Appendix 8 for Math 632: 2025 Winter Notes

Chunye Yang

Jan. 2025

In this chapter we study maps to projective spaces and line bundles in more depth. The central goal of this chapter is to introduce ampleness of a line bundles, which is some sort of positivity conditions. It will help us figure out when a line bundle has a lot of sections available and might even give a morphism to projective spaces. The fundamental facts of ampleness takes some time to establish, but after that, you will appreciate the hard work there. For example, we will see how to notion can be applied to the study of curves.

1 Globally Generated Quasicoherent Sheaves

We begin with a result that allows us to get all the finite type quasicoherent sheaf $\mathcal{F}$ on $ProjS_\bullet$, for a finitely generated graded ring $S_\bullet$ over A, generated in degree 1.

Theorem 1.1. Any finite type quasicoherent sheaf $\mathcal{F}$ can be presented in the form for some $m \in \mathbb{Z}$:

$$\bigoplus_{\text{finite}} \mathcal{O}(-m) \to \mathcal{F} \to 0$$

Rather than going directly to the proof, this theorem is a good excuse for us to introduce some notions that is relevant in wider circumstances, such as global generation, base-point-freeness and ampleness, and their interrelationships.

The first thing we want to mention is an important classical result, called the **Hilbert Syzygy Theorem**. Given any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_k$, which is locally Noetherian, the coherent is the same as finite-type, then the theorem above gives you a surjection:

$$\phi : \bigoplus_{\text{finite}} \mathcal{O}(-m) \to \mathcal{F}$$

Recall that since $\mathcal{O}(-m)$ is also coherent, thus the direct sum is coherent, thus the kernel of the morphism between coherent sheaves is also coherent, then we can present the kernel using the above theorem, thereofore we have constructed an infinite resolution of $\mathcal{F}$ using finite direct sum of line bundles. The Hilbert Syzygy Theorem states that:

Theorem 1.2. There exists a finite such resolution, of length at most $n + 1$.

We won't prove this theorem right now since we won't use it later. But a proof will be given later after we introduce the notion of flatness. The original proof was given by Hilbert in his paper where he also proved the Hilbert basis theorem and introduced the notion of Hilbert polynomials and proved they are eventually polynomials.

Now we begin our real business and introduce global generation. Suppose that X is any scheme, and $\mathcal{F}$ is an arbitrary $\mathcal{O}$-module, the most important definition of this section is the following:

Definition 1.3. $\mathcal{F}$ is **globally generated**, or **generated by global sections** if it admits a surjection from a free sheaf on X:

$$\mathcal{O}^{\oplus I} \to \mathcal{F} \to 0$$

Here I is some index set, possibly finite or infinite. If you think about the meaning of this surjection, since taking stalk commutes with direct sums, finite or infinite, this just states that there is a surjection $\mathcal{O}_{X,p}^{\oplus I} \to \mathcal{F}_p$ or in other words, $\mathcal{F}_p$ is generated by the images of global sections of $\mathcal{F}$ that corresponds to the images of 1 via the global section map $\mathcal{O}(X) \to \mathcal{F}(X)$. We say that $\mathcal{F}$ is finitely globally generated or generated by a finite number of global sections if I is a finite set.

Definition 1.4. We say that $\mathcal{F}$ is globally generated at a point p, or sometimes generated by global sections at p of we can find $\phi : \mathcal{O}^I \to \mathcal{F}$, such that after we take the stalk at p, it's surjective.

This definition is just saying that $\mathcal{F}$ is globally generated at p if every germ at p can be written as a linear combination, over $\mathcal{O}_{X,p}$, of germs of the global sections.

Remark 1.5. Notice that a sheaf $\mathcal{F}$ is globally generated iff it's globally generated at all points: surjectivity is just a surjection at each point, then if $\mathcal{F}$ is globally generated at each point, we can take the index set to be the union of index sets at each point p.

Proposition 1.6. A line bundle $\mathcal{L}$ is globally generated iff it's base-point free.

Proof. If it's globally generated, say the surjection is given by:

$$\mathcal{O}^I \to \mathcal{L}$$

then at each p, pick a germs of the images of $|I|$ 1's, they can't all line in the submodule of $m_{X,p}\mathcal{L}_p$, otherwise any linear combination can't escape this submodule, then we don't get a surjection at p.

On the other hand, if it's base point free, then we know that at each p, there is a global section x_p such that it's germ is not in $m_{X,p}\mathcal{L}_{X,p}$, then I claim this germ generates $\mathcal{L}_{X,p}$ over $\mathcal{O}_{X,p}$, and I will let you check it. □

As a reality check, if we work on an affine scheme, then any quasocoherent sheaf is globally generated, and any finite type quasicoherent sheaf is finitely globally generated, since we always have a surjection $A^{\oplus M} \to M$ for any A-module M and if M is finitely generated, the index set can be taken to be finite. Then apply$\tilde{}$, which is exact. Also, direct sums of globally generated is globally generated, finite direct sum of finitely globally generated is finitely globally generated, and this also works for a generation at a point.

Proposition 1.7. If $\mathcal{F}, \mathcal{G}$ are globally generated, then $\mathcal{F} \otimes \mathcal{G}$ are globally generated.

Proof. Recall that if $\mathcal{F}$ and $\mathcal{G}$ are invertible sheaves, we have showed that tensor product of base point free sheaf is base point free.

More generally, the assertion follows from our understanding the germ of the tensor product global section $x \otimes y$ for global sections x and y. Moreover, if M and N are generated by m_i and n_j, then $M \otimes N$ is generated by $m_i \otimes n_j$. □

The following are some easy but important observations for finite type quasicoherent sheaf $\mathcal{F}$ on X:

Remark 1.8.

- $\mathcal{F}$ is globally generated at p iff it's fiber at p is globally generated, i.e. $\mathcal{F}_p$ is spanned by germs of global sections over $\mathcal{O}_{X,p}$ iff $\mathcal{F}|_p$ is spanned by fibers of global sections over $k(p)$. Recall that since $\mathcal{F}$ is finite type quasicoherent, we can assume that it's finitely globally generated at p, thus there are finitely many global sections whose germ span $\mathcal{F}_p$, thus they also span the fiber since it's the quotient. Conversely, we can also the fiber is a finite dimensional $k(p)$-vector space, thus there are finitely many global sections whose fiber span the vector space, then we can use the geometric Nakayama's lemma, there is an affine open $SpecA$ of p, such that the restriction of the finitely many elements to $SpecA$ generates $\mathcal{F}(SpecA)$ over A, thus they also generated $\mathcal{F}_p$.
- The above point further shows that if $\mathcal{F}$ is globally generated at p, it's locally globally generated, i.e. if it's globally generated at p, then there is an open neighborhood of p such that for every q in that open neighborhood, $\mathcal{F}$ is globally generated.

- Recall that Noetherian or nonempty quasicompact schemes have closed point, then I claim that if $\mathcal{F}$ is over a quasicompact scheme such that it's globally generated at all closed points, then it's globally generated at all points. This is because a closed subset quasicompact space is quasicompact, then suppose p is closed subset and $\mathcal{F}$ is globally generated, then there is an open neighborhood of p such that $\mathcal{F}$ is globally generated, take the union, I claim that this is all of the space, otherwise, the complement is a closed subset, quasicompact, thus there is a closed point, then there is an open neighborhood of that point where $\mathcal{F}$ is globally generated, a contradiction.

- If X is quasicompact, then $\mathcal{F}$ is globally generated iff it's finitely globally generated. Suppose that it's globally generated, then it's globally generated at all closed points, then since $\mathcal{F}$ is finite type, we can assume it's finitely generated at p, then it's finitely generated locally in that neighborhood by geometric Nakayama's lemma. Notice that we know that such opens cover X, then since X is quasicompact, then we can choose finitely many of them, then $\mathcal{F}$ is finitely globally generated.

We are now finally able to state a celebrated result of Serre:

Theorem 1.9. Suppose that $S_\bullet$ is finitely generated in degree 1 over $A = S_0$, then if $\mathcal{F}$ is a finite type quasicoherent sheaf on $ProjS_\bullet$, then there is some m_0 such that for all $m \geqslant m_0$ $\mathcal{F}(m)$ is finitely globally generated.

We can prove this theorem right now, but we will again use it as an excuse to introduce more useful ideas, and the proof of it will be developed in the process. However, assuming this theorem, we can prove the theorem in the beginning of the section:

Proof. Suppose that $\mathcal{F}(m)$ is finitely globally generated by $s_1, \cdots, s_n$, then we have a surjective morphism:

$$\mathcal{O}_{ProjS_\bullet}^{\oplus n} \to \mathcal{F}(m)$$

defined by e_i is mapped to s_i. Recall that by definition, $\mathcal{F}(m)$ is just $\mathcal{F} \otimes \mathcal{O}(m)$, then tensor both sides with $\mathcal{O}(-m)$ gives you the desired result since $\mathcal{O}(-m)$ is a line bundle for $S_\bullet$ generated in degree 1. $\square$

2 Ample and Very Ample Line Bundles

We now introduce ampleness, a central "positivity" property of line bundles that will prove important for a number of disparate reasons-geometric, algebraic, topological and computational, and more.

Definition 2.1. Suppose that X is an A-scheme with structure morphism $\pi : X \to SpecA$, then a line bundle $\mathcal{L}$ on X is called very ample over A, or π very ample, or relatively very ample if $X \simeq ProjS_\bullet$ for some finitely generated A-graded ring $S_\bullet$, generated in degree 1, and $\mathcal{L} \simeq \mathcal{O}_{ProjS_\bullet}(1)$.

We will always just say it's very ample if the structure morphism is clear from the context and notice that existence of a very ample line bundle on X implies that X is a projective A-scheme.

Proposition 2.2. Suppose that $\pi : X \to SpecA$ is proper and $\mathcal{L}$ is an invertible sheaf on X, it's very ample iff there is a finite number of global sections $s_0, \cdots, s_n$ with no common zeros, such that the corresponding morphism:

$$[s_0 : \cdots : s_n] : X \to \mathbb{P}^n_A$$

is a closed embedding

Proof. We don't have to worry about the proper assumption too much, since if $\mathcal{L}$ is ample, then X is projective over A, thus is proper. On the other hand, if you have a closed embedding $X \hookrightarrow \mathbb{P}^n_A$ of A-schemes, then since composition of proper morphism is again proper, we know that X is proper over A.

Suppose that $\mathcal{L}$ is very ample, then it's isomorphic to $\mathcal{O}_{ProjS_\bullet}(1)$. Suppose that the homogeneous elements $f_0, \cdots, f_n$ of degree 1 generates $S_\bullet$ over A, then we know that f_i's are global sections of $\mathcal{O}_{ProjS_\bullet}(1)$ with no common zeros since the locus f_i doesn't vanish is just $D_+(f_i)$, and you can check that the morphism induced by these sections is just the morphism induced by the surjection:

$$A[x_0, \cdots, x_n] \to S_\bullet$$

Therefore, since we know a surjective graded ring morphism gives you a closed embedding via proj construction, we are done.

On the other hand, suppose that we have $s_0, \cdots, s_n$ of $\mathcal{L}$ such that it gives a closed embedding:

$$X \hookrightarrow \mathbb{P}^n_A$$

Then since we know that all closed embeddings of $\mathbb{P}^n_A$ comes from a homogeneous prime ideal $I_\bullet$, we know that we have a commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "\simeq"] \arrow[dr, hook] & & ProjS_\bullet \arrow[dl, hook'] \\ & \mathbb{P}^n_A & \end{tikzcd}$$

for $S_\bullet = A[x_0, \cdots, x_n]/I_\bullet$ and the closed embedding is given by applying proj to the surjective ring morphism. Then to prove $\mathcal{L} \simeq \mathcal{O}_{ProjS_\bullet}(1)$, we only have to show that the restriction of $\mathcal{O}_{\mathbb{P}^n_A}(1)$ to $ProjS_\bullet$ is isomorphic to $\mathcal{O}_{ProjS_\bullet}(1)$. But if you recall that the closed embedding can also be defined using $\mathcal{O}_{ProjS_\bullet}(1)$ and global sections $x_0, \cdots, x_n$ on $S_\bullet$, then we are done. □

Corollary 2.3. A very ample line bundle $\mathcal{L}$ on a proper A-scheme is base point free.

