.

Proof. We claim that we can consider $\mathcal{O}(m)$ and the section s^m, which is the one corresponding x_0 and we claim that $mD = div(s^m)$. If this holds, then we already know that:

$$\mathcal{O}(mD) = \mathcal{O}(div(s^m)) \simeq \mathcal{O}(m)$$

and the isomorphism can be taken such that the canonical section is mapped to s^m.

$mD = div(s^n)$, we can do computation on the standard affine cover $D_+(x_i)$. Recall that x_0 correspond to the rational function $\frac{x_0}{x_j}$ in $D_+(x_j)$ and x_0^m correspond to the rational function $(\frac{x_0}{x_j})^m$ in $D_+(x_j)$ then since div is additive upon multiplication, we are done. $\square$

Remark 1.68. Of course, we can replace x_0 by any linear term and get the same conclusion with s^m replaced by the m-th power of the linear form.

We can start with any homogeneous polynomial of degree m, say f, view is as a global section of $\mathcal{O}(m)$, take the divisor $div(f)$ and conclude that $\mathcal{O}(div(f)) \simeq \mathcal{O}(m)$.

Moreover, if we start with a homogeneous polynomial f of degree m and a homogeneous polynomial of degree n, we can also consider fg, then we claim that $\mathcal{O}(div(s) + div(g)) = \mathcal{O}(m+n)$. To see this, you only have to prove that $div(fg) = div(f) + div(g)$, which is easy to show by passing to the standard affine open and consider the rational functions there.

Using this we prove our promise that the Picard group of the projective space over a field k is $\mathbb{Z}$:

Proposition 1.69. Let $X = \mathbb{P}^n_k$, then $PicX \simeq \mathbb{Z}$ and the isomorphism is given by:

$$\mathbb{Z} \to PicX \qquad m \mapsto \mathcal{O}(m)$$

Proof. We already see this is injective, thus we only have to show it's surjective.

Suppose that $\mathcal{L}$ is a line bundle on X, pick a nonzero global section we can consider $div(s)$, which is a linear combination of irreducible divisors, which correspond to irreducible homogeneous polynomials up to scalar multiplication. Say $div(s) = \sum n_i[f_i]$ where f_i stands for the irreducible divisor determined by $[f_i]$ and the degree of f_i is d_i.

Recall that $\mathcal{O}(div(s)) = \mathcal{O}(\sum n_i[f_i])$. However, from the previous remark, since $[f_i]$ is just $div(f_i)$, then $\sum_i n_i[f_i] = \sum[f_i^{n_i}] = [\prod_i f_i^{n_i}] = div(\prod_i f_i^{n_i})$, thus we know that $\mathcal{L} \simeq \mathcal{O}(div(s)) = \mathcal{O}(div(\prod_i f_i^{n_i})) \simeq \mathcal{O}(\sum_i d_i n_i)$. $\square$

Remark 1.70. There is a different notation convention you can find in the literature, which is when we define $\mathcal{O}(D)$, we can require $div|_U(t) \geqslant D|_U$ instead of $div|_U(t) + D|_U \geqslant 0$. There is not much different about this notation, the only reason we use our convention here is because we want $div(\mathcal{O}(D), 1) = D$ instead of $-D$, the dual sheaf when D is locally principal.

Recall that factoriality is stronger than normality, and if we assume the scheme is factorial, we can show that any Weil divisor D gives you an invertible sheaf.

Proof. Let X be Noetherian and factorial and irreducible, then for any Weil divisor D, $\mathcal{O}(D)$ is an invertible sheaf, in particular the group homomorphism div is now surjective, thus an isomorphism. ☐

Proof. The important property we will use for UFD is that all the codimension 1 primes are principal. For the motivation, let's first think about the case when $D = [Y]$ is an irreducible divisor.

To recall that the corresponding sheaf $\mathcal{O}_X(D)$ is an invertible sheaf iff D is locally principal. That is to show that there is an open cover of X, such that on each X, $Y \cap U$ is determined by a single rational function. To see this, since Y is closed, we can take one the open cover to be $U = X - Y$. And you know that $\mathcal{O}_X(D)$ on U is indeed isomorphic to $\mathcal{O}_U$. Then for a point $p \in Y$, we pick an affine open V containing p, consider $\mathcal{O}_{X,p}$, which is a UFD, and notice that the generic point q of Y is in V and correspond to a codimension 1 prime in $\mathcal{O}_{X,p}$ since intersecting with an open doesn't change the dimension and codimension, then we know that it's a principal ideal, generated by f i.e. the unifomizer. By shrinking to a smaller open, we can assume that it's a regular function of $V = SpecA$ for a Noetherian normal integral domain. Then f obviously has no poles on U, and f has a zero of order one only at the generic point of $Y \cap U$. This is because that all the codimension 1 points are distinct and none of which is contained in another, thus if f has a zero at q, the generic point of another codimension 1 prime, we know that $f \in q$, but this implies that $p' \subset q$ in the local ring $\mathcal{O}_{X,p}$. The same argument can be used to show that if the divisor is of the form $n[Y]$ then it's locally principal by taking the uniformizer f and use f^n.

Now suppose we have a general divisor $D = \sum n_i[Y_i]$ for a finite collection of codimension 1 Y_i. The strategy is similar to the case of an irreducible divisor. We can take one the the U to be the complement of the union of Y_i. For the notation, we will use p_i to be the generic point of all the Y_i. Then for $p \in Y_i$, pick an affine open $SpecA$ notice that all the p_i that is in $SpecA$ is of codimension 1, then consider $\mathcal{O}_{X,p}$, p_i's are all codimension 1, then we can take the uniformizer f_i for each of them and then by shrinking $SpecA$ assume that f_i are regular functions on $SpecA$. Notice that for f_i, it has no poles on all of $Speca$ and only has a zero at p_i, no zeros at other p_j for $j \neq i$. This is because we know that p_i's are distinct and no one is contained in another. If f_i has a zero at p_j, then f_i is in p_j, then p_i will be contained in p_j which is a contradiction. Then we will take the rational function to be $\prod_i f_i^{n_i}$, you can check that it satisfies the condition. ☐

Now we define the important notion of **class group**: Define the class group of a Noetherian normal irreducible scheme to be the quotient group of Weil divisors by the principal Weil divisors:

$$Cl(X) = Weil(X)/Prin(X)$$

Recall that $(\mathcal{L}, s) - (\mathcal{L}, u) = (\mathcal{O}, \frac{s}{u})$ is mapped by div to a principal Weil divisor, then since we know that the kernel of the map $div : \{(\mathcal{L}, s)\}/iso \to Weil/PrinX = Cl(X)$ consists of elements $(\mathcal{O}, f)$ where f is a rational function, by universal property of the quotients, it induces an inclusion:

$$div : PicX = \{\mathcal{L}\}/iso \hookrightarrow Cl(X)$$

Recall that div factors through the locally principal divisors: $div : \{(\mathcal{L}, s)\}/iso \xrightarrow{\simeq} LocPrin(X) \hookrightarrow WeilX$ Then it gives us the following diagram:

$$\begin{tikzcd}
 & \{(\mathcal{L},s)\}/iso \arrow[r, "div:\quad\simeq"'] \arrow[d] & LocPrin(X) \arrow[l, bend right, "D\mapsto(\mathcal{O}(D),1)"'] \arrow[r, hook, "= \text{ if X factorial}"] \arrow[d] & Weil(X) \arrow[d] \\
PicX \arrow[r, "="] & \{\mathcal{L}\}/iso \arrow[r, dashed, "\simeq"] & LocPrin(X)/PrinX \arrow[l, bend left, "D\mapsto\mathcal{O}(D)"] \arrow[r, hook, "= \text{ if X factorial}"] & Cl(X)
\end{tikzcd}$$

Remark 1.71. In particular, if A is a UFD, then all the codimension 1 primes are principal, then you can prove that all Weil divisors on $SpecA$ is principal by taking generators of these codimension 1 primes, thus $Cl(SpecA) = 0$, which implies that $Pic(SpecA) = 0$.

Remark 1.72. Also, form the above diagram, if D satisfies $\mathcal{O}(D) \simeq \mathcal{O}$, it's first a locally principal divisor, as promised we want to show that it's principal.

We know that under the isomorphism:

$$Pic(X) \simeq LocPrin(X)/Prin(X)$$

$\overline{D}$ is mapped to $\mathcal{O} = 0$, then it's again mapped to $\overline{D} = 0$, thus D is a principal divisor, we are done. Therefore, D is principal iff $\mathcal{O}(D) \simeq \mathcal{O}$.

Example 1.73. For example, we know $k[x_1, \cdots, x_n]$ is a UFD, then $Cl(\mathbb{A}_k^n) = 0$, then $PiC(\mathbb{A}_k^n)$, thus all the line bundles on $\mathbb{A}_k^n$ are trivial.

This coincides with our intuition from differential geometry since a contractible space only has trivial vector bundles. Base on this, you might want to prove that all the vector bundles of $\mathbb{A}_k^n$ are trivial. We already showed this when $n = 1$, however, the generalization is hard which is formerly known as Serre's conjecture, proved later by a the Quillen-Suslin Theorem, which leads to a Fields Medal.

Now we study what happens to Weil divisors and class groups if we take out a closed subset.

First, removing a closed subset of pure codimension at least 2 doesn't affect class group. Since you are just restricting to an open subset U which contains all the codimension 1 points, which is dense, thus the principal divisors remain the same, thus the class group remain the same. Formally speaking, we have an isomorphism of SES:

$$\begin{tikzcd}
0 \arrow[r] & PrinX \arrow[r, hook] \arrow[d, "res"] & WeilX \arrow[r] \arrow[d, "res"] & Cl(X) \arrow[r] \arrow[d, dashed, "res"] & 0 \\
0 \arrow[r] & PrinU \arrow[r, hook] & WeilU \arrow[r] & Cl(U) \arrow[r] & 0
\end{tikzcd}$$

The last vertical arrow induced by restriction is an isomorphism since the formal two vertical arrows are isomorphisms and we can apply five lemma.

Now, if we remove a irreducible closed subset Z of codimension 1, the Weil divisor group change in a contrable way. First, clearly we have such a SES:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{1 \mapsto [Z]} WeilX \xrightarrow{res} Weil(X - Z) \longrightarrow 0$$

But if we want to quotient out by the principal divisors, we know that $Cl(X) \to Cl(X - Z)$ is still well defined and induced by restriction since the principal divisors on X restricts to principal divisors on $X - Z$. But the sequence:

$$\mathbb{Z} \xrightarrow{1 \mapsto [Z]} Cl(X) \xrightarrow{res} Cl(X - Z) \longrightarrow 0$$

may lose exactness on the left, for example, when $[Z]$ is a principal divisor. This is usually called the **excision exact sequence of class groups**.

The above discussion works similarly for taking out finitely many irreducible closed subset of codimension 1, say $Z_1, \cdots, Z_n$, then you get a short exact sequence:

$$0 \longrightarrow \mathbb{Z}^n \xrightarrow{e_i \mapsto [Z_i]} WeilX \xrightarrow{res} Weil(X - \cup_i Z_i) \longrightarrow 0$$

and quotient out by the principal divisors you still can lose exactness on the left and get the sequence:

$$\mathbb{Z}^n \xrightarrow{e_i \mapsto [Z_i]} Cl(X) \xrightarrow{res} Cl(X - \cup_i Z_i) \longrightarrow 0$$

And you simultaneously take out some closed subset of pure codimension at least 2 and some irreducible closed subset of codimension 1, basically you only have to care about those codimension 1 stuff since the codimension 2 closed subsets don't affect Weil divisors and class groups. Recall that if X is an irreducible space then any open subspace is also irreducible for trivial reasons, then we don't have to worry about $X - \cup_i Z_i$ is not irreducible.

Example 1.74. Let $X = \mathbb{A}_k^n$, let U be any open subscheme of X, notice that the irreeucible components of $X - U$ is finitely many and the codimension is at least 1, then we can consider the corresponding excision sequence and conclude from $Cl(X) = 0$ that $Cl(U) = 0$, thus $Pic(U) = 0$.

