Appendix 2 for Math 632: 2025 Winter Notes

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1 A Review on Proj Construction

This section is basically an review on the construction of proj of a graded ring, before we take a look at quasicoherent graded modules over a projective scheme and the global curly proj construction.

This proj construction is of importance of for other reasons, as you will begin to see later. Probably one of the most immediate thing we can see is that it gives us something called the **projective space**, which resembles the classic picture you might have in mind. Schemes with desired properties that arise naturally are often projective schemes which are open or closed subschemes of projective spaces(we will review all of these later). Even you you can work very hard to give a finite type scheme over a field which is separated that is not projective, they are highly artificial. They are also related to a lot other notions such as compactness which is properness. But these are later stories, let's first focus on the construction first.

1.1 First Easy construction of $\mathbb{P}^n$

Suppose that k is a field and we have $n+1$ rings indexed by $i = 0, 1, \cdots, n$ of the form $R_i = k[x_{0/i}, \cdots, x_{n/i}]/x_{i/i} - 1$. What we actually have here is $n+1$ polynomial rings of n variables since you find that in each of the above rings, x_{ii} is set to be 1. The reason we use this notation is that we want to borrow our intuition from classic projective space, where each ring R_i will correspond to the projective coordinate where the i-th coordinate is not zero. Therefore, in what follows for each i, we will just use x_{ii} as something that only makes our notation easier, but it's always interpreted as 1, we won't bother to write down $x_{ii} - 1$ in the quotient. Then for each pair of (i, j), we will glue R_i and R_j via the distinguished open $D(x_{i/j})$ and $D(x_{j/i})$. The isomorphism is given by:

$$k[x_{0/i}, \cdots, x_{n/i}, \frac{1}{x_{j/i}}] \cong k[x_{0/j}, \cdots, x_{n/j}, \frac{1}{x_{i/j}}]$$

where the map is given by $x_{k/i} \mapsto x_{k/j}/x_{i/j}$ with inverse $x_{k/j} \mapsto x_{k/i}/x_{j/i}$, which automatically how we are going to match the variable we inverted.

The claim is that this gluing process we defined above cocycle condition. It's not to hard to check this, as we know that intersections will still be affine and every map corresponds to canonically defined k-algebra homomorphisms. The only thing I want to mention here is that checking cocycle condition includes checking that $\Phi_{U_i \cap U_k} \subset U_j \cap U_k$, but here again, everything is affine, and you can check this condition pretty easily.

In conclusion, we get our first definition of the **projective n-space over a field k** , denoted by $\mathbb{P}^n_k$. Also notice that the above argument actually doesn't involve using the fact that k is a field, we can replace k by an arbitrary ring A and the same argument works out verbatim, thus we also have similar notions of projective n-space over a ring A, denoted by $\mathbb{P}^n_P$.

Here is a fun exercise that you can think of, which can be interpreted as some intuition of the Algebraic Hartog's theorem we introduced in the first part of the Appendix.

Proposition 1.1. Show that the global section of $\mathbb{P}^n_k$ is k, thus $\mathbb{P}^n_k$ is not affine.

Proof. You may think this is hard to prove since you have many affine opens glued together, but actually, you only need two of them, as you will see below.

Notice that global functions correspond to a collection of sections on an open cover such that they agree on any two intersections. However, even if we only consider x_0 and x_1 in the coordinate ring, we see that sections on $D_+(x_0)$ and $D++(x_1)$ that restrict to the same element on the intersections are only constants. To see this, the sections on $D_+(x_0)$ is $k[x_{1,0}, \cdots, x_{n,0}]$ and the sections on $D_+(x_1)$ are $k[x_{0,1}, x_{2,1}, \cdots, x_{n,1}]$ and on the intersections since the transition morphism is induced by:

$$k[x_{1,0}, \cdots, x_{n,0}, x_{1,0}^{-1}] \to k[x_{0,1}, x_{2,1}, \cdots, x_{n,1}, x_{0,1}^{-1}]$$

where $x_{j,0} \mapsto \frac{x_{j,1}}{x_{0,1}}$, thus if we have a polynomial $f(x_{1,0}, \cdots, x_{n,0})$ in $x_{1,0}, \cdots, x_{n,0}$, and a polynomial $g(x_{0,1}, \cdots, x_{0,n})$ in $x_{0,1}, \cdots, x_{0,n}$, then we see that on intersections, if we want them agree, we need to set:

