Appendix 13 for Math 632: 2025 Winter Notes

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In this part of the notes, we will develop a little bit towards intersection theory, and in this section X will be a variety, and in most of the applications X will be projective.

1 Intersecting n Line Bundles with a n Dimensional Variety

The central tool in this section is the following:

Definition 1.1. Suppose that $\mathcal{F}$ is a coherent sheaf on X with proper support (which means that it's scheme theoretic support is proper over k and it's automatically true when X itself is proper) of dimension at most n, let $\mathcal{L}_1, \cdots, \mathcal{L}_n$ to be line bundles on X.

Let $(\mathcal{L}_1 \cdot \cdots \cdot \mathcal{L}_n \cdot \mathcal{F})$ be the signed sum over the 2^n subsets of $\{1, \cdots, n\}$, i.e.:

$$\sum_{\{i_1,\cdots,i_m\}\subset\{1,\cdots,n\}} (-1)^m \chi(X, \mathcal{L}_{i_1}^\vee \otimes \cdots \otimes \mathcal{L}_{i_m}^\vee \otimes \mathcal{F})$$

We call this the **intersection of** $\mathcal{L}_1, \cdots, \mathcal{L}_n$ **with** $\mathcal{F}$. There is a question of well-definedness of this quantity, since we defined the Euler characteristic to be the alternating sum of $h^i(X, \cdots)$ up untile i is the dimension of X and if X is not projective or proper this may not be finite.

However, this is not really a question since we assumed that the scheme theoretic support of $\mathcal{F}$ is proper, and we know that $\mathcal{L}_{i_1}^\vee \otimes \cdots \otimes \mathcal{L}_{i_m}^\vee \otimes \mathcal{F}$ is actually a pushforward of a coherent sheaf $\mathcal{G}$ on Y, thus $h^i(X, \mathcal{L}_{i_1}^\vee \otimes \cdots \otimes \mathcal{L}_{i_m}^\vee \otimes \mathcal{F}) = h^i(Y, \mathcal{G})$, but recall that by Grothendieck's coherence theorem, this is finite dimensional.

The most useful case of this definition is when $\mathcal{F}$ is the structure sheaf of a closed subscheme of X of dimension at most n, in this case, we will write it as $(\mathcal{L}_1 \cdots \mathcal{L}_n \cdot Y)$ and call it the **intersection of** $\mathcal{L}_1, \cdots, \mathcal{L}_n$ **with** Y. And if Y is X, we will only write $(\mathcal{L}_1 \cdot \mathcal{L}_2 \cdots \mathcal{L}_n)$ and call it the **intersection of** $\mathcal{L}_1, \cdots, \mathcal{L}_n$, with X left implicit.

Finally if all $\mathcal{L}_i$ are the same, we will write it as $(\mathcal{L}^n \cdot \mathcal{F})$ or $(\mathcal{L}^n \cdot Y)$ or $(\mathcal{L}^n)$, be careful since $\mathcal{L}^n$ doesn't mean $\mathcal{L}^{\otimes n}$.

There are also conventions which omit the parentheses and we also call $(\mathcal{L}_1 \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F}), (\mathcal{L}_1 \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot Y), (\mathcal{L}_1 \cdot \mathcal{L}_2 \cdots \mathcal{L}_n)$ the **intersection number** or **intersection product**.

Remark 1.2. Whenever we write $(\mathcal{L}_1 \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F})$ we implicitly mean that the support of $\mathcal{F}$ is at most n and the reason for this extra requirement will be clear in the following discussion, where we prove many things for the intersection product, with only one thing left for after we study flatness.

Proposition 1.3. When $\mathcal{L}_1 = \mathcal{O}_X$, then $(\mathcal{L}_1 \cdot \mathcal{L}_2 \cdots \mathcal{L}_n \cdot \mathcal{F}) = 0$.

Proof. This is just a simple combinitorics problem since if we have a term:

$$(-1)^m \chi(X, \mathcal{L}_{i_1}^\vee \otimes \cdots \mathcal{L}_{i_m}^\vee \otimes \mathcal{F})$$

and none of the line bundles is equal to the structure sheaf, we know that you can use a cardinality $m+1$ subset by adding $\mathcal{O}_X$ and obtain the same Euler characteristic while changing the sign to $(-1)^{m+1}$, thus the sum cancels. $\square$

Proposition 1.4. Intersection is independent of field extensions of k, i.e. if K/k is a field extension then the intersection product of $(\mathcal{L}_1 \otimes K \cdots \mathcal{L}_n \otimes K, \mathcal{F} \otimes K)$ on X_K is the same as the intersection product of $(\mathcal{L}_1, \cdots, \mathcal{L}_n, \mathcal{F})$ on X.

Proof. First we need to show the support of $\mathcal{F} \otimes_k K$ on X_K is still at most dimension n, and I claim that the dimension of the support is the same as the dimension of the support of $\mathcal{F}$ on X. To see why, we consider:

$$\begin{array}{ccccc} \mathrm{Supp}\mathcal{F}_K & \hookrightarrow & X_K & \longrightarrow & K \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Supp}\mathcal{F} & \hookrightarrow & X & \longrightarrow & k \end{array}$$

where $\mathrm{Supp}\mathcal{F}_K$ is the inverse image of $\mathrm{Supp}\mathcal{F}$ in X_K. Then say C_i is an irreducible component of $\mathrm{Supp}\mathcal{F}$, we know it's of dimension at most n, and the extension, or the inverse image is also of dimension at most n and is pure dimensional, this proves that $\mathrm{Supp}\mathcal{F}_K$ is the union of irreducible closed subsets of dimension at most n, therefore the dimension of $\mathrm{Supp}\mathcal{F}_K$ can not exceed n. Then we only have to show that the support of $\mathcal{F} \otimes K$ is the inverse image of $\mathrm{Supp}\mathcal{F}$ and we are done. We have discussed this before, but let's prove it again here. Two key facts are the following: 1. if $A \to B$ is a ring morphism such that q is a prime of B with preimage p, then for any A-module M, we have a canonical isomorphism of B_q-modules:

$$(M \otimes_A B)_q \simeq M_p \otimes_{A_p} B_q$$

You can prove this by showing that the stuff on the right hand side and the left hand side satisfies the same universal property, i.e. they represent the same object, thus canonically isomorphism. Or you can construct an explicit isomorphism:

$$\frac{m \otimes b}{s} \mapsto \frac{m}{1} \otimes \frac{b}{s} \qquad \frac{m}{t} \otimes \frac{b}{s} \mapsto \frac{a \otimes b}{s \cdot t}$$

which are again, defined by universal properties. 2. For any field extension K/k it's a faithfully flat ring morphism. To prove this, we need the fact that faithfully flat is equivalent to the corresponding morphism on spectra is surjective and the ring morphism is flat. We know this trivially holds for each K/k. Finally, we know that for a faithfully flat ring map, it preserves 0 objects, i.e. if $A \to B$ is a faithfully flat map, then for any A-module M, $M \otimes_A B$ is zero iff $M = 0$. We also need the fact that if $R \to R'$ is a ring map and M is faithfully flat over R, then $M \otimes_R R'$ is faithfully flat over R', we apply this to the morphism $k \to A$, then we know that since K is faithfully flat over k, then $K \otimes_k A$ is faithfully over over A, therefore we can apply what we proved for $A \to K \otimes_k A$ and conclude that everything in the inverse image of the support of M is the support of the pullback of M. Suppose that q is in the support of the pullback of M, we know that it's image p in $SpecA$, should be in the support of M, therefore, we have prove that for a faithfully flat morphism of rings, the inverse image of the support is the support of the pullback. Then recall that on schemes, support of quasicoherent sheaves can be defined locally, thus we proved that the inverse image of the support of $\mathcal{F}$ on X is the support of the sheaf $\mathcal{F} \otimes K$, we are done.

Now suppose that K/k is any field extension, we already showed that the dimension of cohomology groups is independent of field extensions, also the above discussion shows that the intersection number of the pullback makes sense, thus we are done. $\square$

Proposition 1.5. If X is a projective curve and $\mathcal{L}$ is an invertible sheaf on X, then $(\mathcal{L} \cdot X) = \deg_X \mathcal{L}$.

Proof. We know nothing about the intersection product rather than the definition now, thus let's try to directly use the definition:

$$(\mathcal{L} \cdot X) = \chi(X, \mathcal{O}_X) - \chi(X, \mathcal{L}^\vee)$$

Recall that degree on a curve is additive upon tensor products, therefore we know that $\chi(X, \mathcal{L}^\vee) = -\deg\mathcal{L} + \chi(X, \mathcal{O}_X)$, plug this in the above equality, we are done. $\square$

Proposition 1.6. Suppose that k is an infinite field and $X = \mathbb{P}^N$, Y is a dimension n subvariety of X, if $H_1, \cdots, H_n$ are generally chosen hypersurfaces of degree $d_1, \cdots, d_n$ respectively, then we already know that:

$$\deg(H_1 \cap \cdots \cap H_n \cap Y) = d_1 \cdots d_n \deg Y$$

I claim that:

$$(\mathcal{O}(H_1) \cdots \mathcal{O}(H_n) \cdot Y) = d_1 \cdots d_n \deg Y$$

Proof. Recall that by a Bertini Type argument, we showed that there is a dense open in $(\mathbb{P}^{N\vee})^n$ such that for each closed point $(H_1, \cdots, H_n)$ of U, the scheme-theoretic intersection of $H_1, \cdots, H_n$ with X is only a finite number of points, i.e. of dimension 0 for a dimension n variety X in $\mathbb{P}^N$.

Recall that as a subvariety, Y is reduced, thus there is no embedded point, thus if $H_1 \cap \cdots \cap H_n \cap Y$ is a dimension 0 stuff, each the successive intersection doesn't contain the associated points, i.e. components of the previous one, therefore, we can apply Bezout's theorem inductively, this is where the first equality comes from.

To prove the second equality, we first claim that:

$$(\mathcal{O}_X(H_1) \cdots \mathcal{O}_X(H_n) \cdot Y) = (\mathcal{O}_Y(H_1) \cdots \mathcal{O}_Y(H_{n-1}) \cdot Y \cap H_n)$$

where the line bundles $\mathcal{O}_Y(H_i)$ are the pullback of $\mathcal{O}_X(H_i)$ to Y. This is because H_i is going to restrict to an effective Cartier divisor on $Y \cap H_1 \cap \cdots \cap H_{i-1}$, which is proved in the next proposition. Moreover, this procedure can be continued and in the end we get:

$$(\mathcal{O}_X(H_1) \cdots \mathcal{O}_X(H_n) \cdot Y) = (\mathcal{O}_{Y \cap H_2 \cap \cdots \cap H_n}(H_1) \cdot H_2|_{Y \cap H_3 \cap \cdots \cap H_n}) = \deg_C(\mathcal{O}_X(H_1)|_C)$$

where C is the projective curve $Y \cap H_2 \cap \cdots \cap H_n$. This is $\deg_C(\mathcal{O}_X(d_1)|_C)$, recall that degree is additive upon tensor, thus this is $d_1 \cdot \deg_C(\mathcal{O}_X(1)|_C)$, recall that by Bertini's Theorem we can choose the intersection to be smooth, in particular regular, therefore we know that since $\mathcal{O}_X(1)|_C$ is the line bundle that embeds C to the projective space, it's the degree of C, thus we get in the end:

$$d_1 \cdot \deg(C)$$

But we know how to compute inductively the degree of C using Bezout's theorem, i.e. it's $d_1 \cdot d_2 \cdots d_n \cdot \deg Y$, we are done. $\square$

Now we prove the claim in the above proposition:

Proposition 1.7. Suppose that D is an effective Cartier divisor on X that restricts an effective Cartier divisor $D|_Y$ on a closed subvariety Y of dimension at most n, then:

$$(\mathcal{L}_1, \cdots, \mathcal{L}_{n-1} \cdot \mathcal{O}_X(D) \cdot Y) = (\mathcal{L}_1 \cdots \mathcal{L}_{n-1} \cdot D|_Y)$$

More generally, if D doesn't contain any associated point of $\mathcal{F}$, we have:

$$(\mathcal{L}_1 \cdots \mathcal{L}_{n-1} \cdot \mathcal{O}_X(D) \cdot \mathcal{F}) = (\mathcal{L}_1 \cdots \mathcal{L}_{n-1} \cdot \mathcal{F}|_D)$$

Proof. In this proof, we will use two suggestive notaions, suppose that Y is a subvariety of X, we will use $(\mathcal{L}_1, \cdots, \mathcal{L}_n, Y)_X$ to implies that the sheaves in the intersection number are sheaves on X, in particular $\mathcal{O}_Y$ are interpreted as $i_*\mathcal{O}_Y$. On the other hand, we will use the notation $()_Y$ to suggest all the sheaves are line bundles on Y, i.e. we pull line bundles back to Y and $\mathcal{O}_Y$ are just the structure sheaf. In the statement, everything is $()_X$ and $D|_Y$ means that $j_*\mathcal{O}_D \otimes i_*\mathcal{O}_Y$ and $\mathcal{F}|_D$ means that $\mathcal{F} \otimes j_*\mathcal{O}_D$ for i, j closed embeddings of Y, D respectively.

First, we notice that if you have a SES of coherent sheaves on X with support at most dimension n:

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

Then we have:

$$(\mathcal{L}_1 \cdots \mathcal{L}_n \cdot \mathcal{G})_X = (\mathcal{L}_1 \cdots \mathcal{L}_n \cdot \mathcal{F})_X + (\mathcal{L}_1 \cdots \mathcal{L}_n \cdot \mathcal{H})_X$$

The reason is simple since these intersection numbers are defined by Euler characteristics and Euler characteristics are additive upon SESs.

Recall that by definition, the computation of the following quantity consists of two parts, the first one is the terms with no $\mathcal{O}_X(D)$ involved and the second part is the part that has $\mathcal{O}_X(D)$, then by projection formula we have:

$$(\mathcal{L}_1, \cdots, \mathcal{L}_{n-1} \cdot \mathcal{O}_X(D) \cdot Y)_X = (\mathcal{L}_1 \cdots \mathcal{L}_{n-1} \cdot Y)_Y - (\mathcal{L}_1 \cdots \mathcal{L}_{n-1} \cdot i_*\mathcal{O}_Y \otimes \mathcal{O}_X(-D))_X$$

$$= (\mathcal{L}_1 \cdots \mathcal{L}_{n-1} \cdot Y)_Y - (\mathcal{L}_1 \cdots \mathcal{L}_{n-1} \cdot i^*\mathcal{O}_X(-D))_Y$$

Let's consider the SES of effective divisors on Y:

$$0 \to \mathcal{O}_Y(-D|_Y) \to \mathcal{O}_Y \to \mathcal{O}_{D|_Y} \to 0$$

Then I claim that $\mathcal{O}_Y(-D|_Y) \simeq i^*\mathcal{O}_X(-D)$, if this is true then we use the additivity of intersection number to conclude that:

$$(\mathcal{L}_1, \cdots, \mathcal{L}_{n-1} \cdot \mathcal{O}_X(D) \cdot Y)_X = (\mathcal{L}_1 \cdots \mathcal{L}_{n-1} \cdot \mathcal{O}_{D|_Y})_Y$$

Then we are done since we know by projection formula:

$$(\mathcal{L}_1 \cdots \mathcal{L}_{n-1} \cdot i_*\mathcal{O}_Y \otimes j_*\mathcal{O}_D)_X = (\mathcal{L}_1 \cdot \mathcal{L}_{n-1} \cdot i^*j_*\mathcal{O}_D)_Y$$

We also know that $\mathcal{O}_{D|_Y}$ actually means that $k_*\mathcal{O}_{D|_Y}$ where k is the closed embedding of $D|_Y$ to Y. Then we only need to show that $i^*j_*\mathcal{O}_D \simeq k_*\mathcal{O}_{D|_Y}$, we know that applying i^* is exact to the SES of D, therefore we know that $i^*j_*\mathcal{O}_D$ is the cokernel of:

$$\mathcal{O}_Y(-D|_Y) \to \mathcal{O}_Y$$

therefore, is $k_*\mathcal{O}_{D|_Y}$.

Therefore, we only need to prove that $\mathcal{O}_Y(-D|_Y) \simeq i^*\mathcal{O}_X(-D)$, i.e. to prove that if the restriction of D to a closed subscheme correspond to the line bundle which is the pullback. Recall that we know how to affine locally describe:

$$D|_Y \hookrightarrow Y \hookrightarrow X$$

Thus is we assume that X locally is $SpecA$, Y is $SpecA/I$ and D locally in X is cut out by f, we know that $D|_Y$ in Y is cut out by $(\overline{f})$ in A/I. On the other hand, we know that the morphism:

$$\mathcal{O}(-D) \to \mathcal{O}_X$$

is locally $(f) \hookrightarrow A$ and after we pullback, locally on A/I, it's $(\overline{f}) \to A/I$, thus we are done.

Another perspective you should have is that we can consider $(\mathcal{L}_1 \cdots \mathcal{L}_{n-1} \cdot D|_Y)$ as on Y, thus we have translated the computation of intersecton number on X to computation of intersection number on Y.

Now we begin to prove the general case where we assume that D is an effective Cartier divisor that doesn't contain any associated points of $\mathcal{F}$, we again consider the SES of effective Cartier divisors:

$$0 \to \mathcal{O}(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

We know that tensor by $\mathcal{F}$ still gives you a sequence:

$$0 \to \mathcal{F}(-D) \to \mathcal{F} \to \mathcal{F}|_D \to 0$$

It's just we don't know if it's exact on the left, i.e. not sure whether if it's injective or not. But if we assume D doesn't contain any associated point of $\mathcal{F}$, we can work affine locally, say $\mathcal{F}$ on $SpecA$ is a module M and D is locally cut out by (f) a nonzero divisor, then we know that this is locally:

$$(f) \otimes M \to M$$

This is isomorphic to $M \xrightarrow{\cdot f} M$, since we assume that f is not contained in associated point of M in $SpecA$ (associated points are defined affine locally), then $\cdot f$ is injective, we are done.

Then, again, we use a similar strategy, we know that:

$$(\mathcal{L}_1, \cdots, \mathcal{L}_{n-1} \cdot \mathcal{O}(D) \cdot \mathcal{F})$$

consists of two parts, one is those with $\mathcal{O}(D)$ and one is those without $\mathcal{O}(D)$, and we have:

$$(\mathcal{L}_1, \cdots, \mathcal{L}_{n-1} \cdot \mathcal{O}(D) \cdot \mathcal{F}) = (\mathcal{L}_1 \cdots \mathcal{L}_{n-1} \cdot \mathcal{F}) - (\mathcal{L}_1 \cdots \mathcal{L}_{n-1} \cdot \mathcal{F}(-D))$$

Again, we use the additivity of intersection number to conclude. $\square$

Remark 1.8. The first case is indeed a special case of the second, as we know that by assumption, D restricts to an effective divisor in Y (locally a nonzero divisor), this is equivalent to D doesn't contain any component of Y, i.e. D doesn't contain any associated point of Y.

Therefore the proof of the first assertion can be done in a similar manner as the second assertion by tensoring;

$$0 \to \mathcal{O}(-D) \to \mathcal{O}_X \to j_*\mathcal{O}_D \to 0$$

with $i_*\mathcal{O}_Y$ and obtain a SES:

$$0 \to i_*\mathcal{O}_Y(-D) \to i_*\mathcal{O}_Y \to i_*\mathcal{O}_Y \otimes j_*\mathcal{O}_D \to 0$$

This proof is essentially the same as what we did above since in the above proof, we are just using the projection formula to convert the calculations on X using the sequence here to the calculations on Y using the SES on Y. They just differ by how you want to think about this intersection.

Proposition 1.9. Assume that X is projective, for each fixed $\mathcal{F}$, the intersection product is a symmetric multilinear form on $\mathcal{L}_1, \cdots, \mathcal{L}_n$.

Remark 1.10. This proposition holds as well with projective replaced by proper, but this requires more than a direct application of Chow's lemma, so we don't prove it here.

Remark 1.11. Due to the multilinearity of this intersection product, sometimes it's helpful to write tensor products of invertible sheaves additively since this matches more with the multilinear form.

This is helpful but sometimes causes some confusion, some people try to use divisors to make them feel better, this is good when you work with nice enough objects as line bundles correspond to divisors, but you will need to worry about when you are allowed to do it.

Now let's prove this proposition:

Proof. By definition, since tensor products are commutative, it's easy to see that this is symmetric. We only have to prove multilinearity, and we have proved that intersection product is independent of field extension, thus we can assume that k is algebraically closed, in particular k is infinite. We will prove the general case by induction on n.