As another application, let $X = \mathbb{P}_k^n$ and let Z to be the divisor determined by x_0, a global section of $\mathcal{O}(1)$, then you can check $U = X - Z = \mathbb{A}^n$, then we can conclude from the excison sequence that:

$$\mathbb{Z} \to Cl(X) \to Cl(\mathbb{A}^n) = 0 \to 0$$

$Cl(\mathbb{P}_k^n)$ is generated by $[Z]$, possibly with torsion and $Pic(\mathbb{P}_k^n)$ is a subgroup of $Cl(\mathbb{P}_k^n)$. But notice that we have the following sequence:

$$Pic(\mathbb{P}_k^n) \simeq LocPrin(\mathbb{P}_k^n)/Prin(\mathbb{P}_k^n) \hookrightarrow Cl((\mathbb{P}_k^n))$$

Under the isomorphism, the class of $[Z]$ is mapped to $\mathcal{O}(1)$ and $n[Z]$ is mapped to $\mathcal{O}(m)$, which is not trivial, thus $n[Z]$ is not 0, thus $[Z]$ is not a torsion element, thus $Pic(\mathbb{P}_k^n) \hookrightarrow Cl(\mathbb{P}_k^n) \simeq \mathbb{Z}$ is an isomorphism, thus $\mathcal{P}_k^n$ is also isomorphic to $\mathbb{Z}$, generated by $\mathcal{O}(1)$.

This is another way of proving $Pic((\mathbb{P}_k^n)) \simeq \mathbb{Z}$.

Remark 1.75. Recall that we prove that if X is Noetherian irreducible and factorial, then any Weil divisor on X is locally principal, hence the map $PicX \to ClX$ is an isomorphism.

Now we consider a mild generalization: twisting line bundles by divisors. Let $\mathcal{L}$ be an invertible sheaf on a Noetherian normal irreducible scheme, then we define $\mathcal{L}(D) = \mathcal{L} \otimes \mathcal{O}(D)$. If D is locally principal, then $\mathcal{L}(D)$ is also a line bundle. In this case, there are two different ways of interpreting sections of $\mathcal{L}(D)$ over an open subset U: You can view them just as sections of the new line bundle $\mathcal{L}(D)$, or as rational sections of $\mathcal{L}$ with zeros and poles constrained by the divisor D, as shown in the following proposition:

Proposition 1.76. There is a sheaf associated to $\mathcal{L}$, still denoted by $\mathcal{L}(D)$, with sections on each open U defined by:

$$\Gamma(U, \mathcal{L}(D)) = \{t \text{ a nonzero rational section of } \mathcal{L}: \quad div|_U t + D|_U \geqslant 0\} \cup \{0\}$$

It's isomorphic to $\mathcal{L} \otimes \mathcal{O}(D)$ for a locally principal divisor D, thus in particular this new sheaf is also a line bundle.

Proof. Let's first prove that this is indeed a a sheaf. The restriction function is still just inclusion and identity, gluability axioms are also checked similar to $\mathcal{O}(D)$.

To show this is isomorphic to the tensor, we will define a morphism:

$$\mathcal{L} \otimes \mathcal{O}(D) \to \mathcal{L}(D)$$

which is induced by an isomorphism from a presheaf on the basis to a sheaf on a basis, which induces a morphism from a presheaf to a sheaf such that the maps on the stalks are isomorphism. Then this map induces an isomorphism from the sheafification to the sheaf.

To do this, recall that since D is locally principal, we can find a base of X, say U_i such that on each U_i, $D|_{U_i}$ is determined by a rational function g_i on U_i, $\mathcal{L}$ is trivial on each U_i. Moreover, we know that for any $t \in \mathcal{O}(D)(u_i)$, there is a unique $a \in \mathcal{O}(U_i)$ such that $ag_i = t$ Then we can define:

$$\mathcal{L}(U_i) \otimes \mathcal{O}(D)(U_i) \to \mathcal{L}(D)(U_i) \qquad s \otimes t \mapsto t \cdot s$$

defined on the intersection of the domain of s and t(you should check $div(st) + D|_{U_i} \geqslant 0$!). We claim that this is an isomorphism. To show this, we construct an inverse.

For a section s in $\mathcal{L}(D)(U_i)$, notice that $g_i s$ defined on the intersection has positive valuation, thus extends to a unique section s' on $\mathcal{L}(U_i)$. Now we define:

$$\mathcal{L}(D)(U_i) \to \mathcal{L}(U_i) \otimes \mathcal{O}(D)(U_i) \qquad s \mapsto s' \otimes \frac{1}{g_i}$$

Here you probably will feel confused about $\frac{1}{g_i}$ since $\frac{1}{g_i}$ almost never will lie in $\mathcal{O}(D)(U_i)$. To solve this problem, we enlarge the space and consider the inclusion $\mathcal{O}(D)(U_i) \hookrightarrow K(X)$, since $\mathcal{L}(U_i)$ is trivial on each U_i, tensor by $\mathcal{L}(U_i)$ preserves inclusion, then we can think of $\mathcal{L}(U_i)\otimes\mathcal{O}(D)(U_i)$ as inside $\mathcal{L}(U_i)\otimes K(X)$. We have such a diagram:

$$\begin{tikzcd}
\mathcal{L}(U_i)\otimes\mathcal{O}(D)(U_i) \arrow[r, hook] \arrow[d, "\simeq"'] & \mathcal{L}(U_i)\otimes K(X) \arrow[d] \\
\mathcal{O}(D)(U_i) \arrow[r, hook, "inclusion"'] & K(X)
\end{tikzcd}$$

where we think of $\mathcal{L}(U_i)$ as $\mathcal{O}(U)$ via the trivialization.

$$\mathcal{O}(U_i) \simeq \mathcal{L}(U_i)$$

Then you can see that $s' \otimes \frac{1}{g'}$ is identified with $s' \cdot \frac{1}{g'}$, thus indeed is in $\mathcal{O}(D)(U_i)$. Since we know that s' restricts to sg_i and sg_i has positive valuation.

Finally, we need to prove the composition is the identity for both sides. But this is essential trivial if you figure out what each of the morphism is doing. I must admit there are quite a lot of identifications going on, but you don't have to worry too much since all these local identification is essentially to show that the explicit map:

$$\mathcal{L}(U_i)\otimes\mathcal{O}(D)(U_i) \to \mathcal{L}(D)(U_i)$$

is an isomorphism, once this is done, we can forget all the annoying identifications.

Then you only have to see that this explicit map is compatible with restrictions to conclude that it induces an isomorphism on the base, and we are done. $\square$

Proposition 1.77. If using the same assumption as above, if D_1 and D_2 are locally principal, then $(\mathcal{O}(D_1))(D_2) \simeq \mathcal{O}(D_1 + D_2)$

Proof. We are trying to show that when $\mathcal{D}_1$ and $\mathcal{D}_2$ are locally principal, then tensor product of the sheaf of the divisor is the sheaf of the sum of the divisor.

Let's first take a look at what are the sections of $(\mathcal{O}(D_1))(D_2)$ using the identification we proved in the above proposition:

$$\Gamma(U, (\mathcal{O}(D_1))(D_2) = \{\text{rational sections } t \text{ of } \mathcal{O}(D_1)\colon \quad div|_U(t) + D_2|_U \geqslant 0\}$$

Notice that rational sections of $\mathcal{O}(D_1)$ are just sections defined on some dense open of X and in natural are still rational sections of $\mathcal{O}_X$ such that it's poles and zeros satisfy some constraints. Then you have a natural morphism:

$$(\mathcal{O}(D_1))(D_2) \to \mathcal{O}(D_1 + D_2)$$

where on each U it's just defined to be $t \mapsto t$. To see this makes sense, recall that the way we compute $div|_U(t)$ is via a trivialization:

$$\mathcal{O}_V \to \mathcal{O}(D_1)|_V$$

then I will let you figure out why if we view it directly as a rational function and compute the divisor, we indeed have $div|_U(t) + D_1|_U + D_2|_U \geqslant 0$, thus this morphism makes sense. It's easy to see that this is compatible with restrictions, thus indeed defines a morphism of sheaves.

We claim this is an isomorphism. It's not hard to see that it's injective on each U. If t is 0 as an element of $\mathcal{O}(D_1 + D_2)$, then it's indeed just the zero rational function, then t as a rational section of $\mathcal{O}(D)_1$ is also 0.

We also have to show it's surjective. If t is a rational function such that $div|_U(t)+D_1|_U+D_2|_U \geqslant 0$, then we claim that if we view t as a rational section of $\mathcal{O}(D_1)$, then the line bundle divisor $div|_U(t) + D_2|_U \geqslant 0$. This is essentially a local question, suppose that $t = ag_i$ for g_i determine $D_1|_U$ locally, then we know that $div(a)$ should at least be larger than $D_2|_U$, then via the trivialization we know that the line bundle divisor of t is indeed larger than $D_2|_U$, we are done. $\square$

Before we end this section, we state another version of qcqs lemma that we will use later:

For the notation we will use, suppose that X is qcqs, let $\mathcal{L}$ be a line bundle on x with a regular section s. Let $\mathcal{F}$ be a quasicoherent sheaf on X. We can generalize the definition of X_f for a regular function f and define X_s to be the open subscheme where s doesn't vanish. (You can use trivialization to prove the locus where is vanishes is closed.)

Now notice that for each $n \geqslant 0$, we have $\Gamma(X, \mathcal{L}^{\otimes n})$, and for m, n, and x, y elements in $\mathcal{L}^{\otimes m}$ and $\mathcal{L}^{\otimes n}$, you have a natural element $x \otimes y$ in $\mathcal{L}^{\otimes (m+n)}$, this makes $R(\mathcal{L})_\bullet = \oplus_n \Gamma(X, \mathcal{L}^{\otimes n})$ a graded ring where s is a homogeneous element of degree 1. Similarly we have a graded $R(\mathcal{L})_\bullet$ module $\oplus_n \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$.

Proposition 1.78. There is a natural isomorphism:

$$((\oplus_n \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}))_s)_0 \to \Gamma(X_s, \mathcal{F})$$

Proof. Let's first define the natural morphism. What we notice is that $\Gamma(X_s, \mathcal{F})$ is a $R(\mathcal{L})_\bullet$ module. To define the action of the ring, let's define it on each component. Suppose that $a \in \Gamma(X, \mathcal{L}^{\otimes n})$, we first consider it's restriction to: $s \in \Gamma(X_s, \mathcal{L}^{\otimes n})$, but notice that we know that for each n, we have a canonical isomorphism:

$$\mathcal{L}^{\otimes n}|_{X_s} \leftarrow \mathcal{O}_{X_s}$$

where 1 maps to the section corresponding to $s^{\otimes n}$. Let's denote the inverse by $\alpha_n : \mathcal{L}^{\otimes n}|_{X_s} \to \mathcal{O}_{X_s}$. Then we know that a section $a|_{X_s}$ is mapped to a section $\alpha_n(a)$ in $\Gamma(X_s, \mathcal{O}_X)$, then we define the action on $\Gamma(X_s, \mathcal{F})$ by multiplying $\alpha_n(a)$. Then we define the general action by extension by linearity Let's check that this definition indeed a module action. We only need to check for elements in each component, for example, we need to check that if $x, y \in \Gamma(X_S, \mathcal{F})$, then for $a \in \Gamma(X, \mathcal{L}^{\otimes n})$, we have:

$$a(x+y) = ax + ay$$

which is just $\alpha_n(a)(x+y) = \alpha_n(a)x + \alpha_n(a)y$, which is the assumption that $\mathcal{F}$ is an $\mathcal{O}_X$-module. We also need to check that for example, if we have $a \in \Gamma(X, \mathcal{L}^{\otimes n})$ and $b \in \Gamma(X, \mathcal{L}^{\otimes n})$, then we have:

$$(a+b)x = ax + bx$$

But this is again a tautology. What is interesting is that if $a \in \Gamma(X, \mathcal{L}^{\otimes m}), b \in \Gamma(Y, \mathcal{L}^{\otimes n})$, then:

$$(a \cdot b)x = a \cdot (b \cdot x)$$

By definition, we need to show:

$$\alpha_{m+n}(a \otimes b)x = \alpha_n(a) \cdot (\alpha_m(b) \cdot x)$$

Then I will let you see that the equality above is just to check:

$$\begin{tikzcd} & \mathcal{L}^{\otimes m} \otimes \mathcal{L}^{\otimes n} \arrow[dl, "\alpha_m \otimes \alpha_n"'] \arrow[d, "\alpha_{m+n}"] \\ \mathcal{O}_X \otimes \mathcal{O}_X \arrow[r, "\times"'] & \mathcal{O}_X \end{tikzcd}$$

Recall that α_i's are defined to be the inverse image of certain morphisms, therefore what we need to check can be done by using those more explicit maps, but I will let you check is. Finally, we need to show that the identity acts as expected, which is trivial.