$$g(x_{0,1}, \cdots, x_{0,n}) = f(1/x_{0,1}, \cdots, x_{0,n}/x_{0,1})$$

as the polynomial rings are domains, we can clear all the denominators. Therefore, if they want to be equal, we can have any $\frac{1}{x_{0,1}}$ factors, forcing f to be constant, thus g should also be constant. Thus, we see that if they want to correspond to a global section, it should be a constant function, and it's easy to see that all constants do correspond to a global section, thus the global sections are all constants. □

1.2 Some preliminaries on Graded rings

We won't be super scrupulous in this part, we just want to state and prove several important notions and results we will use later.

Definition 1.2. Suppose that $S_\bullet$ is a $\mathbb{Z}$-graded ring. An ideal I is called a **homogeneous ideal** if it can be generated by homogeneous elements.

Several equivalent characterization of homogeneous ideals are:

Proposition 1.3. An ideal I is homogeneous iff for any $a \in I$, since we can write a uniquely as sums of homogeneous elements which we will call the degree n piece of a, I contains each degree n piece of a.

Proof. Suppose that I is homogeneous, then suppose that (f_i) generates I each f_i is homogeneous, therefore each a can be written as $a = \sum b_i f_i$, by decompose b_i into homogeneous pieces, we can assume that each b_i is also homogeneous, thus since a is uniquely decomposed, terms in $\sum_i b_i f_i$ are the homogeneous pieces of a, since $f_i \in I$, then each homogeneous piece of a lies in I. Conversely, we know that I is generated by the set $(f_{a,i})$ where $a_i \in I$ are arbitrary elements and f_{a_i} are the homogeneous pieces of a_i. □

Corollary 1.4. An ideal I in $S_\bullet$ is homogeneous iff there is a direct sum decomposition $I = \bigoplus_n I_n$ where $I_n \subset S_n$.

Proof. If $I = \bigoplus_n I_n$, obviously I is homogeneous. Suppose that I is homogeneous, then if we define $I_n = S_n \cap I$, we see that $I_n \cap I_m = 0$ since $S_n \cap S_m = 0$, on the other hand, we see I is the sum of all I_n since for each $a \in I$, we can write it as sum of homogeneous pieces and each of them is in I_n. This implies that $I = \bigoplus_n I_n$, we are done. □

Therefore, in light of the above result, $S_\bullet/I$ will also have a natural $\mathbb{Z}$-grading, this is why we call I homogeneous ideal.

Proposition 1.5. The set of homogeneous ideals in $S_\bullet$ is closed under sum, product and taking radical.

Proof. Closed under sums is easy to show and we omit it. Product is also easy. To show this also holds for radicals, suppose that $a \in Rad(I)$, then there is a n such that $a^n \in I$, then we write a as sum of homogeneous pieces a_i then easy computation helps us to see that each $a_i^n \in I$, we are done. □

Suppose that T is a multiplicative subset of $S_\bullet$ that only contains homogeneous element, $T^{-1}S_\bullet$ has a natural $\mathbb{Z}$-grading structure. We define the degree n-piece to be all elements of the form $\frac{s}{t}$ where $degs - degt = n$, if it's a direct sum decomposition, it's easy to see that this is a grading. Thus, we only have to show this is indeed a direct sum decomposition, which is not hard and omitted here.

Now continue our discussion of graded rings, now fix a ring A, which will be called a **base ring**, if $S_0 = A$, we will say that $S_\bullet$ is a **graded ring over** A, if $S_\bullet$ is generated by S_1 as a A-algebra, we will say that $S_\bullet$

is **generated in degree** 1, the subset S_+ which is defined by $S_+ = \oplus_{i>0} S_i \subset S_\bullet$ is called the **irrelevant ideal**. If the irrelevant ideal S_+ is a finitely generated ideal, we say that $S_\bullet$ is finitely **generated graded ring over** A. A quick observation is that $S_\bullet/S_+$ is isomorphic to $A = S_0$, which we will use later.