Proof. Just a consequence of the above proposition. □

Now we are going to prove some frequently used fact about tensor product of very ample line bundles, but before that, this is a good place to investigate the box product. Recall that we defined the so called $\mathcal{O}(a, b)$ on $\mathbb{P}^m \times \mathbb{P}^n$ and said using $\mathcal{O}(1, 1)$, we can describe the Segre embedding, but we didn't expand there. We will try to understand it right now. Recall that $\mathbb{P}^m \times \mathbb{P}^n$ can be covered by $D_+(x_i) \times D_+(y_j)$, which is affine, and we will try to understand $\mathcal{O}(a, b)$ on $D_+(x_i) \times D_+(y_j)$.

Let me first state the strategy: similar to we understand $\mathcal{O}(m)$ on $\mathbb{P}^n$ by define it's trivial on $D_+(x_i)$ and define transition functions. I claim that it's trivial on each $D_+(x_i) \times D_+(y_j)$ and then I will give the transition function on the intersection. To see the first claim, notice that tensor product is compatible with restrictions, and if some sheaf is trivial on an open subscheme, then in a smaller open subscheme, it's also trivial. Using the two observations, you only have to prove that, say the projection to $\mathbb{P}^m$ is pr_1 and to $\mathbb{P}^n$ is pr_2, you only have to prove that on $D_+(x_i) \times \mathbb{P}^n$, $pr_1^*\mathcal{O}(a)$ is trivial and similarly for $pr_2^*\mathcal{O}(b)$. Since $D_+(x_i) \times \mathbb{P}^n$ is the inverse image of $D_+(x_i)$ under pr_1, then we only have to prove $\mathcal{O}(a)$ is trivial on $D_+(x_i)$, which is the definition.

Then we need to figure out the transition functions, recall that for tensor products, the transition function is going to be the product and for pullbacks, the transition function is going to be the pullback of the transition function. Then I will let you do the details, but the final result should give you that the transition function is just $\frac{x_i^a}{x_k^a} \times \frac{y_j^b}{y_l^b}$ and this result should let you identify the global sections of $\mathcal{O}(a, b)$ to be the product fg where f is a homogeneous polynomial in variables x_i's of degree a while g should be a homogeneous polynomial in variables y_j's of degree b. Some special cases are when a or b is 0, then the global section is just homogeneous polynomials in some degree in variables x_i or y_j, and if a or b is < 0, there is no global sections.

I also claim that the locus where fg doesn't vanish is $D_+(f) \times D_+(g)$, to check this you only have to check that in $D_+(x_i) \times D_+(y_j)$, the nonvanishing locus is $D_+(x_i f) \times D_+(y_j g)$ for each x_i, y_j, and I will again let you fill the details. The following observation might be useful: if $X \to Y$ is a morphism of schemes, and y a global function is mapped to a global function x, then if you restrict this morphism to $U \subset X$, the the restriction of x to U is equal to the image of y under the new morphism $U \to Y$.

Using this interpretation, we can prove that the Segre embedding is given by the sections $x_i y_j$ in $\mathcal{O}(1, 1)$. We only have to check this locally, and you find that the locus where $x_i y_j$ is not zero is just $D_+(x_i) \times D_+(y_j)$ and the morphism is given precisely by the ring morphism which gives the Segre embedding, thus we are done.

Proposition 2.4. Suppose that $\mathcal{L}, \mathcal{M}$ are invertible sheaves on a proper A-scheme, then if $\mathcal{L}$ is very ample and $\mathcal{M}$ is base point free, then $\mathcal{L} \otimes \mathcal{M}$ is very ample. In particular, very ample times very ample is very ample.

Proof. Since $\mathcal{M}$ is base-point-free and X is proper over A, we know that $\mathcal{M}$ is globally generated, thus it's finitely globally generated since being proper means quasicompact. Then $\mathcal{M}$ gives a morphism $X \to \mathbb{P}^n$.

On the other hand, consider the closed embedding given by the very ample line bundle $\mathcal{L}$, say $X \hookrightarrow \mathbb{P}^m_A$, then consider the morphism:

$$X \to \mathbb{P}^m_A \times_A \mathbb{P}^n_A$$

I claim that this is a closed embedding. To show this, first recall that $\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^m$ is proper, since we know that both $\mathbb{P}^m \times \mathbb{P}^n$ and $\mathbb{P}^m$ are proper over $SpecA$, then the cancellation tells you via the following diagram that X is a closed embedding:

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr, hook] & & \mathbb{P}^m \times \mathbb{P}^n \arrow[dl, "proper"] \\
& \mathbb{P}^n_A &
\end{tikzcd}$$

Then consider the Segre embedding and compose it with $X \hookrightarrow \mathbb{P}^m \times \mathbb{P}^n$, which gives a closed embedding:

$$X \hookrightarrow \mathbb{P}^m \times \mathbb{P}^n \hookrightarrow \mathbb{P}^{mn+m+n}$$

Now notice that the pullback of $\mathcal{O}(1)$ on $\mathbb{P}^{mn+m+n}$ is $\mathcal{O}(1,1)$ on the product, and I will let you check that it's pullback is $\mathcal{L} \otimes \mathcal{M}$, this tells you that $\mathcal{L} \otimes \mathcal{M}$ is very ample. $\square$

Remark 2.5. One quick observation is that if $X = ProjS_\bullet$, then $\mathcal{O}(1)$ is very ample, therefore the above proposition gives you that $\mathcal{O}(m)$ for all $m \geqslant 1$ is very ample. These are the first examples of very ample line bundles we can get a hold of.

Similar to the above proposition, we can prove that box product of very ample line bundle is also very ample.

Proposition 2.6. Let X, Y be proper A-schemes and let $\mathcal{L}, \mathcal{M}$ be very ample line bundles on X, Y, if we let π_X and π_Y to be the natural projections from $X \times_A Y$ to X and Y, then $\mathcal{L} \boxtimes \mathcal{M}$ is a very ample line bundle on $X \times_A Y$.

Proof. The only thing new beyond what we talked about in the previous proposition that you need to notice is the following diagram:

$$\begin{tikzcd}
X \times_A Y \arrow[r] \arrow[d] \arrow[dr, dashed] & Y \arrow[dr] & \\
X \arrow[dr] & \mathbb{P}^m \times_A \mathbb{P}^n \arrow[r] \arrow[d] & \mathbb{P}^n \arrow[d] \\
& \mathbb{P}^m \arrow[r] & A
\end{tikzcd}$$

and notice that the induced morphism $X \times_A Y \to \mathbb{P}^m \times_A \mathbb{P}^n$ is a closed embedding. $\square$

Now let's introduce the notion of a ample line bundle, which is less restrictive and more flexible than very ampleness:

Definition 2.7. We say that a line bundle $\mathcal{L}$ on a proper A-scheme X is **ample over** A, or π**-ample** where $\pi : X \to SpecA$ is the structure morphism, or relatively ample if one of the following equivalent conditions in the following theorem holds.

Theorem 2.8. Let $\pi : X \to SpecA$ be a proper A-scheme and $\mathcal{L}$ is an invertible sheaf on X, then the following conditions are equivalent:
Projective Geometry:

- (a): For some $N > 0$, $\mathcal{L}^{\otimes N}$ is very ample over A.
- (a'): For all $n \gg 0$, $\mathcal{L}^{\otimes n}$ is very ample over A.

Global Generation:

- (b): For all finite type quasicoherent sheaves $\mathcal{F}$, there is an n_0 such that for all $n \geqslant n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated.

Topology:

- (c): As f runs over all the global sections of $\mathcal{L}^{\otimes n}$ over all $n > 0$, the open subsets $X_f = \{p \in X : f(p) \neq 0\}$ forms a base for the topology of X.

- (c'): As f runs over all the global sections of $\mathcal{L}^{\otimes n}$ over all $n > 0$, the open subsets $X_f = \{p \in X : f(p) \neq 0\}$ which are affine (we don't mean that these are all affine, we only pick those that are affine) forms a base for the topology of X.

- (c"): As f runs over all the global sections of $\mathcal{L}^{\otimes n}$ over all $n > 0$, the open subsets $X_f = \{p \in X : f(p) \neq 0\}$ which are affine covers X.

One quick observation is that except for the first two conditions, the other conditions makes no reference to the structure morphism π, and later we will meet a cohomological criterion for ampleness due to Serre. These different flavor conditions for ampleness indicates that ample behaves better than very ample. Before we head to the proof, let's see some sample applications:

Remark 2.9. The first observation is that since $\mathcal{O}(1)$ is very ample, it's ample as tensor product of very ample is very ample. Then the fact that (a) implies (b) tells you that Serre's Theorem we mentioned before: For any finite type quasicoherent sheaf $\mathcal{F}$, $\mathcal{F}(m)$ is globally generated for large enough m.

Proposition 2.10. Suppose $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on X with $\mathcal{L}$ ample, then $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is very ample for $n \gg 0$.

Proof. Recall that tensor product is associative and $\mathcal{L}$ being ample means that for large enough n, $\mathcal{L}^{\otimes n}$ is very ample and $\mathcal{L}^n \otimes \mathcal{M}$ is globally generated and for line bundles, globally generated is the same as base-point free, then a base point free tensor with a very ample is very ample, thus we are done. ☐

Proposition 2.11. Any line bundle on a projective A-scheme is the difference of two very ample line bundles, i.e. for every line bundle $\mathcal{L}$, there are two very ample line bundles $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{L} \simeq \mathcal{M} \otimes \mathcal{N}^\vee$.

Proof. Projective implies proper, thus we don't have to worry too much on the projective condition, the only reason we want projective is that for a projective A-scheme, we indeed have a ample line bundle $\mathcal{O}(1)$ while we don't yet know this for proper A-schemes.

Then notice that from the above proposition, we know that for larger enough m $\mathcal{O}(m)$ is very ample and $\mathcal{L}(m)$ is very ample, then:

$$\mathcal{L} \simeq \mathcal{L}(m) \otimes \mathcal{O}(m)^\vee$$

we are done. ☐

The following proposition is something we will use repeatedly, which says that pullback of ample line bundles sometimes gives you ample line bundles:

Proposition 2.12. Suppose that $\pi : X \to Y$ is a finite morphism of proper A-schemes and $\mathcal{L}$ is an ample line bundle on Y, then $\pi^*\mathcal{L}$ is ample on X.

Proof. Let's first make it clear why we assume finite here. In fact, in the following proof, the only thing that is useful is π being affine, however, if you still remember this, we mentioned that in locally Noetherian settings, proper plus affine is finite, we will prove this much later using Zariski's Main theorem.

Notice that since $\mathcal{L}$ is ample, then as f runs over all the global sections of $\mathcal{L}^{\otimes n}$, those f such that X_f are affine covers Y. This implies that for $\pi^* f \in \pi^*(\mathcal{L}^{\otimes n})$, $\pi^{-1}X_f$ covers X and they are affine since π is an affine morphism. Recall that the locus where $\pi^* f$ doesn't vanish is just $\pi^{-1}X_f$, thus we are done by using criterion (c"). ☐

Similar to being very ample, tensor product of ample line bundles also behaves well:

Proposition 2.13. Suppose that $\mathcal{L}, \mathcal{M}$ are line bundles on a proper scheme X and $\mathcal{L}$ is ample, then $\mathcal{L} \otimes \mathcal{M}$ is ample if $\mathcal{M}$ is ample or base-point-free.

