Appendix 1 for Math 632: 2025 Winter Notes

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1 Easy Properties Quasicoherent $\mathcal{O}_X$-modules

The first work we want to do is to look at another definition of a quasicoherent $\mathcal{O}_X$-module over an affine scheme. This will be done in much the same fashion when we defined the structure sheaf for an affine scheme.

Definition 1.1. First suppose that M is an A-module. We define a sheaf on a distinguished based on $SpecA$ by the following construction. We define: $\widetilde{M}(D(f))$ to be the localization of M at the multiplicative set of all functions that do not vanish outside of $V(f)$, denoted by S_f. Notice that when $D(g) \subset D(f)$, there is natural $\widetilde{M}(D(f)) \to \widetilde{M}(D(g))$ that will be used as the restriction(since $S_f \subset S_g$). This defines a sheaf on the distinguished basis of $SpecA$, thus giving a sheaf on $SpecA$, denoted as $\widetilde{M}$. Thus we can interprete $M_f \cong S_f^{-1}M$ as isomorphism as $S^{-1}R$-modules or as R_f-modules.

Now let's justify our construction in the definition.

Remark 1.2. First, let's take a look at why we define $\widetilde{M}(D(f))$ in terms of a wired localization, not in terms of M_f directly, which seems much more natural.

Again, this is for the same reason why we did it when constructing the sheaf $SpecA$. Recall even if $f \neq g \in A$, it can happen that $D(f) = D(g)$ and since the equivalent condition for this is $D(f) \subset D(g)$ and $D(g) \subset D(f)$, which is again equivalent to $f^n = r_1 g$ and $g^m = r_2 f$ for m, n integers and $r_1, r_2 \in R$. Notice that if we define $\widetilde{M}(D(f))$ as M_f, it obviously depends on f, though we do have a way to identify M_g and M_f when $D(f) = D(g)$ by a canonical isomorphism, we can do it much elegantly by constructing a way that is independent of f, g in the first place, this is our definition. And as the following proposition shows, we have a natural way to identity our definition here with M_f, which matches you intuition.

Proof. First, notice that f doesn't vanish outside $V(f)$ since the complement of $V(f)$ is $D(f)$ and by definition, we have $f \notin p$ if $p \in D(f)$. Therefore we have a canonical map from $M_f \to S^{-1}M$ where S is the multiplicative set consisting of all elements in R such that they don't vanish outside $V(f)$. On the other hand, suppose that g is an element that doesn't vanish outside of $V(f)$, then $D(f) \subset D(g)$, thus we know that f is a nilpotent element of R/g, thus we know $f^n = rg$, thus we see that g is invertible in R_f, thus there is a unique map of rings from $S^{-1}R$ to R_f, thus M_f is a $S^{-1}R$-module, thus by the universal property of localization of modules, there is a unique morphism from $S^{-1}M$ to M_f. Easy to check that these above two maps are inverses of each other. $\square$

Remark 1.3. Similar to the case of the structure sheaf on $SpecR$, we still have to verify the definition above is indeed a sheaf of a basis. Again, we first do this for the entire space $SpecR$, then do modification on $D(f)$. We are trying to show the following sequence is indeed exact:

$$0 \longrightarrow M \longrightarrow \prod_i M_{f_i} \longrightarrow \prod_{ij} M_{f_i f_j}$$

Of course, since we didn't define $\widetilde{M}(D(f))$ to be M_f, here you need to check that the corresponding sequence in terms of the S^{-1} is exact iff the above exact, but since everything is canonically defined, everything here can be checked easily. There is a classical argument on this kind of stuff, we omit it here.

Then we use the equivalence of category between sheaf on a basis and sheaf to get a unique up to isomorphism sheaf. However, besides this way, recall the definition we gave in Note 1, we will comment on that definition and show that it's isomorphic by a unique isomorphism to the above construction.

Well, the good point of the definition we gave is that it's as direct as possible, we don't have to do any kind of local to global procedure to go from a sheaf of basis to a global sheaf, and you can see from the local nature of the definition that this is directly a sheaf, as you will see soon when we go to details below. However, the detriment of this construction is the following: when we are working on a sheaf, we still want to have a grasp on the distinguished affine and we would like to have a simple description, this definition doesn't posses this property at first glance, though we will show later that since this is isomorphic to the above construction by a unique isomorphism, locally on a distinguished open $D(f)$, the obscure definition goes back to the familiar M_f.

Proposition 1.4. The definition we gave in part (1) of the note, actually is a sheaf.

Proof. The fact that it satisfies the identity axiom is easy to check. The reason why is satisfies the gluability axiom is exactly the same as why the constant presheaf is not a sheaf in general but if we require locally constant, it's a sheaf. We won't dive into this too deep, and it's left for you to figure out why this holds. ☐

The most important fact is that this is isomorphic to our construction in the beginning of this section:

Proposition 1.5. The two definition is isomorphic up to a unique isomorphism given by the isomorphism of sheaves on a basis.

Proof. We need a isomorphism on the basis. If we denote F and G by the sheaves on a basis restricted from the first and second definition. On each $D(f)$, the map goes from M_f to $G(D(f))$ is easy, just map $\frac{u}{f^n}$ to the map that maps each p to $\frac{u}{f^n}$. The hard part is to show this definition assembles to an isomorphism of basis. It's easy to see that it commutes with restrictions. Thus we only have to prove that this is indeed an isomorphism on each $D(f)$. We do this by defining it's inverse.

The basic idea is to figure out what element in M_f we expect to get out of s. We know that it maps each p to an element in M_p, in a locally compatible way. It's left for you to see why we can assume that there is an open cover of $D(f)$ again by distinguished opens $D(g_i)$ such that s on each of these smaller opens maps every p to the same element in M_{g_i}. Then the question again reduces to show that:

$$0 \longrightarrow M \longrightarrow \prod_i M_{f_i} \longrightarrow \prod_{ij} M_{f_i f_j}$$

is exact, which we already showed.

In particular, we see that the global section of $\widetilde{M}$ is isomorphic to M in a canonical way. ☐

It's left as an exercise to show that, using the above proposition, that the stalk $\widetilde{M}_p$ is isomorphic to M_p in a way that is compatible with induced map on stalks. You can try to show this use both definitions we provided above:

Proposition 1.6. Suppose that p is a prime of A, there is a canonical isomorphism $\widetilde{M}_p \to M_p$, in particular the stalk of $\mathcal{O}_{SpecA}$ at p is A_p.

Proof. First, you need to notice that by definition, $\widetilde{M}_p$ is a colimit, over all the opens around p, can you simplify it to a colimit of the distinguished opens around p, by giving a natural and canonical isomorphism?

Then how are you going to interpret this new colimit, can you give it a R_p-module structure, by using a explicit description of it? Then how are you going to define the morphism from the colimit to M_p, and after you find this morphism, it's fairly easy to show it's indeed an isomorphism. ☐

The following exercise shows that an important result can be understand from algebraic point of view or geometric point of view differently:

Proposition 1.7. Suppose that M is an A-module, show that the natural map:

$$M \to \prod_p M_p$$

where p ranges among all primes of A, is injective. Do it separately by showing the kernel is zero agebraically and geometrically using stalks.

Proof. *** later ∎

Remark 1.8. This is an important remark. The above exercise shows that M can be viewed as a submodule of $M_{f_1} \times \cdots \times M_{f_n}$. Even though each of the map $M \to M_{f_i}$ is not inclusion, the map to the product is an inclusion. This will be useful later when we try to use Prop 1.11.

An important result below shows that $\widetilde{M}$ is like affine schemes, morphisms between them is determined by maps of modules:

Proposition 1.9. There is a bijection between maps of A-modules $M \to N$ and maps of $\mathcal{O}_{SpecA}$-modules $\widetilde{M} \to \widetilde{N}$. This means that the category of A-modules is a full subcategory of the category of $\mathcal{O}_{SpecA}$-modules.

Proof. This is another way to say the universal property of localization of modules. The key fact we use is that, a morphism of sheaves on a basis gives a morphism of sheaves and if $M \to N$ is a map of A-modules, there is a unique map from $M_f \to N_f$ as maps of A_f-modules such that they commute with restrictions. We leave the details omitted. ∎

Remark 1.10. In particular, the above proposition shows that we have an **equivalence of categories** between A-modules and quasicoherent $\mathscr{O}_{SpecA}$-modules since the functor we described above is fully-faithful and by definition any quasicoherent $\mathscr{O}_{SpecA}$-module is isomorphic to $\widetilde{M}$ for some A-module M.

The reason we care this kind of construction is: instead of working with general $\mathcal{O}_X$-modules, we would like to work with quasicoherent modules, which is again an abelian subcategory that behaves better. You will see more reasons as we proceed.

Before this, now is a good place to introduce the **affine communication lemma**.

Proposition 1.11. Let X be a scheme and P be a property enjoyed by some affine open cover $SpecA_i$ of a scheme of X. Suppose that each affine open $SpecA$ in X (not just the affine opens in the cover) satisfies:

1. If $SpecA \hookrightarrow X$ satisfies P, then for any $f \in A$, $SpecA_f \hookrightarrow X$ does too.
2. If $(f_1, \cdots, f_n) = A$ and $SpecA_{f_i} \hookrightarrow X$ has P, then $SpecA$ has P too.

Then if each $SpecA_i \hookrightarrow X$ has P, then each affine open cover of X will have P. In other words, if in any affine, you can switch back and forth from local to global or the reversed direction and you have an open cover such that each one in this cover has this property, every open cover, including those that might not be in the cover will have P.

Remark 1.12. Such a property in the above proposition is called affine local: A property is **affine local** if we can check it on an open cover in the sense that if an affine cover has this property then every affine has this property. We also have **stalk local** property: A scheme has property P iff all it's stalk has such property, we will see an example of normality later when we do the review or discuss divisors.

The proof of the above theorem is possible by the following lemma, which is also sometimes called **affine communication lemma**, in the covering sense:

Proposition 1.13. Suppose that $SpecA$ and $SpecB$ are affine open subschemes of X, then $SpecA \cap SpecB$ is the union of affine opens that are distinguished simultaneously in both $SpecA$ and $SpecB$.

Proof. Given a point $p \in SpecA \cap SpecB$. Let $SpecA_f$ be a distinguished open in $SpecA$ containing p and is contained in $SpecA \cap SpecB$. Then let $SpecB_g$ be a distinguished open in $SpecB$ contained in $SpecA_f$ which also contains p. Notice that $g \in \Gamma(SpecB, \mathcal{O}_X)$ restricts to an element in $g' \in \Gamma(SpecA_f, \mathcal{O}_X)$. Notice that since the stalk of g and g' will have the same stalk on $SpecA_f$, then we know that in vanishing set of g' in $SpecA_f$ is the same as the vanishing set of g on $SpecA_f$, thus since $SpecB_g \subset SpecA_f$ we have:

$$SpecB_g = SpecA_f - \{[p] : g' \in p\} = Spec((A_f)_{g'})$$

where if $g' = g''/f^n$, then $(A_f)_{g'} \cong A_{fg''}$, we are done.

In conclusion, the key idea here is that a further localization of A_f is still a localization of A. ∎

Now, let's prove Prop 1.11,

Proof. Let $SpecA$ be an affine subscheme of X. If we cover $SpecA$ with a finite distinguished open in both $SpecA$ and some $SpecA_i$, we are done. This is possible by the quasicompactness of $SpecA$ and Prop 1.13. $\square$

Now we introduce the main character of our show in this section:

Definition 1.14. Let X be a scheme, then an $\mathcal{O}_X$-module $\mathcal{F}$ is called a quasicoherent sheaf if for each affine open subscheme $SpecA \subset X$, we have $\mathcal{F}|_{SpecA} \cong \widetilde{M}$ for some A-module M. This is a category $Qcoh_X$, if we take the morphisms to be morphisms of $\mathcal{O}_X$-modules.

As an immediate result about this above definition and as an application of the affine communication lemma, we can check that we only have to check $\mathcal{F}|_{SpecA} \cong \widetilde{M}$ on an affine cover, not every affine in X.

Theorem 1.15. Let X be any scheme, and $\mathscr{F}$ is an $\mathscr{O}_X$-module. Suppose that P is the property of affine open subschemes $SpecA \subset X$ such that $\mathscr{F}|_{SpecA} \cong \widetilde{M}$ for some A-module M. Then P satisfies the condition of affine communication lemma.

Proof. In particular, the theorem says that to check that $\mathscr{F}$ is quasicoherent, we only have to check it on an arbitrary affine cover.

As usual, the first part of the condition in the affine communication lemma is easier to check. Suppose that $\mathscr{F}|_{SpecA} \cong \widetilde{M}$, we claim that for any distinguished open $SpecA_f$ in $SpecA$, we have $\mathscr{F}|_{SpecA_f} \cong \widetilde{M_f}$. We can assume we are working on an affine in the beginning and we only have to prove that $\widetilde{M}$ restricted to a distinguihsed open satisfies the property above. As usual, we only have to check that they agree on a basis with restrictions. But this is just the following diagram, suppose that $D(g) \subset D(g) \subset D(f)$:

$$\begin{tikzcd}
M_f|_{\frac{g}{1}} \arrow[r] \arrow[d] & M_g \arrow[d] \\
M_f|_{\frac{h}{1}} \arrow[r] & M_h
\end{tikzcd}$$

You still need to figure out what are the maps in the diagram, but it should be easy.