The base case is $n = 1$ and in this case, we need to show that:

$$(\mathcal{L}_1 + \mathcal{L}_2 \cdot \mathcal{F}) = (\mathcal{L}_1 \cdot \mathcal{F}) + (\mathcal{L}_2 \cdot \mathcal{F})$$

By definition, we know that the left hand side is:

$$\chi(X, \mathcal{F}) - \chi(X, \mathcal{F} \otimes \mathcal{L}_1^\vee \otimes \mathcal{L}_2^\vee)$$

and the right hand side is

$$2\chi(X, \mathcal{F}) - \chi(X, \mathcal{F} \otimes \mathcal{L}_1^\vee) - \chi(X, \mathcal{F} \otimes \mathcal{L}_2^\vee)$$

Therefore, we are trying to show that:

$$\chi(X, \mathcal{F} \otimes \mathcal{L}_1^\vee \otimes \mathcal{L}_2^\vee) - \chi(X, \mathcal{F}) = \chi(X, \mathcal{F} \otimes \mathcal{L}_1^\vee) - \chi(X, \mathcal{F}) + \chi(X, \mathcal{F} \otimes \mathcal{L}_2^\vee) - \chi(X, \mathcal{F})$$

Recall that we have proved the Riemann-Roch theorem for nonreduced curves and we know that the stuff on the left is the sum over all the irreducible components of C_i of C of the degree of $(\mathcal{L}_1^\vee \otimes \mathcal{L}_2^\vee)$, but we know that the degree is additive, thus it's the sum of the degree of $\mathcal{L}_1^\vee$ plus the degree of $\mathcal{L}_2^\vee$, we are done.

Now we assume linearity for smaller values of n, we now wish to show that, for arbitrary $\mathcal{L}_1, \mathcal{L}_1', \mathcal{L}_2, \cdots, \mathcal{L}_n$,

$$(\mathcal{L}_1 \cdots \mathcal{L}_n \cdot \mathcal{F}) + (\mathcal{L}_1' \cdots \mathcal{L}_n \cdot \mathcal{F}) - (\mathcal{L}_1 \otimes \mathcal{L}_1' \cdots \mathcal{L}_n \cdot \mathcal{F}) = 0$$

We first show that

$$(\mathcal{L}_1 \cdots \mathcal{L}_n \cdot \mathcal{F}) + (\mathcal{L}_1' \cdots \mathcal{L}_n \cdot \mathcal{F}) - (\mathcal{L}_1 \otimes \mathcal{L}_1' \cdots \mathcal{L}_n \cdot \mathcal{F}) = (\mathcal{L}_1 \cdot \mathcal{L}_1' \cdots \mathcal{L}_n \cdot \mathcal{F})$$

This is basically just the definition since the right hand side consists of two parts, the first part is the one where $\mathcal{L}_1$ and $\mathcal{L}_1'$ doesn't appear simultaneously, which is equal to $(\mathcal{L}_1 \cdots \mathcal{L}_n \cdot \mathcal{F}) + (\mathcal{L}_1' \cdots \mathcal{L}_n \cdot \mathcal{F}$, the second part should be those terms where $\mathcal{L}_1$ and $\mathcal{L}_1'$ show up simultaneously, and it's equal to $-(\mathcal{L}_1 \otimes \mathcal{L}_1' \cdots \mathcal{L}_n \cdot \mathcal{F})$.

Now, for $(\mathcal{L}_1 \cdots \mathcal{L}_n \cdot \mathcal{F}) + (\mathcal{L}_1' \cdots \mathcal{L}_n \cdot \mathcal{F}) - (\mathcal{L}_1 \otimes \mathcal{L}_1' \cdots \mathcal{L}_n \cdot \mathcal{F})$, if $\mathcal{L}_n$ is an effective Cartier divisor that doesn't pass any associated points of $\mathcal{F}$, we are done since we know that we can use the previous result to reduce one of the line bundles and use inductive hypothesis.

In particular, if $\mathcal{L}_n$ is very ample, we are done, as we know that the field is infinite thus we can find a hyperplane not passing through any finite collection of points. But there is a variant of this that still works even if we don't do the field extension and assume k is finite. We know that if k is finite, even if we may not find a hyperplane that missing a finite collection of points, we can find a hypersurface of high enough degree that does this, thus we know that there will be a hypersurface H such that $\mathcal{O}(N) \simeq \mathcal{L}^N$.

Therefore, we know that $(\mathcal{L}_1 \cdot \mathcal{L}_1' \cdots \mathcal{L}_n \cdot \mathcal{F})$ if $\mathcal{L}_n$ is very ample, but by symmetry, we know that it also vanishes if $\mathcal{L}_1$ is very ample, i.e. it's indeed linear in the first slot if $\mathcal{L}_1$ is very ample.

Then the first thing we can prove is that if $\mathcal{L}^1$ is very ample, then:

$$(\mathcal{L}_1^\vee \cdots \mathcal{L}_n \cdot \mathcal{F}) = -(\mathcal{L}_1 \cdots \mathcal{L}_n \cdot \mathcal{F})$$

This is we know that:

$$(\mathcal{L}_1 \otimes \mathcal{L}_1^\vee \cdots \mathcal{L}_2 \cdot \mathcal{F}) = 0$$

and $\mathcal{L}_1$ is very ample, thus linear in the first term. We will use this in the following proof.

However, we know that any line bundle on X is the difference of two very ample line bundles (since very ample times base point free is very ample), therefore, we will write:

$$\mathcal{L}_1 = \mathcal{A} \otimes \mathcal{B}^\vee \qquad \mathcal{L}_1' = \mathcal{A}' \otimes \mathcal{B}' \vee$$

where $\mathcal{A}, \mathcal{B}, \mathcal{A}', \mathcal{B}'$ are all very ample.

Therefore, we know that:

$$\begin{aligned}
(\mathcal{L}_1 \otimes \mathcal{L}_1 \cdot \cdots \mathcal{L}_n \cdot \mathcal{F}) &= ((\mathcal{A} \otimes \mathcal{A}' \otimes \mathcal{B}^\vee \otimes \mathcal{B}^{'\vee}) \cdots \mathcal{L}_n \cdot \mathcal{F}) \\
&= ((\mathcal{A} \otimes \mathcal{A}') \cdots \mathcal{L}_n \cdot \mathcal{F}) - ((\mathcal{B} \otimes \mathcal{B}') \cdots \mathcal{L}_n \cdot \mathcal{F}) \\
&= (\mathcal{A} \cdots \mathcal{L}_n \cdot \mathcal{F}) + (\mathcal{A}' \cdots \mathcal{L}_n \cdot \mathcal{F}) - (\mathcal{B} \cdots \mathcal{L}_n \cdot \mathcal{F}) - (\mathcal{B}' \cdots \mathcal{L}_n \cdot \mathcal{F}) \\
&= (\mathcal{L}_1 \cdots \mathcal{L}_n \cdot \mathcal{F}) + (\mathcal{L}_1' \cdots \mathcal{L}_n \cdot \mathcal{F})
\end{aligned}$$

We are done since symmetry and linearity in the first slot suffices to prove multilinearity. □

An added bonus in the proof is:

Corollary 1.12. Suppose that X is projective, then if $\dim \mathrm{Supp} \mathcal{F} < n+1$ and $\mathcal{L}_1, \mathcal{L}_1', \cdots \mathcal{L}_n$ are invertible sheaves on X, then:

$$(\mathcal{L}_1 \cdot \mathcal{L}_1' \cdots \mathcal{L}_n \cdot \mathcal{F}) = 0$$

That is to say, if you want to intersect $n+1$ line bundles with some sheaf whose support has dimension less than $n+1$, then you get 0.

Proof. Well recall that:

$$(\mathcal{L}_1 \cdot \mathcal{L}_1' \cdots \mathcal{L}_n \cdot \mathcal{F}) = (\mathcal{L}_1 \cdots \mathcal{L}_n \cdot \mathcal{F}) - (\mathcal{L}_1' \cdots \mathcal{L}_n \cdot \mathcal{F}) - (\mathcal{L}_1 \otimes \mathcal{L}_1' \cdots \mathcal{L}_n \cdot \mathcal{F})$$

But we already know from linearity that this is 0.

Since we know that this linearity also holds for proper k-varieties, this actually holds for proper K-varieties as well. □

Proposition 1.13. Suppose that X is projective, then the intersection product depends on on the numerical class of $\mathcal{L}_i$'s.

Before we prove it, we actually have something left to prove for numerical equivalence classes when we discussed it for the first time. On of the properties we didn't prove is:

Proposition 1.14. If $\pi : X \to Y$ is a proper morphism, and $\mathcal{L}$ is numerically trivial on Y, then $\pi^*\mathcal{L}$ is numerically trivial on X.

Proof. To prove this, we know that we only need to prove that for every integral curve $i : C \hookrightarrow X$ the restriction of $\pi^*\mathcal{L}$ to C has degree 0.

Notice that by definition:

$$\deg(i^*\pi^*\mathcal{L}) = \chi(C, i^*\pi^*\mathcal{L}) - \chi(C, \mathcal{O}_C)$$

We also have the following diagram:

$$\begin{tikzcd} X \arrow[r, "\text{Proper}"] & Y \\ C \arrow[u, hook, "i"] \arrow[r, "\text{Proper}"'] & C' \arrow[u, hook] \end{tikzcd}$$

where C' is the scheme theoretic image of C, and we know that since $C \to Y$ is proper, the image is close, since C is Noetherian, we know that the underlying set of C' is indeed the set theoretic image of $C \to Y$.

I claim that C' is either a point of a one dimension irreducible stuff not necessarily reduced. If this is true, let's first assume that C' is a curve and we know that $f : C \to C'$ is a surjective morphism between integral projective curves (proper curves are all projective), then it's necessarily finite, in particular affine, therefore, by projection formula, and cohomology is invariant under pushforward of affine morphisms:

$$\chi(C, i^*\pi^*\mathcal{L}) = \chi(C, f^*j^*\mathcal{L}) = \chi(C', f_*(f^*j^*\mathcal{L} \otimes \mathcal{O}_C)) = \chi(C', f_*\mathcal{O}_C \otimes j^*\mathcal{L})$$

On the other hand, we know that:

$$\chi(C, \mathcal{O}_C) = \chi(C', f_*\mathcal{O}_C)$$

Therefore we know that:

$$\deg(i^*\pi^*\mathcal{L}) = \chi(C', f_*\mathcal{O}_C \otimes j^*\mathcal{L}) - \chi(C', j_*\mathcal{O}_C)$$

Then by the Riemann-Roch for nonreduced curves (which also works for reduced curves of course), we know that this is the sum of degree of $j^*\mathcal{L}$ to all the irreducible components of C', but since C' is irreducible, we are just restricting to it's reduction and compute the degree of $\mathcal{L}$, but since $\mathcal{L}$ is numerically trivial on Y, we are done by definition.

If it's just a point, then things are even easier, the restriction of $\mathcal{L}$ to C' can only be the structure sheaf as it only has one point. Then $\chi(C, i^*\pi^*\mathcal{L}) - \chi(C, \mathcal{O}_C)$ reduces $\chi(C, \mathcal{O}_C) - \chi(C, \mathcal{O}_C)$, we are done.

To prove our assertion on the dimension, we consider the reduction of C' and we have the diagram:

$$\begin{tikzcd} C \arrow[r] \arrow[dr] & C' \\ & C'^{red} \arrow[u, hook] \end{tikzcd}$$

Then we know that $C \to C'^{red}$ is also finite, and C' is an integral subvariety of Y, thus it's dimension is the transcendence degree of the function field, but we know that we have a field extension:

$$K(C') \hookrightarrow K(C)$$

and we have the following diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ K(C') \arrow[r, hook] & K(C) \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d] & A/\ker \arrow[r, hook] \arrow[d] & B \arrow[d] \\ K(C') \arrow[r, "\simeq"'] & K \arrow[r, "finite"'] & K(C) \end{tikzcd}$$

and the stuff on the right shows that $K(C')$ to $K(C)$ is actually a finite extension of fields, thus the transcendence degree of $K(C')$ over k is the same as the transcendence degree of $K(C)$ over k, thus the dimension of C' is actually 1, not 0. $\square$

Therefore, the key of the proof is actually knowing when you can factor a morphism through a closed embedding, which we didn't really prove before. So we say something in the following discussion:

Proposition 1.15. Suppose that $i : Z \hookrightarrow X$ is a closed embedding of schemes which is defined by a quasicoherent ideal sheaf $\mathcal{I}$, then a moprhism of schemes: $f : Y \to X$ factors through i uniquely in the diagram before iff f vanishes on $\mathcal{I}$:

$$\begin{tikzcd} Z \arrow[r, hook] & X \\ & Y \arrow[u, "f"'] \arrow[ul, dashed, "\exists!"] \end{tikzcd}$$

Proof. It's easy to prove that if f has a factorization it must vanish on $\mathcal{I}$ this is because if it has such a factorization, then the set theoretic image of f should line in Z and to compute the set theoretic image, we can work locally and assume that f is actually $SpecB \to SpecA$ by choosing an affine open $SpecA$ in X and cover the inverse image by affine opens. This is a ring morphism:

$$A \to B$$

and we know that by our assumption that this ring morphism factors through A/I:

$$\begin{tikzcd} SpecA & f^{-1}SpecA \arrow[l, "f"'] \arrow[dl] & SpecB \arrow[l, hook'] \arrow[dll] \\ SpecA/I \arrow[u, hook] & & \end{tikzcd}$$

thus you proved that f indeed vanishes on $\mathcal{I}$ for each affine in X, and this suffices.

Now this factorization is also unique, and it's unique locally, thus patches together uniquely and this is just a restatement of the universal property of quotients.

Now we come to the more interesting statement, let's suppose that f vanishes on $\mathcal{I}$ and we want a unique factorization. Again, if it factors, it's unique since we can still argue locally like the above discussion using universal property of quotients. Therefore, the only question is how to find such a factorization. First, we can prove the image of Y is in Z, therefore we know that there is a unique topological space map g from Y to Z as a factorization. Thus we only need to define a morphism:

$$\mathcal{O}_Z \to g_*\mathcal{O}_Y$$

What we have at our disposal is:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[dr] & i_*\mathcal{O}_Z \arrow[r] & 0 \\ & & & f_*\mathcal{O}_Y & \end{tikzcd}$$

and we know that $\mathcal{I} \to \mathcal{O}_X \to f_*\mathcal{O}_Y$ is 0, thus the universal property of cokernels gives a morphism from $i_*\mathcal{O}_Z$ to $f_*\mathcal{O}_Y$:

$$i_*\mathcal{O}_Z \to f_*\mathcal{O}_Y$$

This morphism is going to be a morphism of sheaves of rings, then we apply i^{-1} which gives you:

$$\mathcal{O}_Z \to i^{-1}f_*\mathcal{O}_Y$$

Notice that $f_* = i_* \circ g_*$, then we know that $i^{-1}f_* = g_*$, therefore we get a morphism of sheaves of rings from:

$$\mathcal{O}_Z \to g_*\mathcal{O}_Y$$

We are done. Even though we defined this in an abstract way, the uniqueness argument lets us see an explicit definition locally on affines. ∎

Remark 1.16. Apply the previous proposition, we can prove that there is a unique factorization from $f : X \to Y$ to the scheme theoretic image of f as the scheme theoretic image is defined to be the sums of all the quasicoherent ideal sheaves that vanish after you compose:

$$\mathcal{I} \to \mathcal{O}_Y \to f_*\mathcal{O}_X$$

Thus by definition f vanishes on that ideal sheaf. This is how we defined $C \to C''$.

Moreover, we know that C is reduced, thus there is no nonzero nilpotents, then recall that C'^{red} is defined by the nilpotent ideal sheaf, therefore we know that on each affine, the nilradical is mapped to 0 as for each open in C, there is no nonzero nilradical functions. Therefore, we are done. Then we have actually proved that if $f : X \to Y$ is a morphism of scheme with X reduced, then it factors uniquely through the reduction of Y.

Now we go back to the intersection product prove the fact that intersection product only depends on numerical equivalence class:

Proof. Suppose that $\mathcal{L}_1, \mathcal{L}_1'$ are in the same equivalence class, we want to prove that:

$$(\mathcal{L}_1 \cdots \mathcal{L}_n \cdot \mathcal{F}) = (\mathcal{L}_1' \cdots \mathcal{L}_n \cdot \mathcal{F})$$

And we can proceed by induction on n. For $n = 1$, we need to prove that:

$$(\mathcal{L}_1 \cdot \mathcal{F}) = (\mathcal{L}_1' \cdot \mathcal{F})$$

The left hand side is, by definition $\chi(X, \mathcal{F}) - \chi(X, \mathcal{F} \otimes \mathcal{L}_1^\vee)$, since we assumed that the support of $\mathcal{F}$, say S, is at most 1, it's either 0 or 1. If it's 0, we know that it's just a bunch of closed points, and we know that $\chi(X, \mathcal{F}) = \chi(X, i_*\mathcal{G})$ and $\chi(X, \mathcal{F} \otimes \mathcal{L}_1^\vee) = \chi(X, i_*\mathcal{G} \otimes \mathcal{L}_1^\vee)$ where $\mathcal{G}$ is a coherent sheaf on S, therefore, we know that $i_*\mathcal{G}$ only supports on a finite collection of closed points, therefore we know that $i_*\mathcal{G} \otimes \mathcal{M} \simeq i_*\mathcal{G}$ for any line bundle $\mathcal{M}$, therefore the intersection product is 0, for all line bundles $\mathcal{M}$.

If the assumption is the support is dimension 1, then it's a finite collection of closed points and a closed curve C which doesn't contain those closed points, not necessarily reduced. In this case, we know that you can write $\mathcal{F}$ as a direct sum sheaf whose support is on the finitely many closed points and one sheaf whose support whose support is on the curve. Euler characteristic turns direct sums into sums, thus those that support on closed points doesn't matter by our discussion on the dimension 0 case. Therefore, we can assume that the scheme theoretic support of $\mathcal{F}$ is pure dimensional of dimension 1, denoted by $i : C \hookrightarrow X$. Then we know that in this case, the Riemann-Roch for nonreduced curves proves what we want.

Then let's assume $n > 1$ and this result holds for smaller n. Recall that $\mathcal{L}_n$ is the difference of two very ample line bundles, then by multilinearity, we only need to prove this when $\mathcal{L}_n$ is very ample. Notice that it suffices to prove thatfor a nonzero integer m such that:

$$m(\mathcal{L}_1 \cdots \mathcal{L}_n \cdot \mathcal{F}) = m(\mathcal{L}_1' \cdots \mathcal{L}_n \cdot \mathcal{F})$$

Then by multilinearity, we can assume that we work with $\mathcal{L}_n^{\otimes m}$, thus we have already explained that in this case we can assume that $\mathcal{L}_n^{\otimes m}$ is $\mathcal{O}_X(D)$ for an effective Cartier divisor on X that miss all the associated points of $\mathcal{F}$ (by taking a hypersurface with high enough degree m), then we can use the inductive hypothesis. □

Next topic we want to show for this intersection product is sometimes called the **asymptotic Riemann-Roch**:

If Y is a projective curve, we know from the Riemann-Roch for nonreduced curves and degree is additive for tensors that:

$$\chi(Y, \mathcal{L}^{\otimes m}) = m\deg\mathcal{L} + \chi(Y, \mathcal{O}_Y)$$

You find that it's a polynomial in m, whose leading term is the intersection product $(\mathcal{L} \cdot Y)$. It has the following generalization, which is known as asymptotic Riemann-Roch:

Proposition 1.17. Suppose that X is a projective k-scheme and $\mathcal{F}$ is a coherent sheaf on X with $\dim \mathrm{Supp}\mathcal{F} \leqslant n$, then if $\mathcal{L}$ to be a line bundle on X, we have:

$$q(m) = \chi(X, \mathcal{L}^{\otimes m} \otimes \mathcal{F})$$

is a polynomial in m of degree at most n, the coefficients of m^n, i.e. the coefficient of the leading term is $(\mathcal{L}^n \cdot \mathcal{F})/n!$.

Proof. You might think that this proof will be like what we did for Hilbert polynomials where you pick ample line bundles which is $\mathcal{O}(D)$ for an effective Cartier divisor D and use additivity and so on. Well, I claim that this proof is actually not that hard since we have already established quite a lot of properties for intersection products which basically turns this into a combinatorics problem, in particular we know that if the support of $\mathcal{F}$ is at most n, then:

$$(\mathcal{L} \cdots \mathcal{L} \cdot (\mathcal{L}^{\otimes m} \otimes \mathcal{F})) = 0$$

if we have at least $n+1$ $\mathcal{L}$ in this intersection product.