What is worth-noting is that $s \in R(\mathcal{L})_\bullet$ acts by identity, which is an automorphism, therefore we know that this is actually a $(R(\mathcal{L})_\bullet)_s$ module via the expected $(R(\mathcal{L})_\bullet)_s$ action. Then we know from the universal property of localization that a morphism from:

$$(\oplus_{n \geqslant 0} \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}))_s \to \Gamma(X_s, \mathcal{F})$$

It's equivalent give a $R(\mathcal{L})_\bullet$ map:

$$\oplus_{n \geqslant 0} \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) \to \Gamma(X_s, \mathcal{F})$$

Well, we still define it on each component, therefore we only need to define it for:

$$\Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) \to \Gamma(X_s, \mathcal{F})$$

The way you do is is still simple, we do it by first restrict to X_s, then use the isomorphism α_n to get a morphism in $\mathcal{F} \otimes \mathcal{O}_X \simeq \mathcal{F}$. We need to check this is compatible with the $R(\mathcal{L})_\bullet$-action. We can check it on homogeneous elements, which is essentially the same as how we check $\Gamma(X_s, \mathcal{F})$ is an $(R(\mathcal{L})_\bullet)_s$-module, so I will again let you figure it out.

This finishes the construction since we can just restrict the above map to the degree 0 piece. So now our job is to show that it's an isomorphism. To do so, recall that since X is quasicompact, let's cover X by finitely many affine opens $SpecA_1, \cdots, SpecA_n$ on which $\mathcal{L}$ is isomorphic to $\mathcal{O}_X$ via $\beta_n : \mathcal{L} \to \mathcal{O}_X$. Let's first show that this is injective, suppose that we have $\frac{x}{s^n}$ where $x \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ we want to show that if it's mapped to 0, then it's 0. First, what is the image of $\frac{x}{s^n}$, first we know that the image of x should be the image of x under:

$$\mathcal{F} \otimes \mathcal{L}^{\otimes n} \to \mathcal{F}$$

then we need to consider the $\frac{1}{s^n}$ action. We know that the s action is the identity action, therefore we know that action by it's inverse is also the identity action, therefore we know that the image is just the image in $\mathcal{F}$, therefore what we are assuming is that after we restrict x to X_x, it's image in $\mathcal{F}$ is 0. What does this mean, for example, on one of the $SpecA_i$. Notice that on $SpecA_i$, we know that $SpecA_i \cap X_s$ is the distinguished open $D(\beta_i(s))$, then we know that the image of x in $X_s \cap SpecA_i$ is computed using the diagram below:

$$\begin{tikzcd}
\mathcal{F} \otimes \mathcal{L}^{\otimes n}|_{X_s} \arrow[r, "\mathrm{Id}\otimes\alpha_n"] \arrow[d, "\mathrm{Res}"'] & \mathcal{F}|_{X_s} \arrow[d, "\mathrm{Res}"] \\
\mathcal{F} \otimes \mathcal{L}^{\otimes n}|_{X_s \cap SpecA_i} \arrow[r, "\mathrm{Id}\otimes\alpha_n"'] & \mathcal{F}|_{X_s \cap SpecA_i}
\end{tikzcd}$$

For simplicity, let's just denote $X_i = SpecA_i$. Recall that except using α_n, there is another way to go from $\mathcal{F} \otimes \mathcal{L}^{\otimes n}|_{X_i}$ to $\mathcal{F}|_{X_i}$, that is by $\beta_i^{\otimes n}$. And I will let you show that using this, we know that on $SpecA_i$, there is a n_i such that $x \otimes s^{\otimes n_i} = 0$ in $\mathcal{F} \otimes \mathcal{L}^{\otimes n_i}$, therefore it's true for all $N \geqslant n_i$ as well. This holds for all i, and we only have finitely many such i, this implies that there is a N such that $x \otimes s \otimes N = 0$ on all $SpecA_i$, thus globally, therefore this proves that $\frac{x}{s^n} = 0$, thus the map is injective.

To prove surjective, suppose that we have something b in $\Gamma(X_s, \mathcal{F})$, consider it's restriction in X_i, we know that there is a n_i such that $f_i^{n_i} \cdot b$ is from some element in $\Gamma(X_i, \mathcal{F})$. We know that For differential i, j, we have differential n_i, n_j such that $f_i^{n_i} \cdot b$ and $f_j^{n_j} \cdot b$ comes from $\mathcal{F}|_{X_i}$ and $\mathcal{F}_{X_j}$.

Let's assume that t_i maps to f_i under α_1. Since the scheme is quasi-separated as well, we know that $X_i \cap X_j$ is quasicompact, then we know that if we consider the following diagram:

$$\begin{tikzcd}
? \arrow[r, mapsto] & \mathcal{F} \otimes \mathcal{L}^{\otimes n_i}|_{X_i} \arrow[r, "\mathrm{Id}\otimes\beta_i^{\otimes n}"] \arrow[d, "\mathrm{Res}"] & \mathcal{F}|_{X_i} \arrow[d, dashed, "\mathrm{Res}"] & ? \arrow[l] \\
b \otimes s^{\otimes n_i} \arrow[r] & \mathcal{F} \otimes \mathcal{L}^{\otimes n_i}|_{X_i \cap X_s} \arrow[r] & \mathcal{F}|_{X_i \cap X_s} & b \times f_i^{n_i} \arrow[l, mapsto]
\end{tikzcd}$$

Notice that on $X_i \cap X_s$ we have $b \otimes s^{\otimes n_i}$ and on $X_j \cap X_s$, we have $b \otimes s^{\otimes n}$. Moreover for any $N \geqslant n_i$, we know that $b \times f_i^N$ also comes from $\mathcal{F}|_{X_i}$, then by tensor with an appropriate power, we can assume that on $X_i \cap X_j$, $b \otimes s^{\otimes N}$ and $b \otimes s^{\otimes N}$ agree, therefore we know that they come from a global section on $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$. We only need to prove that this section maps to $b \in \Gamma(X_s, \mathcal{F})$. We again check on $X_s \cap X_i$ for all i, but since we know that it's restriction on $X_i \cap X_s$ is $b \otimes s^{\otimes N}$ and s acts by identity, therefore we know that after we apply α_N we indeed get b. This works for all i, we are done. □

1.5 Fun examples

In this section, we will use what we developed earlier to discuss a lot of fun examples. Let's start with the so called **projective transformation**:

Proposition 1.79. All the automorphism of projective spaces fixing k correspond to $(n+1) \times (n+1)$ invertible matrices over k modulo scalars, also known as the group $PGL_{n+1}(k)$.

Proof. Suppose $\pi : \mathbb{P}^n_k \to \mathbb{P}^n_k$ is an automorphism fixing k, suppose that this is given by a line bundle $\mathcal{L}$ on $\mathbb{P}^n_k$, then we know that $\pi^*\mathcal{O}(1) \simeq \mathcal{O}(m)$ for some m since $\mathcal{L}$ since the Picard group of $\mathbb{P}^n$ is generated by $\mathcal{O}(1)$, then suppose that the inverse is φ, then we know that:

$$\varphi^*\pi^*(\mathcal{O}(1)) = \mathcal{O}(1) = \varphi^*(\mathcal{O}(m))$$

since we already prove pullback commutes with tensor product, we know $\varphi^*(\mathcal{O}(m)) \simeq (\varphi^*\mathcal{O}(1)^{\otimes m})$, we know that only when $\varphi^*\mathcal{O}(1) \simeq \mathcal{O}(1)$ and $m = 1$, can the above isomorphism holds.

Therefore, we have an isomorphism $\pi^*\mathcal{O}(1) \simeq \mathcal{O}(1)$, and I claim that the pullback map:

$$\pi^* : \Gamma(\mathbb{P}^n, \mathcal{O}(1)) \to \Gamma(\mathbb{P}^n, \mathcal{O}(1))$$

is an isomorphism as well. This is because we know exactly that $\varphi^* : \Gamma(\mathbb{P}^n, \mathcal{O}(1)) \to \Gamma(\mathbb{P}^n, \mathcal{O}(1))$ is the inverse. I will let you check this, which essentially is still a consequence that the pullback functors are inverses of each other. Recall that we can identify $\Gamma(\mathbb{P}^n, \mathcal{O}(1))$ as the space spanned by $x_0, \cdots, x_n$, then an isomorphism is just an invertible $(n+1) \times (n+1)$ matrix. Moreover, from the proof of morphisms to $\mathbb{P}^n$ correspond to line bundles and sections, we know that π is induced by the line bundle $\mathcal{O}(1)$ and section $\pi^*(x_i)$ for $i = 0, \cdots, n$, only with one caveat that multiplications of scalar gives you the same morphism.

On the other hand, given such a matrix P, you have an automorphism of $k[x_0, \cdots, x_n]$, which induces an automorphism of $\mathbb{P}^n_k$. You can either consider this morphism as induced by applying $Proj$ construction, which tells you that this is indeed an automorphism. You can also think of this as given by the line bundle $\mathcal{O}(1)$ as sections $P(x_i)$. This tells you that after you modulo the action of k-scalar multiples, you get the same automorphism.

Then we need to show that the above two constructions are inverses of each other, and I will let you do it, using the fact that we know very well what are the morphism from $\mathbb{P}^n$ to $\mathbb{P}^n$ in terms of line bundles and base-point-free sections. ☐

Remark 1.80. As what we mentioned, automorphisms of projective spaces are called projective transformations and because of the above proposition, they are $PGL_{n+1}(k)$, and are always called **projective change of coordinates**, since in the proof, you see that each such matrix can be viewed as a matrix acting on the coordinate ring $k[x_0, \cdots, x_n]$.

We will use the a lot later, especially when $n = 1$, where the automorphisms will be called fractional linear transformations.

Proposition 1.81. $Aut(\mathbb{P}^1_k)$ is strictly three transitive on k-valued points, i.e., given two triplets (p_1, p_2, p_3) and (q_1, q_2, q_3), each of three distinct k-valued points on $\mathbb{P}^1$, there is precisely one automorphism sending p_i to q_i.

Proof. To figure this out, we need to know how an automorphism π acts on k-valued points.

Say π correspond to an invertible matrix P, and $[a_0 : \cdots : a_n]$ is a k-valued point, then $\pi([a_0 : \cdots : a_n])$ will be another k-valued point $P([a_0 : \cdots : a_n])$. We didn't really do this before, thus let's do this here with a little bit caution. The first thing you should check is that suppose that $\pi : R_\bullet \to S_\bullet$ is a morphism of graded ring, then after we apply proj, a homogeneous polynomial p in S, if it's in the domain of the induced morphism, is mapped to $\pi^{-1}p$. Check this is not had, as long as you can use the following diagram:

$$\begin{tikzcd} S \arrow[r] & S_f & (S_f)_0 \arrow[l] \\ R \arrow[r] \arrow[u, "\pi"] & R_g \arrow[u, "\pi_f"] & (R_g)_0 \arrow[l] \arrow[u, "(\pi_f)_0"] \end{tikzcd}$$

Then you need to use some algebra coming from classical algebraic geometry see the first chapter of Hartshorne for details. For example, you can prove that all k-valued point comes from the homogeneous ideal:

$$p = \{f \text{ homogeneous} : f(a_0, \cdots, a_n) = 0\}$$

You will see that this k-valued point corresponds to the notation $[a_0 : \cdots : a_n]$ we used before. Then you can prove that under the morphism π' induced by the proj construction, $f \in \pi'(p)$ iff $\pi(f)(a_0, \cdots, a_n) = 0$, then when π is given by an invertible matrix P, this is equivalent to $f(P(a_0, \cdots, a_n)) = 0$, thus we know that $\pi'(p)$ is the homogeneous prime consisting of all homogeneous polynomials such that their evaluation at $P(a_0, \cdots, a_n) = 0$, thus we are done.

Then I claim that the assertion is equivalent to prove that if an invertible matrix fixes three k-valued points, it is the identity. This is just because if P_1, P_2 both send the three k-valued points to the specified three k-valued points, then $P_2^{-1}P_1 = id$ then $P_1 = P_2$. (All the identification is done in the group $PGL_2(k)$) Suppose that P fixes the three distinct k-valued points $[a_i : b_i]$ for $i = 1, 2, 3$, since they are distinct, we know that (a_i, b_i) are linearly independent for $i = 1, 2$. Then since P fixes the three points, it's action on (a_1, b_1) is just multiply by a nonzero scalar k_1, so is it's action on (a_2, b_2) is multiplication by a nonzero scalar k_2. We only have to prove $k_1 = k_2$ to claim that this matrix is the identity. Then we use the fact that it also fixes $[a_3, b_3]$, which means that it's action on (a_3, b_3) is multiply by a nonzero scalar k_3, then if $k_1 \neq k_2$, only when $a_3 = b_3 = 0$ can we have such a k_3, a contradiction. □

Remark 1.82. This is also a good point to identify all the k-valued points in $\mathbb{A}^n$ and $\mathbb{P}^n$ for an arbitrary field k, even though it's simple, since we didn't do this before. The more important case is $\mathbb{P}^n$ since the k-valuded points of the affine n-space is really easy.