Proof. Suppose that $\mathcal{M}$ is ample, then we know that for larger enough n, $\mathcal{L}^{\otimes n}$ and $\mathcal{M}^{\otimes n}$ are very ample, then $(\mathcal{L} \otimes \mathcal{M})^{\otimes n}$ is very ample for large enough n since tensor product is associative. $\square$

Proposition 2.14. The previous exercise for $\boxtimes$ also holds for tensor product.

Proof. You still pass to a very ample line bundle by taking high enough tensor powers, then pullback. $\square$

The following two exercises are only for practice and a later exercise for Riemann-Roch of nonreduced curves and it's mainly useful when the field k is a finite field:

Proposition 2.15. Suppose that X is a projective k-scheme and $\mathcal{L}$ is an ample line bundle on X, say $\mathcal{M}$ is any line bundle on X and $q_1, \cdots, q_m$ are points of X, not necessarily closed, then for every $N \gg 0$, there is a section of $\mathcal{L}^{\otimes N} \otimes \mathcal{M}$ such that it doesn't vanish at any q_i.

Proof. This statement should remind you of one proposition we had before about there is a hyperplane avoiding some finite set of points in the projective space.

First, I can assume that q_1 is not in the closure of any other q_i's. This is because if so, say q_i is in the closure of q_2, then if we find a section that doesn't vanish at q_q, it automatically doesn't vanish at q_2, since we have know the locus where a global section doesn't vanish is open. Then we can take out q_2 from the sequence.

Then we consider the open subscheme obtained by taking out all the closures of the other points. Then, since $\mathcal{L}$ is ample, we know that for larger enough n, there is a global section of $\mathcal{L}^n$ such that X_f is in this new subscheme and contains q_1, then also we know that for large enough n, $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$, which is globally generated, thus there is a global section that doesn't vanish at p_1, then we consider $\mathcal{L}^{\otimes 2n} \otimes \mathcal{M}$ and the section obtained by the tensor product of the two global sections, then it vanishes at all the q_i's except at q_1.

Then we do this for all the other q_i's and conclude with the observation that we can take the sum of all these global sections. $\square$

Proposition 2.16. Suppose that C is a reduced projective curve over some field k, $\mathcal{L}$ is a line bundle on C, and $q_1, \cdots, q_m$ are closed points of C, then we can write $\mathcal{L} \simeq \mathcal{O}(\sum n_j p_j)$ where p_j's are regular points distinct from q_i.

Proof. Recall that from the fact that normalization gives you an isomorphism on a dense open, you know that on a 1-dimensional reduced k-scheme is regular except only for a finite set, which is closed. Then notice that for each regular closed point say p, the canonical embedding $p \hookrightarrow C$ is an effective Cartier divisor, this is because that we can assume that there is an open neighborhood of p such that it's normal, then we can use all the stuff form algebraic Hartog's lemma.

Moreover, since C is projective k-curve, we have an ample line bundle $\mathcal{O}(1)$. Now recall that for $\mathcal{L}$, there is N such that there is a section s of $\mathcal{L}(N)$ such that it doesn't vanish at q_i's, we can further assume, since nonregular points are only finite, it doesn't vanish at any nonregular points. Since this is a reduced scheme, finite type over k, we can assume that the associated points are also finite, thus s can also be taken to not vanish at any associated points, thus s gives you an effective Cartier divisor, **** maybe it's more doable assume this curve is normal or regular* $\square$

Now we begin to prove these equivalent conditions, under the assumption that X is Noetherian. There is only one point that uses the Noetherian hypothesis and we will explain how to remove it:

Proof. We will work from the easiest to the hardest. First, notice that (a') implies (a) and obviously (c') implies (c) and (c").

We now show that (c) implies (c'), thus (c) and (c') are equivalent. **** For future reference, the equivalence of (c) and (c') doesn't need the properness hypothesis.* Suppose that we have a point p such that it's in some open subset U of X, we want to find X_f that is affine and is contained in U. By shrinking U, we may assume U is affine. But notice that from (c), there is some f such that X_f is in U, but recall this implies that X_f is affine.

Now we show that (a) implies (c), since we know that $\mathcal{L}^{\otimes N}$ is very ample, then by the previous proposition, some global sections give you an embedding $i : X \hookrightarrow \mathbb{P}^n$ such that $\mathcal{L}^{\otimes N} \simeq i^*\mathcal{O}(1)$. We want to prove that X_f forms a basis. This is equivalent to prove that for any closed subset Z of X and a point p outside Z, there is some f such that it vanishes on I, but not at p. Recall that via this closed embedding, we can identify Z as a closed subset of $\mathbb{P}^n$. Recall that for any closed subset of $\mathbb{P}^n$, we have an operation $I(Z)$, then we can consider $V(I(Z))$, which is the same as Z, thus we only that for those homogeneous polynomials in $I(Z)$, they vanish on I, but they don't vanish at p, since if all of them vanish at p, then $p \in V(I(Z)) = Z$, a contradiction. Now we can assume that f is a global section of $\mathcal{O}(m) = \mathcal{O}(1)^{\otimes m}$, then pull it back gives you $\mathcal{L}^{\otimes Nm}$ and a section i^*f, whose nonvanishing locus is the inverse image under the closed embedding, thus we are done.

Now we prove that (b) implies (c): Suppose that p is a point in an open subset U of X, we need to find a section f of $\mathcal{L}^{\otimes n}$ such that it vanishes at $X - U$ but it doesn't vanish at p. Suppose that we equip $X - U$ with the reduced subscheme structure and denote the corresponding ideal sheaf by $\mathcal{I}$. If we assume X to be a Noetherian scheme, then $\mathcal{I}$ is finite type quasicoherent, thus by (b) there is some N such that $\mathcal{L}^{\otimes N} \otimes \mathcal{I}$ is globally generated. But if you consider the injective morphism:

$$\mathcal{L}^{\otimes N} \otimes \mathcal{I} \hookrightarrow \mathcal{L}^{\otimes N}$$

induced by tensoring $\mathcal{L}^{\otimes n}$ with the natural inclusion $\mathcal{I} \hookrightarrow \mathcal{O}$, we know that there is a global section of $\mathcal{I} \otimes \mathcal{L}^{\otimes N}$ such that it doesn't vanish at p, but notice that any global section of $\mathcal{I}$ will vanish at $X - U$, by the definition of $X - U$ with the reduced structure.

**** This is the only place where the Noetherian hypothesis is invoked, to remove this, we assume that we work with a qcqs scheme since X is proper over A, then you can show that for a qcqs scheme, any ideal sheaf is the union of all of it's finite type subideal sheaves. Indeed, you can show that on a qcqs scheme, if $\mathcal{F}$ is a quasicoherent sheaf, there is a filtered index category $\mathcal{I}$ giving index by $i \mapsto \mathcal{F}_i$, such that each $\mathcal{F}_i$ is finitely presented and in particular finitely generated, and colim$\mathcal{F}_i = \mathcal{F}$. The proof is quite complicated, even though there are not too much novel ideas involved, so we could just accept it's true here. Then after you prove this, you can just choose one of the finite type sub ideal sheaf $\mathcal{I}$. We will also briefly review the reduced subscheme a little bit later. This general argument is easier for ideals since for ideals we can define the sum of arbitrary ideal sheaves.*

Now let's show that (c") implies (b), i.e. we wish to show that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated for large enough n. The first step is to show that $\mathcal{L}^{\otimes N}$ is globally generated for some large N. To do this, notice that X is quasicomapct, thus we can choose finitely many X_{f_i} with each $f_i \in \mathcal{L}^{\otimes N_i}$ such that X_{f_i}'s are affine and covers X. Recall that tensor powers of a global section has the same nonvanishing locus and vanishing locus, then by raising to the power $(\prod_j N_j)/N_i$, we can assume that f_i all comes to the same tensor power. Notice that if we can show that for all finite type quasicoherent sheaf $\mathcal{F}$, we have $\mathcal{F} \otimes \mathcal{L}^{mN}$ is globally generated for large enough m, we are done. Since we can apply this result to $\mathcal{F}, \mathcal{F} \otimes \mathcal{L}, \cdots, \mathcal{F} \otimes \mathcal{L}^{m-1}$, since it's only finitely many times, we can pick the maximum of the threshold.

Therefore, in order to show that (c") implies that (b), we can assume an additional condition that $\mathcal{L}$ is globally generated. Now for each closed point $p \in X$, by (c"), we can choose f, a global section of some tensor powers of $\mathcal{L}$, such that X_f is affine, then we know that $\mathcal{F}|_{X_f}$ is generated by a finite number of global sections. Notice that since X is quasicompact, if we can prove that there is an open subset U_p of each of the closed point p, such that for all the points q in U_p, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated at q for $n \geqslant M(p)$, we are done. This is because that $\mathcal{L}$ is globally generated, then if some invertible sheaf is globally generated at p, then the tensor power is also globally generated at p, then we can tha maximum of $M(p)$, since there are only finitely many of them.

Now recall that since X is qcqs the sections of $\mathcal{F}$, we can use the variation of qcqs lemma and conclude that for each section on $\mathcal{F}_{X_f}$, you can write it as the restriction a global section of $\mathcal{F} \otimes \mathcal{L}^{\otimes M(p)}$ and divide it by $f^{M(p)}$, this tells you that $\mathcal{F} \otimes \mathcal{L}^{\otimes M(p)}$ is globally generated at p by a finite number of global sections. Then since being globally generated at a point for a finite type quasicoherent sheaf is an open condition, we know that there is an U_p such that $\mathcal{F} \otimes \mathcal{L}^{\otimes M(p)}$ is globally generated at all points $q \in U_p$ by a finite number of global sections. Then since $\mathcal{L}$ is also globally generated, we know that $\mathcal{F} \otimes \mathcal{L}^{\otimes M'}$ is globally generated by a finite number of elements at all $q \in U_p$ for $M' \geqslant M(p)$. Then we are done.

Our pennultimate step is to show (c") implies (a). Our goal is to show that you can find some N such that $\mathcal{L}^{\otimes N}$ gives you an embedding into some projective space. First, we can assume that we cover X by finitely many X_{a_i}, where a_i comes from the same tensor power of $\mathcal{L}$, say $\mathcal{L}^{\otimes N}$. Say $X_{a_i} = SpecA_i$ and since π is finite type, we can assume that A_i is a finitely generated A-algebra, thus we assume $A_i = A[a_{i1}, \cdots, a_{ij_i}]/I_i$, and again, we can assume that each a_{ij} is of the form $s_{ij}/a_i^{m_{ij}}$ for some m_{ij} and s_{ij} is a global section of $\mathcal{L}^{\otimes Nm_{ij}}$. Let $m = max_{i,j}m_{ij}$, we can write for each i, j $a_{ij} = (s_{ij}a_i^{m-m_{ij}})/a_i^m$. For convenience we denote $b_i = a_i^m$ and $b_{ij} = s_{ij}a_i^{m-m_{ij}}$. These are all considered as global sections of $\mathcal{L}^{mN}$ Then if you consider the linear series generated by b_i, b_{ij}'s, since $X_{b_i} = X_{a_i}$, then this is a base point free sereis and thus gives a morphism f into $\mathbb{P}^Q$ where Q is $\#b_i + \#b_{ij} - 1$. Let x_i and x_{ij} be the projective coordinates on $\mathbb{P}^Q$ such that $f^*x_i = a_i$, then you find that the morphism $X_{a_i} \to D_+(x_i)$ is determined by the ring morphism where $\frac{x_{ij}}{x_i}$ is mapped to a_{ij} (check this!). One caveat is that this is not a closed embedding, rather a locally closed embedding, since we only know that it can be factored in to:

$$X \to \bigcup_i D_+(x_i) \hookrightarrow \mathbb{P}^Q$$

where the first is a closed embedding and the second is an open embedding, the main reason is we have too many generators, $D_+(x_i)$'s alone doesn't cover $\mathbb{P}^Q$.