Now, notice that if we expand the above quantity, it is:

$$\sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} p(m-j) = 0$$

And I claim that the relation above is enough to show that $p(m)$ is a polynomial of degree at most n and we can also extract the leading coefficient from this.

To see why, here is a general result that's worth knowing: Suppose that $f(x)$ is a function from $\mathbb{Z}$ to $\mathbb{C}$, we can compute the forward difference at m, which is defined to be:

$$\Delta f(m) = f(m+1) - f(m)$$

This operation cam be iterated and there is a formula for the r-th iteration:

$$\Delta^r f(m) = \sum_{j=0}^{r} (-1)^j \binom{r}{j} f(m+r-j)$$

Now I claim that if $\Delta^{n+1} f(m) = 0$ for all m, then f is a polynomial. To prove this, let's define:

$$a_r = \Delta^r f(0)$$

Then we can define a polynomial:

$$P(m) = \sum_{r=0}^{n} a_r \binom{m}{r}$$

Recall that we can compute the forward difference:

$$\Delta \binom{m}{k} = \binom{m+1}{k} - \binom{m}{k} = \binom{m}{k-1}$$

Therefore we know that $\Delta^r \binom{m}{k} = \binom{m}{k-r}$, and if $r \geqslant k$, we only get 0. Now, it's obvious that Δ is additive, thus we can try to apply Δ to $P(m)$ and you find that since the largest number of r is n, $\Delta^{n+1} P(m) = 0$ for all m and we know that:

$$\Delta^{n+1}(f(m) - P(m)) = 0$$

This implies that $\Delta^n f(m) - P(m)$ is constant, moreover, by definition, we know that $\Delta^n f(0) - \Delta^n P(0) = a_n - a_n = 0$, therefore $\Delta^n P(m) = \Delta^n f(m)$ for all m. Here we are using the convention that $\binom{0}{k} = 0$ if $k \neq 0$ and if $k = 0$, then $\binom{0}{0} = 1$. This convention is used here because we want $\binom{0}{k}$ for $k \neq 0$ to be defined by the polynomial equation and if we insist this, then the only possible way to define $\binom{0}{0} = \Delta\binom{0}{1} = \binom{1}{1} - \binom{0}{1} = 1 - 0 = 1$, and this is going to be consistent with our other definitions. By this definition, and the fact that

$\Delta^n(P-f)=0$, we know that $\Delta^{n-1}(P-f)$ is constant, but you you plug $m=0$, you find that the result is $a_{n-1}-a_{n-1}=0$, thus you keep going down, until you reach $f=P$ for all m. We are done.

Now back to our problem, we see that $p(m-n-1)$ is going to be a polynomial of degree at most n, thus $p(m)$ is going to be a polynomial of degree at most n. Finally, to compute the leading coefficient, we already obtained a formula, which is:

$$a_n = \Delta^n f(0)$$

Then put it back to our problem, we know that the leading coefficient of $p(m-n-1)$ is when you plug 0 to:

$$\sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j}p(m-j)=(\mathcal{L}\cdots\mathcal{L}\cdot\mathcal{F})$$

We are done since we know that the leading coefficient of $p(m-n-1)$ is the same with the leading coefficient of $p(m)$. $\square$

Remark 1.18. What we proved in the above proposition works for any function from $\mathbb{Z}$ and Hilbert polynomials can be seen as a special case of this proposition. This is because we prove that:

$$f(m+1)-f(m)$$

is a polynomial of degree d with rational coefficients from $\mathbb{Z}$ to $\mathbb{Z}$, and in this case we can represent this polynomial as a linear combination of binomial coefficients with rational coefficients. In this case, you know that the $d+1$-th difference is 0, which implies that the $d+2$'s difference of f is 0, thus f is a polynomial of degree at most $d+1$ and from the formula we give in the proof, you find that it's actually still a polynomial with rational coefficients that takes value in $\mathbb{Z}$.

Our next goal is try to generalize the result that if $\pi : C \to C$ is a morphism between integral projective curves and $\mathcal{L}$ is a line bundle on C, then the degree of $\pi^*\mathcal{L}$ is the degree of π times the degree of $\mathcal{L}$. To do this, we will need Chevelley's theorem in one special case of our proof. It's generally a good theorem to know so we will first spend sometime discussing it and introducing some useful notions and results down the road:

Generally speaking, Chevalley's theorem is about the image of finite type morphisms. But the way you should think about it is in the following description: If you have a number of polynomial equations in a number of variables with a number of variables. You probably wondered that what sort of conditions you need to impose on the coefficients for a solution to exist. Given the algebraic description of this problem, you will think that the answer of this problem is also purely algebraic, in the sense that it's not random or involve any sort of exotic functions like exponentials or trigonometic functions. This is indeed the case, and can be profitably intrepreted as a question of images of finite type morphisms and can be explained by Chevalley's theorem. As a consequence, we get an immediate proof of the Nullstellensatz.

To begin with, suppose that $f : X \to Y$ is a morphism of schemes, we have seen cases where the image, as a set, is open or closed. But certainly, the image can be stranger, for example: if you consider the morphism:

$$\pi : \mathbb{A}^2_k \to \mathbb{A}^2_k$$

defined by xx and $y \mapsto xy$, you will find that each point (a,b) is mapped to (a,ab), so you find that the closed points on the y-axis is omitted, except for the origin. Therefore, this set is neither open or closed since for each open neighborhood of $(0,0)$, we can assume that if $D(f)$, thus $f(0,0)$ is not 0, if we let $x=0$, then $f(0,y)$, say k is algebraically closed, then it has solutions, thus there are closed points that are not in the image of π, similarly, you can prove that it's not closed. We will make a definition to capture this phenomenon:

Definition 1.19. A **constructible** subset of a Noetherian topological space is a subset which belongs to the smallest family of subsets such that:

1. Every open subset is in this family.

2. Every finite intersection of family members is in the family.

3. The complement of a family member is also in the family.

You can immediately see that this is a larger family than just those that are open or closed since we see that the intersection of an open subset with a closed subset is in this family, i.e. any **locally closed subset** is in this family, and a characterization of constructible subsets is given by locally closed subsets:

Proposition 1.20. A subset of a Noetherian topological space X is constructible iff it's a finite disjoint union of locally closed subsets.

Proof. Obviously that all finite union of family members is in the family. This is because if you want $X_1 \cup \cdots \cup X_n$, you can compute $((X_1^C) \cap \cdots \cap (X_n^C))^C$. Therefore, it's obviously all disjoint union of locally closed subset are constructible.

Conversely, let's consider the collection of finite disjoint unions of locally closed subsets, if we can prove that it's a family that satisfies the conditions of the constructible sets, we are done since by definition, constructible subsets forms the smallest such families and the family of disjoint unions of locally closed subsets is such a family contained in the family of constructible subsets.

First, since all open subsets are locally closed, we know that all opens are in this family. Then suppose that A, B are locally closed subset, it's easy to show that $A \cap B$ is locally closed, then now suppose that $A = \coprod_i A_i$ and $B = \coprod_j B_j$ are disjoint union of locally closed subsets, we now consider it's intersection:

$$A \cap B = \coprod_{ij} A_i \cap B_j$$

Repeat this finitely many times shows that finite intersections is in the family.

Finally, suppose that A is a locally closed subset $A = B \cap C$ where B is closed and C is open, then it's complement is $C^C \coprod (C - B)$ and $C - B$ is locally closed because it $C \cap B^C$, which is open intersect with open, which is open, thus locally closed. Therefore, now if A is a finite union $A = \coprod A_i$ of locally closed subset, we know that the complement is:

$$A^C = \cap A_i^C$$

Let's assume that each A_i is $B_i \cap C_i$ with B_i closed and C_i open, then we have showed that $A_i^C = C_i^C \coprod (C_i - B_i)$, therefore we know that:

$$A^C = \cap (C_i^C \coprod (C_i - B_i))$$

Let's for notational simplicity, denote $C_i^C = D_i$ and $C_i - B_i = E_i$, now let's use the distributive law of intersections and unions, then we conclude that:

$$A^C = \bigcap_i^n (D_i \coprod E_i) = \bigcup_{S \subset \{1, \cdots, n\}} (\bigcap_{i \in S} D_i \cap \bigcap_{i \notin S} E_i)$$

Notice that we have proved that finite intersections of locally closed subsets will be a locally closed subset, and further more this is actually a disjoint union since for two different S_1, S_2, there will be a $i \in S_1$ but $i \notin S_2$ then we know that:

$$\bigcap_{i \in S_1} D_i \cap \bigcap_{i \notin S_1} E_i \subset D_i \qquad \bigcap_{i \in S_2} D_i \cap \bigcup_{i \notin S_2} E_i \subset E_i$$

And there are indeed disjoint. Therefore we are done. $\square$

Remark 1.21. As a consequence, we have proved that if $\pi : X \to Y$ is a continuous map of topological spaces, since preimage preserves open, closed subsets, intersections and disjoint unions, preimage of a constructible subset is also constructible.

As a reality check, you should be able to prove that the generic point of $\mathbb{A}_k^1$ is not a constructible subset since if it is, it's a disjoint union of locally closed subsets, this implies that itself is a locally closed subset, but notice that the only closed subset containing the generic point is the whole space, thus we want to prove that the generic point itself is open, which is false since if $D(f)$ contains the generic point then f is not 0, then f has a unique irreducible decomposition, since we know that there are infinitely many irreducible polynomials, then this implies that there is one of such, say I and f is not in (I), therefore $D(f)$ contains more than the generic point, we are done.

Here is a result that we will use later:

Proposition 1.22. A constructible subset of a Noetherian scheme X is closed iff it's stable under specialization, i.e. if Z is a constructible subset of X, it's closed iff for every pair of points (y_1, y_2) with $y_1 \in \overline{y_2}$, then if $y_2 \in Z$ then $y_1 \in Z$.

Proof. First assume that Z is a constructible subset, then we know that Z is a finite disjoint union of locally closed subsets say $U_i \cap Z_i$ where U_i is open and Z_i is closed. Recall that we work with Notherian spaces, therefore Z_i can be decomposed to irreducible components. Then we can rewrite $Z = \cup(U_i \cap Z_i)$ but this union may not be disjoint any more. This implies that $Z \subset \cup Z_i$. Recall that if we consider the generic point η_i of Z_i, since it's in $U_i \cap Z_i$, we know that it's in Z, therefore by our assumption, we conclude that Z_i is in Z, therefore we know that:

$$\bigcup_i Z_i \subset Z \subset \bigcup_i Z_i$$

Therefore Z is a finite union of closed subsets, it's closed.

Conversely, since any closed subset, it satisfies the stable under specialization property, thus we are done. □

Remark 1.23. Using the above proposition, we can prove that a constructible subset Z of a Noetherian space is open iff it's stable under generalization, i.e. for any pair of (y_1, y_2) with $y_1 \in \overline{y_2}$, if y_1 is in Z, then y_2 is in Z.

This is because a set satisfies stable under specialization iff it's complement satisfies stable under generalization. You can prove it quite easily essentially by definition.

Another topological property we will use in the proof of Chevalley's theorem is the following:

Proposition 1.24. Suppose that Z is a constructible subset in a Noetherian space Y, then Z contains a dense open of it's closure.

Proof. **** to be completed* □

Now we begin to state Chevelley's theorem, but the reason we care about it is because that in general, the image of a morphism of schemes can be quite messy, for example, if you have a scheme Y, then given any subset of Y, there is a scheme morphism $X \to Y$ whose set-theoretic image is that subset, since you can just take the canonical inclusion of each point and then take disjoint union. This is quite pathological and Chevalley's theorem tells you that in reasonable situations, this doesn't happen and the image can only be constructible subsets:

Theorem 1.25. Suppose that $\pi : X \to Y$ is a finite type morphsim of Noetherian schemes, the image of any constructible subset is constructible, in particular the image of π is constructible.

Again, before we prove this theorem, we give several seeming surprising consequences of this theorem, they are surprising because you will find that they are purely algebraic results that follows from geometric or even topological results:

The first one we want to give is the proof of Nullstellensatz:

Proof. Recall that the statement is that if K is a field extension of k that is a finitely generated k-algebra, then K is actually finite dimensional over k. To prove this, we only need to prove, say K is generated by $x_1, \cdots, x_n$, each x_i is algebraic. Say that one of them is not algebraic, then we know that we have a inclusion:

$$k[x_i] \hookrightarrow K$$

Which gives you a dominant morphism of finite type k-schemes:

$$SpecK \to Speck[x_i]$$

therefore this morphism is also finite type, therefore we know that the image of $SpecK$ should be constructible, but the image is the generic point, which is not constructible, thus our assumption is wrong. □

A similar idea can be used to prove the following, which we will use later:

Proposition 1.26. Suppose that $\pi : X \to Speck$ is a quasifinite morphism, then π is finite. This is to say quasifinite morphisms to a field is finite. Recall that quasifinite means finite type and finite fiber.

Proof. Recall that quasifinite is finite type plus finite fiber.

First assume X is affine, i.e. this morphism is:

$$SpecA \to Speck$$

where A is a finitely generated k-algebra, say generated by $x_1, \cdots, x_n$. We want to say that x_i's are all algebraic over k. Again, say if one of them is not, then we have an inclusion:

$$k[x] \hookrightarrow A$$

If A is a field, then we are done by the similar argument as in the Nullstellensatz. Here, it's not necessarily true that A is a field, but we will prove that it's always a finite direct of fields, therefore we can assume that A is a field and this also solves the case where X is not affine since we can cover with finitely many affines and each is a finite direct sum of finite dimensional k-algebras, thus we know that $X \to Speck$ is affine.

I claim that A is actually a discrete topological space with each prime ideal being maximal. To see this, we only need to show that each irreducible component of $SpecA$ is a single point, notice that this irreducible component is also an irreducible closed subset, which only has finitely many points in $\mathbb{A}^n_k$, therefore this is $V(p)$ for some prime ideal p, which only has finitely many points. Now if p is not a closed point, I claim that $V(p)$ has infinitely many points, to see this, we need to show that $SpecA/p$ has infinitely many points, notice that since p is not closed point, the dimension of A/p is actually larger than 1, therefore we know from Noether normalization that there is finite extension:

$$k[x_1, \cdots, x_m] \hookrightarrow A/p$$

which gives you a dominant, finite, hence surjective morphism $SpecA/p \to Speck[x_1, \cdots, x_m]$ and we know that $Speck[x_1, \cdots, x_m]$ has infinitely many points as long as $m \geqslant 1$. Then we know that we can actually assume that A is actually a finite direct sum of A/m, then if we can show that each $k \to A/m$ is finite, we are done, and we know that A/m is a finitely generated extension of k, thus finite over k by our previous argument of Nullstellensatz. $\square$

Proposition 1.27. Suppose that X is a scheme finite type over $\mathbb{Z}$ then if p is a closed point, the residue field $k(p)$ is a finite field.

Proof. We can assume X is affine and say the morphism is:

$$SpecA \to Spec\mathbb{Z}$$

Let p be a closed point of $SpecA$, consider it's image. I claim that this is not the generic point, this is because the image of any constructible set is constructible and we only need to show that the generic point of $\mathbb{Z}$ is not constructible, this is again to show that the generic point is not open, which is easy.

Then the image is a closed point, say (q), then we can consider:

$$\mathbb{Z}/q \to A/p$$

Again, you only need to show that A/p is finite dimension over $\mathbb{Z}/q$, which by Nullstellensatz, is equivalent to show that it's finitely generated, which follows from $\mathbb{A}$ is finitely generated over $\mathbb{Z}$ and the inverse image of p is q. $\square$

The last application we will give is that surjectivity between varieties can be checked on closed points:

Proposition 1.28. Suppose that $\pi : X \to Y$ is a morphism of k-varieties, then it's surjective iff every closed point of Y is in the image of π.

Proof. It's obvious that if it's surjective, then it's surjective on closed points, therefore the only interesting statement is the converse.

Suppose that every closed point is in the image (and we can assume that it's the image of a closed point), recall that the image of a constructible subset is also constructible, then we know that the image of π is a constructible subset of Y that contains all the closed points and we know that closed points are dense in finite type k-schemes. Suppose that $\pi(X)$ is not all of Y, then since it's constructible, the complement is a nonempty constructible subset, and we know that the complement is a nonempty open of the closure, the closure is a nonempty closed subset, thus we know that closed points is also dense, and it can also be equipped with a k-variety structure, then we know that the complement, being a nonempty open, contains such a closed point, but this is a contradiction. ☐

Remark 1.29. The key in the above proof is to notice that is some subset is constructible, then it's a nonempty open of it's closure. This is because if you write it a finite disjoint union of locally closed subsets, $\coprod C_i \cap D_i$ with C_i open and D_i closed. We know that the closure of each $C_i c \cap D_i$ is contained in D_i, therefore, we can write $C_i \cap D_i$ as $C_i \cap E_i$ where E_i is the closure of $C_i \cap D_i$, thus we know that $\coprod C_i \cap D_i = \coprod_i C_i \cap E_i$, then we claim that the closure of $\coprod C_i \cap D_i$ is just $\bigcup E_i$. First we notice that this is indeed a closed subset containing $\coprod C_i \cap D_i$ and each closed subset containing it should contain $\bigcup E_i$. Now $\coprod_i C_i \cap E_i$ is a finite union of open subsets of $\bigcup E_i$, i.e. $\bigcup_j C_i \cap (\bigcup E_i)$.

Now let's prove Chevalley's theorem:

Proof. First, we can assume that Y is affine, this is because we know that Y is covered by finitely such affines and to show that a subset of Y is constructible, it enough to show that it's constructible in each opens of a finite open cover. To see this, if we know that it's constructible in each affine open we know that it's actually constructible in the whole space, this is because if we write it as:

$$\coprod C_i \cap D_i$$

where C_i's are open and D_i's are closed in this affine, say $SpecA$, we know that it's a $SpecA \cap E_i$ for some closed subsets E_i in Y, then we know that this subset is actually:

$$\bigcup_i C_i \cap E_i$$

A finite union of locally closed subsets, thus constructible. Then since we work with a finite affine cover, the image is a finite union of constructible subsets, thus constructible.

Then I claim that we can reduce the problem to X is also affine, this is again because image of functions respect union in the sense that $f(\bigcup C_i) = \bigcup f(C_i)$ and if each $f(C_i)$ is constructible, then so is the union. Therefore, we can assume that $Y = SpecB$ and $X = SpecB[x_1, \cdots, x_n]/I$ for some ideal I. I claim that we can actually assume that $I = 0$, this is because if we can prove that Chevalley's theorem holds for $SpecB[x_1, \cdots, x_n] \to SpecB$, we can compose:

$$SpecB[x_1, \cdots, x_n]/I \hookrightarrow SpecB[x_1, \cdots, x_n]$$

Then we know that any constructible subset in $SpecB[x_1, \cdots, x_n]/I$ is a constructible subset in $SpecB[x_1, \cdots, x_n]$. This is because $SpecB[x_1, \cdots, x_n]/I$ is closed in $SpecB[x_1, \cdots, x_n]$.

Finally, we can reduce this question to $SpecB[t] \to SpecB$. This is because we can consider:

$$\cdots SpecB[x_1][x_2] \to SpecB[x_1] \to SpecB$$

Finally, since each constructible subset is a finite disjoint union of locally closed subsets, we only have to show that a locally closed subset of $SpecB[t]$ is mapped to a constructible subset in $SpecB$.

Let's first prove a special case, i.e. when Z is a closed subset of $X = SpecB[t]$. Suppose that $Z = V(f_1(t), \cdots, f_r(t))$ where:

$$f_i(t) = a_{id_i} t^{d_i} + a_{i1} t + a_{i0}$$

for $a_{ij} \in B$ and let's denote the degree of each f_i by d_i. Now we will decompose Y into a finite union of locally closed subsets $Y_{\overline{e}}$ where $\overline{e} = (e_1, \cdots, e_r)$ and $e_i \in \mathbb{Z}$ and $-1 \leqslant e_i \leqslant d_i$ and you should think of $Y_{\overline{e}}$

as the locus where $f_1(t), \cdots, f_r(t)$ have true degree $e_1, \cdots, e_r$ respectively, or you should think that the coefficients of t^{e_i} in $f_i(t)$ is nonzero and the coefficients of higher degree terms are 0 at the points of $Y_{\overline{e}}$. More precisely, $Y_{\overline{e}} \subset Y$ is the locus where $a_{ie_i} \neq 0$ but $a_{ij} = 0$ for $j > e_i$. Notice that here we require $-1 \leqslant e_i \leqslant d_i$ because we want to allow the existence of the zero polynomial and we define the degree to be -1.