The converse is a little bit annoying, let's first figure out what are the assumptions. Suppose that we have a $SpecA$-module $\mathscr{F}$ and we have $(f_1, \cdots, f_r)$ generating A together with M_i, which are A_{f_i}-modules such that $\mathscr{F}|_{D(f_i)} \cong \widetilde{M_i}$. We want to show that there is an A-module, M such that $\mathscr{F} \cong \widetilde{M}$. Well, how can we find this M, notice that if we really have $\mathscr{F} \cong \widetilde{M}$ on $SpecA$, we already saw that we will have an exact sequence:

$$0 \longrightarrow M \longrightarrow \prod_i M_{f_i} \longrightarrow \prod_{ij} M_{f_i f_j}$$

where the the second map is defined by the intension that we want to show that if elements in M_{f_i} agrees on intersections, then it comes form the global section. Can we find a similar map in our case? The answer is yes, to see it, consider that we already have M_i's as A_{f_i}-modules, they will be the counterparts of M_{f_i}. What about $M_{f_i f_j}$'s that represents the sections on the intersection, well, we have following isomorphism diagram:

$$\begin{tikzcd}
& \mathscr{F}_{D(f_i)\cap D(f_j)} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\widetilde{M_i}|_{D(f_i)\cap D(f_j)} \arrow[rr, leftrightarrow, dashed] & & \widetilde{M_j}|_{D(f_i)\cap D(f_j)}
\end{tikzcd}$$

since the isomorphism is induced by a exsisting $\mathscr{F}$, then it must satisfy the cocycle condition, we will call them Φ_{ij}. These will be the M_{ij}'s. Therefore, we consider:

$$0 \longrightarrow M \longrightarrow \prod_i M_i \longrightarrow \prod_{i<j} M_{ij}$$

where we define M to be the kernel of the map and we identify M_{ij} and M_{ij} via the isomorphism Φ_{ij}.

Now we want to show $\mathscr{F} \cong \widetilde{M}$. How to do it, we still have to do it locally. Notice that we can view $\mathscr{F}$ as glued together by $\widetilde{M_i}$'s via Φ_{ij}, then by the uniqueness of glued sheaves, we only have to show that $\widetilde{M}|_{D(f_i)}$ is isomorphic to $\widetilde{M_i}$ such that the following diagram commutes:

$$\begin{tikzcd} & \widetilde{M}|_{D(f_i)\cap D(f_j)} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \widetilde{M_i}|_{D(f_i)\cap D(f_j)} \arrow[rr, dashed, leftrightarrow, "\Phi_{ij}"] & & \widetilde{M_j}|_{D(f_i)\cap D(f_j)} \end{tikzcd}$$

since by this diagram, we know that $\widetilde{M}$ is also glued together by $\widetilde{M_i}$ via Φ_{ij}, thus there will be a canonical isomorphism between $\mathscr{F}$ and $\widetilde{M}$.

Notice that we only have to define A_{f_i}-module isomorphisms $M_{f_i} \to M_i$ such that the triangles commute(the tilting maps are after the localization):

$$\begin{tikzcd} & M_{f_i f_j} \arrow[dl] \arrow[dr] & \\ (M_i)_{f_j} \arrow[rr, "\Phi_{ij}"] & & (M_j)_{f_i} \end{tikzcd}$$

How are we going to define the isomorphism $M_{f_i} \to M_i$, consider $i = 1$, and since localization is exact, we have an exact sequence:

$$0 \to M_{f_1} \to (M_1)_{f_1} \times \cdots (M_n)_{f_1} \to (M_{12})_{f_1} \times (M_{13})_{f_1} \times \cdots (M_{(n-1)n})_{f_1}$$

Therefore, if we can establish another exact sequence:

$$0 \to M_1 \to (M_1)_{f_1} \times \cdots (M_n)_{f_1} \to (M_{12})_{f_1} \times (M_{13})_{f_1} \times \cdots (M_{(n-1)n})_{f_1}$$

where we take the second map to be the same as in the preceding sequence, we have a canonical isomorphism of kernels: But how are we going to define the first map? Notice that we can rewrite the second map as:

$$(M_1)_{f_1} \times \cdots (M_n)_{f_1} \to (M_1)_{f_2 f_3} \times (M_1)_{f_1 f_2} \times \cdots (M_1)_{f_{n-1} f_n}$$

Then we can define the first map such that it is an exact sequence, thus we have the desired isomorphism, the only question left is why the diagram we mentioned before commute?

The hint if you look at how we defined Φ_{ij}, it's the global section of the isomorphism connection $\widetilde{M_i}$ and $\widetilde{M_j}$ restricting to $D(f_i) \cap D(f_j)$ and the isomorphism defining it satisfies the cocycle condition. Also, the map of modules determined by Φ_{ij} on smaller distinguished opens are localizations of the global map. $\square$

Now we introduce some notions that are probably useful when we are dealing with integral domains:

Recall that if M is an A-module, it's called torsion free if $am = 0$ implies that a is a zerodivisor or $a = 0$. When A is an integral domain, which will be most cases where we use this notion, the definition of torsion free is equivalent of saying that $am = 0$ iff $a = 0$ or $m = 0$. When A is an integral domain, the torsion elements of M is a submodule called the trosion subsmodule, and we will call M torsion if $M = M_{tors}$ this is equivalent of saying $m \otimes_A K(A) = 0$ since this is equivalent of localizaing M at all nonzero elements, and you can see why.

Definition 1.16. If X is a scheme, and an $\mathscr{O}_X$-module $\mathscr{F}$ is called torsion-free if $\mathscr{F}_p$ is a torsion-free $\mathscr{O}_{X,p}$-module for each $p \in X$.

Definition 1.17. Suppose that X now is a reduced scheme we way that a quasicoherent sheaf on X is **torsion** if it's stalk at each generic point of each component is 0.

Now we proceed to a new but not really completely new topic, we will define what is called the **distinguished base of the Zariski topology** on a scheme. It's not the base of a topology in the usual sense, and it will be a **refinement** of the notion of a sheaf on a base.

Definition 1.18. The open sets of the **distinguished affine base** are all the affine open subsets of X, we have already see this is a base of X. However, given any two affine open $SpecA \subset SpecB$, we don't really understand what is this inclusion. Therefore, we only want to remember the open inclusions $SpecA_f \hookrightarrow SpecA$, which are the **distinguished inclusions**.

Therefore, we define the distinguished affine base of a scheme is the data of the affine open sets and the **distinguished inclusions**.

We can define sheaf on a distinguished open base in the obvious way where we associate each open in this distinguished affine base a section and restriction maps. This should satisfy the identity and gluability axiom in the sense that if you have an affine open is covered by it's distinguished opens, then you can figure the identity and gluability axiom in the obvious sense.

Given a sheaf on X, we get a sheaf on the distinguished base, and we are going to show that all the information of a sheaf is contained in the information of the sheaf on a distinguished base. The first step is to recover the stalk:

This is where the weak affine coomunication lemma come into play since if we have U and V are two affine opens containing p in the distinguished bases, we will use the fact that there is a W containing p and $W \subset U \cap V$ such that $W \to U$ and $W \to V$ are still distinguished inclusions.

Now if $\mathscr{F}$ is a sheaf on X, we know that the stalk of $\mathscr{F}$ at p is: $colim(f \in \mathscr{F}(U))$ where $p \in U$, but we claim that we have a canonical isomorphism from this colimit to the colimit of $colim(f \in \mathscr{F}(U))$ where we only consider the opens and inclusions in the distinguished affine base. (Note that the colimit is guranteed to exist if the distinguished opens are filtered!) The map is of course going to be defined by the universal property of colimits, the affine communication lemma is used to show that this definition is independent of the choice affine opens.

Below is the main theorem of the distinghuished affine base:

Theorem 1.19.
1. A sheaf on the distinguished affine base $\mathscr{F}^b$ determines a unique (up to a unique isomorphism) sheaf $\mathscr{F}$ whose restriction to the affine base is $\mathscr{F}^b$.
2. A morphism of sheaves on a distinguished affine base uniquely determines a morphism of sheaves.
3. An $\mathscr{O}_X$-module on the distinguished affine base yields an $\mathscr{O}_X$-module on X.

Remark 1.20. After we prove the theorem, you will find that the precise meaning of part 1 and 2 of the previous theorem will describe an equivalence of categories between sheaves on X (under the Zariski topology) and category of sheaves on distinguished affine bases on X.

The proof is also a generalization of the notion of sheaf on a base, the proof are really similar but we will still write out the proof here since we didn't really include the proof of the sheaf on a base and the proof in our case really shows the essence of why the sheaf on a bases are similar to our proof here and why it can work in the first place. The true reason behind it is, we only need the open subsets and inclusions that are available to us form a **filtered set**. This guarantees we can make sense of stalks (They are guaranteed to exist in this case!) then we can use various stalk local properties. You should try to convince you that the general stalk local properties of sheaves also holds for various kinds sheaves on some kind of basis as long as the stalk makes sense.

Here is the proof of Thm 1.19:

Proof. We still use the notion of **compatible germs**.

Suppose that $\mathscr{F}^b$ is a sheaf on the distinguished affine base. We can define stalks as the we mentioned before. Then for any open subset U of X, we define a sheaf of compatible germs:

$$\mathscr{F}(u) = \{(f_p \in \mathscr{F}^b_p)_{p \in U} : \text{for all } p \in U, \text{ there exists } p \in U_p \subset U \text{ and } s \in \mathscr{F}^b(U_p) \text{ such that } s_q = f_q \text{ for all } q \in U_p\}$$

You can also conceptualize the above elements in the product: $(f_p \in \mathscr{F}^b_p)_{p \in U}$ as a function s that maps each $p \in U$ to $\mathscr{F}^b_p$ such that it satisfies the **locally constant condition** in the sense that around a smaller neighborhood it comes from the image of a section. Let's first show that this is indeed a sheaf. What is the restriction, it's just the projection maps, if $V \subset U$, the projection maps just forgets the points in V that are not in U(if you interpret it by the locally constant function point of view, this is just the restriction of functions). We first need to show that this is a presheaf, which is just to show that after the projection, the result is still a locally constant function or is still a compatible germ depending on how you interpret it. Suppose that $V \subset U$, and we restricted s, the function interpretation, to V. Is it still locally constant, suppose that $p \in V$, now we know that before we restrict to V, there will be a U_p around p, such that s is constant on U_p. But notice that $U_p \cap V$ is still open, and we can definitely find a distinguished open inside $U_p \cap V$.

The identity and gluability axiom is trivial to check. This proves that this is a sheaf. We want to show that if you restrict $\mathscr{F}$ to the distinguished affine base, you recover $\mathscr{F}^b$, at least isomorphic. Notice that there is a natural morphism from $\mathscr{F}^b$ to the restriction of $\mathscr{F}$ to the distinguished affine base. We want to prove this is an isomorphism. How are you going to proceed, notice that all the stalk local properties we proved for topological spaces have counterparts when we are doing this sheaf on some kind of basis work. Then we only have to prove that we have isomorphism on stalks. Then you can see that the stalk of the restriction of $\mathscr{F}$ to the distinguished affine base is naturally isomorphic to the stalk of $\mathscr{F}^b$ in an obvious way. (You should try to figure out why this is obvious since we have the functions are locally constant!)

What about the uniqueness. We need to first make this uniqueness precise, this is unique in the sense that if there is another sheaf $\mathscr{G}$ on X such that the restriction to the distinguished affine base is isomorphic to $\mathscr{F}^b$, then there is a unique isomorphism such that the following diagram commute:

$$\begin{tikzcd}
\mathscr{F}^b \arrow[r] \arrow[d, leftrightarrow] & \mathscr{F} \arrow[d, dashed] \\
\mathscr{G}^b \arrow[r] & \mathscr{G}
\end{tikzcd}$$

We can prove this using a universal property argument, but you can also directly see this. First, notice that there is only one possible morphism of sheaves such that this diagram commutes. How are you going to define it, notice that since the function is locally constant, then we can cover U by affines on which s is constant, then on each of the affine U_i, the function comes from an element in $\mathscr{F}^b(U_i)$, then using the commutativity, we know that if there is a morphism from $\mathscr{F}$ to $\mathscr{G}$ such that the diagram commutes, then images of elements on $\mathscr{F}(U_i)$ are determined. This also determines a unique map on U by the gluability.

Then we nee to show this is an isomorphism. Again, it's determined by the maps on stalks (again you see that being filtered in the sense that we can make use of stalks), therefore using our previous comment on you can compute stalks on this distinguished affine open base, in a way that is compatible with the diagram above, thus we are done. This proves part 1, thus we see that we start with a sheaf on a distinguished affine base, you get an associated sheaf, you restrict you get an isomorphic sheaf. The converse is also true, you start with a sheaf, restrict to the distinguished open base, take the associated sheaf, you get an isomorphic sheaf.

Then consider part 2, let's first try to make this statement precise. There are two ways we can try to interpret this, the first is if you have two morphisms of sheaves $\Phi_1, \Phi_2 : \mathscr{F} \to \mathscr{G}$ and they agree when restricted to the distinguished basis, then they will agree as morphism of sheaves, this is just using that morphism is determined by stalks and we can compute stalks using distinguished affine base. Another way of interpreting it is that if we have two sheaves $\mathscr{F}$ and $\mathscr{G}$ and we have a morphism of them restricted to the distinguished affine base, then we get an induced morphism. This is also easy to prove by appropriate use of gluability, and you can see that this is a unique morphism by the first way of interpreting it.

This plus 1, we know that if we consider the functor which maps a sheaf on the distinguished affine base to the associated sheaf and the functor which restrict a sheaf to the distinguished affine base. This shows that both functors are fully faithful. (You should make sure you can prove this.) We claim that this is even an equivalence of categories, first you can see that since the map of associated sheaves is already fully faithful, and every sheaf on X is isomorphic to some associated sheaf. Thus we know that the functor of associated sheaves is an equivalence of categories, then what is the inverse we **** maybe use yoneda later.*

Finally, let's prove the last assertion. We still have to make sense of the statement first, what is $\mathscr{O}_X$-module on a distinguished affine base. Well, it's just a natural definition where it must be a sheaf on the distinguished affine base first and on each affine in this base, it's has a corresponding module structure such that it's compatible with restriction. Then how can it determines a global $\mathscr{O}_X$-module. The only natural thinking is to take the associated sheaf, but how are we going to define a module structure on each open, which is equivalent of asking how do you define the scalar multiplications such that it's compatible with restrictions. Follow you nose and you will see that there is only one possible way to define this action such that the action after you restricted to the distinguished affine base, you get what you started with. ☐

The most important reason we want to introduce the above construction is that we can use it to give a very important characterization of quasicoherent $\mathscr{O}_X$-module. We claim that over any scheme X, a quasicoherent sheaf is equivalent of the information that on each affine open $SpecA$ of X, we associate it with an A-module and the module on distinguished opens are given by localizations.