Let's first deal with $\mathbb{A}^n$. Recall that a k-valued point correspond to a morphism $Speck \to \mathbb{A}^n$, then this tells you that they correspond to all the ideal of the form $(x_i - a_i)$ for $a_i \in k$. This ideal can be identified by the ideal of polynomials vanishing at $(a_1, \cdots, a_n)$ since you can see that $(x_i - a_i)$ is contained in that ideal, but $(x_i - a_i)$ is already maximal.

Now let's work with the projective case. By definition, this still corresponds to a morphism $Speck \to \mathbb{P}^n$. Now suppose that the one point image is in $D_+(x_i)$, then we know that this morphism factors through $Speck \to D_+(x_i) \to \mathbb{P}^n$, then since $D_+(x_i)$ is just the affine space $Speck[\frac{x_j}{x_i} : j \neq i]$, we know that this correspond to the prime ideal $(\frac{x_j}{x_i} - k_j)$. Then recall that the homogeneous prime ideal $(x_j - k_j x_i)$ is mapped to $(\frac{x_j}{x_i} - k_j)$, via the bijective map we discussed before in the proj construction. Conversely, such a homogeneous prime ideal indeed gives you a k-valued point, thus these are all the k-valued points in $\mathbb{P}^n$.

A more concrete description of such homogeneous ideals is via what polynomials vanish. For example, we can describe $(x_j - k_j x_i)$ by the ideal I consisting all the polynomials f that vanishes at $(k_0, \ldots, 1, \cdots, k_n)$. Clearly $(x_j - k_j x_i)$ is contained in I, conversely, suppose that $f \in I$, first notice that I is prime, and you know that x_i is not in this prime, then you can pass to the affine $D_+(x_i)$, say it correspond to J there, notice that $f \in I$ iff f homogenized at x_0 is in J. Then this tells you that J is $(\frac{x_j}{x_i} - k_i)$ which again tells you that I is $(x_j - k_j x_i)$.

We keep working with projective transformations:

Proposition 1.83. There is a linear fractional transformation $f(t) \in PGL_2(k)$ precisely of order 3 in PGL_2.

Proof. This is telling us to find an automorphism of $\mathbb{P}^1$ such that the order is 3. By our intuition coming complex geometry, we are thinking this as the Mobius transformation, thus $f(t) = \frac{-1}{t-1}$ is a good candidate since apply f for there times you get $t \mapsto t$. This is the matrix:

$$P = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

and you can compute the power $P^2 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, this is obviously not the identity even if you mod out the action by nonzero scalars, then you compute the triple, which is the identity. □

Proposition 1.84. Any two order 3 elements of $PGL(2)$ are conjugate.

Proof. Recall that conjugate doesn't change the order, then suppose that P_1 and P_2 are two order 3 elements.

They are not the identity and of order 3, thus there is p_1 such that $p_2 = P_1(p_1) \neq p_1$, and $p32 = P_1^2 p_1 \neq P_2(p_1)$ and $p_3 = p_1^2 p_1 \neq p_1$, then we have three distinct k-valued points p_1, p_2, p_3, then we can do this for P_2 and get q_1, q_2, q_3 distinct k-valued points.

Then we know from the strict transitivity that there is a unique H such that H takes p_i to q_i. Now we claim that $H^{-1}P_2H = P_1$. This is because that P_1 takes p_i to p_{i+1} where the subscript is in $\mathbb{Z}/3$. Now we know that $H^{-1}P_2H$ takes p_1 to p_2 and so on, then we are done again from strict transitivity. $\square$

Now we can prove a fun fact that there are no nonconstant maps from projective spaces to a smaller dimensional quasi-projective variety, we will need the following lemma:

Lemma 1.85. Suppose that $\pi : X \to Y$ is a morphism of schemes while $\mathcal{L}$ is a line bundle on Y, let s be a global section of $\mathcal{L}$ with vanishing locus $V(s)$, then the vanishing locus of π^*s is $\pi^{-1}V(s)$.

Proof. Suppose that q is a vanishing point of s, this means that the image of s in the fiber is 0:

$$\mathcal{L}|_q = \mathcal{L}_q \otimes_{\mathcal{O}_{Y,q}} k(q)$$

Recall that if $\pi(p) = q$, then the fiber of of $\pi^*\mathcal{L}$ at p is:

$$\mathcal{L}|_q \otimes_{k(q)} k(p)$$

if the image of s in the fiber of q is s', then the image of π^*s at p is:

$$\overline{s}' \otimes 1$$

where $\overline{s}'$ is the image of s' in the fiber. Notice that $k(p)$ and $\mathcal{L}_q$ are both vector spaces over $k(q)$, then $\overline{s'} \otimes 1$ is 0 if and only if one of them is 0, but we know that 1 is not 0, thus we know that $\overline{s'}$ should vanish. Therefore, we know that s vanish at q iff π^*s vanish at p and $\pi(p) = q$. $\square$

Proposition 1.86. Suppose that X is a quasiprojective k-scheme and $\pi : \mathbb{P}^n_k \to X$ is any k-scheme morphism, then either the image of this map in X is of dimension n or it contracts $\mathbb{P}^n_k$ to a point.

Proof. Let's assume that we are embedding X to $\mathbb{P}^N_k$ for some N since X is a quasiprojective k-scheme. I claim that we can assume k is algebraically closed, an in particular, infinite. To see this, we can consider the extension $\pi_{\overline{k}} : \mathbb{P}^n_{\overline{k}} \to X_{\overline{k}}$, and consider the following commuatative diagram:

$$\begin{tikzcd}
\mathbb{P}^n_k \arrow[r] & X \\
\mathbb{P}^n_{\overline{k}} \arrow[r] \arrow[u] & X_{\overline{k}} \arrow[u]
\end{tikzcd}$$

notice that if the image of $\pi_{\overline{k}}$ is a single point, then so is the image of π since $\mathbb{P}^n_{\overline{k}} \to \mathbb{P}^n_k$ is surjective. To prove the assertion about the dimension of the images, we will use more than what we know right now, thus we omit it here and come back later when we know that fact.

Then we can assume we have a morphism $\pi : \mathbb{P}^n_k \to \mathbb{P}^N_k$ over an infinite field k, which is given by $\mathcal{O}(d)$ on $\mathbb{P}^n_k$ and $N+1$ sections not vanishing simultaneously. Recall that if $d = 0$, then the map is constant. This is because that when $d = 0$, every global section of the structure doesn't vanish anywhere, thus we know that $\mathbb{P}^n \to \mathbb{P}^N$, we considered as a map from $\mathbb{P}^n \to D_+(x_K)$, is just the ring map $k[\frac{X_J}{X_K}] \to k$ where you choose to map $\frac{x_J}{x_K}$ to some elements in k. This already tells you that this morphism is constant.

Recall that this morphism is necessarily proper, thus closed, therefore we know that the image of $\mathbb{P}^n$ is closed in $\mathbb{P}^N$. It's also irreducible since $\mathbb{P}^n$ is irreducible. Then we will prove that if the dimension of the image is not n and $d \neq 0$, then there is a contradiction. First, I claim that this dimension of the image can not be higher than n, this is because we are considering a finite type dominant morphism of irreducible k-varieties, which means that the dimension of the target can not grow. If it has dimension less than n, say m, then $m + 1 \leqslant n$, since we are working with an infinite field, one of the propositions before tells you that there are $m + 1$ hyperplanes such that their scheme-theoretic intersection miss X, thus we know that if we assume these hyperplanes comes from $m + 1$ section of $\mathcal{O}(1)$ on $\mathbb{P}^N$, then their common vanishing locus is not on X, thus the pullback also have no common vanishing locus by the lemma. This tells you that there are $m+1$ hypersurfaces of degree d in $\mathbb{P}^n$ with empty scheme theoretic intersection, which is a contradiction, since the dimension of the first m hypersurfaces is at least 1, and a hypersurface in $\mathbb{P}^n$ meets everything of dimension at least 1. $\square$

Remark 1.87. Besides the fact that we can pass to the algebraic closure, this proposition is proved without using any thing we haven't seen before. There are a few facts about scheme theoretic image we used implicitly listed in previous sections, make sure you know what we are using here.

Now, let's actually some Picard group and class group concretely. The first example is a torsion Picard group:

Proposition 1.88. Suppose that Y is a hypersurface in $\mathbb{P}^n_k$ corresponding to an irreducible degree d polynomial. Then $Pic(\mathbb{P}^n_k - Y)$ is $\mathbb{Z}/d$.

Proof. First, consider the excision exact sequence:

$$\mathbb{Z} \to Cl(\mathbb{P}^n_k) \to Cl(\mathbb{P}^n_k - Y) \to 0$$

Recall that we have identified $Cl(\mathbb{P}^n_k)$ as $\mathbb{Z}$ and it's generated by the divisor $\mathcal{O}(1)$, then we know that the map $\mathbb{Z} \to Cl(\mathbb{P}^n_k)$ is just $1 \mapsto d$, thus this identifies $Cl(\mathbb{P}^n_k - Y)$ as $\mathbb{Z}/d$ and $Pic(\mathbb{P}^n_k - Y)$ is a subgroup of this. Now consider the following diagram:

$$\begin{tikzcd}
Pic(\mathbb{P}^n_k) \arrow[r, "\simeq"] \arrow[d, "res"] & Cl(\mathbb{P}^n_k) \arrow[d, "res"] \\
Pic(\mathbb{P}^n_k - Y) \arrow[r, hook] & Cl(\mathbb{P}^n_k - Y)
\end{tikzcd}$$

(check this really commutes by yourself!) Notice that the generator $\mathcal{O}(1)$ of $Pic(\mathbb{P}^n)$ is mapped to the restriction of $\mathcal{O}(1)$, and then mapped to the restriction of $div(s)$ on $\mathbb{P}^n - Y$, which is the generator of $Cl(\mathbb{P}^n - Y)$, thus we know that $\mathcal{O}(1)$ restricted to $\mathbb{P}^n - Y$ is also the generator of $Pic(\mathbb{P}^n - Y)$, which identifies $Pic(\mathbb{P}^n - Y)$ as $\mathbb{Z}/d$. In particular, the restriction of $\mathcal{O}(d)$ to $\mathbb{P}^n - Y$ is isomorphic to the structure sheaf. □

Proposition 1.89. Keep the same notaion of the previous exercise, and assume $d > 1$. Let $H_0, \cdots, H_n$ be the $n+1$ coordinate hyperplanes of $\mathbb{P}^n_k$.

Then $\mathbb{P}^n - Y$ is affine and $\mathbb{P}^n - Y - H$ is a distinguished affine of it. Moreover $\mathbb{P}^n - Y - H_i$ forms an cover of $\mathbb{P}^n - Y$.

Finally, $Pic(\mathbb{P}^n - Y - H_i) = 0$ but $Pic(\mathbb{P}^n - Y)$ is not 0, and $\mathbb{P}^n - Y - H_i$ is the spec of a unique factorization domain, while $\mathbb{P}^n - Y$ is not. In particular, this tells you that being a UFD is not affine local. Since any localization of a UFD is a UFD, then being a UFD only satisfies one the two properties of the affine communication lemma.

Proof. The first few assertions are essentially trivial. Suppose that Y is the hypersurface corresponding to f, then I claim that $\mathbb{P}^n - Y$ is just $D_+(f)$. Recall that how we defined Y, previously, we defined it by the the vanishing locus of the homogenized f on each $D_+(x_i)$, however, suppose that p is a homogeneous prime which contains f, then we know that you homogenize f at some x_i, it vanishes at the homogenized p, then we know that the locus where f vanish as a closed subset is also Y, then $\mathbb{P}^n - Y$ is just $D_+(f)$, the locus where f doesn't vanish. Similarly, you can prove that $Y \cup H_i$ as a closed subset is just the locus where fx_i vanishes, then the complement is just $D_+(fx_i)$. This readily tells you that $\mathbb{P}^n_k - Y - H_i$ covers $\mathbb{P}^n - Y$ and they are distinguished opens in $\mathbb{P}^n - Y$.

