Appendix 11 for Math 632: 2025 Winter Notes

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This will be a short appendix on some further topics using cohomology. We will discuss higher pushforward sheaves, several results due to Serre and the relation between projective and proper morphisms.

We start with higher pushforward.

1 Higher Pushforward (or Direct Image) Sheaves

Throughout this section, the following theorem will be useful:

Theorem 1.1. Let $F : \mathcal{A} \to \mathcal{B}$ be a covariant functor of abelian categories, and $C^\bullet$ is a complex in $\mathcal{A}$.

1. If F is right exact, then there is a natural morphism $FH^\bullet \to$ ‘$H^\bullet F$.
2. If F is left exact, then there is a natural morphism $H^\bullet F \to FH^\bullet$.
3. If F is exact, then the morphism above are inverses and thus isomorphisms, i.e. F commutes with cohomology.

Proof. Let's deal with the first assertion. $FH^\bullet$ means that F applied to the cohomology and $H^\bullet F$ means that the cohomolgy of the complex $FC^\bullet$.

First, consider the following two sequences:

$$C^i \xrightarrow{d^i} C^{i+1} \longrightarrow \mathrm{Coker} d^i \longrightarrow 0$$

$$FC^i \xrightarrow{Fd^i} FC^{i+1} \longrightarrow F\mathrm{Coker} d^i \longrightarrow 0$$

since the top sequence is exact and F is right exact, the sequence at the bottom is also exact then consider:

$$\begin{tikzcd}
FC^i \arrow[r, "Fd^i"] & FC^{i+1} \arrow[r] & F\mathrm{Coker}d^i \arrow[r] & 0 \\
FC^i \arrow[r, "Fd^i"] \arrow[u] & FC^{i+1} \arrow[r] \arrow[u] & \mathrm{Coker}Fd^i \arrow[r] \arrow[u, dashed] & 0 \arrow[u, dashed, "0"]
\end{tikzcd}$$

then 5-lemma tells you that this morphism in the middle is an isomorphism.

Then we can give a natural morphism, or to be precise, an epimorphism, $F\mathrm{im}d^i \to \mathrm{im}Fd^i$ using the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{im}d^i \arrow[r] & C^i \arrow[r] \arrow[d, rightsquigarrow] & \mathrm{coker}d^i \arrow[r] & 0 \\
 & F\mathrm{im}d^i \arrow[r] \arrow[d, dashed] & FC^i \arrow[r] \arrow[d, dashed, "id"] & F\mathrm{coker}d^i \arrow[r] \arrow[d, dashed, "\simeq"] & 0 \\
0 \arrow[r] & \mathrm{im}Fd^i \arrow[r] & FC^i \arrow[r] & \mathrm{coker}Fd^i \arrow[r] & 0
\end{tikzcd}$$

there the morphism is induced using the universal property of kernel. I claim this is an epimorphism and it can be proved by the five lemma (or the two corresponding four lemma) since you can take the kernel of $F\mathrm{im}d^i \to FC^i$ and by the universal property of kernel, if has a natural morphism to 0, which is an epimorphism.

Then consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^i(C^\bullet) \arrow[r] & \mathrm{coker}d^{i-1} \arrow[r] \arrow[d, squiggly] & \mathrm{im}d^i \arrow[r] & 0 \\
 & FH^i(C^\bullet) \arrow[r] \arrow[d, dashed] & F\mathrm{coker}d^{i-1} \arrow[r] \arrow[d] & F\mathrm{im}d^i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^i(FC^\bullet) \arrow[r] & \mathrm{coker}Fd^{i-1} \arrow[r] & \mathrm{im}Fd^i \arrow[r] & 0
\end{tikzcd}$$

where the top sequence is exact and it comes from

$$\begin{tikzcd}
 & \mathrm{coker}d^{i-1} \arrow[r, dashed] \arrow[dr, dashed] & \mathrm{im}d^i \arrow[d] & \\
C^{i-1} \arrow[r] & C^i \arrow[u] \arrow[r] & C^{i+1} \arrow[r] & \mathrm{coker}d^i
\end{tikzcd}$$

and the most intuitive way to think of that is still via the module category and I will let you think why the kernel is the cohomology using the module point of view. Another point you should be aware of is why the square on the right indeed commutes, the way you should think of this is by the following diagram:

$$\begin{tikzcd}
 & F\mathrm{coker}d^{i-1} \arrow[r, dashed] \arrow[dr, dashed] & F\mathrm{im}d^i \arrow[d] & \\
FC^{i-1} \arrow[r] & FC^i \arrow[u] \arrow[r] & FC^{i+1} \arrow[r] & F\mathrm{coker}d^i
\end{tikzcd}$$

which is just by aaplying F to the above diagram and you should imagine that there is another one sitting abovt this diagram, with $F\mathrm{coker}d^{i-1}$ replaced by $\mathrm{coker}Fd^{-1}$ and $F\mathrm{im}d^i$ replaced by $\mathrm{im}Fd^i$, therefore you get a diagram that looks like the skeleton of a cube:

$$\begin{tikzcd}
 & \mathrm{coker}Fd^{i-1} \arrow[r] & \mathrm{im}Fd^i \\
Fc^i \arrow[ur] \arrow[r] & FC^{i+1} \arrow[ur] & \\
 & F\mathrm{coker}d^{i-1} \arrow[uu] \arrow[r] & F\mathrm{im}d^i \arrow[uu] \\
FC^i \arrow[uu, "id"] \arrow[ur] \arrow[r] & FC^{i+1} \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

and you want to show that the morphisms that form the back of the cube commutes. But notice that there is an epimorphism that helps you do this, and I will let you do the details. This is the most general proof where you work with arbitrary abelian categories, but as you know, you can always think of module categories and all of these abstract nonsense can be checked explicitly by chasing the diagram. For example, you can think of we work with $\otimes_R N$, which is right exact and you can reproduce the above abstract arguments by hands.

Situation for when F is left exact is similar. You first show that F preserves kernels, then use the following sequence to give morphisms from $\mathrm{im}Fd^i$ to $F\mathrm{im}d^i$:

$$0 \to \mathrm{ker}d^i \to C^i \to \mathrm{im}d^i \to 0$$

Then notice that we have a SES:

$$0 \to \mathrm{im}d^{i-1} \to \mathrm{ker}d^i \to H^i(C^\bullet) \to 0$$

Then apply F and obtain a morphism from $H^i(FC^\bullet) \to FH^i(C)$.

The final assertion is that when F is exact, it's of course both left and right exact, therefore we can produce these two morphisms simultaneously and we want to show they are inverses of each other. The fact that they are isomorphisms is not hard, since when F is exact, then you can go over the same argument, and you find that when you need to apply five lemma, you obtain and isomorphism due to exactness, the remaining details are left to you.

Finally, to prove that they are inverses of each other, this is actually a statement that makes little sense, as all of these morphism are just relative notions and once kernels and cokernels, images are identified, we can assume that both morphisms are actually identities, thus of course inverses of each other. ☐

Remark 1.2. This is sometimes called the **FHHF theorem** and we will see a lot of applications of that later.

And obviously, this will be a functorial construction, in the sense that if we have $FH^\bullet \to H^\bullet F$ and we apply another right exact functor G to obtain:

$$GFH^\bullet \to GH^\bullet F \to H^\bullet GF$$

this will be exactly the morphism we obtain by directly apply this argument to $G \circ F$. This is because we can consider the following diagram:

$$\begin{tikzcd}
 & GFH^i(C^\bullet) \arrow[r] \arrow[d, dashed] & GF\mathrm{coker}d^{i-1} \arrow[r] \arrow[d] & GF\mathrm{im}d^i \arrow[r] \arrow[d] & 0 \\
 & GH^i(FC^\bullet) \arrow[r] \arrow[d, dashed] & G\mathrm{coker}Fd^{i-1} \arrow[r] \arrow[d] & G\mathrm{im}Fd^i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^i(GFC^\bullet) \arrow[r] & \mathrm{coker}GFd^{i-1} \arrow[r] & \mathrm{im}GFd^i & 0
\end{tikzcd}$$

then we conclude the assertion from the universal property of cokernel.

Remark 1.3. Another functoriality property of this FHHF morphism is that the natural morphism:

$$FH^\bullet \to H^\bullet F$$

can be viewed as a natural transformation of functors, therefore, if we have a morphism of complexes:

$$C^\bullet \to D^\bullet$$

We have the following commutative diagram:

$$\begin{tikzcd}
FH^\bullet(C^\bullet) \arrow[r] \arrow[d] & H^\bullet F(C^\bullet) \arrow[d] \\
FH^\bullet(D^\bullet) \arrow[r] & H^\bullet F(D^\bullet)
\end{tikzcd}$$

and similar diagram also holds for FHHF morphism of left exact functors.

To show this is indeed commutative, let's consider the right exact case since the left exact is a dual proof. And the proof is just the following diagram and the fact that the morphism from the cohomology to kernel is a monomorphism thus to prove commutativity, we can post compose with a monomorphism:

$$\begin{tikzcd}
 & FHC \arrow[dl] \arrow[dr] \arrow[dd] & \\
HFC \arrow[rr, hook] \arrow[dd] & & \mathrm{cokerFd}^{\mathrm{i}}_{\mathrm{C}} \arrow[dd] \\
 & FHD \arrow[dl] \arrow[dr] & \\
HFD \arrow[rr, hook] & & \mathrm{cokerFd}^{\mathrm{i}}_{\mathrm{D}}
\end{tikzcd}$$

Example 1.4. Let's discuss a concrete example of this FHHF theorem which is also going to be the primary way this theorem is used. This discussion will both help you understand how the FHHF theorem are used and put your mind at ease.

We consider the tensor functor $\otimes_A S$ where S is an A-module for a ring A. This functor is always right exact, therefore according to the claim of FHHF theorem, we have a natural morphism from $H^i \otimes S$ to $H^i(\otimes_A S)$. Let's see how to describe them concretely.