To make this precise, we first do one exercise that is not so related to sheaf on the distinguished affine base. It can also be viewed as the **second definition** of a $\mathscr{O}_X$-module being quasicoherent. Consider if $\mathscr{F}$ is already an $\mathscr{O}_X$-sheaf on X, we claim that it's a quasicoherent module if and only if when we consider the following diagram:

$$\begin{tikzcd}
\Gamma(SpecA, \mathscr{F}) \arrow[rr, "res"] \arrow[dr, "loc"'] & & \Gamma(SpecA_f, \mathscr{F}) \\
& \Gamma(SpecA, \mathscr{F})_f \arrow[ur, dashed, "\alpha"'] &
\end{tikzcd}$$

α is an isomorphism for all $SpecA$. Suppose that $\mathscr{F}$ is quasicoherent, then the above diagram is equivalent of saying that $\mathscr{F}|_{SpecA} \cong \Gamma(\widetilde{SpecA, \mathscr{F}})$. But this is easy since we already know that $\mathscr{F}|_{SpecA} \cong \widetilde{M}$ for some A-module M, then we only have to prove that $\Gamma(SpecA, \mathscr{F}) \cong M$ as an A-module. Conversely, if we have the above diagram and α is an isomorphism for all $SpecA$, we still claim that $\mathscr{F}|_{SpecA} \cong \Gamma(\widetilde{SpecA, \mathscr{F}})$. To see this we still work locally, notice that the above isomorphism will be compatible with restrictions in the sense you can see that the below diagram commutes:

$$\begin{tikzcd}
\Gamma(SpecA, \mathscr{F}) \arrow[d, "loc"'] \arrow[dr, "res"] & \\
\Gamma(SpecA, \mathscr{F})_f \arrow[r, dashed, "\alpha_f"] \arrow[d, "loc"'] & \Gamma(SpecA_f, \mathscr{F}) \arrow[d, "res"] \\
\Gamma(SpecA, \mathscr{F})_g \arrow[r, dashed, "\alpha_g"] & \Gamma(SpecA_g, \mathscr{F})
\end{tikzcd}$$

You should really make sure you understand why it commutes! Then in particular, the lower diagram commutes, this is exactly what we need for $\mathscr{F}|_{SpecA}$ to be isomorphic to $\Gamma(\widetilde{SpecA, \mathscr{F}})$.

Now here comes the real content, we will use the result above to show that on a arbitrary scheme X, giving a quasicoherent sheaf is equivalent of giving each affine open $SpecA$ in X and if there is a distinguished affine $SpecA_f \hookrightarrow SpecA$, the module we associate to $SpecA_f$ is the localization of the module we associate to $SpecA$. To make this precise, suppose that $V \hookrightarrow U$ is a distinguished inclusion, then we already have two modules M_U and M_V, then we see that under the isomorphism $U \cong SpecA$, we can think of M_U and M_V as A and A_f-modules, then we need the following diagram to commute:

$$\begin{tikzcd}
M_U \arrow[d, "res"] \arrow[dr, "loc"] & \\
M_V \arrow[r] \arrow[d, "res"] & (M_U)_f \arrow[d, "loc"] \\
M_W \arrow[r] & (M_U)_g
\end{tikzcd}$$

for $W \subset V$ distinguished opens in U.

Proposition 1.21. The above construction gives a quasicoherent $\mathscr{O}_X$-module.

Proof. We claim that the above construction gives a sheaf on the distinguished affine base. To see this, we already specifyed the restriction map, since it comes from localization, then it is really a presheaf on the distinguished affine base. We need to show that it satisfies the identity and gluability axiom, but this is just the content of the construction we give in the beginning of the section. Also, this is obvious an $\mathscr{O}_X$-module on the distinguished affine base, then by the theorem we give above, it gives a unique $\mathscr{O}_X$-module such that the restriction on the distinguished affine base is what we started with.

Why this is quasicoherent, this is where we use the above result which is the **second definition** of quasicoherent modules. We have to show that that the diagram in that definition commutes and α is an isomorphism. Why it commutes? Well, since we are only considering affine opens and distinguished inclusions, we know that we can restrict to $\mathscr{F}^b$, which means we are back to what we started. But according to our definition, the restriction is nothing else but the localization, thus α is of course the identity, which is an isomorphism.

The major difference here is that we can discard all the bad open inclusions and only consider the distinguished ones. $\square$

As an exercise of how to apply the above criterion, let's consider the following proposition:

Proposition 1.22. Let A be a ring, show that there is a natural isomorphism $\mathfrak{N}(A_f) \cong \mathfrak{N}(A)_f$, which can be used to define a quasicoherent sheaf of nilpotents on any schemes X. This is called the **sheaf of nilpotents**.

Proof. Let's first consider what is the natural isomorphism between $\mathfrak{N}(A)_f \cong \mathfrak{N}(A_f)$. Notice that there is a natural morphism from $\mathfrak{N}(A)$ to $\mathfrak{N}(A_f)$ given by restricting $A \to A_f$ to $\mathfrak{N}(A)$. Notice that if a is a nilpotent in A, it's of course mapped to A_f as a nilpotent, this is obvious a map of A-module. Then by the universal property of localization, there is a unique map $\mathfrak{N}(A)_f \to \mathfrak{N}(A_f)$, we only have to show this is isomorphism, which is left to you.

According to our construction above, we define for every affine open $U \cong SpecA$, an $\Gamma(U, \mathscr{O}_X)$-module, which is just the nilpotents of it, $\mathfrak{N}(\Gamma(U, \mathscr{O}_X))$. Then, it's left as an exercise to check that the diagram we defined above commutes. $\square$

Next, we see that the tensor product of quasicoherent sheaves can also be expressed very clearly in terms of this diatinguished affines, we need an algebraic lemma:

Lemma 1.23. Suppose that M, N are A-modules, then for $f \in A$, there is a natural isomrophism:

$$M_f \otimes_{A_f} N_f \cong (M \otimes_A N)_f$$

Proof. Use universal property of both tensor product an localization to see that the isomorphism is given by:

$$(M \otimes_A N)_f \to M_f \otimes_{A_f} N_f$$

where $\frac{m \otimes n}{f^n} \mapsto \frac{m}{f^n} \otimes \frac{n}{1}$ or the other way around. $\square$

Proposition 1.24. Suppose that $\mathscr{F}$ and $\mathscr{G}$ are quasicoherent sheaves on a scheme X, then $\mathscr{F} \otimes \mathscr{G}$ is a quasicoherent sheaf described by the following information: if $SpecA \subset X$ is affine open, and $\Gamma(SpecA, \mathscr{F}) = M$, $\Gamma(SpecA, \mathscr{G}) = N$, then $\Gamma(SpecA, \mathscr{F} \otimes \mathscr{G})$ is just $M \otimes_A N$ and the restriction to distinguished affine opens are given by: $M \otimes_A N \to M_f \otimes_{A_f} N_f \cong (M \otimes_A N)_f$.

Proof. First, we need to figure out what this statement really means. First, we are given $\mathscr{F}$ and $\mathscr{G}$, and we are trying to use a way that doesn't require sheafification to describe the tensor product. Of course, we would like to define a sheaf on the distinguished bases and show that the associated sheaf it's isomorphic to the tensor product.

By the isomorphism described in the above lemma, we can see that the definition indeed describes a sheaf on the distinguished affine base, then how are we going to show that isomorphism. We will show that the restriction of the tensor product to the distinguished affine base is isomorphic to the stuff we just defined. Recall the definition of the sheafification, it's also defined in terms of locally constant functions or compatible stalks. Notice that the stalk can be computed by affine opens and distinguished inclusions, then it's easy to give an isomorphism. $\square$

As you can see that, with this distinguished affine base, there is no issue with sheafification, the only thing we need is just what it is on affines and distinguished opens, in the sense that it behaves well under localization of one element. The sheafification is implicitly taken care of when we do the associated sheaf construction.

The next topic is qcqs lemma, which is useful later to see what sheaves we construct is quasicoherent, and it also has other applications you will see:

Proposition 1.25. Suppose that X is qcqs, and $\mathscr{F}$ is a quasicoherent scheme on X, suppose that $f \in \Gamma(X, \mathscr{O}_X)$ is a function on X, then the following diagram commutes:

$$\begin{tikzcd} \Gamma(X, \mathscr{F}) \arrow[r, "res"] \arrow[d, "loc"'] & \Gamma(X_f, \mathscr{F}) \\ \Gamma(X, \mathscr{F})_f \arrow[ur, dashed, "iso"'] & \end{tikzcd}$$

Proof. Notice that by the qcqs lemma, there is a finite affine open cover U_i and for any $U_i \cap U_j$, it's also covered by a finite number of affines, we denote them by U_{ijk}.

Notice that when X is affine, we know that the assertion is just the second definition of quasicoherent sheaves. Then, consider the following exact sequence we have from the sheaf axioms:

$$0 \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \oplus_i \Gamma(U_i, \mathscr{F}) \longrightarrow \oplus_{ijk} \Gamma(U_{ijk}, \mathscr{F})$$

the last map is defined to map s_i, s_j section on U_i, U_j to the difference on U_{ijk}, we are saying that if they agree on U_{ijk}, since they cover U_{ij}, they will agree on U_{ij}, thus coming from a global section.

We still need another prepration, notice that if $U \subset X$ is any affine, then $X_f \cap U$ is the distinguished open defined by the restriction of f to U, then since we have:

$$\Gamma(U, \mathscr{F})_f \cong \Gamma(U, \mathscr{F})|_{f|_U} \cong \Gamma(U \cap X_f, \mathscr{F})$$

from $\mathscr{F}$ is quasicoherent.

Then we apply $|_f$, the localization functor to the above sequence to get:

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(X, \mathscr{F})_f \arrow[r] \arrow[d, dashed] & \oplus_i \Gamma(U_i, \mathscr{F})_f \arrow[r] \arrow[d, leftrightarrow] & \oplus_{ijk} \Gamma(U_{ijk}, \mathscr{F})_f \arrow[d, leftrightarrow] \\ 0 \arrow[r] & Ker = \Gamma(X_f, \mathscr{F}) \arrow[r] & \oplus_i \Gamma(U_i \cap X_f, \mathscr{F}) \arrow[r] & \oplus_{ijk} \Gamma(U_{ijk} \cap X_f, \mathscr{F}) \end{tikzcd}$$

the last piece is why the diagram in the statement commute under this isomorphism. The reason is not to hard, it's just not so easy to draw the diagram here, you can image that the original sequence before we localize floating above these two sequences, and we know that the later two terms commute, can you conclude that that the first also commutes? $\square$

Now, we proceed to prove that quasicoherent sheaves form an abelian category. Recall that morphism of quasicoherent sheaves are just morphism of usual $\mathscr{O}_X$-modules, then why it forms an abelian category is not obvious since under this definition, it's just a full subcategory of $\mathscr{O}_X$-modules. We tackle this question now.

In general, there is a lot to check if we start with an arbitrary category, however, we already know that $\mathscr{O}_X$-modules form an abelian category, it's much easier to check that as a full subcategory, it's also abelian. This is just like if you start with a set and want to check that it's a group versus if you start with a subset of a group and check that it's a subgroup is a lot easier.

We begin with a fact: If you have a full subcategory of an abelian category, you only have to check the following three conditions to show that it's an abelian subcategory:

1. 0 is in the subcategory.
2. The subcategory is closed under finite sums.
3. The subcategory is closed under kernels and cokernels.

The first one is trivial. The second one is also easy:

Proposition 1.26. Suppose that $\mathscr{F}$ and $\mathscr{G}$ over X is quasicoherent, then so is $\mathscr{F} \oplus \mathscr{G}$.

Proof. Notice that this is just a local property, since you can have an easy isomorphism of $\mathscr{F}|_U \oplus \mathscr{G}|_U \cong \mathscr{F} \oplus \mathscr{G}|_U$. Then we only have to prove $\widetilde{M} \oplus \widetilde{N}$ over $SpecA$ is isomorphic to $\widetilde{M \oplus N}$. We can check on distinguished affines, the key is that:

$$(M \oplus N)_f \cong M_f \oplus N_f$$

and it commutes with localizations. $\square$

The real content is part 3 in the requirements: we will check it using the second definition of quasicoherent sheaves.

Proposition 1.27. Suppose that $f : \mathscr{F} \to \mathscr{G}$ is a morphism of quasicoherent sheaves on X. Then $Kerf$ and $Cokerf$ are quasicoherent.

Proof. We already know that Ker and $Coker$ are $\mathscr{O}_X$-modules, thus we only have to show that this is quasicoherent by showing that on each affine $SpecA$, it behaves well under localization. This is also going to be a proposition that describes the kernel and cokernel of morphisms of quasicoherent sheaves on affine opens and distinguished inclusions.

Notice that on an affine open $U = SpecA$, suppose that f is given by a module morphism $\beta : M \to N$ since $\mathscr{F}$ and $\mathscr{G}$ are affine, then we can consider the kernel K and cokernel P:

$$0 \to K \to M \to N \to P \to 0$$

which is exact, then notice that apply localization is exact, thus:

$$0 \to K_f \to M_f \to N_f \to P_f \to 0$$

is exact, then if we define $Ker(f)(U) = K$ and $Coker(f)(U) = P$, we see that for a distinguished open, $K_f = Ker(\beta_f)$ and $P_f = Coker(\beta_f)$, then it's left to you that this satisfies the requirement in the second definition. Therefore, both definitions defines quasicoherent sheaves $Kerf$ and $Cokerf$, we only have to show that they are really the kernel and cokernel of the morphism.