We can directly see that the class group of $\mathbb{P}^n - H_i$ is 0, thus the class group of $\mathbb{P}^n - Y - H_i$ is 0, thus the Picard group of $\mathbb{P}^n - Y - H_i$ is 0. This makes sense, since we know that $\mathbb{P}^n - Y - H_i$ is also a distinguished open of $\mathbb{P}^n - H_i$, thus is the spec of a localization of a UFD $k[\frac{x_j}{x_i}]$, thus we already proved that it's class group is 0. However, since $\mathbb{P}^n - Y$ has nontrivial class group, even though it's affine, it can't be the spec of a UFD, thus we are done. □

Proposition 1.90. We still use the same notation as the above proposition, then on $\mathbb{P}^n_k - Y$, H_i restricted to this open subset is an effective Cartier divisor that is not cut out by a single equation.

Proof. First, we need to figure out what is the restriction of H_i to this open subscheme $\mathbb{P}^n_k - Y$. Since we want to prove it is an effective Cartier divisor, we are thinking of this as a closed subscheme of $\mathbb{P}^n_k - Y$. You probably will think that since $\mathbb{P}^n - Y$ is just $D_+(f)$, then restrict H_i to this open will just give you the closed subschem that is cut out by a single function on $D_+(f)$, but as you will see in the proof, this is wrong.

But for now, even though we are not sure if it's cut out by a single equation, we can prove it's indeed an effective Cartier divisor, which means that it's locally cut out by a non-zerodivisor. Notice that $D_+(x_i f)$ covers $D_+(f)$ and on each such open this restriction is still cut out by $\frac{x_i}{x_j}$, this is obviously not a zerodivisor, thus we are done. We are doing this because when we defined H, we only described what H looks like on each $D_+(x_i)$, not on an arbitrary $D_+(f)$, but we are quite clear about how to restrict a closed subscheme of an affine open to a distinguished open.

Another way of view the restriction of H_i is via the restriction of the class group, since H_i in $\mathbb{P}^n$ is just $div(x_i)$. This tells you that the restriction of H_i is not the trivial element, instead it's the generator of the class group. However, if it's given by a global nonzero divisor, it's a principal divisor, which means that it's trivial in the class group, a contradiction. $\square$

Now we will try to figure out what is the Picard group of $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$:

Proposition 1.91. Consider $X = \mathbb{P}^1_k \times_k \mathbb{P}^1_k = Projk[w,x,y,z]/(wz - xy)$, a smooth quadric surface, then $PicX \simeq \mathbb{Z} \oplus \mathbb{Z}$.

Proof. The strategy is to first establish a surjection from $\mathbb{Z} \oplus \mathbb{Z}$ to $Cl(X)$. The only way we know right now to do this is via the excision exact sequence, which means that we need to carefully pick two divisors, say L and M, such that the class group of $X - L - M$ is trivial. What is a scheme of trivial class group, the only thing we know right now is the affine space, thus a candidate of $X - L - M$ is $\mathbb{A}^2_k$ and we only have to cleverly choose L, M.

To see how to do this, we define two closed subschemes L, M by setting $w, x = 0$ and $y, z = 0$ respectively. Informally speaking, these two schemes are of the form: $\{\infty\} \times_k \mathbb{P}^1$ and $\mathbb{P}^1 \times_k \{\infty\}$. You can also interpret these two subschems as the fiber over $[0:1]$ via the two natural projections. This way of interpretation is more important in this proposition, so we will only use it.

First, you need to notice that this closed subscheme is indeed of dimension 1 since it's just $\mathbb{P}^1$, then the dimension codimension equality tells you that the codimension is 1, thus indeed two divisors. We now need to show that $X - L - M$ is $\mathbb{A}^2$.

To show this, it's better to use the second interpretation, since L, M are fiber of ∞, then $X - L$ and $X - M$ should be inverse image of $\mathbb{P}^1 - \{\infty\}$, then we have the following diagram:

$$\begin{tikzcd}
X - L - M \arrow[r, hook] \arrow[d, hook] & X - M \arrow[r] \arrow[d, hook] & \mathbb{P}^1_k - \{\infty\} \arrow[d, hook] \\
X - L \arrow[r, hook] \arrow[d] & \mathbb{P}^1_k \times_k \mathbb{P}^1_k \arrow[r] \arrow[d] & \mathbb{P}^1_k \arrow[d] \\
\mathbb{P}^1_k - \{\infty\} \arrow[r, hook] & \mathbb{P}^1_k \arrow[r] & k
\end{tikzcd}$$

Notice that $\mathbb{P}^1_k - \{\infty\}$ is just $\mathbb{A}^1$, then $X - L - M \simeq \mathbb{A}^1 \times_k \mathbb{A}^1 \simeq \mathbb{A}^2$, thus the excision sequence gives us:

$$\mathbb{Z} \oplus \mathbb{Z} \to Cl(X) \to 0 \to 0$$

Then we need to prove this is also injective. Notice that this morphism is given by $(m,n) \mapsto m[M] + n[N]$. I claim that when m, n are not 0, this is not zero, which proves that this is injective.

Recall that if $X \hookrightarrow Y$ is a closed subscheme, then the structure sheaf on Y will pullback to the structure sheaf of X. Then if we have a line bundle $\mathcal{L}$ on Y whose pullback to X is not trivial, then $\mathcal{L}$ is not trivial. Recall that $\mathbb{P}^1 \times_k \mathbb{P}^1$ is factorial since it's covered by $\mathbb{A}^2$ which is factorial. This tells you that $[M]$ and $[L]$ should be locally principal and correspond to $\mathcal{O}(M)$ and $\mathcal{O}(L)$ in the Picard group. Then if $m[M] + n[L]$ is 0, we know that $m\mathcal{O}(M) + n\mathcal{O}(L)$ is the trivial line bundle, which means that it will restrict to the trivial line bundle on L and M. However, I claim that the restriction of $\mathcal{O}(D)$ to D is $\mathcal{O}$ and to L is $\mathcal{O}(1)$. Similarly restriction of $\mathcal{O}(L)$ to D is $\mathcal{O}(1)$ but it's restriction of L is $\mathcal{O}$, which tells you only when $m = n = 0$ the class $m[M] + n[L]$ is trivial.

To see this assertion, let's take a look at how to restrict to M since the other restriction is similar. Notice that M is the inverse image of $\{\infty\}$, and you can view $\{\infty\}$ in $\mathbb{P}^1$ as the divisor determined by $(\mathcal{O}(1), x)$, thus L is the divisor determined by the pullback of x. Let's consider the following diagram:

$$\begin{tikzcd} & L \arrow[r] \arrow[d, hook, "j"] & \{\infty\} \arrow[d, hook] \\ M = \mathbb{P}^1 \arrow[r, hook, "i"] \arrow[d] & X \arrow[r, "pr_2"] \arrow[d, "pr_1"] & \mathbb{P}^1 \arrow[d] \\ \{\infty\} \arrow[r, hook] & \mathbb{P}^1 \arrow[r] & k \end{tikzcd}$$

I claim that $pr_1 \circ i$ is constant and $pr_2 \circ i$ is the identity. This is already the information in the diagram, and I will let you check it. But recall that restriction by identity gives you exactly the same line bundle and restriction by a constant morphism gives you the structure sheaf no matter what line bundle you started with. This proves the assertion. $\square$

Remark 1.92. You should notice that this method can not be generalized to higher dimension since a closed point in dimension 1 can be determined via a single line bundle section, while for higher dimensions, this fails.

This above proposition has an interesting consequence, which gives you the first nontrivial example of birational but not isomorphic smooth projective surfaces:

Proposition 1.93. The two irreducible smooth projective surfaces $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are birational but not isomorphic.

Proof. From the above proposition, we know that $\mathbb{P}^1 \times \mathbb{P}^1$ consists of an $\mathbb{A}^2$ and so does $\mathbb{P}^2$, obviously they are both irreducible and smooth, then we can just use the identity function on $\mathbb{A}^2$ to conclude that they are birational, but they have different Picard group, $\mathbb{Z}^2$ and $\mathbb{Z}$, thus they are not isomorphic. $\square$

Remark 1.94. We will see in the following section that irreducible smooth projective curves of a field k that are birational must be isomorphic, which is quite different for surfaces.

However, the situation for surfaces are not too bad since we will see later after we introduce the notion of blow-up that if X, X' are projective regular and birational, then you can choose X, X' to be the blow-up of something at some points such that the result is isomorphic.

Let's compute one more Picard and class group:

Proposition 1.95. Let $X = Speck[x, y, z]/(xy - z^2)$ a cone where $char(k) \neq 2$. Then $Pic(X) = 0$ while $Cl(X) = \mathbb{Z}/2$.

Proof. The reason we are assuming $char(k) \neq 2$ is because we want to show that $k[x, y, z]/(xy - z^2)$ is normal. This goes back to an old exercise we did, where if A is a UFD with 2 invertible, then if $z^2 - f$ $A[z]$ is irreducible in $A[z]$ such that f has no repeated prime divisors, $A[z]/(z^2 - f)$ is normal. Then it makes sense to talk about the class group and so on.

We will still give a surjection from $\mathbb{Z}$ to $Cl(X)$. Consider the class $Z = V(x)$, by Krull's principal ideal theorem, this is indeed an irreducible closed subset of codimension 1, thus an element in $Cl(X)$. Now, think about it's complement, which is $D(x)$, I claim that it's isomorphic to $Speck[x, z]_x$, which is an open subset of $\mathbb{A}^2$, thus the class group is 0, this gives you:

$$\mathbb{Z} \to Cl(X) \to 0 \to 0$$

Now I claim that Z is not principal, thus a nontrivial element in $Cl(X)$ but $2Z = div(x)$, thus trivial in $Cl(X)$. The assertion $2Z = div(x)$ is easy to check, since we know that x have at $V(x, z)$ and the uniformizer there is z while $x = \frac{z^2}{y}$, thus $div(x) = 2Z$. However, we saw this before that D is not even a locally principal divisor.

To show the Picard group vanishes, suppose that $\mathcal{L}$ is a line bundle, then consider $div(s)$ for s a rational section. Then $div(s)$ is either Z or 0. If it's 0, then this tells us $\mathcal{L}$ is trivial, if it's Z, then we know that $\mathcal{L} \simeq \mathcal{O}$ such that s is mapped to 1, we are done. $\square$

Remark 1.96. If you wish, you can try to show that $X - \{(0,0,0)\}$ is factorial, therefore on this open subscheme, the Picard group is isomorphic to the class group. We know that class group is insensitive to removing a closed subset of codimension greater than 1, thus Cl is still $\mathbb{Z}/2\mathbb{Z}$, but this implies the Picard group is also $\mathbb{Z}/2$, thus the Picard group is not insensitive to removing a "small" closed subset.

If a Weil divisor Z on a normal Noetherian scheme satisfies the property that nZ is locally principal, we say that Z is $\mathbb{Q}$-Cartier, next we give an example of something that is not $\mathbb{Q}$-Cartier, since we won't use this exercise beyond this section, you just do if for fun:

Proposition 1.97. Consider $X = Speck[w,x,y,z]/(wz-xy)$ and let Z be the divisor cut out by $x = w = 0$. Recall that we have shown that this ideal is codimension 1 but not principal and it's not cut out scheme-theoretically by one equation.

Now if $n \neq 0$, then nZ is not locally principal and hence Z is not even set-theoretically cut out by one equation.

Proof. If nZ is locally principal for some $n \neq 0$, then I claim that it's an effective Cartier divisor.

Notice that if it's locally given by a rational function, then this rational function has no poles, thus is extends to a global section, which is either 0 or a nonzero element, since we know that this scheme $X = \mathbb{P}^1 \times \mathbb{P}^1$, which is integral, then we can assume that the regular function is not a zero divisor. This shows that nZ is an effective Cartier divisor. Recall that complement of an effective Cartier divisor on an affine scheme is still affine (just use affine morphism is affine local on the target). Then this tells you that the complement of Z is affine(recall that if you think of nZ as a closed subscheme, then the underlying closed subset is the same as Z), then consider the closed subscheme of the complement cut out by $y = z = 0$, which is still affine. But this is just the open subscheme of $Speck[w,x]$ minus the origin which is not affine, a contradiction. □

Before we end this section, let's take a look at the last two commonly used method of checking a domain is a UFD, which is related to the class group:

Proposition 1.98. Suppose that A is a Noetherian integral domain then $SpecA$ is a UFD iff A is normal and $ClSpecA = 0$.