The first thing you want to do is to show tensor preserves cokernels by considering the following diagram:

$$\begin{tikzcd}
C^i \otimes S \arrow[r] & C^{i+1} \otimes S \arrow[r] & \text{coker} d^i \otimes S \arrow[r] & 0 \\
C^i \otimes S \arrow[r] \arrow[u] & C^{i+1} \otimes S \arrow[r] \arrow[u] & \text{coker} d^i \otimes S \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

therefore, we will actually take the first row as the cokernel and forget about the second row where things are more mysterious.

Then you consider the following diagram:

$$\begin{tikzcd}
\text{coker} d^{i-1} \otimes S \arrow[r] \arrow[d, dashed, "id"] & \text{im} d^i \otimes S \arrow[d, dashed] \arrow[dr] & & \\
\text{coker} d^{i-1} \otimes S \arrow[r] & \text{im}(d^i \otimes S) \arrow[r, hook] & C^{i+1} \otimes S \arrow[r] & \text{coker} d^i \otimes S \\
C^i \otimes S \arrow[u] \arrow[ur] \arrow[urr, "d^I \otimes S"'] & & &
\end{tikzcd}$$

where every arrow is induced by the universal property, then we want to prove that the square with the dotted arrow is commutative. Say $x \otimes s$ is in $\text{cokerd}^{i-1} \otimes S$, then it goes in $\text{im}d^i \otimes S$ by the tensor product say it goes to $y \otimes s$, then it goes to $y \otimes s$ in $C^{i+1} \otimes S$. But say that $x \otimes s$ in $C^i \otimes S$ is mapped to $x \otimes s$ in $\text{coker}d^{i-1} \otimes S$, then notice that $x \otimes s$, via $d^i \otimes S$ goes to $d^i(x) \otimes S$ in $C^{i+1} \otimes S$, then if $y = d^i(x)$, we are done. But if you look at the diagram before we tensor with S:

$$\begin{tikzcd}
\text{coker} d^{i-1} \arrow[r] & \text{im} d^i \arrow[r, hook] & C^{i+1} \\
x \in C^i \arrow[u] \arrow[ur] \arrow[urr, "d^i"'] & &
\end{tikzcd}$$

you will see our assertion is indeed true. Therefore, the morphism:

$$H^i \otimes S \to H^i(\otimes S)$$

is induced by $x \otimes s$ is mapped to the element in $H^i(\otimes S)$ that is mapped to $x \otimes s$ in $\text{coker}d^{i-1} \otimes S$ where we assume that via $H^i \hookrightarrow \text{coker}d^{i-1}$, x is mapped to x.

Now, if S is flat over A, i.e. the tensor product is exact, then in the above discussion, the morphism between images are also going to be isomorphisms, then the remaining stuff is left for you.

Now we begin our study on Higher pushforwards. Recall that we previously defined cohomology for $\pi : X \to SpecA$ when this morphism is qausicompact and separated. We will now try to define a relative version of this notion for quasicompact and separated morphism $\pi : X \to Y$ when Y is not necessarily affine. Previously, for any quasicoherent sheaf $\mathcal{F}$ on X, we got an A-module $H^i(X, \mathcal{F})$ for each i, and here since the higher pushforward will be a relative version of that, the result is going to be a quasicoherent sheaf on Y.

The main motivation for doing this is that this helps us later to show how cohomology groups vary in families, especially with some nice assumptions. And without any further introduction, let's begin constructing it.

Recall that since this $\pi : X \to Y$ is quasicompact and separated, we know that for any quasicoherent sheaf $\mathcal{F}$ on X and any affine open $SpecA$ in Y, we can construct $H^i(\pi^{-1}SpecA, \mathcal{F})$ by restricting $\mathcal{F}$ to $\pi^{-1}SpecA$. We now want to show that these form a quasicoherent sheaf on X. This just means that we need to check this definition behaves well with distinguished opens. In other words, we have to check that the natural map:

$$H^i(\pi^{-1}SpecA, \mathcal{F}) \to H^i(\pi^{-1}SpecA_f, \mathcal{F})$$

This map comes from our earlier discussion that cohomnology is a contravariant functor in the space. Say $i : SpecA_f \hookrightarrow SpecA$ is the open embedding, then this morphism is given by the composition:

$$H^i(\pi^{-1}SpecA, \mathcal{F}) \to H^i(\pi^{-1}SpecA, i_*i^*\mathcal{F}) \to H^i(\pi^{-1}SpecA_f, i^*\mathcal{F})$$

This can be verified easily since if we choose a finite affine open cover $\mathcal{U}$ of $\pi^{-1}SpecA$, I claim that this induces a natural affine open cover of $\pi^{-1}SpecA_f$ as we know that for each affine in $\mathcal{U}$, say V, $V \cap \pi^{-1}SpecA_f$ is the locus where the pullback of f vanishes, thus is still affine, and in particular, is a distinguished affine of V. Then we know that the natural map is actually induced by a complex morphism from the Cech complex of $\mathcal{F}$ on $\pi^{-1}SpecA$ to the Cech complex of $\mathcal{F}$ on $\pi^{-1}SpecA_f$, which is just the restriction map on each component of the complex. Then we take cohomology and want to conclude that the morphism is localization at f. But if you recall that the restiction of $\mathcal{F}$ to V is going to be the localization at f as an A_f-module, therefore, you can think of the Cech complex of $\mathcal{F}$ on $\pi^{-1}SpecA_f$ as the localization of that on $\pi^{-1}SpecA$, then the morphim of complex can be viewed as the natural morphism to localizations, i.e. we get something like the diagram below:

$$\begin{tikzcd}
\cdots \arrow[r] & C^{j-1} \arrow[r] \arrow[d] & C^j \arrow[r] \arrow[d] & C^{j+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C_f^{j-1} \arrow[r] & C_f^j \arrow[r] & C_f^{j+1} \arrow[r] & \cdots
\end{tikzcd}$$

Then we know that the morphism on cohomology is induced by:

$$\frac{\ker d^j}{\operatorname{im} d^{j-1}} \to \frac{(\ker d^j)_f}{(\operatorname{im} d^{j-1})_f}$$

This is because we already seen in FHHF theorem that when the functor is exact (localization is exact), images and kernels are preserved, therefore we know quite well what are the kernels downstairs and the fact that we are working with modules makes it clear what is the morphism and in the end, we know that localization commutes with quotients, therefore, we it's not hard to prove this is just the localization.

So in conclusion, what we did is for each affine open $SpecA$ in Y, we attached to it something that is invariant of of the choice of A, and satisfies the localization condition when you want A to be explicit, then we know that is is enough to define a quasicoherent sheaf on Y:

Definition 1.5. We define the **i-th higher pushforward sheaf** or the **i-th higher direct image sheaf**, denoted by $R^i\pi_*\mathcal{F}$ to be the quasicoherent sheaf defined above using H^i.

Theorem 1.6. Suppose that $\pi : X \to Y$ is a quasicompact separated morphism of schemes then:

1. $R^i\pi_*$ is a contravariant functor $\mathrm{Qcoh}_X \to \mathrm{Qcoh}_Y$.
2. There is a isomorphism of functors between $R^0\pi_*$ and π_*.
3. A short exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to H \to 0$ of quasicoherent sheaves on X induces a long exact sequence:
$$0 \to R^0\pi_*\mathcal{F} \to R^0\pi_*\mathcal{G} \to R^0\pi_*\mathcal{H} \to R^1\pi_*\mathcal{F} \to R^1\pi_*\mathcal{G} \to \cdots$$
4. After we saw Grothendieck's coherence theorem, we can show that if $\mathcal{F}$ is coherent on X, π is a projective morphism and Y is locally Noetherian, then $R^i\pi_*\mathcal{F}$ is coherent on Y.

Proof. Let's first show the functor structure of $R^i\pi_*$ and you will see that the other parts just follows from the corresponding statements on cohomology.

Suppose that $\mathcal{F} \to \mathcal{G}$ is a morphism of quasicoherent sheaves on X, then we want a morphism $R^i\pi_*\mathcal{F} \to R^i\pi_*\mathcal{G}$. Recall that a morphism of quasicoherent sheaves is equivalent to a morphism on the distinguished affine base, therefore, we only need to construct a morphism:

$$H^i(\pi^{-1}SpecA, \mathcal{F}) \to H^i(\pi^{-1}SpecA, \mathcal{G})$$

that commutes with localizations. But we already saw this when we constructed Cech cohomology. Moreover, we even proved this for the corresponding long exact sequences, and in particular the connecting homomorphism, therefore we know that we can use it to construct the connecting homomorphism $R^i\pi_*\mathcal{H} \to R^{i+1}\pi_*\mathcal{F}$, therefore, this proved 1 and 3 in the statement since exactness stated in 3 can be checked affine locally, and for quasicoherent sheaves, affine locally, exactness can be checked by looking at the corresponding sequence of modules.

To prove 2, recall that there is a functor isomorphism between H^0 and Γ, then if we can prove this also commutes with localizations, we know that given a morphism $\mathcal{F} \to \mathcal{G}$, we can associated it a morphism of quasicoherent sheaves $R^0\pi_*\mathcal{F} \to \pi_*\mathcal{F}$ by specifying a morphism on the distinguished affine base, which is just a morphism from H^0 to Γ that commutes with localizations, and we also want the result morphism to be functorial, this is why the functor isomorphism between H^0 and Γ to commute with localizations. (Actually in this case, we don't have to be so scrupulous about localizations and we can directly look at the fact that they they commutes with restrictions.)

The last statement 4 can also be checked affine locally, and it just follows from $H^i(\pi^{-1}SpecA, \mathcal{F})$ is finitely generated for a coherent module $\mathcal{F}$, which is the corresponding cohomology statement. □

Proposition 1.7. If π is affine then $R^i\pi_* = 0$ for $i > 0$.