This is not done in interms of constructing isomorphisms, instead we only have to check that at each point, for example:

$$0 \to Kerf_x \to \mathscr{F}_x \to \mathscr{G}_x \to Cokerf_x \to 0$$

is exact, this is left as an easy exercise. Thus, the stuff we defined satisfies the universal property of kernel and cokernel, thus they are. $\square$

One corallary of the above statement is for quasicoherent sheaves, to check that a morphism is exact, we only have to do it on an affine cover.

Corollary 1.28. Suppose that $\mathscr{F} \to \mathscr{G} \to \mathscr{R}$ is a sequence of quasicoherent modules on X, then it's exact iff the maps on sections on any affine open cover are exact.

Proof. Suppose that it's exact, then we know that this is equivalent to that it's exact at stalks for each point $x \in X$. For any affine cover, we know that the map induced on stalks is:

$$M_p \to N_p \to R_p$$

if $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{R}$ restricted to U is $\widetilde{M}$, $\widetilde{N}$ and $\widetilde{R}$. Notice that being exact is a local property in the sense that if localization at all primes are isomorphism, then the original morphism is an isomorphism.

Conversely, morphism on stalks can be computed by affine opens and localization is an exact functor. $\square$

Now, we introduce finiteness condition on quasicoherent modules: First, we see several finiteness conditions for an A-module M, where A is just a commutative ring. As a fact, when A is noetherian, the following three conditions are equivalent.

The first one is finitely generated, which is equivalent of a surjection from A^r. The second one is finitely presented, which is characterized by an exact sequence: $A^m \to A^n \to M \to 0$

And you can understand this condition as M is generated by a finitely many collection of generators and these relations of these generatots, which is represented by the kernel of the surjection $A^n \to M$ is again finitely generated.

The third condition is less common, which is called **coherent**, we will say that an A-module M is coherent if it's finitely generated and for any not necessarily surjective map $A^n \to M$, the kernel is finitely generated. However, as promised above, when you are working over a noetherian ring, the three notions are equivalent:

Proposition 1.29. The above three notions are equivelent if A is noetherian.

Proof. Notice that coherent impleis finitely presented and implies finitely generated. Then suppose that M is finitely generated, notice that any map $A^n \to M$ has a kernel as a submodule of a finitely generated module, thus finitely generated, we are done. □

It's a general fact that for any ring A, the coherent A-modules form an abelian subcategory of A-modules, but here we only prove it for noetherian rings:

Proposition 1.30. If A is a noetherian ring, then the full subcategory of coherent A-modules is an abelian subcategory of A-modules.

Proof. Recall that we need to show that 0 is in it, it's closed under finite sums, taking kernels and cokernels.

0 is easy and notice finite sums of finitely generated modules are finitely generated and since we are over a noetherian ring, the other condition of being coherent is automatically satisfied. Then, we only have to show that kernels and cokernels are coherent. Notice that as submodules and quotient of finitely generated modules, again since we are over a noetherian ring, they are finitely generated. □

Now, we begin to extend this discussion on modules to quasicoherent sheaves, before this, we first prove a result that allows us to use affine communication lemma later:

Lemma 1.31. Suppose that M is a finitely generated A-module, then M_f is a finitely generated A_f-module. And conversely, if $A = (f_1, \cdots, f_r)$ and M_{f_i}'s are finitely generated A_{f_i}-modules, then M is a finitely generated A-module.

Proof. The first assertion is trivial. To see the second one, on each M_{f_i}, we have:

$$mf_i^{n_i} = \sum \cdots$$

where the $\cdots$ are linear combinations of finitely many elements in M, then we take n to be the maximum, then $(f_1, \cdots, f_r) = A$ shows the desired result. □

The lemma below is a result saying that finitely presented is also a well-behaving property in the sense:

Lemma 1.32. Suppose that M is a finitely presented A-module, which says that there is some surjection such that $A^m \to M$ has a finitely generated kernel, then for any surjection $A^p \to M$, the kernel is finitely generated.

Proof. First, choose a finite presentation of M:

$$\oplus_k^r Af_k \xrightarrow{\beta} \oplus_i^n Ax_i \xrightarrow{\alpha} M$$

then we know that $\alpha(x_i)$ generates M, then consider the surjection:

$$\Phi : \oplus_j^p Ay_j \to M$$

We can choose lifts of $\Phi(y_j)$ in $\oplus_i Ax_i$, suppose that they are g_j. Then we consider the following surjection:

$$\oplus_i^n Ax_i \oplus \oplus_j^p Ay_j \to M$$

which is defined by the universal property of direct sums and the original map we have to M, it's surjective with kernel generated by $\beta(f_1), \cdots, \beta(f_r).y_1 - g_1, \cdots, y_p - g_p$, notice that these are definitely in the kernel, it's left to you that why this is exactly the kernel. A hint is we have $\beta(f_k)$'s, which can help us eliminate relations. Therefore, we have:

$$M = \oplus_i^n Ax_i \oplus \oplus_j^p Ay_j / (\beta(f_1), \cdots, \beta(f_r).y_1 - g_1, \cdots, y_p - g_p)$$

Now, notice that $\oplus_i^n Ax_i$ and $\oplus_j^p Ay_j$ to M is all surjective, thus we can find h_i's in $\oplus_j^p Ay_j$ such that $\alpha(x_i) = \phi(h_i)$, thus, we know that $x_1 - h_1$ and all the way to $x_n - h_n$ are in the kernel, thus we know that:

$$M = \oplus_i^n Ax_i \oplus \oplus_j^p Ay_j / (\beta(f_1), \cdots, \beta(f_r).y_1 - g_1, \cdots, y_p - g_p, x_1 - h_1, \cdots, x_n - h_n)$$

Then, this means that up to a map $h : \oplus_i Ax_i \to \oplus_j^p Ay_j$, the x_i's are actually not playing a role here. Therefore, after you replace everything in the kernel that's in Ax_i by the image of it under h, Ax_i's can be removed, thus we have:

$$M = \oplus_j Ay_j / (h\beta(f_1), \cdots, h\beta(f_r), y_1 - h(g_1), \cdots, y_p - h(g_p))$$

□

Remark 1.33. This is not a hard result, but the way it's letting us to see the kernel is interesting. To be more precise, we are basically playing with generators and relations here, in the first rewriting:

$$M = \oplus_i^n Ax_i / (\beta(f_1), \cdots, \beta(f_r)) = \oplus_i^n Ax_i \oplus \oplus_j^p Ay_j / (\beta(f_1), \cdots, \beta(f_r), y_1 - g_1, \cdots, y_p - g_p)$$

We are adding more generators, so you should expecting we are going to have more relations, which means that more generators in the kernel. But what should we add here? Notice that since the new generators y_j's we choose are exactly g_j's up to a map from $\oplus Ay_j$ to $\oplus Ax_i$, thus we can neutralize the new generators by setting $y_i - g_i = 0$. This is also what's going on in the second rewriting.

Next, let's show that finite presentation and coherence also satisfies the assumption of affine communication lemma:

Lemma 1.34.

1. Suppose that M is a finitely presented module, then M_f is also finitely presented.
2. $(f_1, \cdots, f_n) = A$, M_{f_i} is a finitely presented A_{f_i}-module for all i, then M is a finitely presented A-module.

Proof. The first assertion is just the fact that localization is exact. To show the second assertion, notice that finitely presneted implies finitely generated, thus M is first finitely generated. Then we choose a surjection:

$$\oplus Ax_i \to M$$

Call the kernel N, we need to show that this is finitely generated. Notice that if we localize at f, we get:

$$N_f \to \oplus A_f x_i \to M_f$$

Notice that since M_f is finitely presented, we know that N_f is finitely generated since finitely presented means that always finitely presented, thus N_{f_i} is finitely generated for all i, then N is finitely generated. □

Finally, let's show that being coherent also satisfies the conditions in affine communication lemma:

Proposition 1.35. Being coherence satisfies the conditions in affine communication lemma.

Proof. The first assertion comes from the fact that any map:

$$A_f^{\oplus p} \to M_f$$

comes from a localization:

$$A^{\oplus p} \to M$$

since we know that the $A_f^{\oplus p} \to M_f$ is determined by where 1 goes to, since M_f is over A_f, f is invertible, thus we can assume that 1 goes to the an element in the image of $M \to M_f$, then we are done since localization preserves finitely generation.

Conversely, M is obviously finitely generated since M_{f_i} being coherent over A_{f_i} first requires M to be finitely generated. Then we need to show that the kernel N of $A^{\oplus p} \to M$ is finitely generated. We localize, then N_f is finitely generated for all f, thus we are done. □

Finally, we can give the definition:

Definition 1.36. A quasicoherent sheaf $\mathscr{F}$ is **finite type**, (reps. **finitely presented**, **coherent**) if for any affine open $SpecA$, $\Gamma(SpecA, \mathscr{F})$ is finitely generated, (resp. finitely presented, coherent), A-module. Notice that if we are over a noetherian scheme, all these three notions are the same, and we know that coherent sheaves are always finite type, but finite type maynot be coherent.

Also, we know that coherent sheaves form an abelian category over any scheme, but not for finitely generated. But if we are over a noetherian scheme, finite type sheaves are the same as coherent sheaves, thus is also an abelian category.

Remark 1.37. By the above affine communication lemma argument, all these above notions can be check an an arbitrary affine cover, instead on any affine open.

The remaining content in this section are Jordan-Holder theorems including related notions and the associated points, we first begin with Jordan-Holder:

Let's work with a general abelian category $\mathscr{C}$: We will say that an object M in $\mathscr{C}$ is **simple** or **irreducible** if it's only sub-objects are 0 and itself. A **composition series** for M is a finite filtration:

$$0 = X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_{n-1} \subsetneq X_n = M$$

such that the quotients X_{i+1}/X_i are simple for all i. If M has a composition series, we will say that M has **finite length**.

The main theorem in this section is the **Jordan-Holder theorem**:

Theorem 1.38. If M has a finite composition series, then all composition series for M has the same length and the quotients are all the same with a possible relabeling.

Assuming the above theorem for now, we have the following definition:

Definition 1.39. We will call the length of any composition series of M the length of M, denoted by $l(M)$. If M doesn't have a composition series, we say that it's length is infinite, written as $l(M) = \infty$.

As an easy example, consider:

Example 1.40. In the abelian category of abelian groups, we see that the finite length objects are finite abelian groups, then we apply Jordan-Holder theorem to $\mathbb{Z}/n\mathbb{Z}$, to get a **unique factorization** of n.

For example, we have two composition series for $\mathbb{Z}/12\mathbb{Z}$:

$$(12) \subsetneq (6) \subsetneq (2) \subsetneq (1) \qquad (12) \subsetneq (4) \subsetneq (2) \subsetneq (1)$$

You can see that even though these are different composition series, but they all have the same length and the quotients are the same, except for a relabeling.

From this point of view, this Jordan-Holder theorem generalizes the notions of well-definedness of dimensions of vector spaces, unique factorization of integers, structure theorem of finitely generated abelian groups and other stuff.

Now, let's prove Thm 1.38:

Proof. In this proof, we will use the notion of intersection of two subobjects in an abelina category, this is not a hard notion and you can prove it fairly easily that any two subobjects in an abelian category has their intersection, once you get the definition right. We won't go into this, instead, we will think of intersections as an operation that you can do on something that has a underlying set of some sort, like modules or sheaves.

Suppose that beside the above composition series of M, we have another composition series:

$$0 = Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq X_{p-1} \subsetneq Y_p = M$$

we consider the rectangular array with entries $Z_{i,j} = X_i \cap Y_j$, you may be able to think of this more concretely if you try to apply this to the above example we give for $\mathbb{Z}/12\mathbb{Z}$. This rectangular array is called the **Jordan-Holder table**.

There are several useful observations for this table:

1. If $i \leqslant i'$ and $j \leqslant j'$ $Z_{ij} \subset Z_{i'j'}$.
2. $Z_{n,j} = Y_j$ and $Z_{i,p} = X_i$. Similarly, $Z_{0,j} = 0 = Z_{i,0}$.
3. $Z_{i,j} \cap Z_{i',j'} = Z_{mi,mj}$ where mi is the minimum of i, i' and mj is the minimum of j, j'.

The key ingredient of the proof is the following:

$$Z_{i,j}/Z_{i,j-1} \cong Y_j/Y_{j-1} \text{ or } 0$$

This can be shown by decreasing induction, on i. The base case if when $i = n$, then we see this is obvious, then suppose that it holds for $i+1$, then for i, we need to consider:

$$Z_{i,j}/Z_{i,j-1} = X_i \cap Y_j / X_i \cap Y_j \cap Y_{j-1}$$

By the second isomorphism theorem for abelian categories, we have thsi is isomorphic to:

$$X_i \cap Y_j + Y_{j-1}/Y_{j-1}$$

Notice that this has a natural embedding into Y_j/Y_{j-1}, which is simple, thus it's Y_j/Y_{j-1} or 0. Similarly, we can prove that $Z_{i,j}/Z_{i-1,j} \cong X_i/X_{i-1,j}$ or 0.

The rest of this prove is hard to write down here, but the idea is again very simple. Notice that by the above discussion, we can do the quotients of adjacent objects, which could possibly be 0 or a simple element. Then we will draw a line between them if the quotient is not zero and leave there if the quotient is zero. If we consider any 2×2 subsquare of the array, we see that the patterns of the lines we draw only falls into 5 categories, using the constraint that if the previous one is zero, the latter one can't be nonzero.

Now, what are these lines doing here. It's left to you to see why if you follow the line, starting from the right side of the table, going to the left, turn left when you can't move forward any more, you will get a bijection between the simple objects of the form Y_{j+1}/Y_j and of the form X_{j+1}/X_j, including multiplicity. (The hint is that look that how the five patterns above tells you.) This proves that the length is the same and the multiplicitys of the simple objects are the same. □

We state a few additional results that is useful, whose proof is easy:

Proposition 1.41. Suppose that M is of finite length, and $M' \subset M'' \subset M$ are subojects of M, we have M''/M' is of finite length and in particular all subobjects of M has finite length.