Proof. Recall that if A is a UFD, then $SpecA$ is factorial thus of course normal and $ClSpecA = 0$ since all the codimension 1 primes are principal.

Conversely, if recall that a Noetherian integral domain is a UFD iff all it's codimension 1 primes is principal. Then when $Cl(X) = 0$, this implies that all the codimension 1 prime ideals is a principal divisor, say determined by a rational function, then this tells you that this rational function doesn't have poles on $SpecA$ thus extends to a regular function f. Now I claim that f generates this codimension 1 ideal. First, notice that f generates this ideal p after we localize at p, then suppose that $g \in p$, then there is some a, b such that $f \cdot \frac{a}{b} = g$. Recall that g is an element in A, thus it valuation at each codimension 1 prime is at least 0 while it's valuation at p is at least 1, since f has valuation at p 1 and 0 at other primes, we know that $\frac{a}{b}$ at least has valuation 0 at each point, thus no poles thus is a regular function, thus $g \in (f)$, we are done. □

Using this we can prove Nagata's lemma, which is the last one method to show a Noetherian domain is a UFD, and is least useful:

Proposition 1.99. Let A be a Noetherian domain and (x) be a prime ideal with $x \in A$ such that $A_x = A[\frac{1}{x}]$ is a UFD, then A is a UFD:

Proof. First I claim that if A_x is integrally closed, then A is integrally closed. Suppose that $T \in K(A)$ satisfies an equation:

$$T^n + a_{n-1}T^{n-1} + \cdots + a_1 T + a_n$$

with $a_i \in A$, then since A_x is integrally closed T is r/x^m where if $m > 0$ then $r \notin (x)$. Then if $m > 0$, then by multiplying with an appropriate power of x, I can assume r plus something in (x) is 0, this tells you that $r \in (x)$, a contradiction.

This makes it possible to talk about $Cl(SpecA)$. Then consider the excision sequence:

$$\mathbb{Z} \to Cl(SpecA) \to Cl(SpecA_x) = 0 \to 0$$

we get a surjection of $\mathbb{Z}$ to $Cl(SpecA)$ where 1 is mapped to $[V(x)]$. However, notice that $V(x)$ is $div(x)$, a principal divisor, thus $Cl(SpecA) = 0$, then the previous proposition tells you that A is a UFD. $\square$

As an application of Nagata's lemma, we can prove:

Proposition 1.100. Suppose that k is an algebraically closed field of characteristic not 2, then if $l \geqslant 3$, $A = k[a, b, x_1, \cdots, x_n]/(ab - x_1^2 - \cdots - x_l^2)$ is a UFD.

Proof. Consider the element a, notice that $A/aA = k[b, x_1, \cdots, x_n]/(x_1^2 + \cdots + x_l^2)$, when $l \geqslant 3$, this is an irreducible polynomial, but since k is algebraically closed, when $l = 2$ it's not.

Then localize at a gives you this ring is isomorphic to $k[a, \frac{1}{a}, x_1, \cdots, x_n]$, clearly a UFD since it's a localization of a UFD, we are done. $\square$

1.6 Effective Cartier Divisors and Invertible Ideal Sheaves

In this section we provide a different way of describing invertible sheaves on a scheme. On advantage of this method over Weil divisors is that it can give line bundles on a everywhere nonreduced scheme, where the codimension 1 points are not regular.

Suppose that $D \hookrightarrow X$ is a closed subscheme corresponding to an invertible ideal sheaf $\mathcal{I}$, then $\mathcal{I}$ is locally trivial. Suppose that $U = SpecA$ is a trivializaing affine open, then we know that the closed subscheme exact sequence is:

$$0 \to I \to A \to A/I \to 0$$

with $I \simeq A$ as an A-module. Then plug $I \simeq A$ in the above SES, we get:

$$0 \to A \to A \to A/I \to 0$$

where $A \to A$ is given by the multiplication of a, then since it's injective a is not a zerodivisor, thus I is generated by a, a nonzerodivisor, this is precisely the definition of an effective Cartier divisor we used before. And this definition is obviously reversible, since if $\mathcal{I}$ is locally generated by a nonzerodivisor, $A \to I$ where 1 is mapped to that nonzerodivisor is an isomorphism, thus $\mathcal{I}$ is locally free of rank 1, a line bundle. Therefore we would like to use this as the new definition of effective Cartier divisors:

Definition 1.101. Suppose that $D \hookrightarrow X$ is a closed subscheme, it's an effective Cartier divisor iff the corresponding ideal sheaf $\mathcal{I}_D$ is invertible.

Recall that locally I is isomorphic to A via a nonzerodivisor a, suppose that you have a different trivialization on $SpecA$, where I is isomorphic to A via a different nonzerodivisor b, then a quick observation is that a, b differ by an invertible element of A. You only have to compose $A \to I \to A$ and see where a is mapped to, say it's x, then you can check that x is invertible and $a = bx$. Also, in the case of a locally Noetherian scheme, we can redefine it as a closed subscheme locally cut out by a single equation not vanishing at any associated points of X.

Suppose that $D \hookrightarrow X$ is an effective Cartier divisor, we want to imitate what we did for Weil divisors and associaed an invertible sheaf to D. By the discussion we just had, it's probably natural to define it by $\mathcal{O}(D) = \mathcal{I}_D$, however, we will use the opposite:

Definition 1.102. Let $D \hookrightarrow X$ is an effective Cartier divisor, then we define $\mathcal{O}(D)$ to be $\mathcal{I}_D^\vee$ and the reason for the dual will be clear in a following proposition.

We will also formally define $\mathcal{O}(nD) = \mathcal{O}(D)^{\otimes n}$ since this time we don't have a concrete meaning of nD. We will also define $\mathcal{O}(-D) = \mathcal{O}(D)^\vee$, thus it's canonically isomorphic to $\mathcal{I}_D$.

Because the convention we take, $\mathcal{I}_D$ is $\mathcal{O}(-D)$, therefore the closed subscheme exact sequence is:

$$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0$$

There is a canonical global section s_D of $\mathcal{O}(D)$, obtained by tensoring the inclusion $\mathcal{I} \hookrightarrow \mathcal{O}$ with $\mathcal{I}^\vee = \mathcal{O}(D)$, which gives us:

$$\mathcal{O} \to \mathcal{O}(D)$$

The reason we used $\mathcal{I}_D^\vee$ as $\mathcal{O}(D)$ is in the following proposition:

Proposition 1.103. Recall that a global section on a locally free sheaf cuts out a closed subscheme. Then the canonical section s_D defined above cut out D.

Proof. This is equivalent to show that the ideal sheaf defined by s_D is $\mathcal{I}$.

Before we prove any thing, there is something you need to notice. Suppose that $\mathcal{M}$ $\mathcal{N}$ are both subsheaves of $\mathcal{L}$, how to prove that they are equal. The philosophy here is similar to that of being a subobject of something usually makes it easier to prove a lot. For example, being a subset of a group, it's easier to prove it's a subgroup than directly prove it's a group. So for the question of determining whether $\mathcal{M}$ $\mathcal{N}$ are equal, the only thing we need to do is to find an open cover of X and show that they are equal on each open. If we want to prove two sheaves are isomorphic, without any further information, only finding an open cover such that two sheaves on each open are isomorphic is not enough, there is the compatibility probably of isomorphisms on intersections. But being subsheaves of a given sheaf, the transition function is the identity in this case, thus the cocycle condition is satisfied without doubt. This is the way by which we will prove $\mathcal{I}$ is equal to the ideal sheaf defined by s_D.

First, we can find an affine open cover of X such that on each $SpecA$, the ideal sheaf is given by:

$$\widetilde{I} \hookrightarrow \widetilde{A}$$

You can think of this as induced by the inclusion $(a) = I \hookrightarrow A$ where a is given by the isomorphism $A \to I$ given by $1 \mapsto a$.

Then let's consider the ideal sheaf defined by s_D, at least what is that when restricted to $SpecA$. To see this, we first need a trivialization of $\mathcal{I}^\vee$ on $SpecA$. First, we know that $\mathcal{I}$ is finitely presented(can you see why?), thus $\mathcal{I}^\vee$ is still quasicoherent, and thus on $SpecA$, it's just $\widetilde{Hom(I, A)}$, which is isomorphic to $\widetilde{A}$ via the isomorphism $A \to I$. The following diagram tells you where does s_D goes:

$$\begin{tikzcd}
I \otimes_A Hom(I, A) \arrow[r, "x \otimes f \mapsto x \cdot f"] & Hom(I, A) \arrow[r, "\varphi \mapsto 1"] & A \\
A \otimes_A A \arrow[u, "1 \mapsto a \otimes \varphi"] \arrow[ur, "1 \mapsto a \cdot \varphi"'] & &
\end{tikzcd}$$

notice that s_d restricted to $SpecA$ is just the image of 1 under the map $A \otimes_A A \to Hom(I, A)$, thus we know that s_D is mapped to A under the trivialization, then we know that locally the ideal sheaf is generated by a, thus it's just:

$$\widetilde{(a)} = \widetilde{I} \hookrightarrow \widetilde{A}$$

thus they agree locally, we are done. $\square$

Remark 1.104. For later use, we state explicitly here that he trivialization $Hom(I, A) \to A$ in the above proposition is just the dual trivialization of I.

Remark 1.105. The reason we must pass to the dual $\mathcal{I}^\bullet$ is because we need a global section such that the ideal sheaf generated by that global section is $\mathcal{I}$. If you stick to $\mathcal{I}$, since the local trivialization is $A \to I$ where 1 is mapped to a, you will need $a^2 \in I$ such that the preimage is a, but there is no reason for these local sections to glue to a global section. For example, suppose that you pick $SpecA$ twice and have two trivializations on $SpecA$, then as what we saw, $A \to I$ where $1 \mapsto a$ and $A \to I$ where $1 \mapsto b$ differs by an invertible element of A, which probably is not 1, thus there is no reason for them to glue.

The above construction is reversible:

Proposition 1.106. Suppose that $\mathcal{L}$ is a line bundle and s is a global section such that it's locally not a zero divisor, then s cut out an effective Cartier divisor D such that $\mathcal{O}(D) \simeq \mathcal{L}$.

Proof. Let's first make sense of what does it mean to not be a local zero divisor.

If we work with a locally Noetherian scheme, we can just say that s doesn't vanish at any associated point of X. Then, in this case, given any open cover of X by affines $SpecA$ where $\mathcal{L}$ is trivial on $SpecA$, we know that the function corresponding to s is not a zerodivisor on $SpecA$. If we are in a more general case, we can just define s is not a local zero divisor by pick an arbitrary trivialization of $\mathcal{L}$ using affine opens, and on each such open s doesn't correspond to a zero divisor. This definition is well defined since if we pick another such trivialization, say $SpecB$ is one of the affine opens, then we can cover $SpecB$ by distinguished affine opens such that on each of them the function corresponding to s is not a zero divisor (affine communication lemma and on a common distinguished open, two trivialization differs by an invertible element, thus if one of them is not a zerodivisor, then so is the other). Then I will let you prove that if $SpecB$ is covered by $D(f_i)$ for $i \in I$, possibly infinite, and $x \in B$ is not a zero divisor on B_{f_i} for each i, then x is not a zerodivisor on B.

Then this readily tells you that s cuts out an effective Cartier divisor just by definition. Then we only have to prove $\mathcal{L} \simeq \mathcal{O}_D$. And as you might expect, we prove this by showing that they have the same transition function. Suppose that the transition function of $\mathcal{L}$ from $SpecA$ to $SpecB$ is given by f for some invertible function f on $SpecA \cap SpecB$. Then if under the trivialization $\mathcal{L}|_{SpecA} \simeq \mathcal{O}_{SpecA}$ s is mapped to a and under the trivialization $\mathcal{L}|_{SpecB} \simeq \mathcal{O}_{SpecB}$ s is mapped to b, we know that $a \times f = b$ on $SpecA \cap SpecB$. Recall that for $\mathcal{I}$, the trivialization is given by:

$$\mathcal{O}_{SpecA} \to \mathcal{I}_{SpecA}, \qquad 1 \mapsto a \quad \text{and} \quad \mathcal{O}_{SpecB} \to \mathcal{I}_{SpecB}, \quad 1 \mapsto b$$

Then on the intersection, consider the following diagram:

$$\begin{tikzcd}
\mathcal{O} \arrow[r, "1\mapsto a"] \arrow[d, "1\mapsto \frac{1}{f}"'] & \mathcal{I} \arrow[d, "id"] \\
\mathcal{O} \arrow[r, "1\mapsto b"'] & \mathcal{I}
\end{tikzcd}$$

I claim it's commutative on $SpecA \cap SpecB$, and you should check. But this tells you that the transition function of $\mathcal{I}$ is $\frac{1}{f}$, which tells you the transition function of $\mathcal{I}^{\bullet}$ is f, we are done.