Proof. This essentially follows from the fact that higher cohomology of affine A-schemes vanishes, and the fact that $R^i\pi_*$ is defined affine locally. □

We will be interested in how higher pushforward behave with respect to base changes:

Proposition 1.8. Suppose that $\psi : Y \to Y'$ is any morphism and $\pi : X \to Y$ is quasicompact and separated in the following Cartesian daigram.

$$\begin{tikzcd} X' \arrow[r, dashed, "\psi'"] \arrow[d, dashed, "\pi'"] & X \arrow[d, dashed, "\pi"] \\ Y' \arrow[r, dashed, "\psi"] & Y \end{tikzcd}$$

Suppose that $\mathcal{F}$ is a quasicoherent sheaf on X, there is a natural morphism:

$$\psi^*(R^i\pi_*\mathcal{F}) \to R^i\pi'_*(\psi')^*\mathcal{F}$$

Moreover, when $i = 0$, this is:

$$\psi^*\pi_*\mathcal{F} \to (\pi')_*(\psi')^*\mathcal{F}$$

which is the usual push-pull morphism we saw before.

Proof. Notice that since this is a Cartesian diagram, π' is also quasicompact and separated, therefore $R^i\pi'_*$ makes sense. We will use the FHHF theorem in this proof.

Let's first assume that X, Y are affine and say the morphism ψ is just $SpecB \to SpecA$, then we know that the morphism we want is just:

$$H^i(X, \mathcal{F}) \otimes_A B \to H^i(X', (\psi')^*\mathcal{F})$$

Recall that the complex we use to compute $H^i(X, \mathcal{F})$ is defined using an affine cover of X, and since ψ is now affine, we know that ψ' is also affine, thus it pulls back the affine cover of X to affine cover of X'. We use that affine open cover to compute the cohomology. But notice that if $SpecC$ is such an affine cover in X, then we know that:

$$M \otimes_C (C \otimes_A B) \simeq M \otimes_A B$$

Therefore, we know that the complex we use to compute the cohomology is the complex of $\mathcal{F}$ tensor with B, therefore we know that since $\otimes_A B$ is right exact, the FHHF theorem gives us the desired morphism.

Now the question becomes how to reduce to this local case adn how to patch the local pieces together globally to get the required global morphism. First for any $SpecB$ in Y' whose image is in $SpecA$ in Y, we can consider the following Cartesian diagram:

$$\begin{tikzcd}
\cdots \arrow[r, hook] \arrow[d, hook] & (\psi')^{-1}\pi^{-1}SpecA \arrow[r] \arrow[d, hook] & \pi^{-1}SpecA \arrow[d, hook] \\
(\pi')^{-1}SpecB \arrow[r, hook] \arrow[d] & X' \arrow[r] \arrow[d] & X \arrow[d] \\
SpecB \arrow[r, hook] & Y' \arrow[r] & Y
\end{tikzcd}$$

and since the image of $SpecB$ is in $SpecA$, the bottom right corner can be replaced by $SpecA$, Moreover, notice that the stuff on the top left corner is the inverse image of $SpecB$. Therefore, if we want to give a morphism:

$$\psi^*(R^i\pi_*\mathcal{F})(SpecB) \to R^i\pi'_*(\psi')^*\mathcal{F}(SpecB)$$

which is just $H^i(\pi^{-1}SpecA, \mathcal{F})\otimes_A B \to H^i(\pi'^{-1}SpecB, (\psi')^*\mathcal{F})$ we only have to consider the outer Cartesian diagram and assume that Y', Y are affine.

Recall that if you have a morphism of quasicoherent modules on $SpecA$, that is given by a morphism of modules $M \to N$ of A-modules, and $SpecB \hookrightarrow SpecA$ is an affine open subscheme of $SpecA$, then the restriction to $SpecB$ is given by $M \otimes B \to N \otimes B$.

Go back to our situation, if $SpecD$ is an affine open of $SpecA$, we know that the morphism restricted on $SpecD$ will be given by:

$$H^i(X, \mathcal{F}) \otimes_A B \otimes_B D \to H^i(X', (\psi')^*\mathcal{F}) \otimes_B D$$

But notice that if $SpecD \hookrightarrow SpecB$ is an open embedding, then the ring morphism $B \to D$ is flat since if:

$$0 \to M \to N \to K \to 0$$

is exact, if we want check:

$$0 \to M \otimes D \to N \otimes D \to K \otimes D \to 0$$

is exact, we can check is on each prime of D, but notice that localize at a prime p of D, that is also in B, is isomorphic to directly localize at the prime of B, thus it's exact.

Therefore we know the cohomology tensor with D is going to be isomorphic to first tensor with D and then take the cohomology. However, if you first tensor with D, consider the following diagram:

$$\begin{tikzcd}
X'' \arrow[r, hook] \arrow[d] & X' \arrow[r, "\psi'"] \arrow[d] & X \arrow[d] \\
SpecD \arrow[r, hook] & SpecB \arrow[r] & SpecA
\end{tikzcd}$$

you find the complex is going to be the complex that computes the cohomology of the pullback of $\mathcal{F}$ to X'', thus the FHHF theorem again tells us that cohomology this tensor commutes with the cohomology via the FHHF morphism, if we can prove that the composition is the FHHF morphism on $SpecD$, we are done, but this is exactly the functoriality remark we made after the FHHF theorem.

The last one minor issue (which is not really a issue if you think it through) comes from the fact that if we first restrict to $SpecB$ then to $SpecD$, we will get a module morphism:

$$H^i(X, \mathcal{F}) \otimes_A B \otimes_B D \to H^i(X', (\psi')^*\mathcal{F}) \otimes_B D \to H^i(X'', (\psi')^*\mathcal{F})$$

But if you directly restrict to $SpecD$, you get:

$$H^i(X, \mathcal{F}) \otimes_A D \to H^i(X'', (\psi')^*\mathcal{F})$$

We only know that the first row is the FHHF morphism assocaited to $\otimes_A B \otimes_B D$ and the second morphism is the FHHF morphism associated to $\otimes_A D$. But notice that since as functors: $\otimes_A B \otimes_B D \simeq \otimes_A D$, their FHHF morphism will be isomorphic, then we will have a commutative diagram:

$$\begin{tikzcd} H^i(X,\mathcal{F}) \otimes_A B \otimes_B D \arrow[r, "FHHF"] \arrow[d, dashed, "\simeq"] & H^i(X'', (\psi')^*\mathcal{F}) \arrow[d, dashed, "\simeq"] \\ H^i(X,\mathcal{F}) \otimes_A D \arrow[r, "FHHF"] & H^i(X'', (\psi')^*\mathcal{F}) \end{tikzcd}$$

this is enough to show the two morphisms on $SpecB$ are equal and I will let you see why.

The last assertion is that when $i = 0$, then this morphism is the usual push-pull map we saw for quasicoherent sheaves. We only have to check this locally. The usual push-pull map on $SpecB$ is just $\Gamma(X,\mathcal{F}) \otimes B \to \Gamma(X', (\psi')^*\mathcal{F})$, which is defined by adjoint property. We start by the identity on the pullback $(\psi')^*\mathcal{F} \to (\psi')^*\mathcal{F}$, then the adjoint property gives $\mathcal{F} \to \psi'_*(\psi')^*\mathcal{F}$, then we apply π_*, which gives you a morphism on $SpecA$, determined by the global section maps on $\mathcal{F} \to \psi'_*(\psi')^*\mathcal{F}$, i.e. $\Gamma(X,\mathcal{F}) \to \Gamma(X', (\psi')^*\mathcal{F})$. But notice that we can use the adjointness of ψ again, to obtain a morphism on $SpecB$ by tensor with B, which is:

$$\Gamma(X,\mathcal{F}) \otimes B \to \Gamma(X', (\psi')^*\mathcal{F})$$

On the other hand, the morphism defined by FHHF theorem is:

$$\begin{tikzcd} \Gamma(X,\mathcal{F}) \otimes B \arrow[r] \arrow[d, dashed] & (\mathrm{coker} d^{-1} = C^0) \otimes B \arrow[d, dashed, "id"] \\ \Gamma(X', (\psi')^*\mathcal{F}) \arrow[r] & C^0 \otimes B \end{tikzcd}$$

I will let you see why this dotted morphism in the diagram is equal to the one defined by the adjointness property (check it locally and notice that global section is determined locally but affine locally we know the adjoint map very well). $\square$

In the above proposition, we allow ψ to be any morphism, thus what we get is just a morphism between two quasicoherent sheaves, and we will see that if ψ is good, then the morphism between the two quasicoherent sheaves will be good:

Proposition 1.9. If ψ in the above proposition is affine and for an affine cover of Y, say $\{SpecA_i\}$, the preimages $SpecB_i$ is flat over $SpecA_i$ in the sense that $A_i \to B_i$ is flat, then the push-pull morphism defined above is an isomorphism.

Proof. We check a morphism is an isomorphism locally, and we know that affine locally this is defined by the FHHF morphism, but if B_i is flat over A_i, since the tensor product is exact, the FHHF morphism is an isomorphism, thus we are done. $\square$

Remark 1.10. You probably have notice that if we assume that the inverse image of $SpecA$ can be covered by $SpecB$ which is flat over $SpecA$, the conclusion also holds. This is the preview of the theorem that cohomology commutes with flat base change.

We used the fact that we work with a Cartesian diagram in a crucial way when we prove the exsistence of the push-pull morphism. When we conclude that the tensor product of a certain complex is the complex that computes the pullback, we need the Cartesian diagram. However, when we say push-pull morphisms, we typically don't require a Cartesian diagram, we want get a push-pull morphism for any commutative square. And we will prove it below:

Proposition 1.11. There is still a natural push-pull morphism:

$$\psi^*(R^i\pi_*\mathcal{F}) \to R^i\pi'_*(\psi')^*\mathcal{F}$$

for any commutative diagram:

$$\begin{tikzcd} X' \arrow[r, "\psi'"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ Y' \arrow[r, "\psi"] & Y \end{tikzcd}$$

and any quasicoherent sheaf $\mathcal{F}$ on X, if we assume π' is quasicompact and separated, which lets the higher pushforward make sense.

Proof. We will use the result when this diagram is Cartesian to prove the result when the diagram is not Cartesian.