More precisely still, we actually defined $Y_{\overline{e}}$ to be $SpecB_{\overline{e}}$ where:

$$B_{\overline{e}} = \frac{B_{\prod_i a_{ie_i}}}{(a_{ij})_{j>e_i}}$$

So we actually have $Y_{\overline{e}} =$

$$(\bigcap_i D(a_{ie_i})) \cap (\bigcap_{i \in \{1,\cdots,r\}, j>e_i} V(a_{ij}))$$

If we have some $e_i = -1$, this means that we don't want any of the coefficients of f_i to be nonzero, thus there is no term of $D(a_{ie_i})$ in the intersection and $V(a_{ij})$ shows up for all $j > -1$. If every index is -1, we know that this is actually the intersection of all $V(a_{ij})$ where a_{ij} are all the coefficients of f_i. Notice that $D(0) = \varnothing$ and $V(0) = Y$.

Then using this notation, we know that Y is indeed the union of all these $Y_{\overline{e}}$ since each point p in Y, we can find a a_{ie_i} for each i such that a_{ie_i} is not in p and it's the coefficient of the largest power that is not in p. It's a disjoint union also for every obvious reasons. Then, it's obvious that for $\overline{e} = (-1, \cdots, -1)$, the scheme-theoretic preimage of $B_{\overline{e}}$ is $SpecB_{\overline{e}}[t_1, \cdots, t_n] \to SpecB_{\overline{e}}$ since every coefficient of the polynomials vanish. This is a surjective morphism. If $\overline{e}$ is not all -1, we know that the scheme theoretic preimage is shown in the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & Spec(B_{\prod a_{ie_i}}/(a_{ij})) \arrow[d] \\
SpecB[x_1, \cdots, x_n] \arrow[r] & SpecB
\end{tikzcd}$$

therefore the preimage is:

$$Spec(B_{\prod a_{ie_i}}/(a_{ij})) \to Spec(B_{\prod a_{ie_i}}/(a_{ij}))[t]/(t^{e_1} + \frac{a_{1(e_1-1)}}{a_{1e_1}} t^{e_1-1} + \cdots, \cdots, t^{e_r} + \frac{a_{1(e_r-1)}}{a_{re_r}} t^{e_r-1} + \cdots)$$

And you notice that since the leading terms all have coefficients 1, this is a finite morphism, thus the image is closed. Recall that a closed subset of a locally closed subset is again locally closed. We have proved that the image of X in Y is disjoint unions of locally closed subsets, thus constructible. $\square$

To prove this for locally closed subsets, we introduce a notation only for this proof: Suppose that Z is a closed subset of X, then we will denote the locus of Y where the entire fiber of of that point is contained in Z, i.e. $\{q \in Y : \pi^{-1}q \subset Z\}$ by $FL(Z)$, the fibral locus for Z. Notice that this is a closed subset of Y. This is because that if p is in some $D_+(a_{ie_i})$, then the inverse image of p is $Speck(q)[x_1, \cdots, x_n]$, but it's image in Z is $Speck(q)[x_1, \cdots, x_n](\cdots)$ where we are modding out something that is not a zero polynomial, since $V(f)$ is not the whole space for $f \neq 0$, we are done.

Now we can go back to the proof:

Proof. If Z is locally closed, say it's $U \cap U'$ where U is open and U' is closed, we can write Z as $U \cap \overline{Z}$ where $\overline{Z}$ is the closure of Z. We already proved that $\pi(\overline{Z})$ is constructible in $Y = SpecB$. Define $\delta Z = \overline{Z} - Z$, which is a closed subset of $\overline{Z}$, thus a closed subset of X, thus the image is also constructible with $FL(\overline{Z})$, $FL(\delta Z)$ both closed in $Y = SpecB$. Notice that by definition: $FL(\overline{Z}) - FL(\delta Z)$ is contained in $\pi(Z)$. Since if p is in $FL(\overline{Z}) - FL(\delta Z)$, it's fiber is contained in $\overline{Z}$ but not totally contained in $\overline{Z} - Z$, therefore there is a q in Z such that the image is p. Also notice that $FL(\delta Z)$ is contained in the complement of $\pi(Z)$ since if q has fiber contained in $\overline{Z} - Z$, then it's not in the image of Z.

Now consider the open subset $Y' = Y - (FL(\overline{Z}) \cup FL(\delta Z))$ and the following diagram:

$$\begin{tikzcd}
Z' \arrow[r] \arrow[d, hook] & X' \arrow[r] \arrow[d, hook] & Y' \arrow[d, hook] \\
Z \arrow[r] & X \arrow[r] & Y
\end{tikzcd}$$

I claim that if you can prove that the image of Z' in Y' is constructible, then we are done. This is because that this plus Y' is open, implies that the image in Y is also constructible, and the image of Y is $FL(\overline{Z}) - FL(\delta Z)$ union the image of Z' in Y. This is because if p is in $\pi(Z)$ but it's not in $FL(\overline{Z}) - FL(\delta Z)$, we know that p will be in $Y - (FL(\overline{Z}) \cup FL(\delta Z))$.

Now I will denote the new morphims $X' \to Y'$ by π' and $\delta Z' = \overline{Z'} - Z' \subset X'$. Now, $\pi' : X' \to Y'$ is again finite type as finite type us preserved by base change. I claim that $\delta Z'$ in X' doesn't meet any generic fiber of π', i.e., for a generic point of an irreducible component of Y', $\delta Z'$ doesn't meet the preimage. If this is true, notice that the image of $\delta Z'$, which is the image of a closed subset is constructible and it doesn't contain any generic point of Y', therefore the complement is a constructible subset that contains each generic point of X. Recall that in Noetherian space any constructible subset contains a dense open of it's closure, which is the whole space, thus it contains a dense open of Y', therefore contains all the generic points. Therefore, we have found a dense open U in Y' such that it doesn't intersect with the image of $\pi(\delta Z')$.

Now over this dense open U, consider the following Cartesian diagram:

$$\begin{tikzcd}
Z'' \arrow[r] \arrow[d, hook] & X'' \arrow[r] \arrow[d, hook] & U \arrow[d, hook] \\
Z' \arrow[r] & X' \arrow[r] & Y'
\end{tikzcd}$$

I claim that the image of Z'' is constructible and if this is true, we have prove the Chevalley's theorem over a dense open of Y'. This is because we know that $\overline{Z'} \cap \pi'^{-1}(U) = Z \cap \pi'^{-1}U$ (since U doesn't intersect the image of $\delta Z'$), therefore Z'' in X'' is closed as it's the intersection $\overline{Z'} \cap \pi'^{-1}(U)$, which is closed. Then our previous result for closed subsets, the image of Z'' in U is constructible.

Now we will need Noetherian induction. Recall that works for properties of closed subsets of Y'. And here we will use the property that for a closed subset W of Y', $\pi'(Z') \cap W$ is constructible in W, and we want to show that $\pi'(Z') \cap Y'$ is constructible in Y'. To show this, Noetherian induction tells you that you first need the case for the empty subset, but there is nothing to prove there. Now we need to prove that if W is a closed subset and the property holds for all the proper closed subsets, then it holds for W. But notice that what we argued above is just this step since we can choose the proper closed subset to be $FL(\overline{Z} \cup FL(\delta Z))$ and we have proved that the image of Z' in the complement (which is a union) is also constructible, therefore in Y, the image is the union of two constructible subsets (constructible subsets in a open or closed subset of the larger space is also constructible in the larger space.) Thus we are done.

Therefore, the only thing left is to prove that the image of $\delta Z'$ doesn't meet any generic point of π'. Now we only care about the fiber of the generic point, that we can further restrict to an affine open of the generic point of Y', and thus we can use a distinguished open, say $SpecB_f$ of the generic point, and the morphism now looks like:

$$SpecB_f[x] \to SpecB_f$$

And we can choose this affine open to be an affine open complement contained in the irreducible component. By our choice of Y', since $\overline{Z'}$ is contained in $\overline{Z}$, we know that the fiber of each point in Y' is not contained in $\overline{Z'}$ or in $\delta Z'$. **** to be completed* ☐

This is basically all about Chevalley's theorem we want to talk about at this point. But before we proceed, we need another result:

Proposition 1.30. Suppose that $\pi : X \to Y$ is an affine, finite type generically finite morphism of locally Noetherian schemes and Y is reduced, then there is an open neighborhood of each generic point of Y over which π is finite. Or in other words, the slogan is that generically finite usually implies generally finite.

Proof. Notice that this is a local question on Y, therefore, we can first take an affine open around each generic point of Y. Say it's $SpecB$, then I claim that we can assume that B is a Noetherian integral domain. This is obtained by throwing out the other irreducible components which makes $SpecB$ irreducible as well.

Then we assume that $X = SpecA = SpecB[x_1, \cdots, x_n]/I$. The condition we assumed implies that:

$$A \otimes_B K(B)$$

is finite over $K(B)$, or in other words, a finite dimensional space over $K(B)$. Then if you consider the following extension of rings:

$$K(B) \hookrightarrow A \otimes_B K(B)$$

it's a finite extension, therefore, if we consider the image of x_i in $A \otimes_B K(B)$, there is a monic polynomial $f_i(t)$ in $K(B)[t]$ such that $f_i(x_i) = 0 \in K(B)$. Now, let b be the product of all the denominators of the coefficients of each f_i, then if you consider the following diagram:

$$\begin{tikzcd}
SpecB_g[x_1, \cdots, x_n]/I \arrow[r] \arrow[d, hook] & SpecB_g \arrow[d, hook] \\
SpecB[x_1, \cdots, x_n]/I \arrow[r] & SpecB
\end{tikzcd}$$

If we use $SpecB_b$ instead of $SpecB$, we can assume that f_i's are in $B[t]$. Now we can reinterpret the condition that $f_i(x_i) = 0$ by $f_i(x_i)$ is annihilated by some nonzero element b_i in B since we know that $A \otimes_B K(B)$ is just the localization of A at all the nonzero elements of B. Let g be the product of all the b_i's, and again if we use B_g instead of B, we can assume that $f_i(x_i)$ is annihilated by an invertible element of B, thus $f_i(x_i) = 0$ in A.

This is precisely the condition we need for A to be finite over B since finitely generated plus integral implies that finite. □

Remark 1.31. The assumption can be weakened considerably , for example, the morphism only need to be finite type and generically finite, it doesn't have to be affine.

To prove this, again choose a $SpecB$ in Y, and choose an affine open $SpecA$ in the inverse image of $SpecB$ (we can require $SpecA$ to be irreducible but it doesn't matter so we don't have to), the above argument shows that $SpecA \to SpecB$ is finite, but we want to conclude that $\pi^{-1}SpecB \to SpecB$ is finite. We are good if we can find an affine open of $SpecB$ such that the inverse image of that affine open is contained in $SpecA$, therefore this will be our goal. This requires a small lemma, which states that if $\pi : X \to Y$ is a quasicompact morphism, and η is a generic point of Y such that η is not in the image of π, then there is an open neighborhood V of η such that $\pi^{-1}V = \varnothing$. This follows from the fact that for quasicompact morphisms, the underlying set of scheme theoretic image is the closure of $\pi(X)$, thus the factorization $X \to Im(\pi)$ is dominant, this implies that all generic points of $Im(\pi)$ is in the image of π (since $X \to Im(\pi)$ is quasicompact and this property holds for all quasicompact morphisms). Our goal is to make sure that η is not in $Im(\pi)$, if it's in $Im(\pi)$, we know that η will be the generic point of some irreducible component of $Im(\pi)$ and I will let you see why. This is a contradiction.

Now go back to our question, we consider $SpecA$ in the inverse image of $\pi^{-1}SpecB$, and $\pi^{-1}SpecB \to SpecB$ is quasicompact due to finite type, then consider the composition:

$$\pi^{-1}SpecB - SpecA \hookrightarrow \pi^{-1}SpecB \to SpecB$$

which is again quasicompact. Now I claim that we can choose such a $SpecA$ that contains $\pi^{-1}\eta$ where η is generic point of $SpecB$. This is because the inverse image of η is a finite set by assumption, therefore if they are not the generic points, then say that p is a generic points whose closure contains one of the inverse images, then we know that this point should also be mapped to the generic point. Then we know that for quasiseparated schemes, if $p_1, \cdots, p_n$ are generic points of irreducible components, then there is an affine containing all of them.

Then combine all of these we know that there is an affine open of $SpecB$ such that the inverse image under $\pi^{-1}SpecB - SpecA \hookrightarrow \pi^{-1}SpecB \to SpecB$ is empty, thus we are done.

For further use and memorization, we single out the results we used in the above remark:

Proposition 1.32. Suppose that $\pi : X \to Y$ is a quasicompact morphism of schemes, then it's dominant if and only if all the generic points of Y are in the image of π.

Proof. See the above remark. □

Proposition 1.33. If X is a quasiseparated scheme and $\eta_1, \cdots, \eta_n$ are generic points of irreducible components of X, then there is an affine open in X containing all η_i.

Proposition 1.34. Suppose that $\pi : X_1 \to X_2$ is a morphism of integral projective k-schemes of the same dimension n and $\mathcal{L}_1, \cdots, \mathcal{L}_n$ are invertible sheaves on $\mathcal{L}_2$, then:

$$(\pi^*\mathcal{L}_1 \cdots \pi^*\mathcal{L}_n) = \deg(X_1/X_2)(\mathcal{L}_1 \cdots \mathcal{L}_2)$$

Notice that when you only have one $\mathcal{L}_1$ and π is a surjective (or dominant) morphism between projective curves, this generalizes the fact that the pullback of the degree of $\mathcal{L}$ is the degree of the morphism times the degree of $\mathcal{L}_1$.

Proof. Let's divide the proof into two cases, the first one is when π is dominant, and the second case is when π is not dominant and we need to use Chevalley's theorem. First, from multilinearity and the fact that pullback commutes with tensor products, we only need to prove this for very ample line bundles since on a projective variety, every line bundle is the difference of two very ample line bundles.

Let's first think about the case when π is dominant, i.e. the generic point of X_1 is mapped to the generic point of X_2. In this case, $\deg(X_1/X_2)$ is just defined to be the degree of π, which is defined to be the length of the generic fiber, i.e. if the generic point of X_2 is η, then this is defined to be:

$$\text{length}(X_1 \times_{X_2} k(\eta))$$

Notice that this is a finite type morphism and I claim that it's also generically finite. This is because we know that by choosing affine opens around each generic points, we can assume that the fiber is:

$$Spec A \otimes_B K(B)$$

Since A is finitely generated over b, we know that $A \otimes_B K(B)$ is finitely generated over $K(B)$, moreover, by dimensional reasons, we know that the transcendence degree of $K(A)/K(B)$ is 0, thus algebraic, then since we have the following inclusion:

$$K(B) \hookrightarrow A \otimes_B K(B) \hookrightarrow K(A)$$

we know that everything in $A \otimes_B K(B)$ is algebraic over $K(B)$, therefore we know that $A \otimes_B K(A)$ is finitely generated and integral over $K(B)$, thus is finite over $K(B)$, therefore this is a generically finite morphism, therefore by the previous remark, there is a dense open V over X_2 such that:

$$\pi^{-1}V \to V$$

is indeed finite. Moreover, we can choose V to be integral as well, therefore, the pushforward of the structure sheaf is finite type over V and since V is Noetherian, we know that finite type is the same as finitely presented, therefore we know that there is a dense open U of V where $\pi^*\mathcal{O}_{X_1}$ is locally free, say of rank d where d is the same as the dimension of $A \otimes_B K(B)$ over $K(B)$.

Now assume that all $\mathcal{L}_i$ are very ample, for later use, we first notice that on an integral projective variety, we know that if a line bundle $\mathcal{L}$ has a section s that doesn't vanish at the generic point, then the closed subscheme cut out by s is an effective Cartier divisor D and $\mathcal{L} = \mathcal{O}(D)$. Now since $\mathcal{L}_i$'s are very ample, for example since $\mathcal{L}_1$ is very ample, there is a section of $\mathcal{L}_1$, which doesn't vanish at the generic point and doesn't contain any generic point of $X_2 - U$ (you can choose it to be the pullback of a hyperplane) **** to be completed* □

Proposition 1.35. Suppose that X is a projective k-variety and $\mathcal{L}$ is an ample line bundle on X, then for any subvariety Y of dimension n, we have $(\mathcal{L}^n \cdot Y) > 0$.

Proof. To prove:

$$(\mathcal{L}^n \cdot Y) > 0$$

Recall that this intersection number is a multilinear symmetric function, therefore, we know that:

$$(\mathcal{L}^{2n}) = 2^n \cdot (\mathcal{L}^n \cdot Y)$$

Therefore, we can prove that $(\mathcal{L}^{2n}) > 0$, keep doing this, and we conclude that we can assume that $\mathcal{L}$ is very ample.

Now I claim that when $\mathcal{L}$ is very ample, $(\mathcal{L}^n \cdot Y)$ is the degree of Y using the embedding of Y into the projective space by the complete linear series $|\mathcal{L}|$. To see why, we can use the asympototic Riemann-Roch, which says that:

$$q(m) = \chi(X, \mathcal{L}^{\otimes m} \otimes j_* \mathcal{O}_Y)$$

is a polynomial of degree n with the leading coefficient $(\mathcal{L}^m \cdot Y)$. Recall that by projection formula we have an isomorphism $\mathcal{L}^{\otimes m} \otimes j_* \mathcal{O}_Y \simeq j_*(j^* \mathcal{L}^{\otimes m})$, therefore, from the fact that cohomology is preserved by finite morphisms, we know that:

$$q(m) = \chi(Y, j^* \mathcal{L}^{\otimes n})$$

However, when $\mathcal{L}$ is very ample, say it induces an embedding to the projective space $i : X \to \mathbb{P}^N$, the pullback of $\mathcal{L}$ to Y induced an embedding of Y to $\mathbb{P}^N$, thus under this embedding $h : Y \hookrightarrow X \hookrightarrow \mathbb{P}^N$ we know that $j^* \mathcal{L} \simeq \mathcal{O}(1)$, therefore we know that $q(m)$ is actually the Hilbert polynomial of Y, therefore, the leading coefficient times $n!$ is $(\mathcal{L}^n \cdot Y)$ and is the degree of Y, we are done. $\square$

Remark 1.36. Latter, after we prove the famous theorem called Nakai-Moishezon criterion, you will find that the condition $(\mathcal{L}^n \cdot Y) > 0$ for any dimension n subvariety Y characterizes ample line bundles.

2 Intersection Theory on a Surface

In this section, we will apply what we developed in the previous section to a regular projective surface (a pure dimension 2 k-variety) over k. Recall that by Auslander-Buchsbaum theorem, any regular surface is factorial, therefore the homomorphism $\mathrm{Pic}X \to \mathrm{Cl}X$ is an isomorphism. We start our discussion by the following proposition:

Proposition 2.1. Suppose that C and D are effective divisors on X (figure out whether this means Cartier or Weil divisor). Then:

$$\deg_C \mathcal{O}_X(D)|_C = (\mathcal{O}_X(C) \cdot \mathcal{O}_X(D) \cdot X) = \deg_D \mathcal{O}_X(C)|_D$$

Proof. The first thing we want to say is that we actually don't have to distinguish effective Cartier or Weil divisors in this case. If D is an effective Weil divisor, I claim that there is an effective Cartier divisor D' on X such that $\mathcal{O}_X(D) \simeq \mathcal{O}_X(D')$. Regular implies factorial which implies normal, therefore, we know that we only have to give an effective Cartier divisor D' whose associated Weil divisor is D and then the corresponding invertible sheaves are automatically isomorphic. Recall factorial means that each Weil divisor is locally principal, therefore, there will be an open cover U_i of X on each U_i, $U_i \cap D$, which is still effective, is cut out by a rational function on U_i. Since then this regular function has no poles, therefore, using the normality condition, it's a regular function f_i on U_i, and it's not a zero divisor, since U_i is reduced, thus no embedded points, then we only have to show that f_i doesn't vanish at generic points of U_i, but since f_i cut out $U_i \cap D$ set-theoretically, it only vanish at some codimension 1 points, not the generic point.

So now let's assume that everything is an effective Cartier divisor. Now we have:

$$(\mathcal{O}_X(C) \cdot \mathcal{O}_X(D) \cdot X) = (\mathcal{O}_X(C) \cdot i_* \mathcal{O}_D) = (\mathcal{O}_X(C)|_D \cdot D) = \deg_D \mathcal{O}_X(C)|_D$$

All of the equalities follows from the previous section. Finally, since the intersection product is symmetric, we know:

$$(\mathcal{O}_X(C) \cdot \mathcal{O}_X(D) \cdot X) = (\mathcal{O}_X(D) \cdot \mathcal{O}_X(C) \cdot X)$$

Then you apply the same argument to conclude that:

$$\deg_C \mathcal{O}_X(D)|_C = (\mathcal{O}_X(C) \cdot \mathcal{O}_X(D) \cdot X) = \deg_D \mathcal{O}_X(C)|_D$$

$\square$

Definition 2.2. We call the above number $(\mathcal{O}_X(C) \cdot \mathcal{O}_X(D) \cdot X)$ the **intersection number** of C and D and denote it by $C \cdot D$.