Proposition 1.42. Length is additive in SES, in the sense that if the following is an exact sequence:

$$0 \to M' \to M \to M'' \to 0$$

then $l(M) = l(M') + l(M'')$, even if some of them is not of finite length.

Proposition 1.43. Every filtration of a finite length object M can be refined to a composition series.

Our application of this stuff is basically in the setting of modules over rings:

Proposition 1.44. Simple objects in A-modules are precisely of the form A/m for some maximal ideal m in A.

Proof. First, notices that submodules of A/m are those ideals containing m, but since m is maximal, then it can only be m or A.

Conversely, suppose that M is simple, consider $A \to M$ where $1 \mapsto m$ for any m not zero, then consider the kernel of this map, we have $A/Ker \hookrightarrow M$, which is injective. Notice that since Ker is not A for this map is not zero, thus we must have $A/Ker \cong M$ and not zero. We claim that Ker is a maximal ideal, otherwise, this is not a simple module. □

Proposition 1.45. Suppose that M is a module of finite length, then we can choose a composition series M_i of M, with $M_i/M_{i-1} \cong A/m_i$, then M is annihilated by $m_1 \cdots m_n$, or equivalently M is an $A/m_1 \cdots m_n$-module.

Proof. Suppose that $m \in M$ and $m \neq 0$, from the composition series, we can find the first M_i in the composition series such that $m \in M_i$ but $m \notin M_{i-1}$, then we know that if we consider the image of m in M_i/M_{i-1}, we know that it's annihilated by $m_1 \cdots m_n$, thus we know that the product of m with element in $m_1 \cdots m_n$ will gets m into M_{i-1}, then we keep going down until we reach 0. □

Notice that if we know that this list of maximal ideals $(m_1, \cdots, m_n)$ consists of distinct maximal ideals $(n_1, \cdots, n_s)$, with multiplicities $l_1, \cdots, l_s$. Then notice that by the Chinese Remainder theorem, we have:

$$A/m_1 \cdots m_n = A/n_1^{l_1} \cdots n_s^{l_s} \cong A/n_1^{l_1} \times \cdots \times A/n_s^{l_s}$$

More precisely, this means that there are $e_1, \cdots, e_s$ in A such that $e_i = 1$ after we mod $n_i^{l_i}$ and $e_i = 0$ after we mod $n_j^{l_j}$ for $j \neq i$. Then we claim:

Proposition 1.46. $M \cong e_1 M \times \cdots \times e_s M$ and each $e_i M$ is a finite length module whose composition series only consists of simples quotients A/n_i. Therefore any finite length module is a product of finite length modules, whose simple quotients only has one type simple objects.

Proof. First, notice that any $m \in M$, it can be written as a sum of elements in $e_i M$, since $m = 1m$ and 1 has an image from $A/m_1 \cdots m_n \to A/n_1^{l_1} \cdots n_s^{l_s} \cong A/n_1^{l_1} \times \cdots \times A/n_s^{l_s}$, which can be written as combinations of e_i. Then, we also know that $e_i M \cap e_j M = 0$ for $i \neq j$, this is because if m is in the intersection, we know that it's $e_i x = e_j y$ where $x, y \in M$, notice that we know $e_i e_i x = e_i x$, this is because that e_i is 1 in the product rings, but $e_i e_j y = 0$ since $e_i e_j$ is 0 in the product ring. Thus, $e_i x = 0$, which implies that the intersection is 0, thus this proves that M is the direct sum or since this is a finite direct sum, it's also the finite direct product of these factors.

Now consider $e_i M$, since it's a submodule of this finite length module, it's of finite length, we claim that the composition series can't consist of A/n_j for $n_j \neq n_i$. If there is such a factor, suppose that $x \in e_i M$ is the element in $e_i M$ such that it's image in N_i/N_{i-1}, the simple quotient is 1, then we know that since if the 1 in A/n_j is also 1 in $A/n_j^{l_j}$, then 1×1 is not zero, but we know that any elements in $A/n_j^{l_j}$ annihilates x, even if 1, thus we get a contradiction. □

Proposition 1.47. Suppose that M has finite length, then M is finitely generated and it's support only consists of finitely many closed points in $SpecA$.

Proof. $e_i M$ is finitely generated since we know that if M/M' is finitely generated and M' is finitely generated, then M is finitely generated. Then we take the composition series of M, do induction to show that M is finitely generated.

Notice that it's support must be primes that contains $m_1 \cdots m_n$ since this ideal annihilates it and is $p \subset m_1 \cdots m_n$, the localization multiplicative system consists of zero divisors, thus the localization is 0, thus it must contain at least one in $n_1, \cdots n_l$. Thus we know that the support is contained in $\{n_1, \cdots, n_l\}$. □

If we apply this notion to a ring:

Definition 1.48. Suppose that A has finite length as a module over itself, then A is called Artinian. If it's futhermore a local ring, we say that it's an Artinian local ring.

Remark 1.49. Notice that if A is Artinian, then we see that $m_1 \cdots m_n$ annihilates A, thus $A \cong A/m_1 \cdots m_n \cong A/n_1^{l_1} \times \cdots \times A/n_s^{l_s}$, each of which is obvious a local ring, and Artinian, thus we proved the structure theorem of Artinian local rings, which says that each Artinian ring is a finite product of Artinian local rings.

Then, we apply this notion of length to quasicoherent sheaves on a scheme X:

Proposition 1.50. Simple objects of $Qcoh(X)$ is the structure sheaf of closed points.

Proof. First, it's easy to show that skyscraper sheaf of the local ring of closed points are simple.

Conversely, suppose that $\mathscr{F}$ is a quasicoherent module on X, which is simple, but not 0. We first claim that it can only be supported at one point, otherwise, the skyscraper sheaf supported at that point with the stalk of $\mathscr{F}$ is a subobject. We only have to show that this is a quasicoherent sheaf. Based on our characterization of quasicoherent sheaves, suppose that $SpecA$ is an affine open, if it doens't contain the point, there is nothing to prove. Then suppose that $SpecA$ contains that point and it correspond to a prime in A, then choose any $f \in A$, we see that if $D(f)$ contains that point, thus we see that $f \notin p$, therefore, $\mathscr{F}_x$ localized at f is isomorphic to itself since we know that $\mathscr{F}_x$ is a A_p-module, and since f is not in p, it's invertible in A_p. Then if $D(f)$ doesn't contain p, we see that $f \in p$, thus we see that the localization of $\mathscr{F}_x$ at f is zero since $D(f)$ doesn't contain the support of $\mathscr{F}_x$ **** later, need more details*

Then we claim that this point must be closed, and in particular, if the scheme is without closed points, there is no simple objects in $Qcoh(X)$. To see this, suppose that x is a point in the support, which is not closed, then consider the closure of x, which consists of at least two points, then there is an affine open of that addition point, and we know that it also contains x. Then we know that in that affine open x and that addition points represents two prime ideals, $x \subset y$, y is the additional point. Then since x is in the support, y is also in the support from just general module theory. ☐

Proposition 1.51. Suppose that $\mathscr{F}$ is a finite length element in $Qcoh(X)$, then it's finite type and the support of $\mathscr{F}$ are only finitely many closed points of X, in particular, if there is no closed points, the sheaf is 0!

Proof. **** later* ☐

Definition 1.52. The **length of a scheme** X, is the length of it's structure sheaf $\mathscr{O}_X$, a scheme is called **finite length** or **Artinian** if $\mathscr{O}_X$ has finite length.

The last section in this section is **associated points**:

2 Review on Properties of Schemes

We gather some loose properties of schemes here, first we focus on some general useful topological properties, then we discuss **normality** and **factoriality**.

Proposition 2.1. A scheme is quasicompact iff it can be written as the union of finitely many affine open subschemes.

Proof. Notice that finite union of quasicomapct spaces are quasicompact. The converse is trivial. ☐

Proposition 2.2. A scheme is quasiseparated if and only if the intersection of any affine open is a finite union of affine open subsets.

Proof. Suppose that X is quasiseparated, then since we know that affine opens are quasicompact, then the intersection should be quasicomapct, which is equivalent of having a finite affine cover.

Conversely, suppose that any intersections of two affine opens has a finite cover of affine. Now suppose that U and V are quasicompact, then they can be covered by finitely many affine, then the intersection can be covered by finitely many intersections of affines, which plus the fact that each of the intersection of affine can be covered by finitely affines, the intersection can be covered by finitely many affines, thus quasicompact. ☐

Proposition 2.3. Affine schemes are quasiseparated.

Proof. Suppose that we have two affine opens in $SpecA$, notice that each can be covered by distinguished opens in $SpecA$, and we know that affine opens are quasicomapct, thus you can choose finitely many of them, then the intersection can be covered by finitely many intersection of affine opens, thus it's affine again. We are done. ☐

Proposition 2.4. Finite union of Noetherian topological spaces are Noetherian.

Proof. You can directly look at the definition, where you want to show that every chain of closed subsets in $Y = \cup_i X_i$ for some $i \in I$ finite index subset. Then, you can take intersection with each X_i, which is still closed in X_i, thus will terminate. Then since we only have finitely many such sets, we can take maximum. ☐

3 Review on Nice classes of Morphisms

The thing we need to do in this section is to do a review on some special classes of morphism between schemes, and show that they have several nice properties, which is going to help us build the theory of fibered products later. As you will see soon, one of these morphisms is the class of open embeddings

First, what we mean by nice classes of morphisms. In general, we want then to satisfy the following several requirements:

Definition 3.1. Suppose that $\mathcal{P}$ is a class of morphisms (always, this will be a class of morphisms with certain properties) that contains all the isomorphisms between schemes, we will call $\mathcal{P}$ nice if the following there conditions are satisfied:

1. The class is presered by **composition** in the sens that if $\pi : X \to Y$ and $\rho : Y \to Z$ is in $\mathcal{P}$, then $\rho \circ \pi$ is in $\mathcal{P}$.
2. The class is preserved by **base change**, or sometimes called **pullback** or just fibered product. In the sense that, if $\pi : X \to Y$ is in the class $\mathcal{P}$, for any morphism $Y' \to Y$, then the morphism in the fibered product $X \times_Y Y' \to Y'$ is also in $\mathcal{P}$. In other words, the lower morphism in the Cartesian square is in $\mathcal{P}$ the the top morphism that's pullback should be in $\mathcal{P}$.

$$\begin{tikzcd} X \times_Y Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

3. The class is local on the target or in other words all the morphisms in this class $\mathcal{P}$ are local on the target in the sense that the two following requirements are satisfied: (1): If $X \to Y$ is in the class, then the restricted morphism $\pi^{-1}V \to V$ is in the class for any open $V \subset Y$. (2): If there is an open cover $\{V_i\}$ of Y and each restricted morphism $\pi^{-1}V_i \to V_i$ is in this class, π is in the class.

Remark 3.2. need more remarks on stalk local and affine local

The first consequence satisfied by all nice classes $\mathcal{P}$ is that it's also **preserved by product**, before we do this, a minor lemma is useful:

Lemma 3.3. Suppose that we assume that all the fibered product involved in this statement exist in the category we are wokring with, then suppose that we have morphism $X \to Y$ and $Z \to Y$. Then if we also have a moprhism $W \to$, then we have a canonical isomorphism between $(X \times_Y Z) \times_Z W \cong X \times_Y W$. We call this cancellation law since the two Z' are canceled.

Proof. The proof only uses the fact that the stack of Cartesian squares is still Cartesian, then the following diagram will tell you the answer:

$$\begin{tikzcd} (X \times_Y Z) \times_Z W \arrow[r] \arrow[d] & W \arrow[d] \\ X \times_Y Z \arrow[r] \arrow[d] & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

$\square$

Proposition 3.4. Requirements 1,2 in Def 3.1 implies that: suppose that two morphism of S-schemes, $X \to Y$ and $X' \to Y'$ are with property P. We will first assume that the fibered products exists for $X \times_S X'$ and $Y \times_S Y'$, then the natural morphism $X \times_S X' \to Y \times_S Y'$ has that property P.

Proof. Consider the following diagram:

$$\begin{tikzcd} X \times_S X' \arrow[r] \arrow[d] & X \times_S Y' \arrow[r] \arrow[d] \arrow[dr] & Y \times_S Y' \arrow[dr] & \\ X' \arrow[r] & Y' & X \arrow[r] & Y \end{tikzcd}$$

Where the left square is Cartesian from Prop 3.3, so the the right square is Cartesian by a similar argument. Then since property P is preserved by pullback, then the morphism $X \times_S X' \to X \times_S Y'$ and $X \times_S Y' \to Y \times_S Y$ have P, since it's preserved by composition, we are done. $\square$

Another important property coming from being preserved by pullback and composition is called the Cancellation theorem. Since the formulation of this theorem involves the introduction of another of morphisms, we will save the theorem for now. But the following 5 properties are the properties that any reasonable classes of morphisms should have, an if you invent a new notion of morphism which doesn't satisfies some of the requirements, you should try to figure if you can make the definition better in order to fix it. We will give examples later.

An an small example we can play with, here are an exercise showing that we included all the isomorphisms in the requirement of nice classes of morphisms is actually reasonable:

Exercise 1. All isomorphisms satisfies $1, 2, 3$ in Def 3.1.

Proof. We only check that it's preserved by pullbacks. Notice that for an isomorphism, it's easy to construct it's pullback square:

$$\begin{tikzcd} Z \arrow[r, "id"'] \arrow[d, "f^{-1}\circ g"'] & Z \arrow[d, "g"'] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

You can check that it does satisfies the universal property. $\square$

The second not so trivial example is the open embeddings:

Definition 3.5. A morphism $\pi : X \to Y$ of schemes is called an **open embedding** or **open immersion**. If it's an open embedding of ringed spaces, in other words it can be decomposed as:

$$(X, \mathcal{O}_X) \xrightarrow{\rho} (U, \mathcal{O}_Y|_U) \xhookrightarrow{\tau} (Y, \mathcal{O}_Y)$$

where the first is an isomorphism and the second is the open inclusion of open subschemes.