**** For future reference, let's record what we proved here: We proved that if $X \to SpecA$ is finite type and there is a line bundle $\mathcal{L}$ on X satisfying (c"), then there is a locally closed embedding from X to $\mathbb{P}^Q_A$ for some Q.*

Now, we know that X and $\mathbb{P}^Q$ are proper over A, thus the locally closed embedding is proper, and in particular it's a closed morphism, thus the image is closed, recall that a locally closed embedding whose image is closed is a closed embedding, we are done.

At this step, recall that we have proved the following diagram:

$$\begin{tikzcd}
 & & (a) \arrow[dll, bend right] & (a') \arrow[l] \\
(c) \arrow[r, bend left] & (c') \arrow[l, bend left] \arrow[rr] & & (c'') \arrow[ul] \arrow[dl] \\
 & & (b) \arrow[ull, bend left] &
\end{tikzcd}$$

Therefore, all the conditions besides (a') are equivalent. Finally, we observe that (a) and (b) together implies (a') since we know that $\mathcal{L}^{\otimes n}$ will be globally generated for large enough n globally generated tensor with very ample is very ample. $\square$

Now that we have finished this important theorem, I would like to review some technical tools we used. The first one is the closed subscheme with the reduced structure. There are a few equivalent ways of defining it. The first two more intuitive and the last one kind of crazy at first sight, but turns out to be reasonable. First, suppose that X is just a closed subset of Y, we define a scheme structure on X such that $X \hookrightarrow Y$ becomes a closed embedding. To see how to do this, suppose that $SpecA$ is any affine open of Y, we notice that as a closed subset of $SpecA$, $X \cap SpecA$ correspond to a radical ideal $I(X \cap SpecA)$, then we define a quasicoherent ideal sheaf by define it to be $I(X \cap SpecA)$ on each affine open. This is compatible with distinguished restrictions since you can prove that localization commutes with arbitrary intersections and for those p not in $D(f)$, p_f is just 0. Then you can check that this indeed gives you a reduced scheme by checking affine locally. And it indeed gives you $X \hookrightarrow Y$. This way of understanding it is useful in a lot of situations, especially when we want to prove some local propositions.

The second way of defining it is via defining the quasicoherent ideal sheaf directly to be those functions vanishing identically on X. You can see easily that this definition indeed gives you a subsheaf of $\mathcal{O}$ and is indeed an ideal sheaf. The only question is that it may not be qusicoherent, thus we check that on affine open $SpecB$, then we know that the functions that vanish on $SpecB \cap X$ are exactly $I(SpecB \cap X)$ (can you see why?). This tells you that this is indeed quasicoherent by the first way of defining it. This is the way of understanding we used in the above proof. You can also interpret this way of understanding to be the smallest closed subscheme of Y such that the underlying closed subset contains X. To interpret it this way we need to understand the SES of a closed embedding better. Say $Z \hookrightarrow Y$ is a closed embedding and the corresponding SES is:

$$0 \to \mathcal{I} \to \mathcal{O}_Y \to i_*\mathcal{O}_Z \to 0$$

notice that the functions in $\mathcal{I}$ all vanish on Z, since you can again check this locally. Thus, if Z contains X, then we know that the functions in the ideal sheaf all vanishes on Z, then the sum of these ideals vanish on X, thus we can define the reduced subscheme structure to be the intersection of all the closed subschemes that contains X.

I claim that this again gives you the same closed subscheme as the above two definitions. The way you check it is still to do it locally, say on $SpecA$, we know that the ideal of the reduced subscheme is $I(SpecA \cap X)$, then you notice that if J is an ideal of A such that every element in that vanishes on $SpecA \cap X$, then we know that $J \subset I(SpecA \cap X)$. On the other hand, we know that elements in $I(SpecA \cap X)$ vanishes identically on $SpecA \cap X$, thus the sum of such ideals is just $I(SpecA \cap X)$, we are done.

The final crazy way of interpreting it is to define it as some kind of scheme theoretic image. But after our discussion of the previous two interpretations, you will find this not that crazy anymore. To start with, recall that for each point p in Y, we have a canonical way to include this point in X, i.e. there is a canonical morphism $Speck(p) \to X$ inducing isomorphism on the residue field. Now for each $p \in X$, we have such an inclusion, and we take the disjoint union U which maps to Y, and we define the reduced subscheme of X to be the scheme theoretic image of $U \to Y$. To see this agrees with the previous two definition, we know that if $\mathcal{I}$ defines a closed subscheme Z such that $U \to X$ lies in Z, then the functions in $\mathcal{I}$ vanishes identically on Z the converse also holds, thus we know that the ideal defining the scheme theoretic image is the sum of all $\mathcal{I}$ such that functions of this ideal sheaf vanishes on X.

In particular, if $X = Y$ as a set, then the reduced scheme structure of X is often called the reduction of Y.

There is also an absolute notion of ampleness which we won't use later thus won't introduce, but at least this tells you that instead of the definition of very ampleness which can only be defined by reference to some structure morphism, the notion of ampleness behaves better in some ways.

3 Application to Curves

We will now apply what we have learned to sutdy curves. Since this word curve has a lot of meansing, we will try to be pedantice about hypothesis in the theorems.

We begin with the result that says every integral curve has a birational model that is regular and projective.

But before we do it, this is a great place to revist some facts of Normalizations, as we will need some facts from there in this section as well.

Generally speaking, normalization is a way to turn a reduced scheme into a normal scheme. Here the reduced assumption is not necessary, and is only included to avoid distraction. We will explain later how to remove it.

A normalization of a reduced scheme X is a morphism $v : \widetilde{X} \to X$ from a normal scheme $\widetilde{X}$ where v induces a bijection of irreducible components of $\widetilde{X}$ and $\widetilde{X}$. You will see later that in reasonable cases, it will also induces a birational morphism on each of the irreducible components. We will define it using a universal property, thus if a normalization exists it's unique up to isomorphism, thus we will just use the normalization, not just a normalization. We first begin with the case where X is reduced and irreducible, thus integral.

Definition 3.1. If X is integral, the normalization $v : \widetilde{X} \to X$ is a dominant morphism from an irreducible normal scheme (also integral) to X such that any other morphism from a normal irreducible scheme to X factors uniquely through $\widetilde{X}$:

$$\begin{tikzcd} \text{normal} = Y \arrow[rr, "\exists!"] \arrow[dr, "\text{dominant}"'] & & \widetilde{X} = \text{normal} \arrow[dl, "\text{dominant}"] \\ & X & \end{tikzcd}$$

Thus if the a normalization exists, it's unique to a unique isomorphism, we only have to show that it exists. If X is affine, say $X = SpecA$ with an integral domain A, let $\widetilde{A}$ be the integral closure of A in $K(A)$, we have a canonical inclusion $A \hookrightarrow \widetilde{A}$, which gives you $Spec\widetilde{A} \to SpecA$, I claim that this is the normalizations of A and satisfies the universal property.

To show this, suppose that $Y \to X$ is another dominant morphism from a normal irreducible scheme to X, consider the following commutative diagram:

$$\begin{tikzcd}
A \arrow[rr, hook] \arrow[d, hook] & & \Gamma(Y, \mathcal{O}_Y) \arrow[dl, hook] \arrow[d, hook] \\
\widetilde{A} \arrow[r, hook] \arrow[d, hook] \arrow[urr, dashed, "\exists!"] & K(\Gamma(Y, \mathcal{O}_Y)) \arrow[dr, hook] & B \arrow[d, hook] \\
K(X) \arrow[rr, hook] \arrow[ur, hook] & & K(Y)
\end{tikzcd}$$

If we can prove that $\Gamma(Y, \mathcal{O}_Y)$ is integrally closed in it's field of fractions, there is a unique morphism $\widetilde{A} \to \Gamma(Y, \mathcal{O}_Y)$ such that the diagram still commutes. This is because, consider an element x of $\widetilde{A}$ in $K(Y)$, then we know that it satisfies a monic equaition with coefficients in A, and A is in $\Gamma(Y, \mathcal{O}_Y)$, however x is also in $K(\Gamma(Y, \mathcal{O}_Y))$, thus it's also in $\Gamma(Y, \mathcal{O}_Y)$.

Now we have to show that our assertion that $\Gamma(Y, \mathcal{O}_Y)$ is normal, say $x \in K(\Gamma(Y, \mathcal{O}_Y))$ satisfies a monic equation with coefficients in $\Gamma(Y, \mathcal{O}_Y)$, it's also in B, thus x is also in B, notice that this holds for each affine open $SpecB$ in Y, thus choose an affine cover of Y by affine opens, thus x is an element of each of them, thus x is an global function, we are done.

**** For future reference, we have proved that for a normal integral scheme, the global sections ring, is also normal, i.e. integrally closed. Actually, we don't even have to pass to $K(\Gamma(Y, \mathcal{O})Y))$, we can directly show that $\Gamma(Y, \mathcal{O})Y)$ is integrally closed in $K(Y)$: suppose that x is in $K(Y)$ such that it satisfies a monic equation with coefficients in $\Gamma(Y, \mathcal{O})Y)$, then since for each affine open $SpecA$, we have an inclusion $\Gamma(Y, \mathcal{O})Y) \hookrightarrow A \hookrightarrow K(Y)$, we can assume that x is in A for all A and they of course agree on intersections, thus it's a global section.*

Now we can show that normalizations of integral schemes exists in general using our idea from the existence of fibered products. We will prove that if $v : \widetilde{X} \to X$ is the normalization, then for any open subscheme of X, say U, it's normalization also exists, i.e. the inverse image of U, I will let you check this. Now using this, suppose that you cover X by affine opens, and for $SpecA$ and $SpecB$, we know that the normalization of these two exists, and the normalization of the intersections also exists, which are given by the inverse image, and is joined via a unique isomorphism, thus we define the gluing isomorphisms to be this unique isomorphism, we only have to check that it satisfies the cocycle condition. But notice that on the triple intersection, the composition gives you an isomorphism between two normalizations of $SpecA \cap SpecB \cap SpecC$, then by the universal property, it's unique, thus agrees with the one we want to to equal to. We are done. The only thing left is to show that $v : \widetilde{X} \to X$ satisfies the universal property, i.e., say Y is an integral normal scheme with a dominant morphism $Y \to X$, it factors through $\widetilde{X}$ uniquely. Notice that if it factors, then argue locally tells you that the factorization is unique. Therefore we only nee to show there is such a factorization. Locally, this is easy, since we cover X by affine opens and can just use the universal property locally, the fact that they agree lcoally again comes from the universal property and intersections commutes with inverse image.

One easy property of the normalization of an integral scheme is that this normalization morphism is both integral and surjective. To show it's integral, since this is an affine local property, and $A \hookrightarrow \widetilde{A}$ is always integral, we are done. To show that it's surjective, we know that the morphism of affine schemes $Spec\widetilde{A} \to SpecA$ is surjective due to the lying over property, i.e. integral extension always gives you a surjective morphism of affine schemes.

Furthermore, we will see later that the normalization of an integral finite type k-scheme is always a birational morphism, i.e. an isomrophism on a dense open of the normalization.