Proposition 2.3. If C doesn't contain any associated point of D then:

$$C \cdot D = h^0(C \cap D, \mathcal{O}_{C \cap D})$$

where $C \cap D$ stands for the scheme theoretic intersection of C and D.

Proof. We already saw that:

$$(\mathcal{O}_X(C) \cdot \mathcal{O}_X(D) \cdot X) = (\mathcal{O}_X(C)|_D \cdot D) = \deg_D \mathcal{O}_X(C)|_D = \chi(D, \mathcal{O}_D) - \chi(\mathcal{O}_X(C)|_D)$$

If C doesn't contain any associated points of D, then we know that $\mathcal{O}_X(C)|_D$ cuts out the effective Cartier divisor $C \cap D$, therefore we have the SES of effective Cartier divisor, therefore since Euler characteristic is additive upon SES, we have;

$$\chi(D, \mathcal{O}_D) - \chi(\mathcal{O}_X(C)|_D) = \chi(D, j_* \mathcal{O}_{C \cap D})$$

but we know that $C \cap D$ is actually a finite collection of points, i.e. a dimension 0 stull, therefore, $\chi(D, j_* \mathcal{O}_{C \cap D}) = \chi(C \cap D, \mathcal{O}_{C \cap D}) = h^0(C \cap D, \mathcal{O}_{C \cap D})$. $\square$

Remark 2.4. There is a small issue in this proof, i.e. it's not hard to see that the pullback of the canonical element in $\mathcal{O}_X(C)$ to D cut out an effective Cartier divisor $C \cap D$ in D (not a zero divisor), we still need to prove that $C \cap D \hookrightarrow D \hookrightarrow X$ is indeed the $C \cap D \hookrightarrow X$ which is cut out by $\mathcal{O}_X(C) + \mathcal{O}_X(D)$.

We do this affine locally say on $SpecA$, we know that C, D is cut out by $(f), (g)$, then we know that in $SpecA/f$, $C \cap D$ is actually cut out by $(\overline{g})$ and we know that, since closed embeddings are affine, affine locally we can describe pullback very explicitly, the details are left for you.

Remark 2.5. The hypothesis that C doesn't contain ant associated points of D is actually excessive, since after we learn what is Cohen-Macaulay, you will find that D has no embedded points. Therefore, we can just assume that C doesn't contain any generic point of D.

In particular, if C, D doesn't have any component in common, then $C \cdot D \geqslant 0$, this means that we can not hope to compute or understand intersection numbers using moving lemmas as in topology if we have some irreducible curve E with $E^2 < 0$, then we can not move E to something F which intersects "transversely" with E that is equivalent to E to compute $E \cdot E$ as $E \cdot F$. Since the equivalent of transverse intersection is sharing no component, but this implies that $E \cdot F \geqslant 0$.

Definition 2.6. The degree of the normal bundle to C in X:

$$\deg_C \mathcal{N}_{C/X} = \deg_C \mathcal{O}_X(C)|_C$$

is called the self intersection $C \cdot C$ of C.

Remark 2.7. From the above discussion, we saw that there are four descriptions of the intersection number:

$$(C \cdot D \cdot X) = \deg_D \mathcal{O}_X(C)|_D = \deg_C \mathcal{O}_X(D)|_C = h^0(C \cap D, \mathcal{O}_{C \cap D})$$

There are advantages and disadvantages to each of them. For example, the description $(C \cdot D \cdot X)$ using Euler characteristics is useful for theoretic reasons, but the geometry is not cleat. The description using degree is good but it's not obviously symmetric in C, D. Finally, $h^0(C \cap D, \mathcal{O}_{C \cap D})$ is local since $C \cap D$ is just a finite collection of points and we can compute this on C by computing the dimension of the vector space associated to each point in $C \cap D$.

Remark 2.8. It's obvious that the intersection product gives a bilinear form:

$$\mathrm{Pic}X \times \mathrm{Pic}X \to \mathbb{Z}$$

Before we start doing some interesting explicit examples, let's develop a few theoretic result which is useful, they are basically just unpack the definition plus simple applications of the dualizing sheaf:

Proposition 2.9. We will assuming Serre duality for a geometrically irreducible smooth projective surface X in this proposition.

Let K_X be the divisor corresponding to the dualizing sheaf ω_X. Then for any curve C in X, we have:

$$C \cdot (K_X + C) = 2p_a(C) - 2$$

This is sometimes called the **adjunction formula**.

Proof. Every we use K_X in the formula, it means ω_X and $-K_X$ is the dual of the dualizing sheaf, therefore we know that $\mathcal{O}_X(K_X)$ is ω_X. Then we need to compute:

$$(\mathcal{O}_X(C) \cdot \mathcal{O}_X(C + K_X) \cdot X)$$

The trick is to compute:

$$(\mathcal{O}_X(C) \cdot \mathcal{O}_X(-C - K_X) \cdot X)$$

By what we computed earlier, this is:

$$\chi(X, \mathcal{O}_X) - \chi(X, \mathcal{O}_X(-C)) - \chi(X, \mathcal{O}_X(C) \otimes \omega_X) + \chi(X, \omega_X)$$

Notice that by Serre duality:

$$\chi(X, \mathcal{O}_X(C) \otimes \omega_X) = h^2(X, \mathcal{O}_X(-C)) - h^1(X, \mathcal{O}_X(-C)) + h^0(X, \mathcal{O}_X(-C)) = \chi(X, \mathcal{O}_X(-C))$$

Similarly, we get:

$$\chi(X, \omega_X) = \chi(X, \mathcal{O}_X)$$

Let's consider the SES of the effective Cartier divisor C:

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to i_*\mathcal{O}_C \to 0$$

Therefore we know that $\chi(X, \mathcal{O}_X) = \chi(X, \mathcal{O}_X(-C)) + \chi(C, \mathcal{O}_C)$, plug this into the formula, we get;

$$\chi(C, \mathcal{O}_C) + \chi(X, \omega_X) - \chi(X, \mathcal{O}_X(C) \otimes \omega_X)$$

Now we know that for the last two terms, we can take out the ω_X and compute χ for their dual, then it becomes:

$$\chi(C, \mathcal{O}_C) + \chi(X, \mathcal{O}_X) - \chi(X, \mathcal{O}_X(-C)) = \chi(C, \mathcal{O}_C) + \chi(C, \mathcal{O}_C)$$

By definition, $p_a(C) = 1 - \chi(C, \mathcal{O}_C)$, thus $2\chi(C, \mathcal{O}_C) = 2 - 2p_a(C)$, then we use the multilinearity to conclude what I want. □

Remark 2.10. A warning is that this proposition has nothing to do with the previous descriptions where C, D are effective because here we don't even know if $K_X + C$ is an effective divisor or not, so here everything is just unpack the definition.

Proposition 2.11. This proposition is often called the **Riemann-Roch for surfaces**. We still assume Serre duality here and use the assumptions in the previous proposition, then:

$$\chi(X, \mathcal{O}_X(D)) = D \cdot (D - K_X)/2 + \chi(X, \mathcal{O}_X)$$

Proof. Let's compute $(\mathcal{O}_X(D)\cdot\mathcal{O}_X(D-K_X)\cdot X)$, by definition it's:

$$\chi(X,\mathcal{O}_X)-\chi(X,\mathcal{O}_X(-D))-\chi(X,\mathcal{O}_X(-D)\otimes\omega_X)+\chi(X,\omega_X\otimes\mathcal{O}(-2D))$$

Again, by the fact that when we compute Euler characteristic on the surface, we can first drop the dualizing sheaf and then use the dual, the above formula is:

$$\chi(X,\mathcal{O}_X)-\chi(X,\mathcal{O}_X(-D))-\chi(X,\mathcal{O}_X(D))+\chi(X,\mathcal{O}(2D))$$

Then I will let you figure why it's:

$$\deg\mathcal{O}(D)-\deg\mathcal{O}(D)+2\deg\mathcal{O}(D)=2\deg\mathcal{O}(D)$$

Therefore:

$$D\cdot(D-K_X)/2=\deg\mathcal{D}=\chi(X,\mathcal{O}(D))-\chi(X,\mathcal{O}_X)$$

□

Remark 2.12. Notice that a smooth surface is regular thus normal and factorial, therefore we know that you have a bijective relation between the Picard group and class group, therefore every line bundle is the associated line bundle of some Weil divisor thus the above result actually holds for any line bundles $\mathcal{L}$ on X. You can also see this from the proof, the same proof goes through if you switch $\mathcal{O}(D)$ to $\mathcal{L}$.

Now we are ready to do some explicit examples and one of them is going to be the first example of blow-up:

Example 2.13. The first example is something we saw before, we first briefly review it.

Let $X=\mathbb{P}^1\times\mathbb{P}^1$. We know that it's a projective surface, which is regular. Recall that we have proved before that the Picard group of X is $\mathbb{Z}\times\mathbb{Z}$. We showed it by considering two divisors:

$$L=\{\infty\}\times\mathbb{P}^1\subset X \qquad M=\mathbb{P}^1\times\{\infty\}$$

Then we consider the the open subscheme $\mathbb{P}^1\times\mathbb{P}^1-L-M$, which we found that to be $\mathbb{A}^2$ since we have the following diagram:

$$\begin{tikzcd} \mathbb{P}^1\times\{\infty\} \arrow[r] \arrow[d, hook] & \{\infty\} \arrow[d, hook] \\ \mathbb{P}^1\times\mathbb{P}^1 \arrow[r] & \mathbb{P}^1 \end{tikzcd} \qquad \begin{tikzcd} \mathbb{P}^1\times\mathbb{P}^1-\{\infty\} \arrow[r] \arrow[d, hook] & \mathbb{P}^1-\{\infty\} \arrow[d, hook] \\ \mathbb{P}^1\times\mathbb{P}^1 \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

Then we know from the excision exact sequence that we have a surjection morphism:

$$\mathbb{Z}\oplus\mathbb{Z}\to \mathrm{Cl}X$$

which is defined to be $aL+bM\to a[L]+b[M]$, and we know that we have an isomorphism $\mathrm{Cl}X\to\mathrm{Pic}X$ defined by $D\to\mathcal{O}(D)$, thus we know that $a[L]+b[M]$ is mapped to $\mathcal{O}_X(L)^{\otimes a}\otimes\mathcal{O}_X(M)^{\otimes b}$. This is actually injective since if you assume $\mathcal{O}_X(L)^{\otimes a}\otimes\mathcal{O}_X(M)^{\otimes b}$ is trivial, then their restriction to M and to L should be trivial. But we saw that $\mathcal{O}_X(L)$ restricts to $\mathcal{O}$ on L but restricts to $\mathcal{O}(1)$ on M and you have similar argument for $\mathcal{O}(M)$. And the claim for the restriction comes from the fact that we can compute the transition functions pretty easily (you should compute the transition function your self to make sure that you indeed understand the computation!).

Therefore, the discussion shows that $\mathrm{Pic}X$ is generated by two independent generators $\mathcal{O}(L)$ and $\mathcal{O}(M)$. Then in this example, we will show that the intersection product, as a bilinear form, is determined by:

$$L\cdot L=M\cdot M=0 \qquad L\cdot M=1$$

Let's first compute $L\cdot M$. Recall that the we already proved that if L,M has no components in common (actually we didn't prove it, only mention this is true due to Cohen-Macaulayness), then the intersection number is:

$$h^0(L\cap M,\mathcal{O}_{L\cap M})$$

It's clear that the intersection is just a k-valued point, therefore the dimension of the global sections is 1, we are done.

Then let's compute the self-intersection. We know that the self-intersection is the degree of the normal bundle, but we have already proved that the restriction of $\mathcal{O}(L)$ to L is trivial and the restriction of $\mathcal{O}(M)$ to M is trivial, thus the degree is 0, we are done.

Our next example is the first example of blow-up:

Example 2.14. Let's first describe what this space is. The space is denoted by $\mathbb{B}\lessdot_{(0,0)}\mathbb{A}^2$, which is a subspace of $\mathbb{A}^2_k \times_k \mathbb{P}^1_k$. Let's use coordinate x, y on the affine space and u, v on the projective space. In formally speaking, this is cut out by the equation $xv = yu$. Formally speaking, this is the closed subscheme cut out by a particular regular section of the line bundle:

$$q^*\mathcal{O}(1)$$

You can interpret this line bundle as $p^*\mathcal{O} \otimes q^*\mathcal{O}(1)$ where p, q are the projections to $\mathbb{A}^2$ and $\mathbb{P}^1$. The regular function we will use will be $p^*x \otimes q^*v - p^*y \otimes q^*u$. By definition, if we cover $\mathbb{A}^2 \times \mathbb{P}^1$ by two $\mathbb{A}^3$, then on each of the $\mathbb{A}^3$, the closed subscheme is cut out by $x\frac{v}{u} - y$ and $y\frac{u}{v} - x$.

One way to think about this blow-up is to think that the points in $\mathbb{P}^1$ parametrizes lines in $\mathbb{A}^2$ that passes through the origin. The reason is simple since we know that a point in $\mathbb{P}^1$ is $\frac{u}{v} = k$ or $\frac{v}{u} = k$, therefore it correspond to the line $kx = y$ or the line $ky = x$. Therefore, we know that the lines passing through the origin actually gets lifted and they don't just intersect in a nondistinguishable way at the origin.

We can describe the fiber of the morphism $\mathrm{Bl}_{(0,0)}\mathbb{A}^2_k \to \mathbb{P}^1_k$ as a formal explanation of the above discussion:

$$\begin{tikzcd}
\cdots \arrow[r, hook] \arrow[d] & \mathbb{A}^2 \times [a:b] \arrow[r] \arrow[d, hook] & {[a:b]} \arrow[d, hook] \\
\mathrm{Bl}_{(0,0)}\mathbb{A}^2_k \arrow[r, hook] & \mathbb{A}^2_k \times \mathbb{P}^1_k \arrow[r] & \mathbb{P}^1_k
\end{tikzcd}$$

Suppose $b \neq 0$ for example, I claim that the dotted stuff at the top left is $\mathbb{A}^2_k/(\frac{a}{b}y - x)$. For example, we can compute this by an extended diagram:

$$\begin{tikzcd}
\mathbb{A}^2/(y\frac{a}{b} - x) \arrow[r, hook] \arrow[d, hook] & \mathbb{A}^2 \arrow[r] \arrow[d, hook] & a/b \arrow[d, hook] \\
\mathbb{A}^3/(y\frac{u}{v} - x) \arrow[r, hook] \arrow[d] & \mathbb{A}^2 \times \mathbb{A}^1 \arrow[r] \arrow[d, hook] & \mathbb{A}^1 \arrow[d, hook] \\
\mathrm{Bl}_{(0,0)}\mathbb{A}^2_k \arrow[r, hook] & \mathbb{A}^2_k \times \mathbb{P}^1_k \arrow[r] & \mathbb{P}^1_k
\end{tikzcd}$$

Everything is affine here and I will let you do the algebra yourself.

We can also think about the morphism $\mathrm{Bl}_{(0,0)}\mathbb{A}^2_k \to \mathbb{A}^2$. I first claim that this is an isomorphism away from the closed point $(0,0)$. To see this, we need to be extra careful. We first use the chart $v \neq 0$, then $D_+(x)$ and $D_+(y)$, and you probably will think that the morphism to $D_+(x)$ is;

$$Speck[x, \frac{1}{x}, y, t]/(x = yt) \to Speck[x, \frac{1}{x}, y, t] \to Speck[x, \frac{1}{x}, y]$$

which is given by ring maps:

$$k[x, \frac{1}{x}, y] \to k[x, \frac{1}{x}, y, t] \to k[x, \frac{1}{x}, y, t]/(x = yt)$$

where x is mapped to x and y is mapped to y. But I will let you see that this is not an isomorphism! Therefore something must went wrong in the argument, and this is indeed true. Let's write out the diagram carefully and see what went wrong.

The following diagram is what we are really doing:

$$\begin{tikzcd}
Speck[x,\frac{1}{x},y,\frac{u}{v}]/(x=y\frac{u}{v}) \arrow[r, hook] \arrow[d] & Speck[x,y,\frac{u}{v}]/(x=y\frac{u}{v}) \arrow[r, hook] \arrow[d] & \mathrm{Bl}_{(0,0)}\mathbb{A}^2 \arrow[d, hook] & \\
D_+(x)\times D_+(v) \arrow[r, hook] & \mathbb{A}^2\times D_+(v) \arrow[r, hook] \arrow[d] & \mathbb{A}^2\times\mathbb{P}^1 \arrow[r] \arrow[d] & \mathbb{A}^2 \arrow[d] \\
 & D_+(v) \arrow[r, hook] & \mathbb{P}^1 \arrow[r] & k
\end{tikzcd}$$

and you see that but the stuff on the left top corner is indeed the inverse image of $D_+(x)\times D_+(v)$, but we are actually not sure what is the morphism to some parts in $\mathbb{A}^2$ we are interested. The next diagram is what we really need to do:

$$\begin{tikzcd}
? \arrow[r, hook] \arrow[d] & D_+(x)\times\mathbb{P}^1 \arrow[r] \arrow[d, hook] & D_+(x) \arrow[d, hook] \\
\mathrm{Bl}_{(0,0)}\mathbb{A}^2 \arrow[r, hook] & \mathbb{A}^2\times\mathbb{P}^1 \arrow[r] & \mathbb{A}^2
\end{tikzcd}$$

However, due to this annoying $\mathbb{P}^1$ we don't direct have a clear picture about what is $? \to D_+(x)\times\mathbb{P}^1$ like what we did for affine fibered product. But we can try to glue. We know that ? is glued by these two pieces:

$$Speck[x,\frac{1}{x},y,\frac{u}{v}]/(x=y\frac{u}{v}) \qquad Speck[x,\frac{1}{x},y,\frac{v}{u}]/(y=x\frac{v}{u})$$

where the isomorphism on the intersection is in the following diagram:

$$\begin{tikzcd}
k[y,\frac{1}{y},\frac{u}{v},\frac{v}{u}] \arrow[rr, "y\mapsto x\frac{v}{u},\frac{u}{v}\mapsto\frac{u}{v}"] & & k[x,\frac{1}{x},\frac{u}{v},\frac{v}{u}] \\
k[y,\frac{u}{v}]_{y\frac{u}{v}} \arrow[u, "\simeq"'] & & k[x,\frac{1}{x},\frac{v}{u}] \arrow[u] \\
k[x,\frac{1}{x},y,\frac{u}{v}]/(x=y\frac{u}{v}) \arrow[u, "\simeq"'] & & k[x,\frac{1}{x},y,\frac{v}{u}]/(y=x\frac{v}{u}) \arrow[u, "\simeq"']
\end{tikzcd}$$

This makes the final object $Speck[x,\frac{1}{x},\frac{v}{u}]$ and the morphism to $D_+(x)$ is actually $k[x,\frac{1}{x},y]\to k[x,\frac{1}{x},\frac{v}{u}]$ where x is mapped to x and y is mapped to $x\frac{v}{u}$, thus an isomorphism since x is invertible and I will left the details to you.

In the above discussion, you see an interesting fact that if you want to consider the inverse image of $D_+(x)$, i.e. where x is not 0, u is never zero, or you can think that if x is not 0, then only assuming v is not 0 is not enough, u must also be nonzero, otherwise x is not invertible, therefore what we are actually i.e. the chart $D_+(v)$ in $D_+(x)$ is contained in $D_+(x)\times D_+(x)\times D_+(u)$, this is where the tricky part is.