In particular, if X is actually a subset of Y and ρ is the identity, we say that $(X, \mathcal{O}_X)$ is an open subscheme of $(Y, \mathcal{O}_Y)$. The difference between an open embedding an an open subscheme is minor and we will just interprete all open embeddings to be open subschemes.

There is a useful property of open embeddings, which is the following one:

Proposition 3.6. Suppose that we have $X \hookrightarrow Y$ is an open embedding, then suppose that $f : W \to Y$ is any morphism of schemes, then there exist a morphism from $X \to W$ such that the following diagram commutes iff $f(W) \subset X$ in set-theoretic sense.

$$\begin{tikzcd} & W \arrow[d, "f"] \arrow[dl, dashed] \\ X \arrow[r, hook] & Y \end{tikzcd}$$

Proof. First that if there is such a map, then our condition is trivially satisfied, thus consider the converse. Notice that this is true in the category of topological spaces. For schemes, we also don't have any choices, since this f gives a $f^{\#} : \mathcal{O}_W \to f_*\mathcal{Y}$, which means that for each $U \subset Y$, we have $\mathcal{O}_Y(U) \to \mathcal{O}_W(f^{-1}U)$. Then for any $U \subset X$, if we can really find a morphism such that the above diagram commutes, we have $\mathcal{O}_X(f^{-1}U)$, if we want the composition matches, suppose that $V \subset Y$, then we have $\mathcal{O}_Y(V) \to \mathcal{O}_W(f^{-1}V) = \mathcal{O}_Y(V) \to \mathcal{O}_X(X \cap V) \to \mathcal{O}_W(f^{-1}(X \cap V))$, one thing to notice that is $f^{-1}V = f^{-1}(X \cap V)$ since $f(W) \subset X$. Then we have such diagram:

$$
\begin{tikzcd}
\mathcal{O}_Y(V) \arrow[rr, "res"] \arrow[d, "id"'] \arrow[drr, "f"] & & \mathcal{O}_Y(V \cap X) \arrow[d, "f"] \\
\mathcal{O}_Y(V) \arrow[r, "res"'] & \mathcal{O}_X(X \cap V) \arrow[r] & \mathcal{O}_W(f^{-1}(V \cap X))
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{O}_Y(V) \arrow[r, "res"] \arrow[d, "f"'] & \mathcal{O}_Y(V \cap X) \arrow[d, "f"] \\
\mathcal{O}_W(f^{-1}V) \arrow[r, "id"'] & \mathcal{O}_W(f^{-1}(X \cap V))
\end{tikzcd}
$$

The left diagram commute if we set $\mathcal{O}_X(X \cap V) \to \mathcal{O}_W(f^{-1}(V \cap X))$ to be f according to the right diagram. It can only be f if we take V to be $V \cap X$ in the first place. □

Easy to see that open embedding is preserved under composition and it's local on the target. For example, if we want to show that $\pi : X \to Y$ is an open embedding if there is an open cover $\{V_i\}$ of Y and $\pi : \pi^{-1}V_i \to V_i$ is, then we can prove that it's the composition of $\rho : X \to \pi(X) \hookrightarrow Y$. It's easy to see that $\pi(X)$ is indeed open, and this ρ, which is considered the restriction of $\pi : X \to \pi(X)$, is an isomorphism since you can check isomorphism locally on the target. Moreover, it's also preserved by pullback, which is the content of the next theorem:

Proposition 3.7. Suppose that $U \hookrightarrow Z$ is an open embedding, then for any $\rho : Y \to Z$, $U \times_Z Y$ exists and the morphism $U \times_Z Y \to Y$ is an open embedding. The detail is in the next diagram:

$$
\begin{tikzcd}
\rho^{-1}(U) \arrow[r, hook] \arrow[d, "\rho|^{U}_{\rho^{-1}(U)}"] & Y \arrow[d, "\rho"] \\
U \arrow[r, hook] & Z
\end{tikzcd}
$$

Proof. We only have to show that this is square is Cartesian. Suppose that we are given such a diagram where the inner square and the outer square commute:

$$
\begin{tikzcd}
W \arrow[drr, "f"] \arrow[ddr, "g"'] & & \\
& \rho^{-1}(U) \arrow[r, hook] \arrow[d, "\rho|^{U}_{\rho^{-1}(U)}"] & Y \arrow[d, "\rho"] \\
& U \arrow[r, hook] & Z
\end{tikzcd}
$$

The fact that the outer square commutes tells us that $\rho \circ f$ has image inside U, thus this implies again that f has image inside $\rho^{-1}(U)$, thus by Prop 3.6, we see that there is only a unique morphism from $\psi : W \to \rho^{-1}(U)$ such that the upper triangle in this diagram commutes:

$$
\begin{tikzcd}
W \arrow[drr, "f"] \arrow[ddr, "g"'] \arrow[dr, dashed, "\psi"] & & \\
& \rho^{-1}(U) \arrow[r, hook] \arrow[d, "\rho|^{U}_{\rho^{-1}(U)}"] & Y \arrow[d, "\rho"] \\
& U \arrow[r, hook] & Z
\end{tikzcd}
$$

Thus, now we only have to make sure the lower triangle commutes:

This diagram commutes in on the level of topological spaces since we know that on the level of topological spaces, ψ is just f and then $\rho \circ f = g$ since we have the fact that the outer square commutes. Then only the level of sheaves, suppose that $V \subset U$ is open, we again use the outer square commutes to show that the morphism of sheaves also agree. We are done. □

We discuss more on open embeddings before we move to the next class of morphisms.

Recall, if X can be covered by **finitely many** $SpecA_i$'s where each A_i is Noetherian, then X is called a **Noetherian Scheme**. If X can be covered by not necessarily finitely many $SpecA_i$ where A_i's are Noetherian, then X is called **locally Noetherian**.

Proposition 3.8. Suppose that $X \hookrightarrow Y$ is an open embedding, then if Y is locally noetherian or noetherian, then X is. But if Y is quasicomapct, X maynot be.

Proof. Notice that localizations of Noetherian rings are Noetherian, this proves the first assertion about locally Noetherian. For noetherian to work, notice that Noetherian schemes are noetherain topological spaces, and we know that open subsets of noetherian spaces are noetherian, then noetherian implies quasicomapct. This proves the second part about noetherian.

For the sake of completeness, we prove that noetherian topological spaces are all quasicompact. Notice that being noetherian is equivalent of every nonempty collection of closed subsets has a minimal element, this is equivalent of every nonempty collection of open subsets has a maximal element. Then suppose that $\mathfrak{U}$ is an open cover of X, the collection of open subsets consisting of all the finite unions in $\mathfrak{U}$. Then, this has a maximal element, we claim that it's X, otherwise it's easy to see that this contradict the maximality.

Finally, we give an example of Y is quasicompact but X is not. Notice that for all rings A, $SpecA$ is quasicompact. Then suppose that $A = k[x_1, \cdots, x_n, \cdots]$, $SpecA$ is quasicomapct, then consider $m = (x_1, \cdots, x_n, \cdots)$, we claim the complement of $V(m)$ is not quasicompact. You can see that $D(x_i)$ for all i will cover the complement since the complement of $D(x_i)$ will be $V(x_i)$, then the intersection of all the complement is $V(m)$, thus it covers the complement of $V(m)$. But this doesn't have a finite subcover since for each finite subcover, we choose it's maximal plus 1, denote it by i, you find $(x_1, \cdots, x_i)$ is not in $V(m)$, but $(x_1, \cdots, x_i)$ is also not in $D(x_j)$'s for the finite subcover for obvious reasons. $\square$

Besides the **local on the target** condition, there is an analogous notion of local on the source:

Definition 3.9. Suppose that P is a property of morphisms of schemes, it's called local on the source if $\pi : X \to Y$ has it and $U \hookrightarrow X$ is an open embedding then $\pi|_U : U \to Y$, π restricted to U has P. Also, conversely, if there is an open cover U_i of X and π restricted to each U_i has P, then π has P. Then you can similarly define **affine local on the target** or **the source**. Obviously, when you want to prove something is affine local on the target, you will find the **affine communication lemma** useful.

But this is not as good as local on the target and you can easily see that open embeddings are not local on the target, for example, if we have the disjoint union of two X, then define a morphism to only one copy of X by on each X it's the identity, then you can see that it's an open embedding on X but clearly not open embedding globally.

Now let's begin to tackle a few more complicated notions than just the open embedding. We will introduce **quasicompact**, **quasiseparated**, **affine**, **finite**, **integral**, **closed embeddings**, **(locally) of finite type** and **quasifinite** morphisms.

We begin with quasicompactness and quasiseparatedness, which are two most straight-forward finite conditions, also for notational simplicity, when quasicomapct and quasiseparated appear simultaneously we will call them **qcqs**:

Definition 3.10. A morphism $\pi : X \to Y$ is **quasicompact**, if for every affine open $U \subset Y$, $\pi^{-1}U$ is quasicompact. Or equivalently, every quasicompact open subset $U \subset Y$ has quasicompact preimage. They are equivalent since every quasicomapct subset in a scheme has a finite affine cover.

Definition 3.11. A morphism $\pi : X \to Y$ is called **quasiseparated** if for every affine open subset $U \subset Y$, $\pi^{-1}U$ is quasiseparated. Recall that a topological space is **quasiseparated** if any intersection of two qussiseparated open subsets are quasicompact. This is called quasiseparated since it's a weakened version of being separated, which implies that any intersection of two affine open subsets are affine.

Remark 3.12. Like the notion of a morphism being quasicompact, this notion of being quasiseparated has an equivalent definition in terms of the inverse image of every quasiseparated open subset is quasiseparated. To prove this, notice that if inverse image of every quasiseparated open is quasiseparated, this implies that every inverse image of affine is quasiseparated since affines are quasiseparated. Conversely, notice that a scheme is quasiseparated if intersection of any two affine opens are a finite union of affine. The converse need to wait for our characterization of quasiseparatedness using the diagonal being quasicomapct later.

Proposition 3.13. Locally Noetherian schemes are quasiseparated, in particular Noetherian schemes are quasiseparated.

Proof. The key here is that every affine open in a locally Noetherian scheme is Noetherian since it can be covered by finitely many spectra of Noetherian schemes, thus Noetherian. In particular, they are Noetherian spaces, and any open subsets of Noetherian spaces is Noetherian and Noetherian space implies being quasicompact. We are done. ☐

Proposition 3.14. Composition of quasicompact morphisms are quasicomapct, and so is true for quasiseparated morphism.

Proof. We still only prove for quasicompact since the latter one is earlier done after we review the notion of separatedness. But notice that the inverse af affine is a finite union of finite affine, then take the inverse image again. ☐

Proposition 3.15. Any morphism from a Noetherian scheme is quasicompact, but this isn't true in general for locally Noetherian schemes.

Proof. Notice that Noetherian schemes are in particular noetherian spaces since it's a finite union of noetherina spaces, thus every open subsets of this space is noetherian thus is quasicomapct. ☐

Proposition 3.16. Any morphism from a quasiseparated scheme is quasiseparated. Then in particular, since (locally) Noetherian schemes are quasiseparated, combined with the above result, we know that any morphism from a Noetherain space is qcqs.

Proof. We need to prove the inverse image of an affine open is quasiseparated, but notice that any open subsets of a quasiseparated space is quasiseparated. This is because that quasicompactness behaves well under open subsets, thus we are done. ☐

Proposition 3.17. Both quasicomapct and quasiseparated morphisms are affine local on the target. That is, a morphism $\pi : X \to Y$ is quasicompact or quasiseparated if there is an affine cover of Y by affine opens $\{U_i\}$ such that the inverse image is quasicomapct or quasiseparated.

Proof. Let's use **affine communication lemma**, and we will deal with quasicomapct first. Notice that if $SpecA$ has quasicompact inverse image, then notice that for any $D(f)$ inside, we want it to have quasicompact inverse image. This follows from the fact that since the inverse image of $SpecA$ is quasicomapct, then it a finite union of affine opens, then restrict to each of them, we see that the intersection of the inverse image with them is distinguished, thus in particular quasicomapct since it's affine, thus finite union of quasicomapct is quasicompact.

Conversely, if $D(f_i)$'s have quasicomapct inverse for finitely many of them and $\cup_i D(f_i) = SpecA$, then the inverse image of $SpecA$ is just finite union of quasicompact, thus quasicompact.

Now for quasiseparated, open subsets of quasiseparated space is quasiseparated, this is the first part of affine communication lemma. Conversely, suppose that the inverse image of $D(f_i)$'s are quasiseparated, then we know that suppose that $SpecB$ and $SpecC$ are affine in the inverse image, then it's intersection is the union of the intersection with the inverse image of $D(f_i)$. Notice that the union of the two intersection is quasiseparated since it's an open subset of the inverse image of $D(f_i)$, thus since these two are affine, then the intersection is finite union of affines, then we can express the intersection of $SpecB$ and $SpecC$ to be finitely many such stuff, we are done. ☐

Remark 3.18. It's true that quasicomapct and quasiseparated morphisms are **stable under base change**, but we will prove this later after we review more on fibered products.

Now we go to the notion of affine morphisms:

Definition 3.19. A morphism $f : X \to Y$ is called an **affine** morphism if for every affine open subset U of Y, $\pi^{-1}U$ is an affine open subscheme of X. Trivially, composition of affine morphism is still affine.

Another fast observation is that:

Proposition 3.20. Affine moprhisms are qcqs.

Proof. Since affine schemes are quasicompact and quasiseparated by our previous remarks, affine morphisms are qcqs. ☐

Proposition 3.21. We will show that affine morphisms are local on the target, i.e. a morphism is affine if there is an affine open cover of Y such that the inverse image is affine.