To use what we proved for Cartesian diagrams, we form the following Cartesian diagram:

$$\begin{tikzcd} X' \arrow[dr, "g"] \arrow[drr, bend left, "\psi'"] \arrow[ddr, bend right, "\pi'"'] & & \\ & X \times_Y Y' \arrow[r, "p_2"'] \arrow[d, "p_1"] & X \arrow[d, "\pi"] \\ & Y' \arrow[r, "\psi"'] & Y \end{tikzcd}$$

we know that using this diagram, we have a push-pull morphism:

$$\psi^* R^i \pi_* \mathcal{F} \to (R^i p_1)_* (p_2^* \mathcal{F})$$

Notice that we have the following diagram:

$$p_2^* \mathcal{F} \to g_* g^* p_2^* \mathcal{F}$$

by the adjoint property, but notice that $g^* p_2^* = (p_2 \circ g)^* = \psi'^*$, therefore, we have:

$$p_2^* \mathcal{F} \to g_* (\psi')^* \mathcal{F}$$

Now apply $(R^i p_1)_*$, we get:

$$(R^i p_1)_* p_2^* \mathcal{F} \to (R^i p_1)_* g_* (\psi')^* \mathcal{F}$$

then if we have a natural functor morphism:

$$(R^i p_1)_* g_* \to (R^i \pi')_*$$

we get the desired morphism.

Let's try to do this, given a $\mathcal{F}$ on X', we want a morphism:

$$(R^i p_1)_* g_* \mathcal{F} \to (R^i \pi')_* \mathcal{F}$$

that is functorial. We still use the distinguished affine base. Notice that on each $SpecA$ in Y', we need a morphism:

$$H^i(p_1^{-1} SpecA, g_* \mathcal{F}) \to H^i(\pi'^{-1} SpecA, \mathcal{F})$$

butnotice that we have a natural morphism:

$$H^i(p_1^{-1} SpecA, g_* \mathcal{F}) \to H^i(g^{-1} p_1^{-1} SpecA, \mathcal{F}) = H^i(\pi'^{-1} SpecA, \mathcal{F})$$

coming from the general property of cohomology and pushforwards.

We need to show that this is compatible with restrictions to distinguished affine opens, then we will be done since the morphism described distinguished opens will be functorial. I will let you do this since we have essentially did it and you can translate this question to the commutativity of a diagram of complexes and the take cohomology. □

Remark 1.12. One observation is that since in general for pushforwards g_*, the morphism:

$$H^i(p_1^{-1} SpecA, g_*\mathcal{F}) \to H^i(g^{-1}p_1^{-1} SpecA, \mathcal{F}) = H^i(\pi'^{-1} SpecA, \mathcal{F})$$

is not necessarily an isomorphism, this natural transformation between $(R^i p_1)_* g_*$ and $(R^i \pi')_*$ is not an isomorphism in general. This turns out to be an interesting question and we will see something similar when we do the Leray spectral sequence.

A useful special case is the following:

Proposition 1.13. If $q \in Y$, then there is a natural morphism $(R^i\pi_*\mathcal{F}) \otimes k(q) \to H^i(\pi^{-1}q, \mathcal{F}|_{\pi^{-1}q})$.

Proof. Recall that the notation on the left hand side stands for the pullback of $R^i\pi_*\mathcal{F}$ to $Speck(q)$. We consider the fiber Cartesian diagram:

$$\begin{tikzcd} \pi^{-1}q \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\pi"] \\ Speck(q) \arrow[r, "g"'] & Y \end{tikzcd}$$

then we already know that there is a natural diagram:

$$(R^i\pi_*\mathcal{F}) \otimes k(q) \to (R^i g_*) f^*\mathcal{F}$$

Recall that since $Speck(q)$ is affine, we know that the higher pushforward is just the higher cohomology, $H^i(\pi^{-1}q, \mathcal{F}|_{\pi^{-1}q})$. ☐

Therefore, we find that even though the fiber of the higher pushforward may not be the the cohomology of the sheaf on the fiber , but there is always a natural morphism we can use to compare them, and in nice situations, we can expect this comparison morphism to be an isomorphism. But we will have to study this later.

Proposition 1.14. Suppose that $\pi : X \to Y$ is quasicompact and separated, say $\mathcal{F}$ and $\mathcal{G}$ are quasicoherent sheaves on X, Y respectively, as shown in the following diagram:

$$\begin{tikzcd} \mathcal{F} \arrow[dr] & & \pi^*\mathcal{G} \arrow[dl] \\ \pi_*\mathcal{F} \arrow[dr] & X \arrow[d] & \mathcal{G} \arrow[dl] \\ & Y & \end{tikzcd}$$

then there is a natural morphism:

$$(R^i\pi_*\mathcal{F}) \otimes \mathcal{G} \to R^i\pi_*(\mathcal{F} \otimes \pi^*\mathcal{G})$$

Moreover, when $\mathcal{G}$ is locally free, this is an isomorphism.

Proof. As you expected, we still define this locally and show that they patch together.

Say on $SpecA \subset Y$, we need to define a morphism of modules:

$$H^i(\pi^{-1}SpecA, \mathcal{F}) \otimes \mathcal{G}(SpecA) \to H^i(\pi^{-1}SpecA, \mathcal{F} \otimes \pi^*\mathcal{G})$$

If you think about what is the stuff on the right hands side, notice that if we cover $\pi^{-1}SpecA$ with a finite cover to compute the cohomology, then the complex we use to compute $\mathcal{F} \otimes \pi^*\mathcal{G}$ is the complex we use to compute $\mathcal{F}$, tensored with $\mathcal{G}(SpecA)$. This is because if you think of an affine open in the cover, or intersections of the affines in the cover, which is still affine, say one of them is $SpecC$, then we know that the section of $\mathcal{F} \otimes \pi^*\mathcal{G}$ on that is just $\mathcal{F}(SpecC) \otimes_C (C \otimes_A \mathcal{G}(SpecA))$, then you can show the complex is indeed just the complex for $\mathcal{F}$, tensor with $\mathcal{G}(SpecA)$. Therefore, we can apply the FHHF theorem, which gives you a natural morphism.

And again, we need to show that this definitions patches, but this time we can use distinguished opens and therefore, we need to show the following diagram commutes:

$$\begin{tikzcd} H^i(\pi^{-1}SpecA, \mathcal{F}) \otimes \mathcal{G}(SpecA) \arrow[r] \arrow[d] & H^i(\pi^{-1}SpecA, \mathcal{F} \otimes \pi^*\mathcal{G}) \arrow[d] \\ H^i(\pi^{-1}SpecA_f, \mathcal{F}) \otimes \mathcal{G}(SpecA_f) \arrow[r] & H^i(\pi^{-1}SpecA_f, \mathcal{F} \otimes \pi^*\mathcal{G}) \end{tikzcd}$$

where the vertical maps comes from restriction maps and the horizontal maps are all FHHF morphisms. We check this by hand, but if you are familiar with the FHHF morphism enough, you will notice that essentially, if we work with modules, morphism between cohomologies are essentially described by what happens in the cokernel and you can use that to check the commutativity more efficiently, but there are indeed quite a lot of confusing stuff going on in the process of proving this.

After we established this morphism, the assertion that the corresponding morphism is an isomorphism when $\mathcal{G}$ is locally free follows from the fact that when $\mathcal{G}(SpecA)$ is free, then tensor is exact, then FHHF morphism is an isomorphism. □

The following fact is useful:

Theorem 1.15. If $\pi : X \to Y$ is a projective morphism and Y is locally Noetherian then the higher pushforwards vanish in degree higher than the maximum dimension of the fibers.

Before we prove it, here is a warning:

Remark 1.16. Recall that the dimensional cohomology vanishing can generalize from projective to quasi-projective to even more broader cases, but here this relative dimensional vanishing result doesn't generalize and the projective requirement is necessary, details are given in the following example:

Example 1.17. Consider the open embedding $\pi : \mathbb{A}^n_k - \{0\} \hookrightarrow \mathbb{A}^n_k$. I claim that $R^{n-1}\pi_*\mathcal{O}_{\mathbb{A}^n_k - \{0\}} \neq 0$. Recall that since $\mathbb{A}^n_k$ is affine, the higher pushforward is just the cohomology, thus we are just trying to show that the $n-1$ degree cohomology is not 0. But this is just the computation we did to compute the cohomology of the projective space.

Notice that the dimension of the fiber of an open embedding is always 1, therefore if the assertion of the relative dimensional vanishing is true, then all higher pushforwards of degree at least 2 vanishes, but for $n \geqslant 3$, the above example tells you that this is false.

Proof. Recall that the vanishing of the higher pushforward is affine local on Y, thus we can assume that Y is affine, therefore, we can assume that the morphism from X to Y factors in the following diagram:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] & \mathbb{P}^n_A \arrow[d] \\ & SpecA \end{tikzcd}$$

for a Noetherian ring A, then this also shows that the dimension of X is at most n, thus the fiber of this morphism is at most n, and we will assume that the maximum of the fiber dimension is m.

Then consider the following diagram:

$$\begin{tikzcd} \pi^{-1}p \arrow[r, hook] \arrow[d] & \mathbb{P}^n_k \arrow[r] \arrow[d] & Speck \arrow[d] \\ X \arrow[r, hook] & \mathbb{P}^n_A \arrow[r] & SpecA \end{tikzcd}$$

we know that since the dimension of $\pi^{-1}p$ is at most m, there are nonempty hypersurfaces $H_1, \cdots, H_{m+1}$ whose intersection missing X. This implies that, if we assume that $f_1, \cdots, f_n$ cut out these hyperplanes, then $D_+(f_1), \cdots, D(f_{m+1})$ covers $\pi^{-1}p$.