Therefore, the fiber of any closed point that is not $(0,0)$ is also that closed point, since we have an isomorphism. Finally, let's think about what is the fiber of $(0,0)$, which is called the **exceptional divisor**, which is a notion that we will spend a lot of time discussing when talking about blow-up later. I claim that this fiber is an effective Cartier divisor and to compute it we still need to look at:

$$\begin{tikzcd}
E\cap Speck[x,y,\frac{u}{v}]/(x=y\frac{u}{v}) \arrow[r, hook] \arrow[d, hook] & Speck[x,y,\frac{u}{v}]/(x=y\frac{u}{v}) \arrow[r, hook] \arrow[d, hook] & \mathbb{A}^3 \arrow[r] \arrow[d, hook] & \mathbb{A}^1 \arrow[d, hook] \\
E \arrow[r, hook] \arrow[d] & \mathrm{Bl}_{(0,0)}\mathbb{A}^2_k \arrow[r, hook] \arrow[d] & \mathbb{A}^2\times\mathbb{P}^1 \arrow[r] \arrow[d] & \mathbb{P}^1 \arrow[d] \\
(0,0) \arrow[r, hook] & \mathbb{A}^2_k \arrow[r, "id"] & \mathbb{A}^2 \arrow[r] & k
\end{tikzcd}$$

We notice that the stack on the left is Cartesian therefore, and I will let you compute that $E \cap Speck[x, y, \frac{u}{v}]/(x = y\frac{u}{v}) \hookrightarrow Speck[x, y, \frac{u}{v}]/(x = y\frac{u}{v})$ is given by the surjective ring map:

$$k[x, y, \frac{u}{v}]/(x = y\frac{u}{v}) \to k[\frac{u}{v}]$$

where x, y maps to 0 and $\frac{u}{v}$ maps to $\frac{u}{v}$. Moreover, you can think of this as quotient out x, which is not a zero divisor in $k[x, y, \frac{u}{v}]/(x = y\frac{u}{v})$ since $Speck[x, y, \frac{u}{v}]/(x = y\frac{u}{v})$ is integral and we know that x doesn't vanish at the generic point. Similarly, you can compute the other one.

Now we can describe $\mathrm{Bl}\mathbb{P}^2$ at a k-valued point. It's will be a closed subscheme of $\mathbb{P}^2 \times \mathbb{P}^1$, cut out by the regular section $p^*x \otimes q^*v - p^*y \otimes q^*u$ in the line bundle $p^*\mathcal{O}(1) \otimes q^*\mathcal{O}(1)$. Now in the open subscheme $D_+(z) \times \mathbb{P}^1$, we know that the closed subscheme is cut out by $x'v = y'u$ where x', y' stands for coordinate $\frac{x}{z}$ and $\frac{y}{z}$, which is the blowup of the affine plane at the origin we saw. I claim that the map $\mathrm{Bl}_{[0:0:1]}\mathbb{P}^2 \to \mathbb{P}^2$ is an isomorphism away from the point $[0 : 0 : 1]$. We already saw this in the chart $D_+(z) \times \mathbb{P}^1$, then in the other two charts, we know that $[0 : 0 : 1]$ is not in that two charts, therefore, we need to show that the morphism is an isomorphism there, but we know that the morphism is given by:

$$\begin{tikzcd}
\cdots \arrow[r, hook] \arrow[d, hook] & D_+(x) \times \mathbb{P}^1 \arrow[r] \arrow[d, hook] & D_+(x) \arrow[d, hook] \\
\mathrm{Bl}_{[0:0:1]}\mathbb{P}^2 \arrow[r, hook] & \mathbb{P}^2 \times \mathbb{P}^1 \arrow[r] & \mathbb{P}^2
\end{tikzcd}$$

recall that we know that the closed subscheme is but out by $v = \frac{y}{x}u$, therefore we know that the inverse image of $D_+(x) \times D_+(u)$ covers the closed subscheme as if $u = 0$ then v is automatically 0. Then we know that the morphism is given by:

$$Speck[\frac{y}{x}, \frac{z}{x}, \frac{v}{u}]/(\frac{y}{x} = \frac{v}{u}) \to Speck[\frac{y}{x}, \frac{z}{x}]$$

and isomorphism. Similarly, you can argue for $D_+(y)$. Then we know that the inverse image of $[0 : 0 : 1]$ can be again interpreted as an effective Cartier divisor, again denoted by E and we call it the **exceptional divisor**. It's an effective Cartier divisor since we already saw that it's an effective Cartier divisor in the inverse image of $D_+(z)$ and in the inverse image of $D_+(x)$ or $D_+(y)$, we can think of it's cut out by 1, obviously not a zero divisor.

Now let's compute the Picard group of $X = \mathrm{Bl}_{[0:0:1]}\mathbb{P}^2$ and then we will compute the intersection product. We still need to use the excision exact sequence, and for this purpose, we need two divisors. Of course, the first one will be the exceptional divisor and we know that besides the exceptional divisor, this is just $\mathbb{P}^2$ minus the point $[0 : 0 : 1]$. Then we will take a line in $\mathbb{P}^2$, not passing the point $[0 : 0 : 1]$. For example, we can pick the hyperplane cut out by $z = 0$, and we take the inverse image in $\mathrm{Bl}_{[0:0:1]}\mathbb{P}^2$, then we know that the blow-up minus these two divisors is just the affine 2-plane minus one closed point. Recall that we already know that the affine 2-plane has trivial Picard group and taking out something of codimension larger than 1 doesn't affect the Picard group, thus the Picard group is 0 for the punctured affine 2 plane. Therefore, we have proved that the Picard group is generated by the line bundles corresponding to E and L where L is the inverse image of the line.

I claim that the Picard group is actually $\mathbb{Z}E \oplus \mathbb{Z}L$, i.e. these two line bundles are independent. To see this, we need to use the intersection number. I will prove that $L \cdot L = 1$ while $E \cdot E = -1$ and finally $L \cdot E = 0$. Therefore, if $L = aE$ for some $a \neq 0$, we know that $L \cdot E$ is not 0, a contradiction. The reason why $E \cdot L$ is 0 is easy since we know that E, L doesn't have any component in common, and in fact they are disjoint, therefore we know that:

$$E \cdot L = h^0(E \cap L, \mathcal{O}_{E \cap L}) = 0$$

To compute the other two intersection numbers. Let's consider a line passing through the origin, say M is the scheme theoretic preimage of the hyperplane $x = 0$ in $\mathbb{P}^2$. What you notice first is that the since the hyperplane $x = 0$ and $z = 0$ intersect at the only closed point $[0 : 1 : 0]$, therefore M and L still only intersects at one closed point. Therefore $M \cdot L = h^0(M \cap L, \mathcal{O}_{M\cap L})$. Consider the following diagram:

$$\begin{tikzcd}
& L \cap M \arrow[ld] \arrow[dd] \arrow[rr, dashed] & & V(z,x) \arrow[ld, hook] \arrow[dd, hook] \\
L \arrow[rr] \arrow[dd, hook] & & z=0 \arrow[dd] & \\
& M \arrow[rr] \arrow[ld] & & x=0 \arrow[ld, hook] \\
\mathrm{Bl}_{[0:0:1]}\mathbb{P}^2 \arrow[rr] & & \mathbb{P}^2 &
\end{tikzcd}$$

where the dotted arrow is induced by the universal property of fibered product, I will let you see that the face on the top and in the bact are both Cartesian, therefore we know that $L \cap M$ is also a k-valued point. This proves $L \cdot M$ is also 1.

(Here is a side remark, in general, the two squares consisting of the dotted arrow may not be commutative, but here it's indeed commutative becase $z = 0 \hookrightarrow \mathbb{P}^2$ and $x = 0 \hookrightarrow \mathbb{P}^2$ are two closed embeddings thus monomorphisms, I will leave the details to you)

$L \cdot L = 1$ is also easy to prove. To see this, I claim that if you take another line in the projective plane that doesn't contain the point $[0 : 0 : 1]$, the preimage in the blow-up is L', by a similar argument, we know $L \cap L'$ is a k-valued point, therefore we know that $L \cdot L'$ is 1. Then if we can show that $\mathcal{O}(L) \simeq \mathcal{O}(L')$ we are done. First consider the pullback section of z, we know that it cuts out L set-theoretically, but the blowup will be regular, therefore we don't have to think about the difference between effective Weil divisors and Cartier divisors, therefore we know that $\mathcal{O}(L) \simeq \pi^*\mathcal{O}(1)$, the same argument works for another line.

Now using the same line of reasoning, we know that $\mathcal{O}(L) = \mathcal{O}(M)$. Now we need to use a fact that we will prove when we discuss blow-ups in general, i.e. for a line passing through the origin $[0 : 0 : 1]$, say $x = 0$, the inverse image is the scheme theoretic union of aE for some nonnegative integer a and m, which is called the proper transformation of $x = 0$ or the strict transformation and it doesn't contain any component of the exceptional divisor E. Now I claim that $L \cdot m = E \cdot m = 1$. To prove this, we don't have to understand m globally. For example, L, we know that it only meet $E \cup m$ at one closed point set-theoretically and the intersection happens in the inverse image of $D_+(y)$ (as it's the inverse image of $[0 : 1 : 0]$). Therefore, we only need to understand L and m in the inverse image of $D_+(y)$ and how they intersect. We know that L in $D_+(y) = Speck[\frac{x}{y}, \frac{z}{y}]$ is $\frac{z}{y} = 0$. We know that m in the inverse image of m is just the inverse image of $x = 0$ since E is completely outside the inverse image of $D_+(y)$ and it's $\frac{x}{y} = 0$, therefore, the scheme theoretic intersection is a k-valued point, thus we know that $h^0(m \cap L, \mathcal{O}_{m\cap L}) = 1$, so is the intersection number. Then we need to understand $aE \cap m$, and we know that the intersection can be figured out in the inverse image of $D_+(z)$, thus we only need to consider the blow-up of the 2-plane:

$$\mathrm{Bl}_0\mathbb{A}^2 \to \mathbb{A}^2$$

And we are trying to figure out what is the inverse image of $x = 0$, i.e. we are considering this diagram:

$$\begin{tikzcd}
& \cdots \arrow[r, hook] \arrow[d, hook] & Speck[x,y]/x \times \mathbb{P}^1 \arrow[r] \arrow[d, hook] & Speck[x,y]/x \arrow[d, hook] \\
? \arrow[ru, hook] \arrow[rd, hook] & \mathrm{Bl}_0\mathbb{A}^2 \arrow[r, hook] & \mathbb{A}^2 \times \mathbb{P}^1 \arrow[r] & \mathbb{A}^2 \\
& Speck[x,y,\frac{v}{u}]/(x\frac{v}{u} = y) \arrow[r] \arrow[u, hook] & \mathbb{A}^2 \times D_+(u) \arrow[u, hook] &
\end{tikzcd}$$

the far left stuff with a question mark computes the scheme theoretic image of $x = 0$ in one chart, and I will let you show that this is $Speck[x, y, \frac{v}{u}]/(x, x\frac{v}{u} = y)$, i.e you can this of this as cut out by x, which is the same as the exceptional divisor. Then in another chart, I will let you do the computation and you will find that it's $Speck[x, y, \frac{u}{v}]/(x = y\frac{u}{v}, x) = Speck[y, \frac{u}{v}]/(y\frac{u}{v})$, while the exceptional divisor is $Speck[y, \frac{u}{v}]/(y)$.

Therefore, we know that this is the union of E and something cut out by $\frac{u}{v} = 0$, and the intersection is again a k-valued point, therefore $m \cdot E = 1$. Therefore, what you end up with is the inverse image is actually $m \cup E$. Therefore for any line in $\mathbb{P}^2$, it's inverse image, or scheme theoretic

Finally, let's prove $m \cdot m = 0$. Here, we pick another line through the point $[0 : 0 : 1]$, for example $y = 0$ and by the same argument, it's scheme theoretic image is $m' + E$, the union of m' and E. We know that since $x = 0$ and $y = 0$ intersects at $[0 : 0 : 1]$, therefore, the scheme theoretic preimage intersects at E, therefore we know that m and m' are disjoint. Therefore we know that $m \cdot m' = 0$. But I claim that $m' = m$ or in other words $\mathcal{O}(m) \simeq \mathcal{O}(m')$. The way you prove this is that you can think the scheme theoretic preimage of $x = 0$ and $y = 0$ is cut out by the pullback section $\pi^* x$ and $\pi^* y$, which makes the corresponding line bundle of the preimage both $\pi^*\mathcal{O}(1)$, but you can also write it as $\mathcal{O}(E) + \mathcal{O}(m)$ or $\mathcal{O}(E) + \mathcal{O}(m')$ since it's the scheme theoretic union of two effective Cartier divisors, thus we are done.

Finally, the same method shows you that $L = m + E$ as well and the sum is the sum of effective Cartier divisors, and if we want to compute: $E \cdot E = (L^2 - 2L \cdot m + m^2) = 1 - 2 + 0 = -1$, we are done.

Definition 2.15. Notice that in the above example of blow-up of $\mathbb{P}^2$ at a single k-valued point, the exceptional divisor e has self-intersection -1. We will give such curves a name: If X is a surface over k and $C \subset X$ is a curve in X consisting of smooth points with $C \simeq \mathbb{P}^1$ and $C \cdot C = -1$. We will say that C is a (-1)-**curve**.

As C is isomorphic to $\mathbb{P}^1$ and we know that the line bundles on $\mathbb{P}^1$ are $\mathcal{O}(d)$, with degree d, therefore, the assumption that $C \cdot C = -1$ can be restated as the normal bundle is $\mathcal{O}(-1)$.

As part of the reward of the long example we did:

Proposition 2.16. The blow-up of $\mathbb{P}^2$ at the point $[0 : 0 : 1]$ is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ even if they have the same Picard group and they are smooth birational surfaces.

Proof. We first give a simpler approach, which is to consider the self-intersections. We know that if X and Y are isomorphic, then the intersection number induces the same multilinear function on the product of Picard groups. Thus the same self-intersections

But notice that on $\mathbb{P}^1 \times \mathbb{P}^1$, the intersection number on $\mathbb{Z}L \times \mathbb{Z}M$ is given by $L \cdot L = 0 = M \cdot M$ while $L \cdot M = 1$, therefore the image of the self-intersection in $\mathbb{Z}$ is $(aL + bM, aL + bM) = 2ab$ which is $2\mathbb{Z}$. On the other hand, the Picard group of the blowup is $\mathbb{Z}e \times \mathbb{Z}l$ with $l \cdot l = 1$, $e \cdot e = -1$ and $l \cdot e = 0$. Therefore the image of the intersection form is $(al + be, al + be) = a^2 - b^2$ which is $\mathbb{Z}$. Therefore they are not isomorphic.

Now, we give an approach that is more general as you will see later that this method can be generalized later to show that the Hizerbruch surface $\mathbb{F}_0$ is not isomorphic to the Hizerburch surface $\mathbb{F}_1$.

We still want to study it's Picard groups, but with additional information since as you have already seen, they have the same Picard group. What we will do is ti identify those effective elements in the Picard group. Here by effective elements, we mean that a line bundle is effective iff it's $\mathcal{O}(D)$ where the class of D is represented an effective divisor, i.e. $[D] = [E]$ where $E \geqslant 0$. This is well defined as if $\mathcal{O}(D') = \mathcal{O}(D)$, this implies that $[D] = [D']$. Now in the Picard group of $\mathbb{P}^1 \times \mathbb{P}^1$, we know that it's generated by $\mathcal{O}(L)$ and $\mathcal{O}(M)$ while L, M are both irreducible divisors, thus effective. Then we only need to prove that $[aL + bM]$ is represented by an effective divisor iff $a, b \geqslant 0$. To do so, say E is an effective divisor, we consider $\mathcal{O}(E) \simeq \mathcal{O}(aL + bL)$, we know that implies that there is a nonzero section of $\mathcal{O}(E)$ and conversely, if a line bundle has nonzero sections, the class is effective. However, we know that since $\mathcal{O}(aL)$ and $\mathcal{O}(bM)$ don't have any nonzero section when $a < 0$, $b < 0$. To see this, I claim that I can identity $\mathcal{O}(L) = p^*\mathcal{O}(1)$ and identity $\mathcal{O}(M) = q^*\mathcal{O}(1)$ where p, q are going to be the two projections. This is because we know that L is the fiber of $\{\infty\}$, and the and we know that $\{\infty\}$ is the closed subscheme cut out by y in $\mathcal{O}(1)$, therefore we know the fiber is cut out by the pullback section in $p^*\mathcal{O}(1)$. The similar argument works for another one and in conclusion we know that $\mathcal{O}(aL + bM) = \mathcal{O}(a, b)$. Then we will need a result which we will prove later, i.e. the **Kunneth formula** which tells you that the dimension of $\mathcal{O}(a, b)$ is $(a + 1)(b + 1)$ when $a, b \geqslant 0$ and 0 otherwise, therefore the effective elements in the Picard group are those (a, b) with $a, b \geqslant 0$.

Then for the blowup of the projective space, say an element in the Picard group is $a\mathcal{O}(l) + b\mathcal{O}(e)$, we want to identity when it has a nonzero element. What we can be sure is that since $\mathcal{O}(l) = \mathcal{O}(e + m)$, we know that elements of the form: $(a, -a)$ is going to be effective as well since they are represented by am. Moreover, the usual (a, b) where $a, b \geqslant 0$ are effective as well, therefore, we know that as semigroups the effective subsemigroup are not isomorphic as one can generated by 2 elements but in another semigroup any two elements don't generate the semigroup. ☐

Before we are done with the blowup temporarily, let's try to connect to an earlier notion, the nef cone, which we didn't really discussed at length, especially that we didn't give any examples. Here we use this blowup to give more insight on this notion:

Recall that for a proper k-variety the numerically trivial subgroup of line bundles are denoted by $\mathrm{Pic}^\tau X$ and the $\mathrm{N}^1(X)$ group is defined to be $\mathrm{Pic}X/\mathrm{Pic}^\tau X$. Recall that is a line bundle $\mathcal{L}$ is numerically trivial after some tensor power, then $\mathcal{L}$ is itself numerically trivial. Therefore, this group $\mathrm{N}^1(X)$ is abelian and non-torsion. By the highly nontrivial Neron Severi Theorem, we know this is also a finitely generated subgroup, then it's free and finitely generated, with the rank called the **Picard number**, denoted by $\rho(X)$. We define $\mathrm{N}^1_{\mathbb{Q}}(X) = \mathrm{N}^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, which is tentatively called the **$\mathbb{Q}$-line bundles**.

Recall that in this vector space $\mathrm{N}^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, we have a basis which is the basis of $\mathrm{N}^1(X)$ tensor with 1. Therefore any elements in $\mathrm{N}^1_{\mathbb{Q}}(X)$ are written as a unique linear combination of the basis, for example:

$$\alpha = \sum q_i[\mathcal{L}_i] \otimes 1$$

We say that it's numerically effective if for all curves C in X:

$$q_i \sum \deg_C \mathcal{L}_i \geqslant 0$$

Then I claim that the nef $\mathbb{Q}$-line bundles forms a closed cone in $\mathrm{N}^1_{\mathbb{Q}}(X)$, which is called the **nef cone**. To make this precise, we first need to understand what does the word "closed" mean. We will need some sort of topology on $\mathrm{N}^1_{\mathbb{Q}}(X)$. With the basis, we can identify it with $\mathbb{Q}^{\rho(X)}$, which is a subset of $\mathbb{R}^{\rho(X)}$ and we will give it the topology of the usual subspace topology of the Euclidean topology, i.e. it's a closed subset iff it's $C \cap \mathrm{N}^1_{\mathbb{Q}}(X)$ where the subset C is closed in $\mathrm{N}^1_{\mathbb{R}}(X)$. Then we need to understand what dose the word **cone** mean, a cone in a vector space is a subspace C where if x is in C then for any scalar $\lambda \geqslant 0$, $\lambda x \in C$. Now let's see the proof:

Proof. The proof is basically just a definition as it's obvious that if α is nef, for any nonnegative scalar λ, $\lambda\alpha$ is also going to be nef.

Now to prove it's closed, let's actually pass to the real vector space, $\mathrm{N}^1_{\mathbb{R}}(X)$ whose operations descends to $\mathrm{N}^1_{\mathbb{Q}}(X)$. Therefore, we can also define the nef cone in $\mathrm{N}^1_{\mathbb{R}}(X)$ and if we can prove it's closed here and the nef cone in $\mathrm{N}^1_{\mathbb{Q}}(X)$ is just $\mathrm{N}^1_{\mathbb{Q}}(X)$ intersects with the nef cone in $\mathrm{N}^1_{\mathbb{R}}(X)$, we are done. Suppose that $\alpha_i = \sum_j r^i_j[\mathcal{L}_j]$ converges to α, we know that $\alpha = \sum_j(\lim_{n\to\infty} r^i_j)\mathcal{L}_j$, by our assumption, for all i

$$\sum_j r^i_j \deg_C(\mathcal{L}_j) \geqslant 0$$

This implies that if you take the limit, it's also nonnegative. Thus the cone is closed. Finally, it's obvious that the intersection of the nef cone in the real vector space is the nef cone in the rational vector space, we are done. ☐

After these review, we look at three examples:

Example 2.17. In this example, we want to find what is the nef cone of $X = \mathbb{P}^2$. The first step is to find what is the $\mathrm{N}^1(X)$ group. We know that the line bundles on $\mathbb{P}^2$ are $\mathcal{O}(d)$ for $d \in \mathbb{Z}$. I claim that the numerically trivial line bundle is only the trivial line bundle, i.e. $\mathcal{O}(0)$. To see this, first notice that we can only assume $d > 0$ as if $d < 0$, it's the dual. Then for example, we consider $\mathcal{O}(d)$, we need to find a curve in $\mathbb{P}^2$, such that the degree of $\mathcal{O}(d)$ on that curve is not 0. Let's choose an irreducible polynomial of degree d, then we know that the line bundle corresponding to $V(f)$ is $\mathcal{O}(d)$, then recall that we can always find a homogeneous polynomial g of high enough degree which doesn't pass any irreducible components of $V(f)$, plus the fact that $V(f)$ doesn't have any embedded points, we know that the closed subscheme $V(g)$, intersects $V(f)$, but only at a finite number of points, moreover the degree of $\mathcal{O}(d)$ on $V(g)$, is going to be $h^0(V(g) \cap V(f), \mathcal{O}_{V(g)\cap V(f)})$, which is not 0 as it has a finite collection of points.