Proof. Obviously this is still an argument involving affine communication lemma. Suppose that $f : X \to Y$ is a morphism such that $\pi^{-1} SpecB = SpecA$, then we notice that the restriction of π to $SpecA$ is induced by $\varphi : B \to A$, thus we know that for any $s \in B$ the inverse image of $D(s)$ will be $D(\varphi(s))$, affine.

For the converse, there are several ways to do this, one of which is to use an independent criterion we will show later. Or you can directly solves this problem, which essential uses the same idea. It's just one of which is more general and another one is more specific.

We first use the cirterion. Now suppose that $f : X \to SpecB$ is a morphism, where we have $f^{-1}D(f_i)$ are affine and $(f_1, \cdots, f_n) = B$. Notice that this morphism is uniquely determined by a ring morphism $\varphi : B \to \Gamma(X, \mathscr{O}_X)$, thus we know that the inverse image $f^{-1}D(f_i) = X_{\varphi(f_i)}$. This is because the locus where $\varphi(f_i)$ can be checked affine locally. Then we use the cirterion saying that X is affine iff there are finitely many elements in the global section such that the vanishing locus is affine open and these elements generates the global section.

Or you can tackle this problem specifically and the idea of which is also going to be used in the proof of the above criterion. Now again we still have $f : X \to SpecB$ is a morphism, where we have $f^{-1}D(f_i)$ are affine and $(f_1, \cdots, f_n) = B$, still denote by $\varphi : B \to \Gamma(X, \mathscr{O}_X)$ the morphism that induces the scheme morphism, we want to show that X is affine. How can we do it, well, if it's affine, it must be isomorphic to the spec of it's global section, then consider $g : X \to Spec\Gamma(X, \mathscr{O}_X)$ induced by the identity map, we want to show this is an isomorphism. We will show that g is an isomorphism over $D(\varphi(f_i))$, which proves that g is an isomorphism. For notational simplicity, we will use A to denote $\Gamma(X, \mathscr{O}_X)$.

To do this, first, what is the inverse image, $g^{-1}D(\varphi f_i)$, one quick way to see this is by the below commutative diagram:

$$\begin{tikzcd}
 & SpecB \\
X \arrow[ur] \arrow[dr] & \\
 & SpecA \arrow[uu]
\end{tikzcd}$$

This is commutative since it's induced by the same ring map. Now, notice that the inverse image of $D(f_i)$ in $SpecA$ is $D(\varphi(f_i))$, thus we know that the inverse image is just $X_{\varphi(f_i)}$. Then notice that we know since f is qcqs since qcqs is an affine local notion. Then we can apply the **qcqs lemma**, we know that the localization from $\Gamma(X, \mathscr{O}_X)$ to $\Gamma(X_f, \mathscr{O}_X)$ is precisely the localization, (via an isomorphism), then we know that the map that induces the morphism $X_f \to D(\varphi(f_i))$ is going to be:

$$\begin{tikzcd}
 & \Gamma(X, \mathscr{O}_X) \arrow[r, "id"] \arrow[dl, "loc"'] \arrow[d, "res"'] & A \arrow[d, "loc"] \\
\Gamma(X, \mathscr{O}_X)_{\varphi(f_i)} \arrow[r, leftrightarrow, "iso"'] \arrow[rr, dashed, bend right, "iso"'] & \Gamma(X_{\varphi(f_i)}, \mathscr{O}_X) \arrow[r] & A_{\varphi(f_i)}
\end{tikzcd}$$

Then, we are done. ☐

A non-obvious consequence of the affine locality of affine morphism is the following:

Proposition 3.22. Suppose that Z is a closed subset of $SpecA = X$ where Z is locally cut out by a single equation in the sense there is an open cover U_i of $SpecA$ and on each U_i, $U_i \cap Z$ is cut out by a single element in $\Gamma(U_i, \mathscr{O}_X)$, i.e. $U_i \cap Z$ is the vanishing locus of that element. Then the complement of Z is affine.

Proof. It's obvious that if Z is globally cut out by a single element $f \in A$, then the complement of Z is going to be $D(f)$. However, this is not so clear when Z is locally cut out by a single element.

Now consider the open immersion: $SpecA - Z \hookrightarrow SpecA$, the assertion in the statement is equivalent to this is is an affine morphism. Now, notice that we can choose distinguished opens of $SpecA$ such that they cover $SpecA$ and each of them is contained in a U_i we use in the statement. Therefore, we know that on $D(f_i)$, we know by the hypothesis that the inverse image is an affine open, thus we use the affine locality to show that this is a affine, thus inverse image of every affine is affine, thuse the inverse image of $SpecA$ is affine. $\square$

Now we turn to integral and finite morphisms:

Definition 3.23. We say that a morphism $f : X \to Y$ is a **finite** morphism if for **every** affine open $SpecB \subset Y$, we have $\pi^{-1}SpecB$ is the specturm of a **finite** B-algebra. Notice that by definition, finite morphisms are affine.

Again, our first result will show that finite morphism is affine local on the target:

Proposition 3.24. $f : X \to Y$ is finite if there is an affine open cover of Y such that the inverse of those affine opens is the spectrum of a finite algebra.

Proof. First, $f^{-1}SpecB = SpecC$ where $\varphi : B \to C$ makes C a finite B-algebra. Then to prove this first assertion in the affine communication lemma, we only have to prove that if $\varphi : B \to C$ is finite, then $\varphi_f : B_f \to C_{\varphi f}$ is still finite. This follows from that localization preserves finite generation.

Conversely, notice that if $C_{\varphi f_i}$ is finite B_{f_i}-modules, then again use the fact that finite generation is affine local, we get C is finitely generated as an B-module. $\square$

The following examples will give you some intuition of finite morphisms, where you can see that all of them is closed, finite to one as a map of sets.

Example 3.25. Consider the morphism $f : Speck[t] \to Speck[u]$ given by $u \mapsto p(t)$ for a degree n polynomial. We claim that this is finite since you can show that $k[t]$ is a finitely generated $k[u]$-module by $1, t, \cdots, t^{n-1}$. We only have to we can represent t^m for any m as desired. Suppose that $m < n$, then it's trivial. Suppose that $m = n$, then notice that $1 \cdot u = p(t)$, then you make sure that the leading coefficient is 1, then we get t^n by minus all the lower degree terms. Then for all $m \leqslant 2n$, we can consider ut^k then minus all the lower degree terms use induction, higher terms are similarly obtained. This is usually called the **branched cover**.

The second example is **closed embedding**s, we know that $A \to A/I$ is finite, thus we are done.

The third example is the **normalization**. A classical example is the map $Speck[t] \to k[x,y]/(y^2-x^2-x^3)$, the affine line over the nodal cubic. This is defined by: $k[x,y]/(y^2 - x^2 - x^3) \to k[t]$ where $x \mapsto t^2 - 1$ and $y \mapsto t^3 - t$, this is a well defined map by the universal property of the quotient. This is a finite morphism since we have: $k[t]$ is generated as a $k[x,y]/(y^2 - x^2 - x^3)$-module by $1, t$. Again, we only have to show this for t^m, $t < 4$ is trivial, then use induction again, $m = 4$, it's just $ty + t^2$, then for $m + 1$, if that is an even number, we are done, it's that is an odd number suppose that $2k + 1$. Notice that we can use $x^{k-1}y$, then we are done.

The last example is that: if $X \to Speck$ is a finite morphism, then X is a finite union of points with discrete topology. We don't want to quote some fancy theorem, so we give a hands-on argument. First, notice that since it's finite, we can assume that $X = SpecA$ and the map is induced by a ring map $k \to A$. We first show that every prime of A is maximal. To show this, suppose that we have a strict containment of prime ideals $p \subset p'$ but $p \cap k = p' \cap k$, this is a contradiction of the fact that finite morphisms are in particular integral, and for an integral morphism, you can't have strict containment in the target but equality in the source. Then notice that since A is a finite k-algebra, it's a Noetherian ring, thus only finitely many irreducible components, but notice that these components correspond to the closure of minimal primes, which are simultaneously maximal, thus it only has a finite collection of closed points, each of which is an irreducible component. This helps us to prove that this topology is discrete, notice that irreducible components are always closed, since closure of irreducible sets are irreducible. Now, finite union of closed subsets is closed, thus each point is also open. Finally, what is the residue field, which is A/m for a maximal ideal. Notice that we have $f : k \to A/m$, this is still finite since $k \to A$ is finite, thus the residue field is just a finite extension of k.

Now we prove some easy properties of finite morphism:

Proposition 3.26. Composition of finite morphism is finite.

Proof. This basically follows from the fact that composition of finite ring maps is finite. □

Proposition 3.27. This is a proposition that basically shows that a finite morphism is both **affine** and **projective**, though we will have to define what projective means later. To be precise, if R is a finite R-algebra we define a graded ring S by define $S_0 = A$ and $S_n = R$ for all $n \geqslant 1$. We will denote elements in S by $\sum_i a^i_{n_i}$ where the subscript is used to indicate which component it's in and the superscript is used to distinguish that even if we all use a, they can be different elements in R. The multiplication is the multiplication in R, with the corresponding degree kept track of. There is an isomorphism $ProjS \to SpecR$.

Moreover, if R is a finite A-algebra which means that it's a finitely generated A-module. We can conclude that S is a finitely generated graded ring over A, thus $SpecR$ is a projective A-scheme.

Proof. First, let's establish the isomorphism. We still define the morphism locally then show that it glues together to form a global isomorphism.

Suppose that $f_m \in S$, we consider $D_+(f_m)$, which is $SpecS_{(f_m)}$, we claim that it's isomorphic to $D(f^n) = SpecR_f$ via the ring map $R_f \to S_{(f_m)}$ defined by:

$$\begin{tikzcd} R \arrow[d] \arrow[dr] & \\ R_f \arrow[r] & S_{(f_m)} \end{tikzcd}$$

where $R \to S_{(f_m)}$ is defined by $x \mapsto \frac{x_0}{1_0}$. Then you will see that for f^n, it's mapped to $f^n_0/1_0$, it's inverse is just $1_{mn}/f^n_m$, thus by the universal property, we have $R_f \to S_{(f_m)}$. We claim that this is an isomorphism. First, it's obvious it's injective, then it's surjective since if we have $\frac{a_{mn}}{f^n_m} \in S_{(f_m)}$, it's easy to check that $\frac{a}{f^n}$ maps to this element, thus isomorphism.

Therefore, the only thing we need to do is to check the gluability condition, which is to check the commutativity of the following diagram:

$$\begin{tikzcd} & R_{f_m} \arrow[r] \arrow[dl] & S_{(f_m)} \arrow[dr] & \\ R_{f_m g_l} \arrow[rrr] & & & S_{(f_m g_l)} \\ & R_{g_l} \arrow[r] \arrow[ul] & S_{(f_m)} \arrow[ur] & \end{tikzcd}$$

This is left for you to check since everything is defined canonically.

Now, R is a finite A-algebra, suppose that $r_1, \cdots, r_n$ generates R as an A-module. Now, suppose that f_m is a homogeneous element of positive degree in S, notice that $f = \sum a_i r_i$, then it's obvious that S is generated as an A-algebra by $r_1, \cdots, r_n$ treated as degree 1 elements. □

The last important of finite morphism we will prove in this section is finite morphisms have finite fibers. Though we haven't quite defined what is the fiber of a morphisms of schemes as a scheme, we are only doing this as a set theoretic argument in the following proposition and it will be more enlighting to do it here:

Proposition 3.28. Finite morphisms have finite fibers.

Proof. Suppose that $f : X \to Y$ is a finite morphism. First, we can assume that Y is affine, thus being finite means that we can also assume that X is affine. Then suppose that $f : SpecA \to SpecB$ is finite, we first claim that we can reduce the problem to the case of B is an integral domain. To see this, suppose that we want to find what is the preimage of $q \in SpecB$, this means that we are trying to find primes in A such that the inverse image is q. Suppose that $\varphi : B \to A$ is the ring map that induces the morphism of schemes. Then notice that we have a induced ring map: $\overline{\varphi} : B/q \to A/(\varphi q)$, and we know that if a prime in A has inverse image containing q, then it must contain (φq), thus we reduced ourselves to the case of B is a integral domain and we are trying to find which primes has inverse image the generic point. Obviously, this preserves the finiteness.

Then suppose that now we have $SpecA \to SpecB$ where B is an integral domain, we can consider $\varphi_0 : Frac(B) \to S^{-1}A$ where S is the image of $B - \{0\}$. Then we see that this is also finite and moreover suppose that q is a prime in A such that the preimage is (0), then we know that q doesn't intersect the image of $B - \{0\}$, thus we reduced to the case where B is a field, which we already did before, thus we are done. $\square$

Now, we consider two examples where we show that two morphism which have finite fibers that are not finite morphisms:

Example 3.29. The first one is a trivial example: consider the open embedding $\mathbb{A}^2_k - \{0\} \hookrightarrow \mathbb{A}^2_k$, this is clearly a morphism with finite fiber, but this is not affine, thus not a finite morphism.

The next one is the open embedding of $\mathbb{A}^1_{\mathbb{C}} - \{0\} \hookrightarrow \mathbb{A}^1_{\mathbb{C}}$, this is of course affine, and the fiber is finite. However this is not finite since if you think of $\mathbb{C}[t] \to \mathbb{C}[t, t^{-1}]$, this is obviously not finite since the negative terms are not feasible by finitely many generators only using the module structure.

Finally, we get to the notion of **integral morphisms**:

Definition 3.30. A morphism $\pi : X \to Y$ of schemes is **integral** if it's affine and $\pi^{-1}SpecB = SpecA$ where the corresponding ring map $B \to A$ is integral.

The following three propositions are intended to show that integral morphisms is affine local on the target and closed under composition:

Proposition 3.31. Suppose that $\phi : B \to A$ is a ring morphism such that $(b_1, \cdots, b_n) = B$ and $\phi_{b_i} : B_{b_i} \to A_{\phi(b_i)}$ is integral for all i, then ϕ is integral.