Now we want to show that the for a reduced scheme, possibly with more than 1 irreducible components, the normalizations still exists. First, we need to formally define it, here we won't do this in all generality, we will only deal with the case where X is a scheme such that it can be covered with affine opens such that each of them only has finitely many irreducible components. We define the normalization of such a scheme X to be $v : \widetilde{X} \to X$ form a normal scheme which induces a bijection of irreducible components, in the sense that irreducible components of $\widetilde{X}$ is mapped to irreducible components of X in a bijective way which satisfies the universal property that any such scheme Y with a morphism to X and induces a bijection of components factors through $\widetilde{X}$ with a unique morphism.

We again first deal with the affine case, say $SpecA$ is a reduced affine scheme with finitely many minimal primes p_i, we want to find the normalization of $SpecA$. Notice that for each p_i, we have a closed embedding $SpecA/p_i \hookrightarrow SpecA$, and $SpecA/p_i$ is an integral scheme, thus we have the normalization X_i for this closed subscheme, we will define the normalization to be the disjoint union $\bigcup_i X_i$ with the obvious morphism. It clear that $\bigcup_i X_i$ is normal and since each $Spec\widetilde{A/p_i}$ maps to $SpecA/p_i$, it satisfies the bijective components map condition we imposed above. Finally, we have to check it satisfies the universal property, i.e. we want to find a unique morphism:

$$\begin{tikzcd}
Y \arrow[rr, dashed, "\exists!"] \arrow[dr] & & Spec\prod_i \widetilde{A/p_i} \arrow[dl] \\
& SpecA &
\end{tikzcd}$$

To do so, we first notice that Y is a normal scheme with finitely many irreducible components (since $SpecA$ has finitely many components), then I will let you prove form the fact that if a normal ring R has only finitely many minimal prime ideals p_i then $R \simeq \prod_i R/p_i$ via the obvious map that Y is a disjoint union of it's irreducible components, each both open and closed. Then to define a morphism from Y to $Spec\prod_i A/p_i$ we only have to define $Y_i \to SpecA_i$ since different Y_i's don't interplay with each other. Now notice that if Y is both closed and open, then the open embedding is also a closed embedding (a locally closed embedding whose image is closed is closed embedding).

Now, we know that if $f : X \to Y$ is any scheme morphism, and f is a global sections that vanishes on a closed subset Z, then the pullback will vanish in $f^{-1}Z$, then if you consider the morphism:

$$Y_i \hookrightarrow Y \to SpecA$$

we know that the image is in the irreducible component $SpecA/p_i$, thus the elements in p_i pulls back to a global function of Y_i such that it vanishes on all of Y_i, since Y_i is reduced, we can factor this morphism through $SpecA/p_i$ uniquely, then since $Spec\widetilde{A/p_i}$ is the normalization, you obtain a unique morphism to $Spec\widetilde{A/p_i}$. The following diagram summarizes the above discussion:

$$\begin{tikzcd}
Y_i \arrow[rr, dashed] \arrow[dr, dashed, "\exists!"] \arrow[dd] \arrow[ddr] & & Spec\prod_i \widetilde{A/p_i} \arrow[ddl] \\
& Spec\widetilde{A/p_i} \arrow[ur] \arrow[dl] \arrow[d] & \\
SpecA/p_i \arrow[r, hook] & SpecA &
\end{tikzcd}$$

At this point, you probably will ask what if Y_i can goes to $Spec\widetilde{A/p_j}$ for some $j \neq i$. The answer is that this never happens, thus you only have to consider how to map Y_i into $Spec\widetilde{A/p_i}$. To see why, notice that this morphism we just define correspond to a ring morphism $\prod_i \widetilde{A/p_i} \to \widetilde{A/p_i} \to \Gamma(Y_i, \mathcal{O}_Y)$, i.e., it's a ring morphism that mapps all the factors other than $\widetilde{A/p_i}$ to 0.

If there is another ring map such that the required morphism commutes, some 1 in other factors is not mapped to 0, thus there is some elements not in p_j, but in every other p_i, which is not mapped to 0, a contradiction. But the above discussion tells you that on $\widetilde{A/p_i}$, you only have a unique choice. Therefore we are done and the disjoint union of normalizations of $SpecA/p_i$ are indeed normalization of $SpecA$. According to our new definition.

Using this, we can show that in general such a normalizations exists. The first step is to show that this construction is compatible with passing to any open subscheme, which is essentially the same argument as the normalization of integral schemes once you notice that the inverse image is still normal and induces a bijection of irreducible components. Then we only have to cover X by $SpecA$ such that each of them has finitely many minimal primes and the gluing is again just the universal property.

In particular, our assumption covers all the locally Noetherain schemes. Now let's see a few examples:

Example 3.2. Recall the nodal cubic $Speck[x,y]/(y^2 - x^2(x+1))$ we talked about before. We claim that the normalization is $Speck[t] \to Speck[x,y]/(y^2 - x^2(x+1))$ which is defined by $x \mapsto t^2 - 1, y \mapsto t(t^2 - 1)$.

First, notice that $k[t]$ is normal, i.e. integrally closed. On the other hand, I claim that the fraction field of $k[x,y]/(y^2 - x^2(x+1))$ is exactly the fraction field of $k[t]$ via the following diagram:

$$\begin{tikzcd}
k[x,y]/(y^2 - x^2(x+1)) \arrow[r, hook] \arrow[d, hook] & k[t] \arrow[d, hook] \\
K(k[x,y]/(y^2 - x^2(x+1))) \arrow[r, hook] & k(t)
\end{tikzcd}$$

since you notice that t in the fraction field $k(t)$ is just $t(t^2-1)/(t^2-1)$ which is the image fo y/x.

Finally, if we can show that $k[t]$ is contained in the integral closure, we are done since it will just be the integral closure. To see this, we only have to show that t is in the integral closure, i.e. there is a monic equation in t with coefficients in $k[x,y]/(y^2 - x^2(x+1))$ such that it's 0. For example you can just take $t^2 - x - 1 = 0$, we are done.

Remark 3.3. We can also do similar stuff to show that the normalization of $k[x,y]/(y^2 - x^3)$, the cusp, is still $k[t]$, and the normalization is induced by $k[x,y]/(y^2 - x^3) \to k[t]$ where $x \mapsto t^2, y \mapsto t^3$. You still prove that they have the same fraction field via the above ring morphism. Then you only have to show that t is in the integral closure, which is again easy. Similar method also works for the tacnode defined by $y^2 = x^4$

Now we will do an important generalization of the idea of normalization, which is called **normalization in a function field extension**, this is the stuff we will use in this section. The set up is simple, suppose that X is integral, the normalization $v : \widetilde{X} \to X$ of X in a given algebraic field extension L of the function field $K(X)$ of X is a dominant morphism form a normal integral scheme $\widetilde{X}$ with function field L, such that v induces the inclusion $K(X) \hookrightarrow L$, that is universal with respect to the following property indicated in the following diagram:

$$\begin{tikzcd}
SpecL \arrow[r, "\simeq"] \arrow[dr, "\simeq"'] & SpecK(Y) \arrow[r] \arrow[d, "\simeq"] & Y = \text{normal} \arrow[d, dashed, "\exists!"] \arrow[dd, bend left] \\
& SpecK(\widetilde{X}) \arrow[r] \arrow[d] & \widetilde{X} = \text{normal} \arrow[d, "v"] \\
& SpecK(X) \arrow[r] & X
\end{tikzcd}$$

To be more precise the universal for any other integral normal scheme Y and a dominant morphism $f : Y \to X$ which induces a field extension $K(X) \hookrightarrow K(Y) \simeq L$ that makes the following diagram commutes:

$$\begin{tikzcd}
& K(\widetilde{X}) \arrow[dr, "\simeq"] \arrow[dd, "\simeq"] & \\
K(X) \arrow[ur, hook] \arrow[dr, hook] & & L \\
& K(Y) \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

there is a unique morphism from Y to $\widetilde{X}$ that is dominant (automatically dominant if exists) and induced the isomorphism $K(\widetilde{X}) \simeq K(Y)$ in the above diagram.

Again, due the universal property, if it exists, it's unique up to isomorphism, thus we only have to prove the existence. We still begin with the affine case. Suppose that A is an integral domain, with fraction field

$K(A)$, then consider the integral closure of A in L via the inclusion $A \hookrightarrow K(A) \hookrightarrow L$, which we will call B. Then, it gives you a morphism: $SpecB \to SpecA$ that induced the field extension $K(A) \hookrightarrow K(B) = L$. Now to prove it satisfies the universal property, it suffices to consider the following diagram:

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, hook] & B \arrow[r, hook] \arrow[dl, dashed, "\exists!"] & K(\widetilde{X}) \arrow[d, "\simeq"] \\
\Gamma(Y, \mathcal{O}_Y) \arrow[rr, hook] & & K(Y)
\end{tikzcd}$$

The isomorphism $K(\widetilde{X}) \simeq K(Y)$ is the isomorphism in the above diagram. I claim that this is commutative and out job is to show the existence and uniqueness of the dotted arrow such that the diagram still commutes. The reason the diagram commutes is because we can augment it via the following diagram:

$$\begin{tikzcd}
& K(X) \arrow[dr, hook] \arrow[ddr, hook] & \\
A \arrow[ur, hook] \arrow[r, hook] \arrow[d, hook] & B \arrow[r, hook] \arrow[dl, dashed, "\exists!"] & K(\widetilde{X}) \arrow[d, "\simeq"] \\
\Gamma(Y, \mathcal{O}_Y) \arrow[rr, hook] & & K(Y)
\end{tikzcd}$$

and the commutativity is clear here. To show the elements of B actually lies in $\Gamma(Y, \mathcal{O})Y)$, we notice that elements of B in $K(Y)$ satisfies monic equations with coefficients in A, thus also in $\Gamma(Y, \mathcal{O}_Y)$, we are done.

Then we need to show that they exist in general. But we can still choose an affine open and the gluing on the intersection just comes from the universal property, this gives you a normal scheme $\widetilde{X}$ and we need to check it satisfies the universal property, which is still similar to what we did before.

Remark 3.4. Notice that if we consider the identity extension, then you will notice that normalization in the identity extension is just the normalization we discussed in the beginning of this section.

Let's look at some examples of this idea:

Example 3.5. Consider the ring of integers $\mathbb{Z}$ with fraction ring $\mathbb{Q}$, we can consider the algebraic extension $\mathbb{Q}(i)$ and by our construction, to compute the normalization of $Spec\mathbb{Z}$ in the field extension $\mathbb{Q} \hookrightarrow \mathbb{Q}(i)$ is just to compute the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(i)$. This is a purely algebraic problem, but the answer should be understood in terms of geometry.

Another example is when k is a field of characteristic not 2 and we consider $X = Speck[x]$, we know that the fraction field is $k(x)$, and we can consider the field extension: $k(x) \hookrightarrow k(x, y)$ where $y^2 = x^2 + x$, thus an algebraic extension. First, notice that the fraction field of $k[x, y]/(x^2 + x - y^2)$ is $k(x, y)$ and we have a canonical morphism $k[x] \hookrightarrow k[x, y]/(x^2 + x - y^2)$ where $x \mapsto x$ which gives you the field extension $k(x) \hookrightarrow k(x, y)$ on the fraction field. I claim that $k[x, y]/(y^2 - x^2 - x)$ is normal and it's contained in the integral closure of $k[x]$, thus $k[x, y]/(y^2 - x^2 - x)$ is the integral closure of $k[x]$ in $k(x, y)$.

To see why it's normal, recall that this is of the form $y^2 - x^2 - x$, notice that this is irreducible and $x^2 + x$ has no repeated prime factors, thus the result is a normal ring we are done.