Moreover, I claim that under this isomorphism, s is mapped to s_D, which is like what we did for Weil divisors. Recall that the trivialization of $\mathcal{L}$ and $\mathcal{O}_D$ is given by:

$$\mathcal{L}|_{SpecA} \xrightarrow{s\mapsto a} \mathcal{O}|_{SpecA} \xrightarrow{a\mapsto s_D} \mathcal{O}_D|_{SpecA}$$

this trivialization of $\mathcal{O}_D$ is exactly the one that gives the transition functions f. Then the same argument for the similar statement for Weil divisor works here and shows that the global isomorphism takes s to s_D. $\square$

Remark 1.107. The above two propositions tell you that the map $D \mapsto (\mathcal{O}(D), s_D)$ and $(\mathcal{L}, s) \mapsto D_s$ where $\mathcal{L}$ is a line bundle with a section s not locally a zerodivisor and $\mathcal{D}$ is the effective Cartier cut out by s are inverses of each other, if we consider the quotient group $\{(\mathcal{L}, s)\}/iso$ where the isomorphism between $(\mathcal{L}, s)$ and $(\mathcal{M}, t)$ is an isomorphism of line bundles such that s is mapped to t. You only need to show that if $(\mathcal{L}, s) \simeq (\mathcal{M}, t)$, then they cut out the same Cartier divisor. I will let you check this.

Now suppose that $\mathcal{I}, \mathcal{J}$ are two quasicoherent ideal sheaf of $\mathcal{O}$, we can define the product of these two quasicoherent ideal sheaf $\mathcal{I}\mathcal{J}$ to be the quasicoherent sheaf such that on each affine $SpecA$, it's defined to be $IJ \subset A$ where $\mathcal{I}(SpecA) = I$ and $\mathcal{J}(SpecA) = J$. We need to check this is a well defined definition, thus we use the distinguished affine base machinery and define the restriction to be $IJ \to I_f J_f$, then we only have to show the canonical morphism $(IJ)_f \to I_f J_f$ is an isomorphism. I will let you do it.

Proposition 1.108. Suppose that $\mathcal{I}, \mathcal{J}$ are two invertible ideal sheaves corresponding to effective Cartier divisors D, D', then $\mathcal{I}\mathcal{J}$ is also an invertible ideal sheaf. We define the Cartier divisor corresponding to the product ideal sheaf to be $D + D'$, then $\mathcal{O}(D + D') \simeq \mathcal{O}(D) \otimes \mathcal{O}(D')$. Moreover, the canonical section $s_{D+D'}$ is mapped to $s_D \otimes s_{D'}$.

Proof. First, you have a morphism of sheaves on a distinguished affine base to show that $\mathcal{I}\mathcal{J}$ is indeed an ideal sheaf of $\mathcal{O}_X$. To show it's invertible, we only have to consider what is an localization, recall that product of non-zerodivisors is still a non-zerodivisor, then we can define the trivialization to be:

$$A \to IJ \qquad 1 \mapsto a_1 a_2$$

if the trivialization of $\mathcal{I}$ and $\mathcal{J}$ on $SpecA$ is $1 \mapsto a_1$ and $1 \mapsto a_2$. I will let you show $1 \mapsto a_1 a_2$ is indeed an isomorphism.

The fact that $\mathcal{O}(D + D') \simeq \mathcal{O}(D) \otimes \mathcal{O}(D')$ can be proved using multiple methods. For example you can try to show that they have the same transition function, which can be obtained by showing that $\mathcal{I}\mathcal{J}$ has the same transition function as $\mathcal{I} \otimes \mathcal{J}$ (remember transition function of tensors of line bundles is just product). Or you can show that $\mathcal{O}(-(D + D')) \simeq \mathcal{O}(-D) \otimes \mathcal{O}(D')$ by showing $\mathcal{I}\mathcal{J} \simeq \mathcal{I} \otimes \mathcal{J}$, this again comes to the fact that tensor product of ideals is isomorphic to multiplications in a canonical and natural way, and use $\mathcal{I}\mathcal{J}$ and $\mathcal{I} \otimes \mathcal{J}$ are both quasicoherent, thus you only have to prove isomorphism on the distinguished affine base.

To show that this isomorphism can be chosen such that $s_{D+D'}$ is mapped to $s_D \otimes s_{D'}$, we can still use the transition function argument like what we did in the previous proposition. □

Remark 1.109. The above proposition shows that effective Cartier divisors forms an abelian semigroup, that is an abelian group without identity and inverses. If you have two Cartier divisors D, D', the above proposition tells you how to add them up, but in generally, there is not a way to define $D - D'$, even though we can always formally define such stuff on the level line bundles, for example, we can define $\mathcal{O}(D - D')$ to be $\mathcal{O}(D) \otimes \mathcal{O}(D')$, even though $D - D'$ is not an effective Cartier divisor any more.

We also have a map of semigroups, from Cartier divisors to line bundles with sections not locally a zero divisor quotient isomorphisms, defined to be $D \mapsto (\mathcal{O}(D), s_D)$, and the fact that it's indeed a map of semigroups is the above proposition. And if you define the map of semigroups to the Picard group by forgetting the sections, you get a map of semigroups from Cartier divisors to Picard groups where $D \mapsto \mathcal{O}(D)$.

Before we end this section, we would like to give a preview of the definition of normal bundles which we will come back to later:

Definition 1.110. If D is an effective Cartier divisor on a scheme X, then the restriction of the invertible sheaf $\mathcal{O}(D)$ to D via $D \hookrightarrow X$, denoted by $\mathcal{O}(D)|_D$, is called the **normal line bundle** to D, denoted by $\mathscr{N}_{D/X}$.

Remark 1.111. The origin of this name comes from complex geometry since it's really the normal bundle there in the differential or holomorphic setting. We will define this normal bundle in a more general setting later and you will see a lot of the complex intuition breaks down if the algebra is not so nice.

1.7 The Graded Module Corresponding to a Quasicoherent Sheaf

This section can somewhat be viewed as the dual section of the section "The Quasicoherent Sheaf Corresponding to a Graded Module". It answers quite a lot of fundamental questions which we didn't really touch in that section.

Throughout this section $S_\bullet$ is a finitely generated graded algebra generated in degree 1. So in particular, we have $\mathcal{O}(m)$ for all m, also $M_\bullet$ is a graded $S_\bullet$ module and $\mathcal{F}$ is a quasicoherent sheaf on $ProjS_\bullet$.

We have seen that under the assumption above of $S_\bullet$, we can get a quasicoherent sheaf on $ProjS_\bullet$ by taking associated sheaves of a graded module $M_\bullet$. In this section, you will see that these are all of them. The way we do it is to define a functor $\Gamma_\bullet$ from the category of quasicoherent sheaves on $ProjS_\bullet$ to graded $S_\bullet$-modules, trying to reverse the $\tilde{}$ construction. This attempt is not as successful as we might expect but still with good properties, in the sense that $\Gamma_\bullet$ is not really the inverse of $\tilde{}$, since we have already seen that $\tilde{}$ can turn two different modules into the same quaisocherent sheaf. But there is an isomorphism:

$$\widetilde{\Gamma_\bullet(\mathcal{F})} \simeq \mathcal{F}$$

and you will see that for any graded module $M_\bullet$, $\Gamma_\bullet(\widetilde{M_\bullet})$ is a better version of $M_\bullet$, because $Gamma_\bullet(\widetilde{M_\bullet})$ is going to be saturated.

Moreover, there is a saturation function from garded modules to saturated garded modules that's is quite like groupfication and sheafification functor we saw before, it's adjoint to the forgetful functor from saturated graded modules to graded modules. More precisely, the relation between $\tilde{}$ and $\Gamma_\bullet$ is given by the diagram:

$$\begin{tikzcd}
& \text{Graded } S_\bullet\text{-modules} \arrow[dl, "\sim"'] \arrow[dd, bend left, "\text{saturation}"] \\
Qcoh_{ProjS_\bullet} \arrow[dr, "\Gamma_\bullet\text{, equivalence}"'] & \\
& \text{Saturated graded } S_\bullet\text{-modules} \arrow[uu, bend left, "\text{forgetful}"]
\end{tikzcd}$$

where the two vertical arrows (saturation and forgetful) form an adjoint pair. Moreover, $\Gamma_\bullet$ is also adjoint to $\tilde{-}$ where we think $\Gamma_\bullet$ as a functor not to saturated graded modules, but to graded modules.

Let's first give the definition $\Gamma_\bullet$: Recall that on $X = \mathbb{P}^n_k$, we know that there is a natural isomorphism between $\Gamma(X, \mathcal{O}(m))$ and M_n where $M_\bullet = S_\bullet$. Recall that $\mathcal{O}(m)$ in this case is just $\widetilde{S_\bullet(m)} = \widetilde{S}(m)$ Thus you will suspect that in good situations this also holds, thus we will define:

Definition 1.112. With the above assumptions, for a quasicoherent sheaf $\mathcal{F}$ on $X = ProjS_\bullet$ we define $\Gamma_\bullet(\mathcal{F})$ by:

$$\Gamma_m(\mathcal{F}) = \Gamma(X, \mathcal{F}(m))$$

Proposition 1.113. There is a natural S_0-module morphism $M_m \to \Gamma(ProjS_\bullet, \widetilde{M(m)_\bullet})$

Proof. Recall how we defined $\widetilde{M(m)_\bullet}$, suppose that $f_1, \cdots, f_n$ are the degree 1-elements generating $S_\bullet$, then we know that on each $D_+(f_i)$, $\widetilde{M(m)_\bullet}$ is just $(M_f)_m$ and the restriction to the intersection is given by the natural map to $(M_{fg})_m$. Then we define $x \in M_m$ goes to $\frac{x}{1}$ in $(M_f)_m$, then it's easy to see that they agree on the intersections, thus gives you a S_0-module morphism $M_m \to \Gamma(ProjS_\bullet, \widetilde{M(m)_\bullet})$.

In other words, it's induced by the universal property of the kernel of a certain short exact sequence. $\square$

Proposition 1.114. $\Gamma_\bullet(\mathcal{F})$ is a graded $S_\bullet$-module.

Proof. We already have a direct sum decomposition of $\Gamma_\bullet(\mathcal{F})$, thus we only have to figure out how $S_\bullet$ acts on $\Gamma_\bullet(\mathcal{F})$ and show that action satisfies the hypothesis of a graded module over $S_\bullet$.

To show this, recall that we have a graded ring $\Gamma_\bullet(\mathcal{O}_X) = \oplus_m \Gamma(X, \mathcal{O}(m))$, you can define the ring structure by the tensor product. And we know that $\Gamma_\bullet(\mathcal{F})$ is a $\Gamma_\bullet(\mathcal{O}_X)$-graded module. The action is simply define to be if $x \in \Gamma(X, \mathcal{O}(m))$ and $y \in \Gamma(X, \mathcal{F}(n))$, then their product is defined to be the tensor product in $\Gamma(X, \mathcal{F}(m+n))$. We also have a graded ring map:

$$S_\bullet \to \Gamma_\bullet(\mathcal{O}_X)$$

which is defined to be the direct sum of the natural map $S_m \to \Gamma(X, \mathcal{O}(m))$, and you check that this is indeed a ring morphism. This requires you to understand the isomorphism $\mathcal{O}(m+n) \simeq \mathcal{O}(m) \otimes \mathcal{O}(n)$ quite well and I will let you to figure it out. This naturally makes $\Gamma_\bullet(\mathcal{F})$ as a graded $S_\bullet$-module. $\square$

Remark 1.115. In particular, if $\mathcal{F} = \widetilde{M_\bullet}$ for a graded module, the way $S_\bullet$ acts on $\Gamma_\bullet(\widetilde{M_\bullet})$ can be described on $D_+(f_i)$ by multiplication, which basically comes from your understanding of the isomorphism $\widetilde{M_\bullet}(m) \simeq \widetilde{M(m)_\bullet}$.

Now, suppose you start with a graded $S_\bullet$-module $M_\bullet$, then you apply $\tilde{}$ and then $\Gamma_\bullet$ to get another graded $S_\bullet$-module, together with a map $M_\bullet \to \Gamma_\bullet(\widetilde{M_\bullet})$, I will call this map the **saturation map**.