This tells you that the $\mathrm{N}^1(\mathbb{P}^2)$ group is generated by one element $\mathcal{O}(1)$ as we are only quotienting out the zero element of the Picard group. Then the $\mathrm{N}^1_{\mathbb{Q}}$ is just a one dimensional vector space. Now I claim that the nef cone is just $a\mathcal{O}(1)$ where a is a nonnegative rational number. To proceed, we first prove that $\mathcal{O}(1)$ is numerically effective, i.e. for any curve C in $\mathbb{P}^2$, the degree of $\mathcal{O}(1)$ on C is nonnegative. Recall that we know to prove the degree of $\mathcal{O}(1)$ on a curve C is nonnegative, we can raise it to a high enough power. But for any curve C, we know that we can find a f such that $V(f)$ doesn't pass any irreducible points of C, therefore the degree of $\mathcal{O}(d)$ on C is $h^0(C \cap V(f), \mathcal{O}_{V(f)\cap C})$, which is nonnegative.

In conclusion, the nef cone in $\mathrm{N}^1_{\mathbb{Q}}(\mathbb{P}^2)$ is:

$$\{a(\mathcal{O}(1) \otimes 1) : a \geqslant 0\}$$

Example 2.18. Now we consider the surface $\mathbb{P}^1 \times \mathbb{P}^1$. We already saw that it's Picard group is generated by two free generators $\mathcal{O}(L)$ and $\mathcal{O}(M)$.

Again, let's first find the numerically trivial subgroup. I claim that the only element that is numerically trivial is the trivial line bundle. To see why, let's still use the usual notation $\mathcal{O}(a, b)$. As long as a or b is not 0, for example $a \neq 0$, then we consider the curve M, and since we know that the restriction of $\mathcal{O}(1, 0)$ is $\mathcal{O}(1)$ on M and the restriction of $\mathcal{O}(0, 1)$ to M is $\mathcal{O}$, we know that the restriction of $\mathcal{O}(a, b)$ is $\mathcal{O}(a)$ on $M \simeq \mathbb{P}^1$, thus the degree is a.

Therefore the $\mathrm{N}^1(X)$ group is again generated by two free generators $\mathcal{O}(L) \simeq \mathcal{O}(1, 0)$ and $\mathcal{O}(M) \simeq \mathcal{O}(0, 1)$. Therefore, we know the $\mathrm{N}^1_{\mathbb{Q}}(X)$ is a two dimensional vector space. And I claim that the effective cone is the following:

$$\{\mathcal{O}(a, b) : a, b \geqslant 0\}$$

Again, we prove that $\mathcal{O}(1, 0)$ and $\mathcal{O}(0, 1)$ are numerically effective. For example, we need to show that for any curve C, the degree of $\mathcal{O}(1, 0)$ on C is nonnegative. if it doesn't intersect with M, then we know that the degree is computed by the dimension of the global section of the intersection, which is 0. Assume that C intersect with M, but doesn't contain any component of M, then again, the degree is the dimension of the global section. Therefore, we only have to assume that C contains $M \simeq \mathcal{P}^1$. Since we can assume that C is irreducible (and it's reduced by assumption), thus we can assume C is just $M = \mathbb{P}^1$ and we are done. The same works for $\mathcal{O}(0, 1)$. Then we consider line bundles of the form $\mathcal{O}(a, 0)$. We know that again, we can assume that the curve C is M, then we actually want to compute the restriction of $\mathcal{O}(a, 0)$ to M, thus it's just $\mathcal{O}(a)$ and easy to compute the degree, the same argument works for $\mathcal{O}(0, b)$ as well.

Finally, when a, b are not 0, we are given a irreducible reduced curve C and we want to compute the degree of the restriction of $\mathcal{O}(a, b)$ to C. Recall that the line bundle actually is $\mathcal{O}(a, 0) \otimes \mathcal{O}(0, b)$, thus we can compute the degree separately and then add them up. There are two possibilities, the first one is that C doesn't contain M not L, in this case the degree is always nonnegative. The second case is C is L or M, in this case we know how to restrict and how to compute the degree, thus I will leave the details here.

The last example we would like to do is the blowup of $X = \mathbb{P}^2$ at $[0 : 0 : 1]$:

Example 2.19. The conclusion we would like to draw is that the nef cone of $\mathrm{Bl}_{[0:0:1]}\mathbb{P}^2$ is generated by l and m where m is the strict transform of a line passing through the point $[0 : 0 : 1]$.

Well, first, I claim that the numerically trivial subgroup is still only the trivial bundle. To see why, say the line bundle is given by $a\mathcal{O}(l) + b\mathcal{O}(e)$ where a, b are not both 0. We can first consider the curve l where we know that the degree of $\mathcal{O}(l)$ on l is going to be the self-intersection $(l \cdot l) = 1$ and the degree of $\mathcal{O}(e)$ on l is $(l \cdot e) = 0$, therefore the degree of $a\mathcal{O}(l) + b\mathcal{O}(e)$ on l is a. Similarly, consider e we know the degree of the restriction is going to be $-b$, therefore we are done.

This shows the group $\mathrm{N}^1(X)$ is generated by two free generators, then so is $\mathrm{N}^1_{\mathbb{Q}}(X)$. To identify the nef elements, let's first show that l, m are nef. The fact that l is nef is because l is irreducible and reduced, thus we know that for any irreducible and reduced curve C, it either doesn't contain any associated point of l or is l, then we are done. For m, again, since m is irreducible and reduced, we know that an irreducible, reduced curve C that contains the generic point is m, thus $(m \cdot m) = 0$, we are done.

Then I claim that those not in the cone spanned by l, m is not nef, which finishes the proof. Say $\alpha = al + be = al + b(l - m) = (a + b)l - bm$, assume that this is not in the cone spanned by l, m, then either $a + b < 0$ or $b > 0$. Say $b > 0$, then we consider the intersection with e:

$$(e \cdot (a + b)l - bm) = -b < 0$$

When $a + b < 0$, we consider the intersection with m:

$$(m \cdot (a+b)l - bm) = a + b$$

The next topic we are interested in is **fibrations**, here we won't give the most general results, instead we will only state those that are relevant to us.

First, suppose that $f : X \to B$ is a morphism from a regular projective surface to a regular curve, not contracting any irreducible components of X, which means that if X_i is a component, then $f(X_i)$ is not a point. We use the symbol B here to indicate that we are thinking of B as a base, and X is a a surface parametrized by the base B. Now for a closed point in B, since B is regular, this is an effective Cartier divisor in B, which means that we can consider the scheme-theoretic preimage, which is also a effective Cartier divisor. This is just the result of the following local algebra: Suppose that $SpecB \to SpecA$ corresponds to a ring map $A \to B$ from a regular ring to a regular ring, where f in A a nonzero divisor such that (f) is a maximal ideal. Then if f is mapped to something in B which is not in any minimal primes in B, then the image of f is not a zero divisor. This is because regularity implies being reduced, therefore we only need to show the image is not in any minimal primes. For the sake of contradiction, if it is in some minimal primes p, then the inverse image of p contains (f), but (f) is a maximal ideal, thus the image of p is a closed point, this tells you $SpecB \to SpecA$ contracts points, which is a contradiction.

Then I will let you figure out why this implies $F = \pi^{-1}b$ is an effective Cartier divisor and recall that b is cut cut out by a section in the line bundle $\mathcal{O}_B(b)$, then this implies that F is cut out by the pullback section in $\pi^*\mathcal{O}_B(b)$, therefore we know that $\mathcal{O}_X(F) \simeq \pi^*\mathcal{O}_B(b)$, which is going to be isomorphic to $\mathcal{O}$ on F since we have the following Cartesian diagram:

$$\begin{tikzcd} F \arrow[r, hook] \arrow[d] & X \arrow[d] \\ b \arrow[r, hook] & B \end{tikzcd}$$

and any pullback of line bundle on B to b is the structure sheaf. This implies that $(F \cdot F) = 0$ as the degree of the structure sheaf on a curve is 0.

Now if X is not assumed to be regular, F may not be an effective Cartier divisor any more, so it become difficult to compute the intersection number, this is why we assume X is regular.

Proposition 2.20. Suppose that E is an elliptic curve, i.e. a projective, geometrically integral, geometrically regular curve over k of genus 1 and a choice of base point p. Consider the following difference map:

$$\pi : E \times E \to E$$

which is defined to be the composition of:

$$E \times E \xrightarrow{id \times i} E \times E \xrightarrow{m} E$$

where i is the inverse morphism on the group variety E. Let Δ be the diagonal in $E \times E$, I claim this difference morphism helps us to show $\Delta \cdot \Delta = 0$. Then $\mathrm{N}^1_{\mathbb{Q}}(E \times E)$ has rank at least 3, which helps prove that in general, if you have schemes X and Y, then the map:

$$\mathrm{Pic}X \times \mathrm{Pic}Y \to \mathrm{Pic}X \times Y \qquad \mathcal{L} \to \mathbb{M} \mapsto \mathcal{L} \boxtimes \mathcal{M}$$

is not an isomorphism in general and the example of $X = Y = \mathbb{P}^1$ is misleading.

Proof. The first property we will need about this difference morphism is that $\pi^{-1}p$, the scheme theoretic preimage of p, is the diagonal. Notice that this property is preserved by going to the algebraic closure of k, we have already seen that the multiplication, the inverse or the identity is preserved by base change to the algebraic closure of k, therefore, the extension of π is the difference on the extension E_k. Moreover, the extension of the fiber of p is the fiber of the base point p_K in E_K, recall that the base change of the diagonal of $E \times E$ is the diagonal of $E_K \times E_K$ and we have showed before that closed embeddings which agree after base change agree before the base change, thus all we need to do is assume k is algebraically closed prove the conclusion there. We don't want to say to much about the fact that the preimage of p is indeed the diagonal, but you should have this geometric intuition that since this is the difference, it's fair enough that preimage is indeed seems like the diagonal.

Another advantage of working with an algebraically closed field is that we can avoid a lot of problems that can arise in fibered products. Notice that $E \times E$ is still regular since E is regular and k is algebraically closed and E is locally finite type over k. Now I claim the difference morphism $E \times E$ to E is a fibration, i.e. no irreducible components are contracted. Recall that since E is irreducible, then $E \times_k E$ is irreducible, therefore, we are just trying to prove that the image of $E \times E$ is not a single point. It suffice to prove that the image is not a single closed point, but this is obvious since you can compute the image of the closed points (q_1, p) and (q_2, p) which are q_1 and q_2, not the same. Then by our discussion above, Δ, being the scheme theoretic preimage of p, a k-valued point, has the property $(\Delta \cdot \Delta) = 0$.

To prove the N^1 group has at least rank 3. Here, we consider three divisors $F_1 = E \times p$, $F_2 = p \times E$ and $F_3 = \Delta$. Recall that since F_1 and F_2 only has one scheme theoretic intersection point $p \times p$ at a k-valued point, we know that $F_1 \cdot F_2 = 1$. Now consider the usual projection $E \times E \to E$, this is also a fibration, then we know that F_1 and F_2 being the scheme theoretic preimage of p, has self-intersection 0. Finally, notice that the diagonal intersects with F_i at a k-valued point scheme theoretically:

$$\begin{tikzcd}
\Delta \cap F_1 \arrow[r] \arrow[d] & \Delta \arrow[r] \arrow[d, hook] & p = Speck \arrow[d, hook] \\
F_1 \arrow[r] \arrow[d] & E \times E \arrow[r, "\pi"'] \arrow[d, dashed, "p"] & E \arrow[d] \\
Speck \arrow[r] & E \arrow[r] & k
\end{tikzcd}$$

therefore $(\Delta \cdot F_i) = 1$. Then suppose that $aF_1 + bF_2 + cF_3$ is numerically trivial in the Picard group, we know that $(aF_1 + bF_2 + cF_3 \cdot F_i) = 0$ as the intersection product only depends on the numerical equivalence class. Then this implies that $b + c = a + c = a + b = 0$ i.e. $a = b = c = 0$, therefore we know that F_1, F_2, F_3 are independent in the N^1 group, therefore the rank is at least 3. This helps us to show that:

$$\mathrm{Pic}E \times \mathrm{Pic}E \to \mathrm{Pic}E \times E$$

is not an isomorphism. This is because firstly: $\mathrm{Pic}X \times \mathrm{Pic}Y \to \mathrm{Pic}X \times Y$ is always injective if X, Y are k-varieties over k. That is $p^*\mathcal{L} \otimes q^*\mathcal{M} \simeq \mathcal{O}$ iff $\mathcal{L}$ and $\mathcal{M}$ are both trivial. This is because we can restrict to $X \times \{y\}$ for a k-valued point k and I will let you do the details. Now I claim that any line bundle $\mathcal{L}$ on E is of the form $\mathcal{O}(dp)$ where d is the degree of $\mathcal{L}$. This is because we know that degree 0 line bundles on E is only $\mathcal{O}$! Therefore we know that $\mathrm{Pic}E$ is generated by $\mathcal{O}(p)$, but we know that $p^*\mathcal{O}(p)$ is F_2 where p is the projection and $q^*\mathcal{O}(p) = F_1$, therefore we know that the image of $\mathrm{Pic}E \times \mathrm{Pic}E \to \mathrm{Pic}E \times E$ are all $aF_1 + bF_2$, this can not be surjective as if so, we $\Delta = aF + 1 + bF_2$ in N^1 group, but this is not true as we already saw they are independent. $\square$

Our next goal is to understand intersections of curves on a ruled surface. Recall that if C is a regular curve and $\mathcal{F}$ is locally free of rank 2, then the projectivization of $\mathcal{F}$, $\mathbb{P}\mathcal{F}$ is called the a ruled surface over C, and in particular, we will prove a lot of stuff for the Hizerbruch surface and in particular fulfill our earlier promise that $\mathbb{F}_n$ will be distinct for different n. Before we do this, let's prove a useful proposition:

Proposition 2.21. Suppose that X is a scheme and $\mathcal{V}$ is a locally free sheaf of rank 2. Recall that the universal property of the projectivization tells you that we have a bijection between the set of surjective morphisms $\mathcal{F} \to \mathcal{L} \to 0$ onto line bundles on X and sections $\sigma : X \to \mathbb{P}\mathcal{F}$ of the canonical projection $\mathbb{P}\mathcal{F} \to X$. This gives us a SES:

$$0 \to \mathcal{I} \to \mathcal{V} \to \mathcal{D} \to 0$$

corresponding to a section:

$$\sigma : X \to \mathbb{P}\mathcal{V}$$

where we take $\mathcal{I}$ to be the kernel of the surjective morphism $\mathcal{V} \to \mathcal{D}$. Recall we have showed that since $\mathcal{V}$ and $\mathcal{D}$ are both of finite rank, the kernel is necessarily locally free, of rank 1, thus is also a line bundle.

Now I claim that $\sigma : X \to \mathbb{P}\mathcal{V}$ is an effective Cartier divisor and the normal bundle to $\sigma(X)$ is $\mathcal{D} \otimes \mathcal{I}^\vee$.

Proof. **** to be completed later after another treatment on the relative proj, this is the same for the computations we will have for Hizerbruch surfaces* □

Now we will use the conclusion of this proposition to compute something concrete:

The final topic in this section is the **Hodge index theorem**:

Remark 2.22. The name "index" comes from the following notion of the index of a real quadratic form. Recall that we have classified the quadratic forms over an algebraically closed field k by the technique of completing the squares and this is like a diagonalizing process. We proved that any quadratic form can be written as a sum of squares and the number of the squares only depends on the quadratic form.

Similarly, now, suppose that V is a real vector space of dimension n, together with a symmetric bilinear form $b(\cdot,\cdot)$ on V and an inner product $(\cdot,\cdot)$, then I claim that there are **integers** a_+, a_0, a_- such that given any orthogonal basis $(v_1,\cdots,v_n)$ with respect to the inner product, the number of $(v_i \cdot v_i)$ that are positive (resp. zero, negative) is a_+ (resp. a_0, a_-). The pair (a_+, a_-) is always called the **index** or the **signature** of the symmetric bilinear form. Notice that one special case is when you have an inner product of V, then $a_+ = n, a_0 = 0 = a_-$.

This result is not hard to prove, as you can compute the intersection matrix M for each orthogonal basis $(v_1,\cdots,v_n)$ with respect to the form b. Recall the form is symmetric, therefore M is a symmetric real matrix, by the Spectral Theorem, we know that all the eigenvalues of M are real. Now let's think about what is this matrix M, this is the matrix A such that:

$$b(x,y) = (Mx, y)$$

You can check this on v_i, v_j. Then what are the eigenvalues of this matrix M, recall that the Spetral theorem also tells you that there is an orthonormal basis e_i such that $Me_i = \lambda_i e_i$, then we know that:

$$b(e_i, e_i) = \lambda_i$$

Let's write the number of positive (resp. zero negative) λ_i as a_+ (resp. a_0, a_-). Now, let's try to give meanings to these integers. I claim that a_+ is the dimension of the the subspace where B is positive definite. What we discussed above shows that the dimension is at least a_+. Recall that the space is decomposed to the direct sum of the space where b is positive definite, zero and negative definite. Then we know that suppose that W is a subspace of the positive definite subspace. Now let w be a vector in W, say it's $\sum y_i e_i$, then we know that:

$$b(w,w) = b(w_+, w_+) + b(w_-, w_-) = \sum_{\lambda_i > 0} y_i^2 + \sum_{\lambda_i < 0} y_i^2$$

Now if the dimension of W is larger then a_+, we know that there is a w such that y_i with $\lambda_i < 0$ is not 0, since ae_i with λ_i is in the positive definite subspace, we know that we can eliminate the components of those e_i in w with w still in the subspace, but this implies that:

$$\sum_{\lambda_i < 0} y_i^2 > 0$$

a contradiction. This proves the assertion. And this also shows that for any basis v_i, the number of $b(v_i, v_i) > 0$ is a_+, since we are just counting the number of v_i that is in the positive definite subspace, and since v_i is a basis, there should be exactly a_+ of them in the subspace, as this is the dimension.

By a similar argument, we know that a_0 is the dimension of the "kernel" of b and a_- is the dimension of the negative definition subspace. This finishes the proof.

At first sight, what we will prove, which is called the Hodge index theorem, is not quite related to the index we saw above:

Theorem 2.23. Suppose that X is an geometrically irreducible smooth projective surface over a field k with $\mathcal{L}$ and $\mathcal{H}$ in Pic$X$ such that $\mathcal{H} \cdot \mathcal{H} > 0$ and $\mathcal{L} \cdot \mathcal{H} = 0$, then:

1. $\mathcal{L} \cdot \mathcal{L} \leqslant 0$, and
2. $\mathcal{L} \cdot \mathcal{L} = 0$ iff $\mathcal{L}$ is numerically trivial.

Before we prove this theorem, we show that this is indeed some sort of index type statement:

Proposition 2.24. Assume $\rho(X)$ is finite, then the Hodge index theorem is equivalent to the statement that on the group $\mathrm{N}^1_{\mathbb{R}}(X) = \mathrm{N}^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$, the induced intersection form has index $(1, \rho(X) - 1)$.

Proof. Smoothness implies regularity, so we don't have to worry about this. Assume that $\mathrm{N}^1(X)$ is spanned by the basis $\mathcal{L}_1, \cdots, \mathcal{L}_n$, with $\mathcal{L}_1$ being ample. Let's first assume Hodge index theorem, and we need to prove that the index is as desired.