Now we can consider $\mathbb{P}^1_k$ and the field extension $k(x, y)$, we want to compute the normalization of $\mathbb{P}^1_k$ in $k(x, y)$, by the way we construct the normalization, we still first do is locally. On $Speck[x]$, it's $Speck[x, t]/(x^2 + x - t^2)$ which the stuff we do above and on $Speck[y]$, it's a little bit tricky here and the field extension is given by $k(x, y)$ where $y^2 = x^2 + x$, however, the inclusion is given by:

$$k(y) \to k(x, y) \quad y \mapsto \frac{1}{x}$$

I claim that over this field extension, the normalization is given by:

$$k[y] \hookrightarrow k[t, y]/(t^2 - 1 = y) \quad y \mapsto b$$

To see why, notice that the fraction field of $k[t, y]/(t^2 - 1 - y)$ is $k(t, y)$ where $t^2 = y + 1$, this is isomorphic to $k(x, y)$ where $t^2 = x^2 + x$ since we can define t is mapped to $\frac{t}{x}$ and y is mapped to $\frac{1}{x}$, this is well defined by the universal property of quotient, and it's obviously surjective. And this ring is normal again by the same reason as above, and it's contained in the integral closure of $k[y]$.

Over the intersection, notice that the two local normalization is glued together via $k[y, y^{-1}, t]/t^2 - y - 1 \to k[x, x^{-1}, t]/t^2 - x^2 - x$ where $t \mapsto \frac{t}{x}$ and $y \mapsto \frac{1}{x}$.

Below are some other interesting properties of normalization you might want to know:

Theorem 3.6. Suppose that A is a Noetherian integral domain and $K = K(A)$ with a finite field extension L/K and B is the integral closure of A in L, then:

1. If A is integrally closed and L/K is separable, then B is a finite A-algebra.
2. If A is a finitely generated k-algebra (not necessarily integrally closed) then B is a finite A-algebra.

Thus in particular, the integral closure of a finitely generated k-algebra A is a finite A-algebra.

One remark is that part 2 in the above theorem doesn't hold for Noetherian rings in general and there are even Noetherian rings whose integral closure is not Noetherian, this is a sign that the proof is not easy, we will refer the interested to the literature. Some interesting results of the above theorem are listed below:

Proposition 3.7. If X is an integral finite type k-scheme, then it normalization $v : \widetilde{X} \to X$ is a finite morphism.

Proof. Check locally and use the above theorem. □

Proposition 3.8. Suppose that X is a locally Noetherian integral scheme and X is normal, then if $L/K(X)$ is a finite separable field extension, then the normalization of X in $L/K(X)$ is a finite morphism.

Similarly, if X is an integral finite type k-scheme, then the normalization of X in a finite field extension of filed is finite.

Proof. Again check locally since we defined the nornalization locally to be the integral closure in the field extension and use the above theorem. □

In particular we know that the normalization of an irreducible variety in a finite extension is again integral and finite type over k and finite morphism is affine, thus separated, thus the normalization is still a variety.

Proposition 3.9. Suppose that X is an integral finite type k-scheme then the normalization morphism is birational. Moreover, is it's an isomorphism on a dense open of X, thus the normal points of X forms a dense open.

Proof. Suppose that $v : SpecB \to SpecA$ is the restriction of the normalization map where B is the integral closure of A, and the morphism is induced by $A \hookrightarrow B$. Now since B is a finite A-algebra, we can assume that B is generated by $\frac{f_i}{g_i}$ where $f_i, g_i \in A$, say $g = \prod_i g_i$, then consider the natural morphism $A_g \to B$, this is both injective and surjective, thus an isomorphism, then we know that $v : SpecB \to SpecA_g$ is an isomorphism, thus birational. □

Proposition 3.10. Suppose $\rho : Z \to X$ is a finite birational morphism from an integral finite type scheme over k to a normal integral finite type k-scheme, then ρ is an isomorphism.

Proof. Recall that finite implies integral, then suppose that locally this morphism is given by $SpecB \to SpecA$ and $A \to B$, since this is a birational morphism, then we know that $K(B) = K(A)$ and is induced by the above ring morphism $A \to B$. However, since B is finite, thus integral over A, and A is normal thus B is in A, we are done. □

Before we finish the discuss of normalization, another construction of a similar vein is worth revisting, i.e. the reduction of a scheme. We will also try to define it via a universal property like what we did for normalizations and show that it agrees with the definition we had before:

Definition 3.11. Suppose that X is a scheme, we say that $\rho : X^{red} \to X$ is the reduction of X where X^{red} is reduced if any morphism $W \to X$ from a reduced scheme W factors uniquely through ρ:

$$\begin{tikzcd} W \arrow[r, dashed, "\exists!"] \arrow[dr] & X^{red} \arrow[d] \\ & X \end{tikzcd}$$

Now similar to fibered product and normalization, if it exists, it's unique up to a unique isomorphism, thus we need to show it exists. Again, we do this for an affine scheme first. Say $SpecA$ is an affine scheme, with the nilradical ideal $\mathfrak{N}(A)$, then I claim $SpecA/\mathfrak{N}(A) \to SpecA$ satisfies the universal property. Notice that the reduced elements of A are mapped to 0 in W since W is reduced. Therefore $A \to \Gamma(W, \mathcal{O}_W)$ uniquely factors through $A/\mathfrak{N}(A)$, we are done. Again, suppose that X has a reduction, then any open subset has a reduction, then for the general existence, you can just consider gluing them via the universal property. And in particular you notice that $X^{red} \hookrightarrow X$ is a closed embedding.

We now show that our earlier construction satisfies the universal property. Recall that we previously defined X^{red} to be the closed embedding determined by the ideal sheaf of nilradicals. Therefore the construction satisfies the universal property locally, thus we only have to show that there is indeed a morphism from W to X^{red}, and it's necessarily unique. But the morphism can also be define locally using universal property and then glue.

Definition 3.12. By the universal property of reduction, say $\pi : X \to Y$ is a morphism of schemes, reduction induces a unique morphism $\pi^{red} : X^{red} \to Y^{red}$, called the reduction of π.

Theorem 3.13. If C is an integral finite type one dimensional k-scheme, then there is a regular projective irreducible k-variety C' of dimension 1 that is birational to C.

Proof. We can assume C is affine since we only care about birational model. Then by Noether Normalization lemma, we can choose some $x \in K(C)$ such that $K(C)/k(x)$ is a finite extension of fields.

Then we can consider the normalization of $\mathbb{P}^1$ in the field extension $K(C)/k(x)$, which is a finite morphism $C' \to \mathbb{P}^1_k$.

I claim that this is still a curve, and more precisely I claim it's integral of dimension 1 and finite type over k. To see why recall that if $SpecB \to SpecA$ correspond to an integral extension $A \hookrightarrow B$ then $dimA = dimB$. Via this, suppose that X is an integral scheme and $v : \widetilde{X} \to X$ is the normalization of X possibly in a finite field extension, then I claim $\dim\widetilde{X} = \dim X$. Recall that when you have an open cover of an scheme with the same dimension, then the dimension of the scheme is the same as the dimension of each of the open cover. Then consider the nomralization, which is just locally given by:

$$A \hookrightarrow B$$

where B is the integral closure of A possible in a finite field extension. Recall the theorem we just had, when A is a finite finitely generated over k, we know that B is a finite A-algebra, thus the dimension of B is the same as the dimension of A, we are done. Thus the dimension of C' is the same as the dimension of $\mathbb{P}^1$, which is 1.

We need to show that C' is regular. First, it's integral, thus regular at the generic point. Then besides the generic point, each point is closed, and it's regular at all closed points because C' is normal by construction thus regular at codimension 1 pts. C' is birational to C since C' is also finite type over k, integral, thus sharing the same function field means they are birational. (Actually we only need the target to be finite type over k)

Finally, since $C' \to \mathbb{P}^1_k$ is finite (thus proper) and $\mathbb{P}^1_k$ is projective, we know that C' is also proper over k, thus since there is a very ample line bundle on $\mathbb{P}^1$, there is a very ample, thus C' is projective, we are done. $\square$

Theorem 3.14. If $C = SpecA$ is irreducible, regular and one dimensional k-variety, then there is an open embedding $C \hookrightarrow C'$ into some projective k-variety which is regular and of pure dimension 1.

Proof. Let's first prove this when C is affine. I claim that we have the following diagram:

$$\begin{tikzcd} C \arrow[d, "\text{finite}"'] & C' \arrow[d, "\text{finite}"] \\ \mathbb{A}^1_k \arrow[r, hook] & \mathbb{P}^1_k \end{tikzcd}$$

where C' is the normalization of $\mathbb{P}^1$ in the finite field extension $K(C)/k(x)$ and $C \to \mathbb{A}^1$ is given by the Noether normalization.

Notice that we assume C is regular, thus it's normal, then notice that the morphism $C \to \mathbb{A}^1_k$ induces the field extension $K(C)/k(x)$ then notice that $k[x] \to A$ is finite, thus A is the integral closure of $k[x]$ in $K(C)$, thus $C \to \mathbb{A}^1_k$ is indeed the normalization of $\mathbb{P}^1_k$ in $K(C)$.

On the other hand, the inverse image of $\mathbb{A}^1_k$ under $C' \to \mathbb{P}^1_k$ is also the normalization of $\mathbb{A}^1_k$ in $K(C)$, this gives the desired open embedding into C'.

Now, let's deal with general C, let C_1 be any nonempty affine open subset of C, then by our discussion above, we have an integral regular projective curve $\widetilde{C_1}$ of dimension 1 together with an open embedding $C_1 \hookrightarrow \widetilde{C_1}$. Recall that C is an irreducible regular k-variety of dimension 1, then any closed subset of C, if not the entire C, is just a finite collection of closed points. Then we know that $C - C_1$ only consists of finitely many closed points, then by successively applying the curve to projective theorem, we conclude that $C_1 \hookrightarrow \widetilde{C_1}$ extends to a birational morphism $C \to \widetilde{C_1}$.

I claim that this morphism, as a morphism of sets, is injective. Otherwise, suppose that p_1, p_2 is both mapped to q, we have a diagram:

$$\begin{tikzcd} \mathcal{O}_{Y,q} \arrow[r, hook] \arrow[d, hook] & \mathcal{O}_{X,p_1} \arrow[d, hook] \\ \mathcal{O}_{X,p_2} \arrow[r, hook] & K(X) \end{tikzcd}$$

since C is regular and so is $\widetilde{C_1}$ and they are birational, we know that $\mathcal{O}_{Y,q}, \mathcal{O}_{X,p_1}$ and $\mathcal{O}_{X,p_2}$ are discrete valuation domains such that the fractions field is $K(X)$, from the valuative criterion of separatedness, we know that on C, distinct regular closed points gives you distinct valuations. Recall that valuation is defined to be the order of vanishing, then suppose that the morphism $\mathcal{O}_{Y,q} \to \mathbb{O}_{X,p_i}$ maps the uniformizer t of $\mathcal{O}_{Y,q}$ to $ut_1^m \in \mathcal{O}_{X,p_1}$ and vt_2^n in $\mathcal{O}_{X,p_2}$ where t_i's are the uniformizers of $\mathcal{O}_{X,p_i}$, but this tells you that the order of t is $1 = m = n$, thus $\mathcal{O}_{X,p_i}$ has the same valuation, thus p_1, p_2 are the same point. Then notice that the complement of the image of C is a finite collection of closed points, thus closed, thus the image of C is open, thus this birational morphism identifies C as an open subset of $\widetilde{C_1}$.