Let's lift $f_1, \cdots, f_{m+1}$ back to $A_p[x_0, \cdots, x_n]$, say they are f_i', and let F be the product of the denominators of the coefficients of f_i'. Then since the denominators of the coefficients each f_i' are not in pA_p, then we know that their, i.e. F is not in pA_p as well, or in other words, $p \in D(F)$. Therefore, we can think f_i' are elements in $A_F[x_0, \cdots, x_n]$. Then consider the following diagram:

$$\begin{tikzcd}
\pi^{-1}p \arrow[r, hook] \arrow[d] & \mathbb{P}^n_k \arrow[r] \arrow[d] & Speck \arrow[d] \\
X' \arrow[r, hook] \arrow[d, hook] & \mathbb{P}^n_{A_F} \arrow[r] \arrow[d, hook] & SpecA_F \arrow[d, hook] \\
X \arrow[r, hook] & \mathbb{P}^n_A \arrow[r] & SpecA
\end{tikzcd}$$

Then we can intersect X' with $V(f_i')$ in $\mathbb{P}^n_{A_F}$ and obtain a new closed subscheme of $\mathbb{P}^n_{A_F}$. Then this closed subscheme, since $X' \to SpecA_F$ is also projective, is mapped to a closed subset of $SpecA_F = D(F)$.

Notice that that closed subset of $D(F)$ doesn't include p by the way we choose f_i'. Therefore, there is another affine open of p, in $D(F)$, missing the image of the intersection. But if we assume that g_i's are the restriction of f_i' to $B[x_0, \cdots, x_n]$, then we know that $D(g_i)$'s cover $\pi^{-1}SpecB$. This implies that $\pi^{-1}SpecB$ is covered by $m+1$ affine opens, thus we know the cohomology is at most of degree m, in other words any other higher cohomology vanishes, thus we are done. $\square$

Proposition 1.18. Suppose that $\pi : X \to Y$ is proper and X, Y are Noetherian, then if $\mathcal{L}$ is a π-ample invertible sheaf on X, for any coherent sheaf $\mathcal{F}$ on X and any $m \gg 0$ we have $R^i\pi_*(\mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0$ for all $i > 0$.

Proof. By the definition of π-ample, we know that for every $SpecB$ of Y, $\mathcal{L}$ restricted to $\pi^{-1}SpecB$ is ample over B, then we know that for high enough power, $\mathcal{L}$ is π-very ample, thus we know that there is the following diagram:

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[dr] & \mathbb{P}^n_B \arrow[d] \\
& SpecB
\end{tikzcd}$$

such that the pullback of $\mathcal{O}(1)$ is $\mathcal{L}^{\otimes N}$, therefore we know from Serre vanishing that all the higher cohomology of $\mathcal{F} \otimes \mathcal{L}^{\otimes m_0 N}$ vanish for large enough m_0.

Notice that we can apply the same argument to $\mathcal{F} \otimes \mathcal{L}$ and obtain m_1, similarly, for all $\mathcal{F} \otimes \mathcal{L}^{\otimes i}$ for $0 \leqslant i \leqslant N-1$, we can apply the argument and obtain $m_0, \cdots, m_{N-1}$, we take m to be the largest one among them.

Therefore, what we get is for all $r \in \{mN + t : m \gg 0, 0 \leqslant t \leqslant N-1\}$, we know the higher cohomology vanishesm thus we obtain all the higher cohomology vanishes for $r \geqslant mN$, we are done. $\square$

2 Serre's Characterization of Ampleness and Affineness

Let's first deal with ampleness. We already know that if $\pi : X \to SpecB$ is a proper morphism, we have a list of equivalent conditions to determine if a line bundle on X is ample or not. Here we will give another criterion due to Serre using cohomological methods, but with Noetherian hypothesis.

Theorem 2.1. Suppose that A is a Noetherian ring and $\pi : X \to SpecA$ is proper, if $\mathcal{L}$ is an invertible sheaf on X, then the following are equivalent:

1. $\mathcal{L}$ is ample on X.

2. For all coherent sheaves $\mathcal{F}$ on X, there is an n_0 such that for all $n \geqslant n_0$:

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$$

Remark 2.2. The first condition in the above proposition doesn't include the condition we stated in the starred section discussing ampleness.

As usual, before we prove it, let's give some applications to motivate it:

Proposition 2.3. Suppose that X is proper over $SpecA$ for a Noetherian ring A, then a line bundle $\mathcal{L}$ on X is ample iff it's ample on X_{red}.

Proof. Recall that we have already proved that pullback of an ample line bundle by a finite morphism is ample, and in particular since closed embeddings are finite, pullback by a closed embedding of an ample line bundle is ample. Therefore, if $\mathcal{L}$ is ample on X, $\mathcal{L}|_{X_{red}}$ is ample on X_{red}.

Conversely, suppose that $\mathcal{I}$ is the ideal sheaf cutting out X_{red}, then since X is Noetherian, we know that there is a positive integer r such that $\mathcal{I}^r = 0$. Then for any coherent sheaf $\mathcal{F}$ on X, consider the following filtration:

$$0 = \mathcal{I}^r\mathcal{F} \hookrightarrow \mathcal{I}^{r-1}\mathcal{F} \hookrightarrow \cdots \hookleftarrow \mathcal{I}\mathcal{F} \hookrightarrow \mathcal{F}$$

Consider the following SES:

$$0 \to \mathcal{I}\mathcal{F} \hookrightarrow \mathcal{F} \to \mathcal{F}/\mathcal{I}\mathcal{F} \to 0$$

we can twist this sequence by any power of $\mathcal{L}$ and it remains exact, then via the LES associated to it, we know that if we can prove that the higher cohomology of $\mathcal{I}\mathcal{F}$ and $\mathcal{F}/\mathcal{I}\mathcal{F}$ vanishes after we twist by powers of $\mathcal{L}$, the same conclusion holds for $\mathcal{F}$.

But to prove this for $\mathcal{I}\mathcal{F}$, we use a similar SES and we keep doing this until we reach $\mathcal{I}^r\mathcal{F} = 0$, therefore, we find that all we need to prove is that if we twist $\mathcal{I}^{i-1}\mathcal{F}/\mathcal{I}^i\mathcal{F}$ by a high enough power of $\mathcal{L}$, higher cohomology vanishes.

However, notice that since each subquotient is annihilated by $\mathcal{I}$, we know that there is a coherent sheaf $\mathcal{G}^i$ on X_{red} such that $i_*\mathcal{G} \simeq \mathcal{I}^{i-1}\mathcal{F}/\mathcal{I}^i\mathcal{F}$ where $i : X_{red} \hookrightarrow X$ is the closed embedding. But as we know, cohomology is stable under pushforward by closed embeddings, therefore we know that:

$$H^i(X, \mathcal{I}^{i-1}\mathcal{F}/\mathcal{I}^i\mathcal{F} \otimes \mathcal{L}^{\otimes n}) = H^i(X, i_*\mathcal{G} \otimes \mathcal{L}^{\otimes n})$$

Recall that via the projection formula, we know that $i_*\mathcal{G} \otimes \mathcal{L}^{\otimes n} \simeq i_*(\mathcal{G} \otimes i^*\mathcal{L}^{\otimes n})$, then we are done. □

Proposition 2.4. Suppose that X is a proper A-scheme over a Noetherian ring A and $\mathcal{L}$ is an invertible sheaf on X. Then $\mathcal{L}$ is ample iff it's ample on each component of X.

Proof. Again, since pullback by closed embeddings of ample line bundles are still ample, we only have to show the converse, i.e. if $\mathcal{L}$ is ample on each component, then $\mathcal{L}$ is ample on X.

Let's first reduce to the case when X is reduced. We use the proposition we just proved and conclude that if X_0 is a component of X, $\mathcal{L}$ is also ample on $(X_0)_{red}$. Notice that if $X_{red} \hookrightarrow X$ is the reduced subscheme of X, then we take know that the reduced closed subscheme on the component X_0 fit in the following diagram:

$$\begin{tikzcd}
(X_{red})_0 \arrow[r, hook] \arrow[dr, hook] & X_{red} \arrow[r, hook] & X \\
& X_0 \arrow[ur, hook] &
\end{tikzcd}$$

by the uniqueness of closed subschemes of X which is reduced and cut out X_0. Therefore we have reduced the problem to X is reduced.

Now suppose that $\mathcal{I}_j$ is the ideal sheaf cutting out the component X_j. I claim that the product of all such ideal sheaves is 0. To see this, we only have to work locally, and conclude that on each $SpecA$ of X, the ideal sheaf correspond to $I_1 \cdots I_n$ and $V(I_1 \cdots I_n)$ is $SpecA$, therefore, we know that $I_1 \cdots I_n$ is in the nilradical, which is 0 since A is reduced!

Then we consider the following filtration:

$$0 = \mathcal{I}_1 \cdots \mathcal{I}_n\mathcal{F} \hookrightarrow mathcal I_1 \cdots \mathcal{I}_{n-1}\mathcal{F} \hookrightarrow \cdots \hookrightarrow \mathcal{I}_1\mathcal{F} \hookrightarrow \mathcal{F}$$

Then similar to what we did in the above proposition, we use the subquotients, associated LES and the fact that cohomology is stable under pushforward of closed embedding to conclude the result as the subquotients are annihilated by $\mathcal{I}_j$. □

Proposition 2.5. Suppose that C is a proper reduced curve over k and $\mathcal{L}$ is a line bundle on C such that $v^*\mathcal{L}$ is ample on the normalization of C via the normalization map:

$$v : \widetilde{C} \to C$$

Then $\mathcal{L}$ is also ample on C.

Before we prove it, we need a few preparations. This result is not used in any essential way later in the development, but we state it here as an excuse to introduce a useful notion, an occasional **right adjoint** to the pushforward functor π_* when $\pi : X \to Y$ is a **finite** morphism. Typically, we think of the left adjoint of the pushforward, i.e. the pullback functor, not the right adjoint. However, when π is finite, it's possible to define such a right adjoint. This can also be seen as the analogue of the fact that occasionally, we also have a left adjoint for the pullback functor, i.e. the extension by zero functor. But we will discuss this later when we need it.

**** to be discussed later*

Now let's go back to prove Thm 2.1:

Proof. The part that ampleness implies cohomological vanishing is the consequence of Serre vanishing and since it's similar to one of the previous argument, I left it for you as an exercise.

To see another implication. We assume the cohomological vanishing criterion and we will show that for any coherent sheaf $\mathcal{F}$, $\mathcal{F} \otimes \mathcal{L}^n$ is globally generated for $n \gg 0$.