Obviously, we have proved that the self-intersection of ample line bundles are positive, therefore the index is at least 1 and let's denote this number $a_1 = \mathcal{L}_1 \cdot \mathcal{L}_1$. Now we need to prove that the self-intersection of all the other $\mathcal{L}_n$ are negative. By the Hodge index theorem, since they are not numerically trivial, the self-intersection is negative iff $\mathcal{L}_1 \cdot \mathcal{L}_i = 0$. But we don't have to stick with this fixed basis and we can do modifications. We can assume that actually that none of the remaining line bundles have zero intersection with $\mathcal{L}_1$. If so, we are done for those line bundles. Now let's denote $a_i = \mathcal{L}_1 \cdot \mathcal{L}_i$ for $i \geqslant 2$ and now we can consider another basis:

$$\mathcal{L}_1, a_1\mathcal{L}_2 - a_2\mathcal{L}_1, \cdots, a_1\mathcal{L}_n - a_n\mathcal{L}_1$$

I will let you check this is indeed a basis, and this satisfies the assumption in the theorem.

Conversely, let's assume the index and try to prove it implies the theorem. We take a basis $\mathcal{L}_i$ of $\mathrm{N}^1(X)$, the first one being $\mathcal{H}$. Again, what we can assume about the basis is actually orthogonal with respect to the intersection form. We can first require $\mathcal{L}_1 \cdot \mathcal{L}_i = 0$ for $i \geqslant 2$ then keep going down. Then let's assume that under this basis:

$$\mathcal{L} = \sum a_i \mathcal{L}_i$$

and we assume $\mathcal{L}_1 \cdot \mathcal{L} = a_1 = 0$, this implies that all the other $a_i = 0$ for $i \geqslant 2$. Then if we compute the self-intersection, you get a nonpositive number and it's 0 iff all the integer coefficients vanish. ☐

Remark 2.25. We haven't proved the fact that $\rho(X)$ is finite, therefore, in the following proof of the Hodge index theorem, we will not use it. But the above result is still useful conceptually because X is projective therefore always admits an ample line bundle, and this is going to be positive, and all the other line bundles are negative.

This is enough motivation right now and let's actually prove it:

Proof. Let's first assume $\mathcal{H}$ is ample. Since we assumed that X is smooth and geometrically irreducible, we can apply Serre duality and in particular, we have a dualizing sheaf ω_X. We will first prove that, for any line bundle $\mathcal{M}$, if $\mathcal{M} \cdot \mathcal{H} > \omega_X \cdot \mathcal{H}$, then the top cohomology vanishes, i.e. $h^2(\mathcal{M}, X) = 0$.

We have actually seen something similar with a curve C, where $h^1(C, \mathcal{L}) = h^0(C, \omega_C \otimes \mathcal{L}^2)$, then if the degree of $\mathcal{L}$ is larger than $2g - 2 = \deg\omega_C$, then this is a negative degree line bundle, thus no global sections.

Similarly, for surfaces, we can prove $h^0(X, \omega_X \otimes \mathcal{M}^\vee) = 0$ by Serre duality. Suppose that it has nonzero global sections, then we consider the divisor it cut out, since this is a global section, we know that the divisor is going to be effective, therefore we consider the corresponding effective Cartier divisor, say D, then we know that:

$$\mathcal{H} \cdot (\omega_X - \mathcal{M}) = (\mathcal{H} \cdot D) > 0$$

This is because we can also choose an global section of $\mathcal{H}$, which corresponds to a hyperplane in the projective space X embeds into or, by raising $\mathcal{H}$ to a high enough power, we can assume that the section gives you a hypersurface, and no matter it's a hyperplane or a hypersurface, it doesn't contain the components of D, then the intersection number can be computed by the global section of the scheme theoretic intersection, thus is nonnegative, therefore:

$$(\omega_X - \mathcal{M}) \cdot \mathcal{H} \geqslant 0 \Rightarrow \omega_X \cdot \mathcal{H} \geqslant \mathcal{M} \cdot \mathcal{H}$$

a contradiction.

We next show that if $\mathcal{H}$ is ample and $\mathcal{L}$ is any line bundle with $\mathcal{L} \cdot \mathcal{L} > 0$ and $\mathcal{L} \cdot \mathcal{H} > 0$, then for $n \gg 0$, $\mathcal{L}^{\otimes n}$ has a nonzero global section. The reason is, first, when $n \gg 0$, we know that:

$$\mathcal{L}^{\otimes n} \cdot \mathcal{H} = n\mathcal{L} \cdot \mathcal{H} > \omega_X \cdot \mathcal{H}$$

Therefore the top cohomology of $\mathcal{L}$ vanishes. Then let's use Riemann-Roch for curves(recall that by the remark after the proposition where we proved the Riemann-Roch, it's valid for any line bundles):

$$\chi(X,\mathcal{L}^{\otimes n}) = h^0(X,\mathcal{L}^{\otimes n}) - h^1(X,\mathcal{L}^{\otimes n}) = \frac{n^2}{2}\mathcal{L}\cdot\mathcal{L} - \frac{n}{2}\mathcal{L}\cdot\omega_X + \chi(X,\mathcal{O}_X)$$

Therefore, we get a polynomial expression:

$$h^0(X,\mathcal{L}^{\otimes n}) = \frac{\mathcal{L}^2}{2}n^2 - \frac{\mathcal{L}\cdot\omega_X}{2}n + \cdots$$

Since $\mathcal{L}^2 > 0$, then when n is large enough, this is larger than 0.

Now we can prove this for ample $\mathcal{H}$. Assume $\mathcal{L}\cdot\mathcal{H} = 0$, we want to prove that $\mathcal{L}\cdot\mathcal{L} \leqslant 0$, for the sake of contradiction, let's assume this is not true and $\mathcal{L}^2 > 0$. Now consider:

$$\mathcal{H}' = \mathcal{L}\otimes\mathcal{H}^{\otimes n}$$

Since $\mathcal{H}$ is ample, then for $n \gg 0$, we know that $\mathcal{H}'$ is very ample, and in particular ample. And we know:

$$\mathcal{L}\cdot\mathcal{H}' = \mathcal{L}^2 > 0$$

Then we know by the above discussion that $\mathcal{L}^{\otimes m}$ is effective for $m \gg 0$, i.e. there is a nonzero section, which implies that:

$$(\mathcal{L}\cdot\mathcal{H}) = \frac{1}{m}(\mathcal{L}^{\otimes m}\cdot\mathcal{H}) > 0$$

because this is the intersection of an effective line bundle with an ample line bundle. A contradiction, thus we are done. The last assertion is that $\mathcal{L}\cdot\mathcal{L} = 0$ iff $\mathcal{L}$ is numerically trivial. To prove this, notice that if it's numerically trivial, then nothing there is to prove. So let's assume $\mathcal{L}\cdot\mathcal{L} = 0$ but it's not numerically trivial. It's not numerically trivial, means that there is a curve C in X, such that $(C,\mathcal{H}) \neq 0$, let's assume that C correspond to a line bundle $\mathcal{D}$, thus the assumption is that $\mathcal{D}\cdot\mathcal{L} \neq 0$ for some line bundle $\mathcal{D}$. Now we can find another line bundle $\mathcal{R}$ such that $\mathcal{R}\cdot\mathcal{H} = 0$ but $\mathcal{R}\cdot\mathcal{L} \neq 0$. Let's take:

$$\mathcal{R} = (\mathcal{H}\cdot\mathcal{H})\mathcal{D} - (\mathcal{D}\cdot\mathcal{H})\mathcal{H}$$

Now take $\mathcal{L}' = \mathcal{L}^{\otimes n}\otimes\mathcal{R}$ so $\mathcal{L}'\cdot\mathcal{H} = 0$. Let's compute:

$$\mathcal{L}'\cdot\mathcal{L}' = 2n\mathcal{L}\cdot\mathcal{R} + \mathcal{R}\cdot\mathcal{R}$$

Then by choosing a suitable $n \in \mathbb{Z}$, we can make $\mathcal{L}'\cdot\mathcal{H} = 0$ but $\mathcal{L}'\cdot\mathcal{L}' > 0$, which is a contradiction, we are done. This concludes the proof for ample line bundles.

For the general case, let's assume $\mathcal{H}$ is any line bundle such that $\mathcal{H}^2 > 0$ and assume $\mathcal{L}$ is a line bundle such that $\mathcal{H}\cdot\mathcal{L} = 0$. Pick an ample line bundle $\mathcal{A} \in \mathrm{Pic}X$, since X is projective, there is always such a line bundle. Now if:

$$\mathcal{A}\cdot\mathcal{L} = 0$$

Then we are done, since $\mathcal{A}$ is ample and we already prove the theorem for ample line bundles. So we will assume $\mathcal{L}\cdot\mathcal{A}$ is not zero(this in particular shows $\mathcal{L}$ is not numerically trivial). Similarly, we can assume that $\mathcal{A}\cdot\mathcal{H}$ is nonzero, otherwise $\mathcal{H}^2 \leqslant 0$, a contradiction to our hypothesis. Now take:

$$\mathcal{M} = (\mathcal{A}\cdot\mathcal{L})\mathcal{H} - (\mathcal{A}\cdot\mathcal{H})\mathcal{L}$$

Notice that $\mathcal{A}\cdot\mathcal{M} = 0$, therefore $\mathcal{M}^2 \leqslant 0$, i.e.,

$$(\mathcal{A}\cdot\mathcal{L})^2\mathcal{H}^2 - 2(\mathcal{A}\cdot\mathcal{H})^2(\mathcal{L}\cdot\mathcal{H}) + (\mathcal{A}\cdot\mathcal{H})^2\mathcal{L}^2 \leqslant 0$$

But $\mathcal{L}\cdot\mathcal{H} = 0$, therefore we know that:

$$(\mathcal{A}\cdot\mathcal{H}^2)\mathcal{H}^2 + (\mathcal{A}\cdot\mathcal{H})^2\mathcal{L}^2 \leqslant 0$$

But $\mathcal{H}^2 > 0$, $(\mathcal{A}\cdot\mathcal{H}^2) > 0$ and $(\mathcal{A}\cdot\mathcal{L}^2) > 0$, thus implies $\mathcal{L}^2 < 0$, we are done. □

3 The Grothendieck Group of Coherent Sheaves and an Algebraic Version of Homology

The goal of this short section is to give some exposition on some more advanced topics which connects to areas that is way beyond just introduction algebraic geometry. You will see Grothendiexk Groups of Coherent sheaves and a version of Chern classes.

Definition 3.1. Let X be a variety, we define the Grothendieck group of coherent sheaves, which is denoted by $\mathrm{K}(X)$ or denoted by $\mathrm{K}_0(X)$ sometimes. Is an abelian group generated by symbols of the form $[\mathcal{F}]$ where $\mathcal{F}$ is a coherent sheaf on X subject to relations $[\mathcal{F}] = [\mathcal{F}']$ if $\mathcal{F} \simeq \mathcal{F}'$ or more generally $[\mathcal{F}'] + [\mathcal{F}''] = [\mathcal{F}]$ if there is a SES:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}' \to 0$$

Or more precisely, this is the abelian group:

$$\mathbb{Z}\{[\mathcal{F}] : \mathcal{F} \in \mathrm{Coh}(X)\}/\langle [\mathcal{F}'] + [\mathcal{F}''] - [\mathcal{F}] \rangle$$

where we first consider a free abelian group and then consider the subgroup generated by $[\mathcal{F}'] + [\mathcal{F}''] - [\mathcal{F}]$.

Now if X is proper, recall that by the Grothendieck Coherence Theorem, the Euler characteristic map $\chi(-) : \mathrm{Coh}_X \to \mathbb{Z}$ is well defined since the Euler Characteristic is finite. Then since Euler characteristic is additive upon SES, we know that the map can be first naturally defined the free abelian group $\mathbb{Z}\{[\mathcal{F}] : \mathcal{F} \in \mathrm{Coh}(X)\}$ and then the kernel clearly contains the subgroup, therefore it descends naturally to $\mathrm{K}(X)$. Or similarly, if X is integral, we can consider the rank of coherent sheaves and again, since rank is also additive in SES, the rank function can also be defined on $\mathrm{K}(X)$.

Definition 3.2. The Grothendieck group is filtered by dimension: let $\mathrm{K}(X)^{\leqslant d}$ be the subgroup generated by those $[\mathcal{F}]$ where $\mathcal{F}$ is supported in dimension at most d. This is called the **dimension filtration**.

Let $\mathrm{A}'_d(X)$ be the d-th graded piece of the filtration, i.e.:

$$\mathrm{A}'_d = \mathrm{K}(X)^{\leqslant d}/\mathrm{K}(X)^{\leqslant d-1}$$

Definition 3.3. Let $\mathcal{L}$ be an invertible sheaf on X, we can define

$$\mathcal{L}\cdot : \mathrm{K}(X) \to \mathrm{K}(X) \qquad [\mathcal{F}] \mapsto [\mathcal{F}] - [\mathcal{L}^\vee \otimes \mathcal{F}]$$

This is well defined because if we have a SES, then tensoring by $\mathcal{L}^\vee$ preserves exactness.

Proposition 3.4. If X is projective and k is infinite, $\mathcal{L}$ sends $\mathrm{K}(X)^{\leqslant d}$ to $\mathrm{K}(X)^{d-1}$.

Proof. Assume that $\mathcal{L}$ is $\mathcal{A} \otimes \mathcal{B}^\vee$, where $\mathcal{A}$ and $\mathcal{B}$ are very ample. Now, since we assume k is infinite, in the embedded projective space, there will be a hyperplane avoiding all the associated points of $\mathcal{F}$ and also avoids the associated points of X, then the corresponding pullback section in $\mathcal{A}$ cuts out an effective Cartier divisor, say D_1, then consider the SES of the effective Cartier divisor:

$$0 \to \mathcal{A}^\vee \to \mathcal{O}_X \to \mathcal{O}_{D_1} \to 0$$

Now we tensor this with $\mathcal{F}$ and obtain a SES:

$$0 \to \mathcal{A}^\vee \otimes \mathcal{F} \to \mathcal{F} \to \mathcal{F}_{D_1} \to 0$$

The left morphism is injective as you can check it locally, recall that $\mathcal{A}^\vee$ locally is generated by one element, which is the restriction of that global section of $\mathcal{A}$, and the morphism is given by multiplication by that restricted section, therefore, since this avoids all the associated points, the map is injective.

This shows that $[\mathcal{F}] = [\mathcal{F}_{D_1}] + [\mathcal{A}^\vee \otimes \mathcal{F}]$. Therefore we know that the image is now:

$$[\mathcal{F}_{D_1}] + [\mathcal{A}^\vee \otimes \mathcal{F}] - [\mathcal{L}^\vee \otimes \mathcal{F}]$$

Moreover, we know that the support of $\mathcal{F}_{D_1}$ is $d-1$ since we know that D_1 avoids all the associated points of $\mathcal{F}$, and in particular all the generic points of the support, thus the support of the restriction is 1 less by Krull's theorem. On the other hand, since $\mathcal{B}$ is also ample, we have a SES:

$$0 \to \mathcal{B}^\vee \otimes \mathcal{F} \to \mathcal{F} \to \mathcal{F}_{D_2} \to 0$$

Tensor with $\mathcal{L}^\vee$, we get:

$$0 \to \mathcal{F} \otimes \mathcal{A}^\vee \to \mathcal{F} \otimes \mathcal{L} \to (\mathcal{F} \otimes \mathcal{L})_{D_2} \to 0$$

Then I will leave the rest of the proof to you. □

For the rest of the section, I will assume that X is projective and k is infinite. Even though these assumptions can be weakened a lot, I will assume this just for simplicity.

Recall that by the above proposition, since $\mathcal{L}\cdot$ will map a sheaf with support of dimension at most a to sheaves with support of dimension at most $a-1$, we know that it descends to the graded piece:

$$\mathrm{A}'_d \to \mathrm{A}'_{d-1}$$

This is because you can first consider:

$$\mathrm{K}^d(X) \to \mathrm{K}^{d-1}(X)/\mathrm{K}^{d-2}(X)$$

Notice that the kernel of this maps contains all those $[\mathcal{F}]$ with support of dimension at most $d-1$, then we can use the universal property of quotient to descend this map. The final map is denoted by $c_1(\mathcal{L})\cap$, and this c_1 here stands for the action of the **first Chern class**.

Proposition 3.5. $c_1(\mathcal{L} \otimes \mathcal{L}')\cap = (c_1(\mathcal{L})\cap) + (c_1(\mathcal{L}')\cap)$.

Proof. We will first prove that:

$$((\mathcal{L} \otimes \mathcal{L}')\cdot) - (\mathcal{L}\cdot) - (\mathcal{L}'\cdot) = -(\mathcal{L}\cdot) \circ (\mathcal{L}'\cdot)$$

This will suffice, since notice that the composition will map something in K^d to K^{d-2} and thus after you pass it to $\mathrm{A}'_d \to \mathrm{A}'_{d-1}$, you just get 0.

To prove this, it's basically just by definition:

$$((\mathcal{L} \otimes \mathcal{L}') \cdot \mathcal{F}) = [\mathcal{F}] - [\mathcal{L}^\vee \otimes \mathcal{L}'^\vee \otimes \mathcal{F}] - [\mathcal{F}] + [\mathcal{L}^\vee \otimes \mathcal{F}] - [\mathcal{F}] + \mathcal{L}'^\vee \otimes \mathcal{F}]$$

On the other hand, on the right hand side, you get:

$$-(\mathcal{L} \cdot ([\mathcal{F}] - [\mathcal{L}'^\vee \otimes \mathcal{F}])) = -([\mathcal{F}] - [\mathcal{L}'^\vee \otimes \mathcal{F}] - [\mathcal{L}^\vee \otimes \mathcal{F}] + [\mathcal{L}^\vee \otimes \mathcal{L}'^\vee \otimes \mathcal{F}])$$

Then you see that they are indeed the same. □

Remark 3.6. This type of trick might remind you of the trick we used when we prove the intersection number is a multilinear function. This is not a coincidence and this remark is here to show you why and how this $c_1(\mathcal{L})$ is connected to the intersection product.

Suppose that $\mathcal{F}$ is a coherent sheaf of support at most dimension n and $\mathcal{L}_1, \cdots, \mathcal{L}_n$ are n invertible sheaves. Now consider:

$$c_1(\mathcal{L}_1) \cap \cdots \cap c_1(\mathcal{L}_n) \cap [\mathcal{F}]$$

This is a class in $\mathrm{A}'_0(X)$ as each time you apply c_1 the dimension of the support drops by 1. If you indeed use the definition to compute this object, for example first in $K(X)$, you will get the class:

$$\sum_{0 \leqslant k \leqslant n} (-1)^k [\mathcal{L}_{i_1}^\vee \otimes \cdots \otimes \mathcal{L}_{i_k}^\vee \otimes \mathcal{F}]$$

which may reminds you of the definition of the intersection product, once you take the Euler characteristic. Before you take the Euler characteristic, this is not a number, but you consider this class, it's represented by class of sheaves whose dimension is at most 0, or in other words, those that are supported in a number of closed points, then recall that Euler characteristic is well defined for classes, thus taking the Euler characteristic of the summation, is the same as taking Euler characteristic of those sheaves whose support is 0-dim. Recall that the cohomology of dimension 0 stuff vanishes if the degree is at least 1, thus in the end, taking Euler characteristic is juts computing the sum of the global dimensions! Therefore, we have upgraded our intersection product from a number to a class in A'_0 and we can recover the old number by just simply taking the Euler characteristic. What we gain here is we initially only have the information of characteristics, but now we actually have some sheaves, meaning we have more information on the intersection rather than just the numbers.

But more on this topic will need you to read more which is beyond the context of this notes.

Now we make some final remark in this section, which are only meant to be enlightening, and we will not and are not able to discuss the details. Let's say something more on this group $\mathrm{A}'_d(X)$. Say you have a morphism of projective varieties:

$$\pi : X_1 \to X_2$$

We can define maps $\pi_* : \mathrm{K}(X_1) \to \mathrm{K}(X_2)$. I will be vague about what this morphism is, but what's important is that it descends to a morphism $\mathrm{A}'_d(X_1) \to \mathrm{A}'_d(X_2)$ and is $\pi_*[\mathcal{F}] = [\pi_*\mathcal{F}]$. This is naturally called the pushforward, and it behaves well with the action of the first Chern class, which can be used to prove the general version of projection formula we talked about before.

If π is flat, then π^* is an exact functor and in this case we can define $\pi^* : \mathrm{K}(X_2) \to \mathrm{K}(X_1)$ and it also play well with Chern classes.

Our group $\mathrm{A}'_d(X)$ is a good approximation of the theory of Chow groups $\mathrm{A}_d(X)$, and there is a surjective map:

$$\mathrm{A}_d(X) \to \mathrm{A}'_d(X)$$

which becomes an isomorphism after you tensor with $\mathbb{Q}$. All of these discussed is just the beginning of the long and interesting story of intersection theory, which need more reading if you want to study it.

4 The Nakai-Moishezon and Kleiman Criteria for Ampleness

References