Now, we need to prove that this morphism is an open embedding of schemes, not just as topological open embedding. We check this locally, i.e. for each $p \in C$, we show that there is an open neighborhood of p such that the restriction of $C \to \widetilde{C_1}$ is an open embedding. For example, we can assume C_2 is an affine open of p, we can repeat the construction we did for C_1 and get the following diagram:

$$\begin{tikzcd} C_1 \arrow[dd, hook] \arrow[dr, hook] & & C_2 \arrow[dd, hook] \arrow[dl, hook'] \\ & C \arrow[dl] \arrow[dr] & \\ \widetilde{C_1} & & \widetilde{C_2} \end{tikzcd}$$

Now notice $\widetilde{C_1}$ is also an integral regular projective k-variety of dimension 1, thus the complement of C_1 in $\widetilde{C_1}$ is also just a finite collection of closed points, thus the morphism $C_1 \to \widetilde{C_2}$, by the curve to projective theorem, extends to a morphism $\widetilde{C_1} \to \widetilde{C_2}$. Similarly, you have a morphism $\widetilde{C_2} \to \widetilde{C_1}$ extending $C_2 \to \widetilde{C_1}$. I claim the composition is an isomorphism, this is because the morphism, when restricted to $C_1 \cap C_2$ is the identity morphism, and from the reduced to separated theorem ($\widetilde{C_i}$ are regular and projective, thus reduced and separated), the composition is the identity on entire $\widetilde{C_1}$ and $\widetilde{C_2}$, thus the two morphisms are isomorphisms.

Since $C_1 \hookrightarrow C$ and $C_2 \hookrightarrow C$ are monomorphisms, we know the triangle $C, \widetilde{C_1}, \widetilde{C_2}$ also commutes. However we see $C_2 \hookrightarrow C \to \widetilde{C_1}$ is an open embedding, we are done ☐

One immediate corollary is that all regular proper curves are projective:

Corollary 3.15. Suppose that C is an integral regular proper k-variety of dimension 1, then C is projective over k.

Proof. Via the above proposition, we have an open embedding $C \hookrightarrow C'$ of k-schemes where C' is an integral normal projective k-variety of dimension 1. This implies that this open embedding is a proper morphism, thus the image is closed, then recall that any locally closed embedding whose image is closed is a closed embedding, thus C' is a closed subscheme of a projective k-scheme, thus C is also projective. ☐

Theorem 3.16. The following categories are equivalent:

1. Integral regular projective dimension 1 k-varieties with surjective k-morphisms.
2. Integral regular projective dimension 1 k-varieties and dominant k-morphisms.
3. Integral regular projective dimension 1 k-varieties and dominant rational maps over k.
4. Integral one dimensional k-varieties and dominant rational maps over k.
5. The opposite category of finitely generated fields of transcendence degree 1 over k and k-homomorphisms.

Proof. It's not hard to go from 1 to 4 since surjective is dominant, dominant is dominant rational and each integral regular projective curve is a quasiprojective curve.

To be more precise, for example to show that the category in 1 and 2 are equivalent, we only have to define the functor that maps X to X and the morphism $X \to Y$ to the morphism $X \to Y$. To show this is an equivalence of category, we notice that the map defined on morphisms is clearly injective, now suppose that $X \to Y$ is a dominant k-morphism, we know that $C_1 \to C_2$ is proper, thus the image is closed, thus the morphism must be surjective, which proves the equivalence.

To go from 2 to 3, we define the functor similarly, and suppose that $C_1 \dashrightarrow C_2$ is a rational dominant map, suppose that it's defined on a dense open subset U of C_1, since we know that the closed subsets of C_1 are just finite collection of closed points, we can apply the curve to projective several times to get a well defined morphism $C_1 \to C_2$ that restricts to the rational dominant morphism we started with.

To get from 3 to 4, we again define the functor similarly, suppose that C is an integral dimension 1 k-variety, notice that we can choose an affine dense open and then pass to the normalization $C' \to C$, we know that the normalization morphism is a birational morphism, then let $C' \hookrightarrow \widetilde{C'}$ be the embedding of C' into a projective integral dimension 1 k-variety, this shows that each object in category 4 is isomorphic to the image of some objects in category 3. The fact that this functor is fully-faithful is easy.

This proved that the four categories are equivalent. To prove 4 is equivalent to 5, this is essentially a exercise we didn't do before. ☐

Proposition 3.17. Suppose that X is an integral k-scheme and Y is also integral and finite type over k. If we are given an extension of fields $\phi : K(Y) \hookrightarrow K(X)$, there is a dominant rational map of k-schemes $\phi : X \dashrightarrow Y$ inducing the field extension ϕ.

Proof. Since we only care about rational maps, we can assume that X, Y are affine with $X = SpecA$ and $Y = SpecB$. Notice that the field extension ϕ induces an inclusion $B \hookrightarrow K(A)$. Recall that B is finitely generated over k, say by $\frac{f_i}{g_i}$ where f_i, g_i are in A, then we have an inclusion induced by $\phi : B \to A_g$ where g is the product of all g_i's. This gives you the desired dominant rational map. ☐

Remark 3.18. Notice that given a dominant rational map over k we can also produce a field extension preserving k by just taking the extension induced by the morphism, you should check this is the inverse construction of the construction in the above proposition.

Proposition 3.19. Suppose that K is a finitely generated field extension of k, then there is an integral affine k-variety X with $K(X) = K$.

Proof. Suppose that K is generated by $t_1, \cdots, t_n$, then consider the ring morphism $k[x_1, \cdots, x_n] \to K$ which is defined by $x_i \mapsto t_i$, then you have $k[x_1, \cdots, x_n]/Ker \to K$ where the kernel is a prime ideal, thus the result is indeed an integral affine k-variety, and the fraction field is contained in K. However, since each t_i is in the image the fraction fields are the same. □

Using the above two propositions:

Proposition 3.20. There is an equivalence of category between the category of integral k-varieties with morphisms being dominant rational maps defined over k and the opposite category of finitely generated field extension of k where morphisms are inclusions that fixes k.

Proof. The functor is defined to be mapping X to it's function field and a dominant rational map is mapped to the field extension induced by it. It's essentially surjective and fully faithful due to the previous two exercises. □

Using this equivalence, we can finish the proof of the theorem:

Proof. Notice that for a integral one dimensional k-vareity, by Noether Normalization, we know the function field is a finitely generated extension of k with transendence degree 1. Conversely, if K is a finitely generated field extension of transcendence degree 1, the irreducible affine variety we constructed in the above propositions is of dimension 1 by our characterization using transcendence degree of the function field, we are done. □

The theorem we just proved has a lot of interesting consequences, for example, we will introduce the notion of **degree** of a projective morphism from a curve to a regular curve:

To give some intuition about what this degree will be, we discuss a little bit on complex geometry. Say you have a map out of a compact Riemann surface, it has a well defined degree, which is the cardinality of the fiber, counted with appropriate multiplicity. We will show below that the algebraic version of this fact is the degree. Suppose that $\pi : C \to C'$ is a surjective map of irreducible regular projective curve over k, we will show that π has a well defined degree.

First, let's show that π is finite. If you know the characterization of a finite morphism in terms of being projective and finite fiber, you are done with this since the preimage of the generic point is the generic point (closed points are mapped to closed points for finite type k-schemes) and the preimage of a closed point is closed, thus a finite collection of closed points. However, this is a theorem we will introduce later, thus we don't use it here. Instead, we consider the following argument: First, notice that we can consider the normalization C'' of C' in the function field of C, since both C' and C are finite type over k and they are of dimension 1, then the field extension is finite (hint: use Noether Normalization). Now since we know that the normalization of an integral finite type over k-scheme in a finite field extension is a finite morphism, the morphism $C'' \to C'$ is finite. However, if you consider the following diagram:

$$\begin{tikzcd} C \arrow[rr, dashed] \arrow[dr] & & C'' \arrow[dl] \\ & C' & \end{tikzcd}$$

where the dotted arrow is given by the universal property normalization in $K(C)$, I claim that it's an isomorphism. To see why, we only have to show that it's an isomorphism on a dense open, since C is reduced and C'' is separated. But if you recall how this morphism is defined locally on affine opens, you will see that is defined by the isomorphism $K(C'') \to K(C)$, then this is an isomorphism by our observation before. Then as the composition of an isomorphism and a finite morphism $C \to C''$ is finite.

We want it to be finite since we can apply the following proposition we will prove later:

Proposition 3.21. Suppose that $\pi : C \to C'$ is a finite morphism where C' is a regular Noetherian scheme of pure dimension 1 and C is a Noetherian scheme of pure dimension 1 with no embedded points, for example if it's reduce. Then $\pi_* \mathcal{O}_C$ is locally free of finite rank.

Before we prove it, let's make some comments and see why it's useful:

Remark 3.22. The hypothesis of no embedding points is the same as requiring each associated point of C maps to a generic point of some components of C', as you will see in the proof below. The hypothesis of C' being regular is also necessary, since if you consider the normalization of the nodal cubic curve, then the preimage of the nodal point is two closed point, while the preimage of other points is just 1 point. As you will see later, the most important hypothesis is the morphism being finite and flat.

Motivated by the above proposition:

Definition 3.23. Suppose that C' is irreducible, then the rank of $\pi_*\mathcal{O}_C$ is called the the degree of π.

Recall that we have defined the degree of a rational map from one integral curve over k to another to be the degree of the the function field extension (it's always a finite number). Now we prove that this is the same as the definition above:

Proposition 3.24. If in the statement of the above proposition, we assume further C, C' are integral over k, then π, considered as a rational map, has degree the same as the the rank of $\pi_*\mathcal{O}_C$.

Proof. Say the induced function field extension is of degree d, which means that the function field extension:

$$K(C') \hookrightarrow K(C)$$

is a finite extension of degree d. Then we know that $\pi_*\mathcal{O}_C$ is locally free of finite rank, then we only have to compute it's rank at the generic point. Say we compute the fiber of $\pi_*\mathcal{O}_C$ at the generic point η, first notice that the inverse image of η is the generic point since C, C' are irreducible, and also recall that integral morphisms are closed, so are finite morphisms, then we know that the stalk of $\pi_*\mathcal{O}_C$ at the generic point is the stalk of $\mathcal{O}_C$ at the generic point(can you see why since this is not generally true), thus it's just $K(C)$, thus the dimension over $K(C')$ is just the degree of the field extension. $\square$

Let's keep use the same hypothesis of C and C' as in the proposition we haven't proved and we assume further p is a closed point of C' with it's scheme-theoretic inverse image a zero dimensional scheme over k.

Proposition 3.25. Suppose that C' is finite type over k and n is the dimension of the k-vector space of global sections of the structure sheaf of $\pi^{-1}p$, then $n = (deg(\pi))(deg(p))$.

If we assume further C is regular and $\pi^{-1} = \{p_1, \cdots, p_m\}$, then suppose t is a uniformizer of $\mathcal{O}_{C',p}$, we have:

$$\deg\pi = \sum_i^m (\text{val}_{p_i}\pi^*t)\deg(k(p_i)/k(p))$$

Proof. Recall that the degree of a closed point p of a finite type over k scheme is defined to be the degree of the field extension $k(p)/k$.

To do this, we can assume that C' and C are affine since fiber is determined only by local conditions. Say the morphism is $SpecA \to SpecB$ and the fiber is just $A \otimes_B k(p)$ considered as a k-vector space via $k \hookrightarrow k(p)$. Then we know that the dimension is just the dimension of $k(p)$ over k time the dimension of $A \otimes_B k(p)$ over $k(p)$, thus we only have to prove that the degree of π is the same as $\dim_{k(p)} A \otimes_B k(p)$, notice that we know that when it's affine, we understand pushforward pretty well, and $\pi_*\mathcal{O}_C$ is just A considered as a B-module, then we know that the fiber at p is just $\otimes_B k(p)$, we are done.