Proposition 1.116. The saturation map $M_\bullet \to \Gamma_\bullet(\widetilde{M_\bullet})$ is a map of $S_\bullet$ graded modules.

Proof. Notice that if $x \in M_m$, then it's send to the element in $\Gamma(X, \widetilde{M(m)_\bullet})$ whose restriction to each $D_+(f_i)$ is $\frac{x}{1}$. Then suppose that $s \in S_n$, since the action of $S_\bullet$ on $\Gamma_\bullet(\widetilde{M_\bullet})$ is also given by multiplication from the previous remark, we are done. $\square$

You probably will assume the saturation map to be an isomorphism, but it's false in general:

Proposition 1.117. The saturation map need not be injective nor surjective.

Proof. Let $S_\bullet = k[x]$ with the usual grading, we claim that the saturation of $M_\bullet = k[x]/x^2$ is 0. Then we know that $ProjS_\bullet$ is just $Speck$.

Now since the associated sheaf of $M_\bullet$ is just 0, we know that the saturation is 0. But the saturation map goes out of something nonzero, thus not injective.

It's also not surjective. For example, consider $M_\bullet = xk[x]$. Then you find that $xk[x]$ is just $k[x]$ and the saturation map $xk[x] \to k[x]$ will map x to x, thus this is indeed not a surjective morphism. $\square$

Now we show that $\Gamma_\bullet$ is a functor from quasicoherent sheaves on $ProjS_\bullet$.

Proposition 1.118. If $\mathcal{F} \to \mathcal{G}$ is a morphism of quasicoherent sheaves on $ProjS_\bullet$, then there is a induced $S_\bullet$ graded modules morphism $\Gamma_\bullet(\mathcal{F}) \to \Gamma_\bullet(\mathcal{G})$, respecting composition and identity.

Proof. We still define this morphism on each degree and show that its indeed a $S_\bullet$ graded module morphism.

For each m, recall that $\Gamma_m(\mathcal{F}) = \Gamma(X, \mathcal{F}(m))$ and $\Gamma_m(\mathcal{G}) = \Gamma(X, \mathcal{G}(m))$, also notice that for this m, apply tensor product and then global section functor to the following morphism:

$$\mathcal{F} \otimes \mathcal{O}(m) \to \mathcal{G} \otimes \mathcal{O}(m)$$

gives you a map φ_m from $\Gamma_m(\mathcal{F})$ to $\Gamma_m(\mathcal{G})$. Then we can define $\Gamma_\bullet(\mathcal{F}) \to \Gamma_\bullet(\mathcal{G})$ using universal property of direct sums. Also, since tensor product is associative, $\varphi_m(x) \otimes s = \varphi_{m+n}(x \otimes s)$ for some s global section of $\mathcal{O}(n)$. This comes from the following diagram:

$$\begin{tikzcd}
\mathcal{F} \otimes \mathcal{O}(m+n) \arrow[r] & \mathcal{G} \otimes \mathcal{O}(m+n) \\
\mathcal{F} \otimes \mathcal{O}(m) \otimes \mathcal{O}(n) \arrow[r] \arrow[u, "\cong"] & \mathcal{G} \otimes \mathcal{O}(m) \otimes \mathcal{O}(n) \arrow[u, "\cong"]
\end{tikzcd}$$

If this is indeed a morphism of $S_\bullet$-module, then this clearly respects identity and composition since on each direct sum component, it's induced by the composition of two functors. This map is indeed a morphism of $S_\bullet$ graded modules, consider any $x \in \Gamma(X, \mathcal{F}(m))$ and $s \in S_n$, for notational simplicity, since we only care the image of s in $\Gamma(X, \mathcal{O}(n))$, we will still denote it by s, then sx is the element in $\Gamma(X, \mathcal{F}(m+n))$ defined by $s \otimes x$, then you apply φ_{m+n} and get $\varphi_{m+n}(s \otimes x) = s \otimes \varphi_m(x)$ by the discussion we have above, this tells you that it's indeed a $S_\bullet$-graded morphism. $\square$

Example 1.119. For example let $M_\bullet = S_\bullet$, you get the saturation map $S_\bullet \to \Gamma_\bullet(\widetilde{S_\bullet})$, since $\Gamma_\bullet(\widetilde{S_\bullet})$ has a ring structure, which is described either using global sections of tensor product or locally described by multiplication. The two ways to describe it is essentially the same. Then you can check that the saturation map is a ring homomorphism.

For example if $S_\bullet = A[x_1, \cdots, x_n]$, then the saturation map is a ring isomorphism, as we already identified the global section $\Gamma(X, \mathcal{O}(m))$ as the degree m homogeneous polynomials with coefficients in A, and you should check the saturation map just maps f to f.

However, as we hinted above, if $S_\bullet$ is not good enough, then this saturation map is not necessarily an isomorphism. For example, let $S_\bullet = k[x,y]/(xy)$ with the usual grading, in this case the saturation map is not an isomorphism since if you compute $\widetilde{S_\bullet(m)}$, you will get the global section is $kx^m \oplus ky^m$, thus the saturation is $k[x] \oplus k[y]$ with direct sum ring structure and the ring map is just:

$$f \mapsto (f_x, f_y)$$

where f_x consist of all the terms of f only with x and f_y consists of all the terms of f only with y. Obviously, they are not isomorphic.

The above discussion tell yous that if you start with a graded module and apply$\tilde{}$then apply $\Gamma_\bullet$, things are kind messy, even if we have a natural transformation between $\Gamma_\bullet \simeq \tilde{}$and the identity functor, it is not a natural isomorphism. However, if you start from another direction, i.e. strat with a quasicoherent sheaf $\mathcal{F}$, apply $\Gamma_\bullet$ and then$\tilde{}$, you also have a natural isomorphism between$\tilde{} \circ \Gamma_\bullet$.

To do this, we need to first define a natural morphism from $\widetilde{\Gamma_\bullet(\mathcal{F})}$ to $\mathcal{F}$. Not surprisingly, we still define it locally on the open cover $D_+(f)$ and then show it behaves well on the overlaps. To define this morphism, we need to get the help of $\mathcal{O}(1)$, for example, we know that if f is a homogeneous element d, then f is in $\mathcal{O}(d)$ and $D_+(f)$ is just X_f. Then we know that $\mathcal{F}(D_+(f))$ is:

$$\oplus_n(\Gamma(X, \mathcal{F}(n))_f)_0 \simeq \Gamma(X_f, \mathcal{F})$$

via the variation of qcqs lemma argument we had before. Here, this isomorphism has a more explicit form thanks to the fact that $D_+(f)$ is affine. For example, if we want to know where $\frac{z}{f^n}$ goes where $z \in \Gamma_{ndegf}(\mathcal{F})$, then what we do is first restrict z to $D_+(f)$, and we can think of it as an element in $\mathcal{F}(D_+(f)) \otimes \mathcal{O}(d)(D_+(f)) \otimes \cdots \otimes \mathcal{O}(d)(D_+(f))$, which is mapped to $\mathcal{F}(D_+(f))$ via $\mathcal{O}(d)(D_+(f)) \simeq \mathcal{O}(D_+(f))$ where $f \mapsto 1$. Then I will let you check using the explicit form that this definition yields compatible maps on the intersection. (Hint, both sides are quasicoherent sheaves, therefore, if any morphism on the intersection makes the diagram commute, it should be the morphism you are looking for, can you find such a natural morphim on the intersection such that the diagram commute?) Then this clearly defines an isomorphism on $D_+(f)$ via the variation of qcqs lemma. I will also let you check this is indeed a natural transformation.

One immediate corollary is that

Corollary 1.120. Up to isomorphism, all quasicoherent sheaves on $ProjS_\bullet$ comes from a graded module $M_\bullet$, in particular, every quasicoherent sheaf on a projective A-scheme, up to isomorphism comes from a graded-module.

Proposition 1.121. Each closed subscheme on $ProjS_\bullet$ comes from a homogeneous ideal $I_\bullet$.

Proof. Consider the SES of the closed embedding $\iota : Z \hookrightarrow X = ProjS_\bullet$:

$$0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

Recall that if we apply $\Gamma_\bullet$ to the above sequence, we get a seqeunce, possibly not exact anymore. However, recall that the definition of $\Gamma_\bullet$, it's defined by tensor a line bundle and then taking global sections, and then direct sum. Taking global sections is only left exact, the result functor is left exact, thus we know that we have a left exact sequence:

$$0 \to \Gamma_\bullet \mathcal{I}_Z \to \Gamma_\bullet \mathcal{O}_X \to \Gamma_\bullet \mathcal{O}_Z$$

Recall that we have a saturation map $S_\bullet \to \Gamma_\bullet \mathcal{O}_X$, which give you the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & Ker \arrow[r] \arrow[d, dashed] & S \arrow[r] \arrow[d] & \Gamma_\bullet \mathcal{O}_Z \arrow[r] \arrow[d, dashed, "id"] & 0 \\
0 \arrow[r] & \Gamma_\bullet \mathcal{I}_Z \arrow[r] & \Gamma_\bullet \mathcal{O}_X \arrow[r] & \Gamma_\bullet \mathcal{O}_Z &
\end{tikzcd}$$

then apply$\tilde{}$, which is exact. However, even though after apply $\Gamma_\bullet$, we may lose exactness, it can be recovered by apply$\tilde{}$, via the natural isomorphism in the previous proposition. Thus in the end, you get the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{Ker} \arrow[r] \arrow[d, dashed] & \widetilde{S} \arrow[r] \arrow[d] & \widetilde{\Gamma_\bullet \mathcal{O}_Z} \arrow[r] \arrow[d, dashed, "id"] & 0 \\
0 \arrow[r] & \widetilde{\Gamma_\bullet \mathcal{I}_Z} \arrow[r] \arrow[d, dashed, "\simeq"] & \widetilde{\Gamma_\bullet \mathcal{O}_X} \arrow[r] \arrow[d, dashed, "\simeq"] & \widetilde{\Gamma_\bullet \mathcal{O}_Z} \arrow[r] \arrow[d, dashed, "\simeq"] & 0 \\
0 \arrow[r] & \mathcal{I}_Z \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{O}_Z \arrow[r] & 0
\end{tikzcd}$$

If we can prove the vertical composition in the middle is the identity, then five lemma tells you the vertical composition on the left is an isomorphism, and we are done.

To check that, we only have to look at those $D_+(f)$ where f is a homogeneous element of degree 1. Recall that $S_\bullet \to \Gamma_\bullet(\widetilde{S_\bullet})$ is given by $s \in S_d$ to the element t in $\Gamma(X, \widetilde{S(d)_\bullet})$ such that it restrict to $\frac{s}{1}$ on each $D_+(f)$. Then on $D_+(f)$, $\widetilde{S_\bullet} \to \widetilde{\Gamma_\bullet \mathcal{O}_X}$ is given by $\frac{s}{f^d} \mapsto \frac{t}{f^d}$ and it's again mapped to $\frac{s}{f^d}$ since t restricted to $D_+(f)$ is just $\frac{s}{1}$ and the isomorphism between $(S_f)_0 \to (S_f)_d$ is given by $1 \mapsto \frac{f^d}{1}$. This tells you the middle vertical morphism is indeed id. There are quite a lot of identifications going on, be careful when you read this part.

**** this proof is wrong, to be corrected later* □

Remark 1.122. For the first time, we see that for every closed subscheme of $ProjS_\bullet$, there is a homogeneous ideal of $S_\bullet$ cutting out the same closed subscheme. Recall the similar result for affine schemes, which is obtained essentially by definition, you will find it surprising that the result here takes so much tedious work.

Moreover, here the result is not as perfect as the result for affine schemes. Here we only proved that for each quasi-coherent ideal sheaf of $\mathcal{O}_{ProjS_\bullet}$, there is a homogeneous ideal such that the associated sheaf is the same ideal sheaf.

However, if we start with a ideal I and take the associated sheaf then apply $\Gamma_\bullet$, we may not be able to get something that is isomorphic to I, rather, we get it's saturated version, or the best version of this ideal I. While in the affine case the correspondence between ideals and closed subschemes is a strict one to one correspondence.

We end this section with two comments we won't use, and thus won't prove:

Remark 1.123. As we promised in the beginning of the section, the saturation functor is adjoint to the forgetful functor, and $\Gamma_\bullet$ is adjoint to $\tilde{}$. We won't prove but the way of doing it is like the way we proved inverse image functor is adjoint to pushforward. We will identify the morphisms of both sides using some compatible collection of morphisms.

References