Let's begin with a special case, where we show that $\mathcal{L}^{\otimes n}$ is globally generated for $n \gg 0$. Since we work with a quasicompact scheme, we know that each closed subset has a closed point, and if we can prove that for each closed point in X, there is a neighborhood of that closed point such that $\mathcal{L}^{\otimes N}$ is globally generated at every point in that open neighborhood for every $N \geqslant n_p$, we are done since these open neighborhoods covers X and by quasicompactness, we can choose finitely many such cover and take n to be the maximum.

Now let p be a closed point of X, and say the ideal sheaf of the canonical closed embedding is $\mathcal{I}_p$, which is still coherent since we work with Noetherian schemes, and by our assumption, there is some n_0 such that for all $n \geqslant n_0$, $H^1(X, \mathcal{I}_p \otimes \mathcal{L}^{\otimes n}) = 0$, and we can consider the following SES:

$$0 \to \mathcal{I}_p \otimes \mathcal{L}^{\otimes n} \to \mathcal{L}^{\otimes n} \to \mathcal{L}^{\otimes n}|_p \to 0$$

obtained by twisting the closed embedding SES with $\mathcal{L}^{\otimes n}$ and the last term is $i_*\mathcal{O}_p \otimes \mathcal{L}^{\otimes n}$, then by the associated LES, we know that $\mathcal{L}^{\otimes n}$ is globally generated at p for all $n \geqslant n_0$, then again, since we work with coherent sheaves, we know that there is a neighborhood V_0 such that $\mathcal{L}^{\otimes n_0}$ is globally generated at all points of V_0 and since tensor product of globally generated sheaves are still globally generated, then we know that $\mathcal{L}^{\otimes kn_0}$ is globally generated at all points in V_0. We apply the same argument for all $1 \leqslant i \leqslant n_0 - 1$ $\mathcal{L}^{\otimes n_0+i}$ for all points in a neighborhood V_i of p, then by tensoring it with $\mathcal{L}^{\otimes kn_0}$ and take the open neighborhood to be $\cap_i V_i$, we obtain that $\mathcal{L}^{\otimes kn_0+n_0+i}$ for all $0 \leqslant i \leqslant n_0 - 1$ are globally generated for all points in that neighborhood, we are done.

If you think through the above proof, you will find that we require p is a closed point so that we will have a canonical closed embedding whose ideal sheaf is coherent.

Now suppose that $\mathcal{F}$ is a coherent sheaf, we need to show that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated. The argument is similar, first we consider a closed point p, and it has again the canonical closed embedding SES, and we tensor it by $\mathcal{F}$, which may not be left exact, thus we take the kernel and obtain the following SES:

$$0 \to \ker \to \mathcal{F} \to \mathcal{F}|_p \to 0$$

Notice that $\mathcal{F}$ and $\mathcal{F}|_p$ is coherent, thus the kernel is coherent, then we twist it by $\mathcal{L}^{\otimes n}$ for $n \geqslant n_0$ and by the cohomology vanishing and the associated LES, we conclude that $\mathcal{F} \otimes \mathcal{L}^{\otimes n_0}$ is globally generated in a neighborhood V_0 of p. Recall that for all n large enough $\mathcal{L}$ is also globally generated, then we take n_0 to the larger threshold. Then you apply the same argument to $\mathcal{F} \otimes \mathcal{L}^{\otimes n_0+i}$ for all $1 \leqslant i \leqslant n_0 - 1$, and obtain V_i, we take $V = \cap V_i$ and use that fact that we can twist by $\mathcal{L}^{\otimes n_0}$ and remain globally generated to conclude the result. $\square$

Next result we will show is Serre's cohomological characterization of affineness:

Theorem 2.6. Suppose that X is a quasicompact and separated scheme, then the following are equivalent:

1. X is affine.
2. For every quasicoherent sheaf $\mathcal{F}$ on X all higher cohomology vanishes.
3. For every quasicoherent sheaf $\mathcal{F}$ on X, $H^1(X, \mathcal{F}) = 0$.

Proof. We only have to show that 3 implies 1 since we already proved 1 implies 2 and 2 implies 3.

Let $A = \Gamma(X, \mathcal{O}_X)$ and we have a natural morphism of schemes:

$$X \to SpecA$$

and we are good if we prove this is an affine morphism. Or in other words, if we can prove that there are f_i's generating A and X_{f_i} is affine.

Let's first produce X_{f_i}'s covering X and then prove that they generate A. Let $p \in X$ be any point, we choose an affine open of p, say U, and consider the canonical reduced closed subscheme $X - U \hookrightarrow X$ with ideal sheaf $\mathcal{I}$. Recall that U is also quasicompact, thus it has a closed point, therefore we can pick any closed point of U in the closure of p. It needn't be a closed point of X.

Consider the closure of q in X, denoted by $\overline{q}$, and consider the SES of the closed embedding:

$$0 \to \mathcal{I}_{\overline{q}} \to \mathcal{O}_X \to \mathcal{O}|_{\overline{q}} \to 0$$

We tensor it with the ideal sheaf $\mathcal{I}$, which is right exact, thus $\mathcal{I} \to \mathcal{I}|_{\overline{q}}$ is surjective, then we can take it's kernel, consider the associated LES and use the cohomology vanishing condition to conclude that the corresponding map on the global sections is also surjective. I claim this is enough to show that there is an element f in A such that $X_f \subset U$ is affine and contains p. First, notice that if we can prove $X_f \subset U$, then it's automatically affine since U is affine. Notice that for any f in the global section of $\mathcal{I}$, for all points not in U, we know that it vanishes there, therefore for every element f in the global section of $\mathcal{I}$, X_f is in U, we now only have to show it contains p. But now we need to use the surjection on the global section. We know that if we can show that f doesn't vanish at q we are done, since this implies that f doesn't vanish at a neighborhood of q, but since q is in the closure of p, that neighborhood contains p.

Now notice that since the closure of q in U is equal to the closure of q in X intersect with U, therefore we know that $\overline{q} \cap U$ is just q itself. Then consider the morphism $\mathcal{I}_{\overline{q}} \to \mathcal{O}_{\overline{q}}$, notice that this is surjective, since we can check it on stalks. If the point is not in U, then the stalk of the second object is 0, if it's in U, notice that only q is the only possible point where the second object is not 0, but at q, we see that the morphism is the identity, thus it's indeed surjection, then for similar reasons, the induced map on global sections is also surjective. Then consider the composition:

$$\mathcal{I} \to \mathcal{I}|_{\overline{q}} \to \mathcal{O}_{\overline{q}}$$

the corresponding map on global sections is surjective, then consider the following diagram:

$$\begin{tikzcd}
\Gamma(X, \mathcal{I}) \arrow[r, two heads] \arrow[d] & \Gamma(X, \mathcal{O}|_{\overline{q}}) \arrow[d, two heads, "id"] \\
\mathcal{O}_q \arrow[r, two heads, "id"'] & \mathcal{O}_q
\end{tikzcd}$$

we conclude that there is some $f \in \Gamma(X, \mathcal{I})$ such that it's mapped to 1, not in the maximal ideal, thus not vanishing, we are done.

Therefore, by quasicompactness, we can find finitely many such f_i's such that X_{f_i} covers X, the only thing left is to prove that they generate A.

Since X_{f_i} covers X, then we know that the following morphism:

$$(f_1, \cdots, f_n) : \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X$$

is surjective. Then as both of them are quasicoherent, the induced morphism on the global section is surjective, we are done. $\square$

Moreover, if you think about the proof above, we can actually prove that:

Proposition 2.7. The equivalent conditions above is also equaivlent to:

$$H^1(X, \mathcal{I}) = 0$$

for all quasicoherent sheaves of ideals.

Proof. We only need to show the hypothesis above implies that X is affine. We use the same strategy and show that essentially everything we used in the proof can be replaced by this condition.

The key is that any quasicoherent sub-$\mathcal{O}_X$-module of a quasicoherent ideal sheaf is still a quasicoherent ideal sheaf, then i will let you identify the two places where we used the cohomological vanishing assumption and see that you are actually applying it for a quasicoherent ideal sheaf. □

3 Chow's Lemma and Grothendieck's Coherence Theorem

The last part of this short section on cohomology is on the interplay between properness and projectiveness. We will prove Grothendieck's result on higher pushforward of coherent sheaves by a proper morphism and just knowing this can help use translate a lot of the results on projective schemes to proper schemes. To prove it, we will need Chow's lemma, which is in itself a useful tool to convert results on projective schemes to proper schemes in general.

Theorem 3.1. Suppose that $\pi : X \to Y$ is a proper morphism of locally Noetherian schemes, then for any coherent sheaf $\mathcal{F}$ on X, $R^i\pi_*\mathcal{F}$ is coherent.

Remark 3.2. Before we prove this, we make a comment here. The proof requires to sophisticated fact: the first one is Lerray spectral sequence which we will not prove and not tell you when you can use it until much later. But at least we can state it here and tell you that it applies to the situations in this theorem.

Suppose that $\pi : X \to Y$ and $\rho : Y \to Z$ are quasicompact and separated morphisms, then for any quasicoherent sheaf $\mathcal{F}$ on X, there is a spectral sequence with E_2 term given by $R^p\rho_*(R^q\pi_*\mathcal{F})$ converging to $R^{p+q}(\rho \circ \pi)_*\mathcal{F}$.

Another fact we need is **Chow's lemma**:

Theorem 3.3. Suppose that $\pi : X \to SpecA$ is a proper morphism and A is Noetherian, then there is a $\mu : X' \to X$ which is surjective and projective such that $\pi \circ \mu$ is also projective and such that μ is an isomorphism on a dense open subset of X:

$$\begin{tikzcd} X' \arrow[rr, "\mu"] \arrow[dr, "\pi\circ\mu"'] & & X \arrow[dl, "\pi"] \\ & Y & \end{tikzcd}$$

In particular, if X is a proper k scheme, it admits a projective birational morphism from a projective k-variety.