Then if C is regular, then we can make sense of valuations on closed points of C. First, notice that we can write $A \otimes_B k(p)$ as $A_p \otimes_{B_p} k(p)$. Then I claim that $A_p \simeq \prod_i A_{p_i}$, to see why, notice that all the maximal ideals of A_p are $p_i A_p$. We know that these maximal ideals are coprime, thus Chinese remainder theorem tells you that $A_p \simeq A_p/p_1A_p \times \cdots \times A_p/p_mA_p$, then use the fact that tensor commutes with direct sums, we get $A_p \otimes_{B_p} k(p) \simeq \prod_i (A_p/p_iA_p \otimes_{B_p} k(p))$, then we know it's dimension as a $k(p)$-vector space is the degree of π, which is the sum of each of the component. Therefore we have:

$$\deg\pi = \sum_i \dim_{k(p)} A_p/p_iA_p \otimes_{B_p} k(p)$$

Notice that each $A_p/p_iA_p \otimes_{B_p} k(p)$ is isomorphic to $A_{p_i} \otimes_{B_p} k(p)$. To see this, we need to think of this a little bit differently, first, consider the following diagram of morphisms of rings:

$$\begin{tikzcd} B_p \arrow[rr] \arrow[dr] & & A_{p_i} \arrow[dl] \\ & A_p \arrow[r] & A_p/p_iA_p \end{tikzcd}$$

we can identify $A_{p_i} \otimes_{B_p} k(p)$ as mod A_{p_i} by the extension of pB_p under the morphism $B_p \to A_{p_i}$. On the other hand, you can think of $A_p/p_iA_p \otimes_{B_p} k(p)$ as moding out the extension of pB_p via the ring morphism $B_p \to A_p \to A_p/p_iA_p$, however, extension of ideals is the same in a commutative diagram, plus the fact that $A_{p_i} \to A_p$ is injective (since C is also normal, we can choose $SpecA$ to be the spec of a normal domain), this suffice to prove the result. Therefore:

$$\deg\pi = \sum_i \dim_{k(p)} A_{p_i} \otimes_{B_p} k(p)$$

notice that as what we said above $A_{p_i} \otimes_{B_p} k(p)$ is just the quotient of A_{p_i} by the extension of pB_p, say the uniformizer t is mapped to us^m where s is the uniformizer of A_{p_i}, we first notice that m is a nonnegative number since you can think uniformizer t is the image of some elements in the affine open. Then notice that this means that $A_{p_i} \otimes_{B_p} k(p)$ is just $A_{p_i}/p_i^m A_{p_i}$, please notice that this is not a $k(p_i)$-space, since $p_iA_{p_i}$ doesn't annihilate it, however, it's still a $k(p)$-space. To figure out the dimension as a $k(p)$-vector space notice that each $p_i^k A_{p_i}/p_i^{k+1} A_{p_i}$ is a $k(p)$-vector space and is a one dimensional $k(p_i)$-vector space by Nakayama's lemma (We are over a Noetherian ring so each ideal is finitely generated), then we have a filtration of $k(p)$-vector spaces:

$$0 = p_i^m/p_i^m \subset p_i^{m-1}/p_i^m \subset \cdot \subset p_i/p_i^m \subset A_p/p_i^m$$

whose successive quotient is a one dimension $k(p_i)$-space, this tells you that $\dim_{k(p)} A_{p_i} \otimes_{B_p} k(p) = m \cdot \deg(k(p_i)/k(p)) = \mathrm{val}\pi^*t) \cdot \deg(k(p_i)/k(p)$. Therefore:

$$\deg\pi = \sum_i \mathrm{val}(\pi^*t) \cdot \deg(k(p_i)/k(p)$$

□

Proposition 3.26. Suppose that C is a irreducible regular projective curve over k and s is a nonzero rational function on C, then the number of zeros of s equals to the number of poles counted with appropriate multiplicity.

Proof. First, consider the open subset R_s where s is regular, or in other words, the maximal domain of definition of s. Similarly we have a rational function s^{-1}, and we can consider the locus $R_{s^{-1}}$ where s^{-1} is regular. Since we work with a regular curve over k, at each closed point, either s or s^{-1} is regular, thus R_s and $R_{s^{-1}}$ cover C.

Notice that on R_s, we can define a morphims from R_s to $\mathbb{P}^1$ use the pair $[1, s]$. Similarly on $R_{s^{-1}}$, we can define a morphism to $\mathbb{P}^1$ using $[s^{-1}, 1]$. Notice that on the generic point of C, s, s^{-1} are both regular, thus there is a dense open where s, s^{-1} are both regular, notice that on the open where they are both defined, you can check The morphisms given by the two pairs agree there, thus this defines a well defined morphism s from C to $\mathbb{P}^1$.

If p is a closed point where s has a zero, then p is a point where s^{-1} has a pole and vice versa. Then we only have to show that the number zeros of s and s^{-1} is the same. Now consider the morphism $s : C \to \mathbb{P}^1$. We know this morphism is either dominant, thus finite, or constant. If it's constant, you will notice that on R_s, where s has no poles, s also has no zeros(you can prove it's invertible there). Similarly, you will find that s^{-1} has no zeros on $R_{s^{-1}}$, thus s has no poles as well, we are done.

On the other hand, if $s : C \to \mathbb{P}^1$ is dominant, it's finite, we have a well defined degree of s, which can be computed using the previous proposition using the inverse image. Notice that by construction, consider the k-valued point $[0 : 1]$ in $\mathbb{P}^1$, I claim it's inverse image is where s has a zero. For example, we only have to consider the morphism on R_s to figure out the zeros of s, then on R_s it's defined by $[1, s]$, thus we only that on the standard open which contains $[0:1]$, it's defined by $k[\frac{y}{x}] \to \mathcal{O}_C(R_s)$ where $\frac{y}{x} \mapsto s$, then you will notice that since s is the pullback of $\frac{y}{x}$, the locus where s is 0 is the inverse image where $\frac{y}{x}$ is 0, thus we are done. Similar arguments works for the poles, or in other words zeros of s^{-1} which lets you say that the poles of s are the closed points lying in the inverse image of $[1:0]$, then you can compute the degree of s using two ways:

$$\deg(s) = \sum_{\text{zeros of } s} \mathrm{val}_p(s)\deg(k(p):k) = \sum_{\text{poles of } s} -\mathrm{val}_p(s)\deg(k(p):k)$$

since we know that the uniformizer $\frac{y}{x}$ in the local ring at $[0:1]$ is mapped to s and the uniformizer $\frac{x}{y}$ is mapped to s^{-1} and $\mathrm{val}(s^{-1}) = -\mathrm{val}(s)$. □

Remark 3.27. In a later section, we will define the notion of degree of a line bundle on a smooth projective curve, the above proposition can be thought to be the first step to that and you will see later the degree of the map s is the same as the degree of a certain line bundle.

Example 3.28. The formula we gave above to compute the degree of a morphism s can be used to interpret some behaviors we have observed about fibers in a more general way. For example, we still assume C' is irreducible and d is the rank of the locally free sheaf $\pi_*\mathcal{O}_C$ where $\pi : C \to C'$ is a dominant morphism thus finite. Then consider the fiber of $\pi_*\mathcal{O}_C$, say the residue field of C' at p is K, then the fiber is d-dimensional, thus the points in the fiber counted with appropriate multiplicity is always d.

To see a concrete example, consider the morphism given by $\mathbb{Q}[x] \to \mathbb{Q}[y] \quad x \mapsto y^2$, the projection of the parabola to the x-axis. This is a finite morphism since notice that $\mathbb{Q}[y]$ is generated as a $\mathbb{Q}[x]$-module by $1, y$. Thus the degree of the morphism is well defined, should be a certain number.

We have already computed the fiber over $x = 1$, which is $Spec\mathbb{Q}[y]/(y^2 - 1)$, two points, $y = -1$ and $y = 1$, each with multiplicity 1, thus multiplicity 2 in total. If you do a detailed computation, you will find that in the local ring $x - 1$, thu uniformizer is mapped to $(y-1)(y+1)$, thus at $y - 1$ and $y + 1$ are both of multiplicity 1 since the field extension is $\mathbb{Q} \to \mathbb{Q}$.

Then if you consider the fiber at $x = 0$, you will find that the fiber is $\mathbb{Q}[y]/y^2$ and in the local ring, the uniformaizer of x is mapped to y^2, thus the valuation of y^2 is 2, thus the multiplicity in total is still 2.

Now if you think about the fiber of $x = -1$, the fiber is $\mathbb{Q}[y]/y^2 + 1$, then the uniformizer $x + 1$ is mapped to the uniformizer $y^2 + 1$, thus the valuation is 1, howevet, the field extension here is $\mathcal{Q} \to \mathbb{Q}[y]/y^2 + 1$, a degree 2 extension, thus the multiplicity is 2 still.

Finally, if you consider the fiber of the generic point, it's $\mathbb{Q}(y)$ with field extension $\mathbb{Q}(x) \to \mathbb{Q}(y)$ where $x \mapsto y^2$. Thus this is a degree 2 extension, since $1, y$ generates the field extension, thus the multiplicity is still 2.

Now let's prove the $\pi_*\mathcal{O}_C$ is indeed locally free before we end this section:

Proof. Still, we will try to use what we know about DVRs to prove this.

$\pi_*\mathcal{O}_C$ being locally free of finite rank is local on the target, thus we assume C' is affine, and since C' is regular and it's a curve, we know that it's normal, thus C' is a disjoint union of it's irreducible components, thus replacing C' by it's irreducible components, we can assume C' is integral.

Now since π is a finite morphism, then we know that $\pi^*\mathcal{O}_C$ is quasicoherent of finite type. Since C is integral and in particular reduces, we only have to prove the rank of $\pi_*\mathcal{O}_C$ is contant on C' to conclude it's locally free of finite rank. We will show this by showing that the rank at any closed point is the same as the rank at the generic point.

Say $C' = SpecA'$ for some normal integral domain A' and since π is finite, we assume $C = SpecA$ for some A. Notice that the rank of $\pi_*\mathcal{O}_C$ at p a closed point is the dimension of A/m as a A'/m vector space if we assume p correspond to the maximal ideal m. Here A/m is actually mA, consider the as an ideal of A' times an A'-module A. The isomorphism $A/m \simeq A_m/mA_m$ still comes from the fact that extension of ideals can be done in any order and . Similarly, the rank of $\pi_*\mathcal{O}_C$ at the generic point is just $\dim_{K(A')}(A'^{\times})^{-1}A$.

Notice that these two operations factor through localizing at m. In the sense that A/m viewed as an A'/m vector space has the same dimension as the A'_m/mA_m vector space A_m/mA'_mA_m. And localizing at $(A'^{\times})$ is a further localization at the maximal ideal m, thus we only have to prove that A'_m is a finite rank free A'_m-module of rank d, then the dimension will be the same since no matter it's a further localization or a quotient, they commute with direct sums.

The key here is to notice A'_m is a DVR, thus we have a uniformizer t in A'_m, then by replacing A' by a suitable localization and repeat the above discussion we can assume t in A'. Then since we know that A is a finitely generated A'-module, then A_m is a finitely generated A'_m-module, since A'_m is a PID, we know that A_m is a finite direct sum of A'_m and A'_m/I for some ideal $I = (t^n)$ of A'_m, thus we only have to show A'_m doesn't appear in the direct sum. If there is such a summand, then under the ring morphism $A'_m \to A_m$, t is a zero divisor in A_m thus a zero divisor in A, then consider $A' \to A$, we know that t is in an associated point of A, and notice that the inverse image of that associated point is just m. To see this, consider the following diagram of rings:

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[d] \\ A'_m \arrow[r] & A_m \end{tikzcd}$$

But this tells you that there is an associated point in $\pi^{-1}p$. On the other hand, we know that $\pi^{-1}p$ are just a finite collection of closed point, and we assumed there is no embedded points, a contradiction. Thus you can also assume each associated point of C maps to a generic point, thus there is no associated point in $\pi^{-1}p$, a contradiction again, we are also good. $\square$

References