Proof. Let's first show that we can replace B by the image $\phi(B)$ to assume that B is a subrings of A. To show this, notice that each of the $\phi_{b_i} : B_{b_i} \to A_{\phi(b_i)}$ factors through $B_{b_i} \to \phi(B)_{\phi(b_i)}$, then it's obvious that we can replace B by $\phi(B)$. Then suppose $t \in A$, consider $\frac{t}{1}$ in each of A_{b_i}, we know that since it's assumed to be integral, we know that there is a monic polynomial with variable $\frac{t}{1}$ and coefficients in B_{b_i} such that it's equal to 0, then we can clearly the denominator to conclude that there is t_i and m_i such that we have:

$$b_i^{t_i} a^{m_i} \in B + Ba + \cdots + Ba^{m_i}$$

By multipliting b_i an a, we can assume from we only have finitely many b_i that this there is t, m that works for all i. Then since b_i generates B, we can use them to write 1, we know that $a^m \in B + Ba + \cdots + Ba^m$, we are done. $\square$

Proposition 3.32. A ring map $\phi : B \to A$ is integral preserved by localization and quotients of B. It's also preserved by quotients of A, but not localizations of A.

Proof. Let's first prove $T^{-1}B \to \phi(T)^{-1}A$ is integral. Let $\frac{a}{\phi(b)}$ is in $\phi(T)^{-1}A$. Notice that since $0 = a^n + b_{n-1}a^{n-1} + \cdots + b_0$, we know that $(a^n + b_{n-1}a^{n-1} + \cdots + b_0)\phi(b)^k = 0$ for all k, thus we know that $\frac{a}{\phi(b)}^n + \frac{a}{\phi(b)}^{n-1} \cdot \frac{1}{\phi(b)} + \cdots + \frac{b}{1} = 0$ since you can clear the denominator.

Also, $B/J \to A/\phi(J)A$ is obviously integral since everything here is just formal. Also, this shows that $B \to A/I$ is also integral. However, $B \to S^{-1}A$ may not be integral. An example is $k[t] \to k[t]_{(t)}$. The fundamental reason is $k[t]$ is a UFD, thus it's integrally closed in $k(x)$. But we can also give a example, $k[t] \to k[t]$ is integral but $k[t] \to k[t]_t$ is not since you can consider $\frac{1}{t}$. $\square$

Proposition 3.33. The property of being an integral extension is preserved by localization of B, but not quotients of B or localization or quotients of A.

Proof. $\square$

The following lemma is useful for a later proposition:

Proposition 3.34. Let $\phi : B \to A$ be a ring homomorphism, then $a \in A$ is integral over B iff a is in a sub B-algebra of A that is a finitely generated B-module.

Proof. □

The consequence of which is the following seemingly trivial result:

Proposition 3.35. Suppose that $f : C \to B$ and $g : B \to A$ are both integral then $g \circ f$ is integral.

Proof. Suppose that $a \in A$, then since g is integral, then a in in a finitely generated B-module, suppose that it's generated by $a_1, \cdots, a_m$, then you can also generate a_i using finitely many elements in b, thus this algebra is also a finitely generated C-module, thus we are done. □

Therefore, you can see that integral morphisms is **closed under composition**.

Proposition 3.36. Integral morphisms are affine local on the target.

Proof. We already see that if $\phi : B \to A$ is integral, then it's preserved by localizations on B, thus we know that $B_f :\to A_{\phi(f)}$ is going to be integral, this is the first half of the affine communication lemma.

Conversely, this is the content of a previous proposition, we are done. □

Now, we fulfill our promise to show that finite morphism is closed by showing that integral morphism is closed:

Proposition 3.37. Suppose that $f : X \to Y$ is an integral morphism, then it's closed in the sense that a closed subset of X is mapped to a closed subset of Y.

Proof. Suppose that C is closed in X, we want to show that $f(C)$ is closed, then we have to show that there is an open cover in Y such that the intersection with $f(C)$ is closed in that open. Thus this can be reduced to the case where Y is affine, thus X is also affine since the morphism is assumed to be integral.

Then suppose that $f : SpecA \to SpecB$ which is induced by $\phi : B \to A$ an integral ring morphism. We want to show that if $V(I)$ is closed in $SpecA$, then it's image is also closed, which means that it's $V(J)$ for some ideal in B. Consider $J = \phi^{-1}I$, which gives us $B/J \to A/I$ which is still integral and most importantly injective, thus an integral extension. Then we know that by the lying over property of integral extension, any prime in B/J, there will be an prime in A/I such that the inverse image is that prime. The same property holds in $B \to A$, thus we know that $V(J)$ is in the image of $V(I)$. On the other hand, if some prime is in the image, we know that it must lie in $V(J)$, thus we know that $V(J)$ is equal to the image, thus the image is closed. □

Again, integral morphisms is still closed under base change, which will be discussed in detail later in the Review section of fibered product, but the below algebraic lemma is useful there:

Lemma 3.38. Suppose that $\phi : B \to A$ is integral, and $B \to C$ is any other ring map, we have the induced map $C \to A \otimes_B C$ is integral.

Proof. The picture is like below:

$$\begin{tikzcd} A \otimes_B C & C \arrow[l] \\ A \arrow[u] & B \arrow[l] \arrow[u] \end{tikzcd}$$

the map $C \to A \otimes_B C$ is $c \mapsto 1 \otimes c$. We want to prove this is integral. Suppose that $\sum a_i \otimes c_i$ is in $A \otimes_B C$, we only have to show that each of them is integral them the summation is going to be integral since we already know that the elements that are integral is a subring of $A \otimes_B C$. Thus suppose that $a \otimes c$ is in $A \otimes_B C$, since A is integral over A, we have $a^n + a^{n-1}b_{n-1} + \cdots + b_0 = 0$, then notice that if we denote the images of b_i in C still by b_i. Then we have $(a \otimes 1)^n + (a \otimes c)^{n-1} \cdot (b_{n-1} \otimes c) + \cdots + b_0 \otimes c^n = (1 \otimes c)^n[(a \otimes 1)^n + (a \cdot b_{n-1} \otimes 1)^{n-1} + \cdots + b_0 \otimes 1] = 0$. □

The next useful finiteness condition is called (locally) of finite type:

Definition 3.39. A morphism $\pi : X \to Y$ is **locally of finite type** if for every affine open $SpecB$ in Y, each affine open $SpecA$ in the inverse of $SpecB$ has a structure of a finitely generated B-algebra via $B \to A$ the map induced by $SpecA \to SpecB$. This π is **of finite type** if it's locally of finite type and quasicompact.

Proposition 3.40. In the definition of locally finite type, we actually can use an affine open cover instead of every affine open.

Proof. This is basically the result of the following assertion: Suppose that B is a ring and A is an B-algebra, then A is a finitely generated B-algebra then A_{f_i} is for any f_i in A. If each A_{f_i} is a finitely generated B algebra, then so is A where $(f_1, \cdots, f_r) = A$.

The first assertion is obvious, you can easily verify that the old elements plus $\frac{1}{f}$ generates A_f. For the converse, suppose that A_{f_i} is generated by $\frac{a_i}{f_i^{n_i}}$ over B, we claim that all the a_i's plus f_i's generated A over B. Consider $\frac{a}{1}$, then by considering it in different A_{f_i}, since we only have finitely many of them, we can assume that af_i^m is a combination of polynomials with variables consisting of a_i's and f_i's and coefficients in B, then use a partition of unity argument to conclude. □

Thus, we know that a finite type morphism is a locally finite morphism such that the inverse image of every $SpecB$ can be expressed as a finite union of $SpecA_i$ and each A_i is a finitely generated B-algebra.

A very trivial example of a finite type morphism is $\mathbb{P}^n_A \to SpecA$ since we know that $\mathbb{P}^n_A$ is covered by $n+1$ affine opens of the form $SpecA[x_0, \cdots, x_n]$.

Now, we show that (locally) of finite type is also affine local on the target:

Proposition 3.41. A morphism $f : X \to Y$ is (locally) of finite type if there is an affine open cover $SpecB_i$ of Y such that $f^{-1}SpecB_i$ is (locally) finite type over $SpecB_i$.

Proof. To show the first assertion of the affine communication lemma, we need to show that if $f : X \to SpecB$ has the property that X is covered by Specs of finitely generated B-algebras. Consider $f^{-1}SpecB_f \to SpecB_f$, notice that if you think about $SpecA \to SpecB$ which is induced by $B \to A$ and A is a finitely generated B-algebra, then $SpecA \cap f^{-1}SpecB_f \to SpecB_f$, the left hand side is just $SpecA_{\varphi(f_i)}$ and the map is induced by $B_f \to A_{\varphi(f_i)}$, which is obviously finite type since it's generated. You can also easily see this is finitely generated by the same elements. They will cover the inverse image and if we can have a finite cover beforehand, we have a finite cover now.

The converse is easier, we still consider $f : X \to SpecB$, and if for each $f^{-1}SpecB_{f_i} \to SpecB_{f_i}$, it (locally) of finite type and we also have $(f_1, \cdots, f_r) = B$, notice that since it's (locally) finite type, there is a $SpecA$ in the inverse image of $SpecB_{f_i}$ such that the morphism is induced by $B_{f_i} \to A$ a finite type ring map, then notice that $B \to B_f$ is also of finite type and finite type is closed under composition. We are done, and notice that the finiteness condition is also satisfied. □

Below is some easy properties finite type morphism enjoys:

Proposition 3.42. Finite morphism is finite type.

Proof. Using the affine locality on the target and affine locality on the inverse imabe, it's just a trivial result of that finite ring is finite type. □

Proposition 3.43. A morphism $f : X \to Y$ is finite iff it's integral and finite type.

Proof. The implication of finite implies finite type and integral is simple.

Conversely, notice that integral plus finite type implies that the inverse image of $SpecB$ is the spec of a finitely generated B-algebra which is also integral. Then, we are just trying to prove a finite type and integral ring morphism is finite. We proceed by induction, suppose that A is generated by 1 elements, thus we have:

$$B \to B[a]$$

Notice that since a is integral, we have:

$$a^n + \cdots + ab_1 + b_0 = 0$$

we claim that $a^n, a^{n-1}, \cdots, a, 1$ generates $B[a]$ as a finite B-module. This is because for any higher term than a^n, you can multiply powers of a to the above expression and then eliminate what you don't need. Then for the induction step, suppose that this holds for $B \to B[a_1, \cdots, a_n]$ and we are considering $B[a_1, \cdots, a_{n+1}]$, notice that this map decomposes as:

$$B \to B[a_1, \cdots, a_n] \to B[a_1, \cdots, a_{n+1}]$$

Both are obviously finite type and integral, thus we are done since finite composed with finite is finite. □

The following properties are also important:

Proposition 3.44. Every open embedding is locally of finite type and thus a quasicompact open embedding is finite type, thus every open embedding into a locally Noetherian scheme is of finite type.

Proof. Suppose $U \hookrightarrow X$ is an open embedding, for every affine open $SpecA$ in X, since $SpecA \cap U$ is open in $SpecA$, it can be covered by distinguished affine opens, and since $A \to A_f$ is finite type, we are done.

When X is locally Noetherian, we only have to show that $U \hookrightarrow X$ is quasicompact. From the affine locality of quasicompactness, we only have to show that there is an affine open cover such that the inverse image is quasicompact, but we can cover X by specs of Noetherian schemes, thus we are done. □

Proposition 3.45. Morphism of locally finite type is closed under composition. Thus since we know that quasicompact morphisms is closed under compositions, morphisms of finite type is also closed under composition.

Proof. This is an easy consequence that finite type ring morphisms is closed under composition. □

Proposition 3.46. Suppose that $f : X \to Y$ is locally of finite type, then if Y is locally Noetherian, then X is locally Noetherian. Similarly, if f is of finite type and Y is Noetherian, then X is Noetherian.

Proof. This follows form the polynomial rings of finitely many variables over a Noetherian ring are Noetherian and quotients of Noetherian rings are Noetherian. □

Finally as an application, we consider the **absolute Frobenius Morphism**:

Example 3.47. We will define this morphism without any choice, of course you can try to define it on affine opens and try to glue them, but we don't have to do it here.

Consider the morphism of schemes $X \to X$ defined by the identity on the underlying topological space and the morphism of structure is defined by $\mathscr{O}_X \to \mathscr{O}_X$ where on each open U, we have $\mathscr{O}_X(U) \to \mathscr{O}_X(U)$ defined by $f \mapsto f^n$. This is a well defined morphism of locally ringed spaces since you can see that the restriction is easily satisfied.

Now if you go to affine opens, you see this is the morphism $A \to A$ defined by $f \mapsto f^p$. This makes sense as you can see that under this ring morphism, a prime $p \subset A$ is mapped back p under the inverse image since if any f is mapped to p, then $f^p \in p$, then $f \in p$. We are done with the definition of the morphism. This morphism is obviously affine.

Now, we claim that if X is locally of finite type over $\mathbb{F}_p$, then F is finite, which means that it's integral and of finite type. First, since X is locally of finite type over $\mathbb{F}_p$, you can cover X by specs of finitely generated $\mathbb{F}_p$-algebras. Consider $SpecA \to SpecA$, where $SpecA$ is in that cover, and you can see that if you write $A = \mathbb{F}_p[x_1, \cdots, x_n]$, the morphism is defined by:

$$\mathbb{F}_p[x_1, \cdots, x_n] \to \mathbb{F}_p[x_1, \cdots, x_n]$$

where $x_i \mapsto x_i^p$, we want to show this is finite, we claim that this is generated by 1 and all the polynomials in variables x_i and the degree of each of which is less than p, again we only have to show that any monomial can be written as a linear combination, but now for a monomial, we took out powers of multiples of p, they came from elements in $\mathbb{F}_p[x_1, \cdots, x_n]$ in the source, we are done.

There are also less important notions of **quasifinite** and **(locally) finitely presented** morphisms, which we will discuss later:

References