We will first use it to prove Grothendieck's result and then prove it. You will find that with the help of Chow's lemma, Grothencieck's result is not that hard to prove once you know what is a spectral sequence. But the proof of Chow's lemma is much trickier.

Proof. This question is local on Y thus we may asssume $Y = SpecA$, where A is a Noetherian ring, we will work by Noetherian induction on Supp$\mathcal{F}$, and the base case is Supp$\mathcal{F} = \varnothing$, which is obvious since $\mathcal{F} = 0$ in this case and we know that the pushforward of any degree is 0.

So we fix Supp$\mathcal{F}$, and we may assume that this holds for all coherent sheaf with a support strictly contained in Supp$\mathcal{F}$. We didn't really find a good place to discuss Noetherian induction properly, so let's use this proof as an excuse to do so.

The usual way of doing Noetherian induction is via the following result: Suppose that X is a Noetherian topological space and P is a property of closed subsets of X. If the following property holds for P, then it holds for X: If Y is a closed subset of X and the property holds for all proper closed subset of Y, then it holds for Y and the property holds for the empty set. To prove this, let's assume that P doesn't hold for X, then we know that by assumption, there is a proper closed subset of X, say X_1 and $X = X_0$, such that P doesn't hold for X_1, then we proceed further, then since the space is Noetherian, we know that after finitely many steps, we reach the empty set, but the property holds for that, thus a contradiction.

Thus, when we want to apply Noetherian induction, we typically first check it for the empty set and after that we need to prove that the property we want satisfies the property that if it holds for all proper closed subsets of some closed subsets Y, then it holds for Y. This is why we can assume that we can fix $\mathrm{Supp}\mathcal{F}$ and assume that it holds for all proper closed subsets. One small remark here is that we can indeed represent every closed subset of X as the support of a coherent sheaf $\mathcal{F}$, and the trick is to use the reduced closed subscheme.

Now enough with the prerequisites, let's head to the real business.

The first step we need to do is to show that we can actually assume $\mathrm{Supp}\mathcal{F}$ is X. Recall that if X is a scheme and $\mathcal{F}$ is a finite type quasicoherent sheaf on X, we can define it's scheme theoretic support to be the scheme-theoretic intersection of all closed subschemes whose ideal sheaf annihilate $\mathcal{F}$. Then we know that this ideal sheaf of the scheme theoretic support also annihilate $\mathcal{F}$ and moreover, the underlying topological space of this closed subscheme is the support of $\mathcal{F}$. The first assertion is easy to prove since the ideal sheaf is just the sum of a bunch of ideal sheaves each of which kills $\mathcal{F}$. To see the second assertion, we can define a quasicoherent ideal sheaf and show that it annihilate $\mathcal{F}$ and that for every ideal sheaf $\mathcal{I}$ that annihilate $\mathcal{F}$, it's going to be a subsheaf of this. We define it on a distinguished base, and notice that if $SpecA$ is affine, then we define the ideal on $SpecA$ to be $\mathrm{Ann}(\mathcal{F}(SpecA))$, then you can easily see this is an ideal then we need to show that this is well behaved when going to a distinguished open. Or in other words, we need to show that $\mathrm{Ann}M_f$ is canonically isomorphic to $(\mathrm{Ann}M)_f$, what is clear is that $(\mathrm{Ann}M)_f$ is always in $\mathrm{Ann}M_f$, conversely, suppose that $\frac{a}{f^n}$ is in $\mathrm{Ann}M_f$, then we know that $f^m ax = 0$ for every $x \in M$, then since M is finitely generated, we can assume that it's generated by $x_1, \cdots, x_n$, then for each of these elements, we know $f^{m_i} ax_i = 0$, therefore we can take f to be the largest and say that $\frac{a}{f^n}$ is actually from $\mathrm{Ann}M$ then localized at f and this is why we need this finite generation condition. We also need to show that locally $V(\mathrm{Ann}M) = \mathrm{Supp}M$ for a finitely generated module M. Clearly, if p is in the support, then p contain $\mathrm{Ann}M$. Conversely, if p contains $\mathrm{Ann}M$, then we know that p is in the support, otherwise if M_p is 0, then we know that, if still we assume $x_1, \cdots, x_n$ generates M, then we know that there are $s_1, \cdots, s_n$ not in p such that s_i annihilates x_i, thus the product s annihilates M, thus $s \in \mathrm{Ann}M \subset p$, contradiction, we are done.

Now if we consider the following diagram:

$$\begin{tikzcd}
\mathrm{Supp}\mathcal{F} \arrow[rr, hook] \arrow[dr] & & X \arrow[dl] \\
& SpecA &
\end{tikzcd}$$

since we know that there is a $\mathcal{G}$, cohenrent module on $\mathrm{Supp}\mathcal{F}$ such that the pushforward is $\mathcal{F}$ and cohomology is stable under pushforward of closed embeddings, then we pnly have to show that the cohomology of $\mathcal{G}$ on $\mathrm{Supp}\mathcal{F}$ is finitely generated over A. Moreover, closed embeddings are proper, thus the composition is again proper, thus the hypothesis is preserved.

We now invoke Chow's lemma and say that there is a projective morphism $\mu : X' \to X$ that is an isomorphism on a dense open U of X, therefore $X - U$ is a closed subset of X strictly smaller than X, such that $\pi \circ \mu : X' \to SpecA$ is still projective. We know that $\mathcal{G} = \mu^*\mathcal{F}$ is coherent on X' since it's the pullback of a coherent sheaf and $\mu_*\mathcal{G}$ is coherent on X since it's the pushforward by a projective morphism and we already know this for a projective morphism.

Now consider the adjunction morphism:

$$\mathcal{F} \to \mu_*\mu^*\mathcal{F}$$

If you restrict this to U, since we know that adjunction commutes with restriction, we know that this is the adjunction map obtained from the identity:

$$\mu^*\mathcal{F}|_V \to \mu^*\mathcal{F}|_V$$

But recall that if $\pi : X \to Y$ is an isomorphism of schemes, we know that the adjunction map take the identity $\pi^*\mathcal{F} \to \pi^*\mathcal{F}$ to an isomorphism for quasicoherent sheaf $\mathcal{F}$, as you can check it locally $id : M \otimes A \to M \to A$ is mapped to $M \to M \otimes A$, which is still and isomorphism since $M \otimes A \simeq M$.

Therefore if you consider the kernel and cokernel of $\mathcal{F} \to \mu_*\mathcal{G}$, as in the following sequence:

$$0 \to \mathcal{E} = \ker \to \mathcal{F} \to \mu_*\mathcal{G} \to \mathrm{coker} = \mathcal{H} \to 0$$

We know that the support of the kernel and cokernel are strictly smaller than the support of $\mathcal{F}$. Therefore, by our hypothesis, we can assume that the higher pushforward of $\mathcal{E}$ and $\mathcal{H}$ are coherent. Then I claim that if all the higher pushforward of $\mu_*\mathcal{G}$ are coherent, we can conclude that the higher pushforward of $\mathcal{F}$ are coherent.

The reason is that we can break the exact sequence of 4 terms into two exact sequences of 3 terms and one of them is:

$$0 \to \mathcal{E} \to \mathcal{F} \to \ker \to 0$$

where the kernel is the kernel of $\mu_*\mathcal{G} \to \mathrm{coker} = \mathcal{H}$, therefore we can first consider the associated LES of the second SES and conclude that, again using the fact that we can break it into a lot of SES, all the higher pushforward of the kernel is coherent, then by looking at another LES to finish the proof.

Therefore, the question is not translated to the question of showing the higher pushforward of $\mu_*\mathcal{G}$ is coherent. This is where we need to use the Leray Spectral sequence. Consider the morphism:

$$X' \xrightarrow{\mu} X \xrightarrow{\pi} SpecA$$

We know that we have a spectral sequence, whose E_2 page is given by $R^p\pi_*(R^q\mu_*\mathcal{G})$ converging to $R^{p+q}(\pi \circ \mu)_*\mathcal{G}$, we know that since μ is projective, then $R^q\mu_*\mathcal{G}$ is coherent. Moreover, since μ is an isomorphism on U, then it's affine there, thus for $q \geqslant 1$, we know that $R^q\mu_*\mathcal{G}$ is 0 on U, thus it's support is smaller than X, therefore by our inductive hypothesis, we know that $R^p\pi_*(R^q\mu_*\mathcal{G})$ is coherent for all $q \geqslant 1$. Therefore, the only possible noncoherent sheaf on E_2 page are in the row $q = 0$, which are the sheaves we want to be coherent.

The final ingredient is that since $\pi \circ \mu$ is projective, then $R^{p+q}(\pi \circ \mu)_*\mathcal{G}$ is coherent for all p, q. We know that since $E_2^{p,q}$ converges to $R^{p+q}(\pi \circ \mu)_*\mathcal{G}$, we know by definition that $E_\infty^{p,q}$ will be coherent, since it will be the quotient of two coherent sheaves. We know that $E_2^{p,q}$ is coherent for $q > 1$, then we know that for $r \geqslant 2$ and $r + 1$, $E_{r+1}^{p,q}$ is the cohomology:

$$\frac{\ker(E_r^{p,q} \to E_r^{p+r,q-r+1})}{\mathrm{im}(E_r^{p-r,q+r-1} \to E_r^{p,q})}$$

which is the quotient of two coherent sheaves, therefore coherent. Now I claim that $E_2^{p,0}$ is coherent iff there is a given $n \geqslant 2$ such that $E_n^{p,0}$ is coherent. If $E_n^{p,0}$ is coherent, then since it's the quotient of $E_{n-1}^{p,0}$ by something coherent, then it's coherent iff $E_{n-1}^{p,0}$ is coherent, then we can keep going down until we reach $E_2^{p,0}$, and vice versa, but we have already saw that $E_\infty^{p,q}$ is coherent for all p, q, we are done. □

Now we prove Chow's lemma and you will see that we will use properness in three of it's components, i.e. finite type, universally closed and separated. You will see that separatedness is going to be used in a particularly tricky way.