Here, this finitely generated graded ring is a little bit unusual since we are characterizing it using it's irrelevant ideal, below is a proposition that makes it more familiar:

Proposition 1.6. $S_\bullet$ over A is a finitely generated graded ring iff $S_\bullet$ is a finitely generated graded A-algebra i.e. generated over A by a finite number of elements of positive degree. In particular, we will see that the element that generates the ideal and the elements that generates S as an algebra can be the same.

Proof. Suppose that S_+ is generated as an ideal by (f_i), by decomposing f_i into it's homogeneous pieces, we can assume that these are all homogeneous elements of positive degree. Then, we claim, these elements generate $S_\bullet$ as an A-algebra. To see this, suppose that $s \in S_+$ is an homogeneous element, here we don't have to worry about $A = S_0$ since we are over A, then since S_+ is generated by a f_i's as an ideal, we can assume that $s = \sum_i f_i y_i$ where y_i's are also homogeneous elements, then we keep writing y_i's as a summation of $f_i z_i$, with z_i of lower degree, then we will reach degree 0 in the end, thus s can be written as a polynomial of f_i's with coefficients in A, thus we are done.

Conversely, this is just a trivial implication. $\square$

One immediate corollary is that:

Proposition 1.7. A graded ring $S_\bullet$ over A is Noetherian iff A is Noetherian and $S_\bullet$ is a finitely generated graded ring.

Proof. If A is Noetherian and $S_\bullet$ is finitely generated, then the above proposition says that it's the quotient of a polynomial ring, thus Noetherian since A is Noetherian by Hilbert's Basis theorem. Conversely, if $S_\bullet$ is Noetherian, then we know that S_+ as an ideal is finitely generated, thus we know that $S_\bullet$ is finitely generated as an A-algebra. Again, notice that $A = S_\bullet/S_+$, thus Noetherian. $\square$

Sometimes we also care the special case of R is not only finitely generated, but it's generated in degree 1. By our above reasoning, we see that the elements that generates S as an algebra also generate the irrelevant ideal, later you will see that they will form an open cover.

Now, let's begin the construction of Proj, where we will follow the normal path to first define it as a set, then topological space, then finally give it a structure sheaf. The first step is to describe it as a set:

Now, we fix a $\mathbb{Z}_{\geqslant 0}$-graded ring $S_\bullet$. For each homogeneous $f \in S_+$, we already know that there is a natural $\mathbb{Z}$-grading on $(S_\bullet)_f$. Consider the 0-th graded piece of $(S_\bullet)_f$, which is also a ring and will be denoted by $((S_\bullet)_f)_0$, this notation is horrible and to simplify this, we will drop the $\bullet$ notation since in this section we will always interpret S as a $\mathbb{Z}_{\geqslant 0}$-graded ring. We will further simplify it by writing $S_{(f)}$, instead of S_f to imply that we are taking the 0-th graded piece. These $S_{(f)}$, when f ranges over all homogeneous elements of positive degree, will be the affine build blocks of $ProjS$ we will see later, but let's wait a bit longer to do some preparations.

Below is a tricky proposition that is essential for the construction, we won't write out every detail, instead, we will gives the steps you can follow:

Proposition 1.8. Suppose that $f \in S$ is homogeneous of positive degree. There is a bijection between homogeneous prime ideals of S_f and prime ideals of $S_{(f)}$.

Proof. Directly prove this with this notation is to confusing and not looking at the nature of this proof. Instead, let's consider the situation where we have a $\mathbb{Z}$-graded ring A with $f \in A$ which is a homogeneous element of positive degree which is invertible. Notice that the inclusion of $A_0 \to A$ gives us a map from the homogeneous ideals of A to primes of A_0 by taking inverse image, or in other words taking intersection. We claim that this is a bijection by giving it's inverse. Suppose that $P_0 \subset A_0$ is a prime ideal, we define $Q \subset A$ to be $Q = \oplus_i Q_i$, where $Q_i \subset A_i$ and $a \in Q_i$ if and only if $\frac{a^{degf}}{f^i} \in A_0$. On of the first thing you should be able to verify is that $Q_0 = P_0$ and this is indeed a direct sum. Then you will want to prove that $a \in Q_i$ if and only if $a^2 \in Q_i$, then if $a_1, a_2 \in Q_i$, $(a_1 + a_2) \in Q_i$. Use this you can prove that Q is indeed an ideal, then since we have a decomposition, it's a homogeneous ideal. Finally, try to show that Q prime. Since it's homogeneous, we only have to check the product of homogeneous elements.