Proof. First, since X is Noetherian, it has finitely many irreducible components, we cover X with finitely many affines, $U_1, \cdots, U_n$. We can assume that each of them intersect with every component of X, this is because we know that if U_i doesn't meet a component Z, we know that since Z is a component and there are only finitely many, there is a nonempty open contained in Z, and we just replace U_i with the union. Then we know that the intersection of U_i denoted by U is dense in X.

Now since U_i is finite type over A, we know that there is a closed embedding $U_i \hookrightarrow \mathbb{A}_A^{n_i}$ and we will use $\overline{U_i}$ to denote the scheme theoretic closure of:

$$U_i \hookrightarrow \mathbb{A}_A^{n_i} \hookrightarrow \mathbb{P}_A^{n_i}$$

Now consider the following morphism:

$$U \to X \times_A \prod \overline{U_i}$$

which is defined by the composition of:

$$U \hookrightarrow U^{n+1} \qquad U^{n+1} \hookrightarrow X \times \prod \overline{U_i}$$

the first one is a closed embedding since $U \to SpecA$ is separated (monomorphisms are all separated and composition of separated morphisms are separated). The second morphism is an open embedding since $U \hookrightarrow X$ and $U \hookrightarrow U_i \hookrightarrow \overline{U}$ are all open embeddings (here we know that $U_i \hookrightarrow \overline{U_i}$ is an open embedding since U_i to the projective space is a locally closed embedding, then it's factorization to the scheme theoretic image is an open embedding, we talked about this in the discussion of scheme theoretic closure and locally closed embeddings).

Let X' be the scheme theoretic closure of U in $X \times_A \prod \overline{U_i}$ thus we have the following diagram:

$$\begin{tikzcd}
X' \arrow[d, hook] \arrow[dr, "\mu"] & \\
X \times_A \prod \overline{U_i} \arrow[r, "\text{proj}"'] \arrow[d, "\text{proper}"'] & X \arrow[d, "\text{proper}"] \\
\prod \overline{U_i} \arrow[r, "\text{proj}"'] & SpecA
\end{tikzcd}$$

Recall that since X is quasicompact, then composition of projective morphisms to X will be projective, therefore, if we can prove that μ is an isomorphism above U and $X' \to SpecA$ is projective, we are done. $\square$

Remark 3.4. Before we continue the proof, we make a small remark on the scheme theoretic closure of a locally closed embedding and show that it's factorization is indeed an open embedding.

Recall that we have proved before that is a morphism $\pi : X \to Y$ is quasicompact then we can compute it's scheme theoretic image affine locally and in particular, the closure of the set theoretic image is the underlying set of the scheme theoretic image. Then suppose that:

$$V \overset{\text{closed}}{\hookrightarrow} X \overset{\text{open}}{\hookrightarrow} Y$$

is a quasicompact locally closed embedding, we want to show that the factorization to it's scheme theoretic image is an open embedding.

This is basically a topology statement, say that V is closed in an open subset X of Y, if we consider it's closure in Y and then intersect it back to X, we obtain V since the intersection is the closure of V in X and V is closed in X, then if you consider the following diagram:

$$\begin{tikzcd}
& \text{im} \cap X \arrow[r, hook, "\text{open}"] \arrow[d, "\text{closed}"'] & \text{im} \arrow[d, hook, "\text{closed}"] \\
V \arrow[r, "\text{closed}"] & X \arrow[r, "\text{open}"] & Y
\end{tikzcd}$$

We know that $\text{im} \cap X$ is the same as a set as V in X, Moreover, if we compute the closed embedding in an affine open of X, say $SpecA$, we know that $\text{im} \cap X$ is cut out by the elements in A that's pulled back to 0 and V, since it's contained in X, is also cut out by these elements, thus we know these are indeed the same closed embedding!

Moreover, recall that if $X \hookrightarrow Y$ is a closed embedding, whose corresponding ideal sheaf is $\mathcal{I}$, then if $X' \to Y$ is any morphism of schemes such that the pullback of $\mathcal{I}$ is 0, then we know that we have a factorization:

$$\begin{tikzcd}
& X \arrow[d, hook] \\
X' \arrow[ur, dashed] \arrow[r] & Y
\end{tikzcd}$$

it's defined locally on X and they patch together. Another way to see it is to form the fibered product:

$$\begin{tikzcd} X' \cap X \arrow[r] \arrow[d, hook] & X \arrow[d, hook] \\ X' \arrow[r] & Y \end{tikzcd}$$

and show that $X' \cap X \hookrightarrow X'$ is an isomorphism, which can be checked locally.

You see that $\mathrm{im} \cap X \hookrightarrow \mathrm{im}$ is just this factorization.

Then we head back to our proof of Chow's lemma, the following proposition is useful, though it's not directly connected to the proof, it's idea is extremely useful:

Proposition 3.5. Suppose that $T_0, \cdots, T_n$ are separated schemes over A with isomorphic open subschemes, say V for every one of them. Then V is a locally closed subscheme of $T_0 \times_A \times_A \cdots \times_A T_n$, if we denote by $\overline{V}$ the closure of this locally closed subscheme, we have the following isomorphisms:

$$\begin{aligned} V &\simeq \overline{V} \cap (V \times_A T_1 \times_A \cdots \times_A T_n) \\ &\simeq \overline{V} \cap (T_0 \times_A V \times_A \cdots \times_A T_n) \\ &\qquad \cdots \\ &\simeq \overline{V} \cap (T_0 \times_A T_1 \times_A \cdots \times_A V) \end{aligned}$$

Proof. First, let's take a look at why V is a locally closed subscheme of $T_0 \times_A \cdots \times_A T_n$. Consider the morphism:

$$V \to \prod_{i=1}^{n} T_i$$

We don't know that kind of morphism is this, but we can consider the graph of this morphism:

$$V \hookrightarrow V \times_A \prod_{i=1}^{n} T_i$$

since we know that T_i is separated over A, then so is the product, then we know that this morphism is going to be a closed embedding, then compose the graph morphism with the following open embedding:

$$V \hookrightarrow V \times_A \prod_{i=1}^{n} T_i \hookrightarrow T_0 \times_A \prod_{i=1}^{n} T_i$$

induced by product of open embeddings, we are done.

Then we know that since everything here is quasicompact, therefore by the remark we just made, if you intersect the scheme theoretic image with $V \times_A \prod_{i=1}^{n} T_i$m you just obtain itself, which is isomorphism to V, and this argument works if we put V at some other components. ☐

Now back to Chow's lemma:

Proof. Using the same idea, we consider $X' \to X \times_A \prod \overline{U_i}$, it's defined to be the scheme theoretic image of $U \to X \times_A \prod \overline{U_i}$, which is a locally closed embedding, therefore we know that we have the following diagram:

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d] & U \times_A \prod \overline{U_i} \arrow[r] \arrow[d, hook] & U \arrow[d, hook] \\ X' \arrow[r, hook] & X \times_A \prod \overline{U_i} \arrow[r] & X \end{tikzcd}$$

and you know that the morphism $U \to U \times_A \prod \overline{U_i}$ is the graph morphism, then the composition with the projection is obviously the identity, thus we are done.

Finally, we are left with proving $X' \to \prod \overline{U_i}$ is a closed embedding, but notice that it's the composition of two closed maps, therefore we only need to show that it's a locally closed embedding. The following exercise is useful: ☐

Proposition 3.6. Let A_i be the closure of U in $B_i = X \times_A \overline{U_1} \times_A \cdots \times_A U_i \times_A \cdots \times_A \overline{U_n}$, where only the i-th bar is removed.

Let C_i be the closure of U in $D_i = \overline{U_1} \times_A \cdots \times_A U_i \times_A \cdots \times_A \overline{U_n}$

Then there is is an isomorphism $A_i \simeq C_i$ induced by the caonical projection $B_i \to D_i$

Proof. First, notice that there is a section of the projection $B_i \to D_i$, which can be considered as the graph morphism:

$$D_i \to X$$

which is defined as $D_i \to U_i \hookrightarrow X$, then since X is separated over A, the graph morphism is a closed embedding, thus the section is also a closed embedding.

Then if you think of $U \hookrightarrow D_i$ as the composition of

$$U \hookrightarrow U^{n+1} \hookrightarrow U_1 \times_A \cdots \times_A U_n \hookrightarrow D_i$$

, then U is a locally closed subscheme of D_i, then use the closed embedding to map it back to B_i, which is the same as directly let U map to B_i, then we know that the closure of U in D_i can be identified as the closure of U in B_i since D_i to B_i is a closed embedding, finally we can use the fact that this is a section to conclude the isomorphism, this is summarized in the following diagram:

$$\begin{tikzcd} U \arrow[r] \arrow[d, dashed, "id"] & B_i \arrow[d, shift left] \\ U \arrow[r] & D_i \arrow[u, hook, shift left] \end{tikzcd} \qquad \begin{tikzcd} & B_i \arrow[d, shift left] \\ \overline{U} \arrow[ur] \arrow[r, hook] & D_i \arrow[u, hook, shift left] \end{tikzcd}$$

and what is implicit in this diagram is that if you have a composition of morphism:

$$U \to X \hookrightarrow Y$$

where the second morphism is a closed embedding, then the scheme theoretic image of the composition can be computed by first computing the image in X and then compose with the closed embedding at least for quasicomapct morphisms and this is basically a consequence of the fact that we can compute the scheme theoretic image locally and the following algebraic fact:

If $A \to B$ factors through $A \to A/I \to B$, then suppose that the kernel of $A/I \to B$ is J/I, then the morphism also factors through $A \to A/J \to B$. $\square$

Back to Chow's lemma again:

Proof. Recall that U_i covers X, then $\mu^{-1}(U_i)$ covers X', but we notice that $\mu^{-1}(U_i) = A_i$.

Recall that locally closed embeddings can be checked local on the target, notice that we have a opens of $\prod \overline{U_i}$, which is D_i, then if we can prove that over D_i we have a closed embedding, we are done. But this is just the closed embedding $A_i \hookrightarrow D_i$, or in other words $A_i \simeq C_i \hookrightarrow D_i$, thus we are done. $\square$

References