Finally, it's easy to check these are inverses of each other, we are done. Finally, we apply our discussion above to S_f and $S_{(f)}$. $\square$

This make it possible to talk about $SpecS_{(f)}$ as a **subset** of $ProjS$: First, we claim that homogeneous prime ideals of S_f correspond to homogeneous prime ideals of S that doesn't contain f. We have seen similar statement in the affine case, here we still consider the map $S \to S_f$, suppose that there is a homogeneous prime ideal in S, we see that the extension is also homogeneous since it's generated by homogeneous elements, it's also prime. Then suppose that there is a homogeneous prime in S_f, it's first correspond to a prime in S that doesn't contain f, it's also homogeneous since if f is in it, we write f in homogeneous prices, then the image of each piece should lie in the homogeneous ideal we started with, thus each homogeneous piece of f is in the inverse image, thus it's homogeneous.

After we have the above proposition, suppose that we define the set $ProjS$ to be the set of all homogeneous prime ideals in S not containing the irrelevant ideal, or in other words, all the relevant homogeneous prime ideals. Suppose that T is a set of homogeneous elements in S, we will define the **vanishing set** of T to be $V(T) \subset ProjS$ to be the homogeneous prime ideals containing T but not S_+, in particular if T contains S_+, $V(T) =$. We will define for a homogeneous element f of positive degree in the obvious way. Let $D_+(f) = ProjS - V(f)$, which are called the **projective distinguished open set**, it will indeed turns out to be open once we defined the topology on $ProjS$.

The reason why we don't allow f to have degree 0 is not clear right now since it completely makes sense if we define $V(f)$ and $D_+(f)$ for degree 0 elements. One thing we can say is that you will see later even if we don't consider those degree 0 elements f, we still have an affine open cover $D_+(f)$, but if we consider $D_+(f)$ for f of degree 0, this may not be affine.

Notice that by our discussion above $D_+(f)$ correspond to $Spec(S_{(f)})$ in a **one to one** fashion.

Proposition 1.9. The set of all $V(I)$'s satisfies the axiom of the closed subsets in a topological space where I's are homogeneous ideals. Thus $ProjS$ is a topological space, we will call this topology the **Zariski Topology** on $ProjS$.

Proof. This is proved in the same manner as in the affine case, the restriction of homogeneous ideals doesn't complicate the argument since we already showed that homogeneous ideals are closed under radicals, products, sums. Also, similar kind of argument show that $D_+(f) \cap D_+(g) = D_+(fg)$ $\square$

Another easy result is that $D_+(f)$ where f runs through all homogeneous elements of S_+ form a basis of the Zariski topology, again, notice that still we don't have to consider degree zero f, even though the definition completely makes sense. The proof is still the same as in the affine case, where $ProjS - V(I)$ is the union of $D(f_i)$ where I is a homogeneous ideal and f_i are all the homogeneous elements in I. Notice that we actually only have to consider f_i with positive degree. Notice that for those $f_i \in I$ such that f_i is in S_0, the set $V(f_0)$ is the homogeneous primes containing f_i but not S_+, notice that (f_i) is not a prime ideal containing S_+, otherwise I will contain S_+, we don't have any to prove. But if (f_i) doesn't contain S_+, notice that if we decompose (f_i) into $(f_i) = \oplus_i F_i$, notice that $F_i \subset I$ for all i, and we can also assume that f_i is not zero, otherwise we don't have anything to prove either. Therefore, we can assume that there is at least one of F_i is not zero, thus choose any one of those that not zero, pick g in it, it's also in I, thus if $p \in V(f_i)$, which means that $f_i \in p$, thus $g \in p$, we see that $p \in V(g)$, therefore, all those $V(f_i)$ where f_i is of degree 0, primes in $V(f_i)$ is also in other $V(g)$ where $g \in I$ and is of positive degree, therefore:

$$V(I) = \bigcap_{f_i \in I, homogeneous} V(f_i) = \bigcap_{f_i \in I, homogenes, deg f_i > 0} V(f_i)$$

This is because that if p is in the right hand side, but not the left hand side, there is a $f \in I$ degree 0 such that $f \notin p$, but $f \in I \subset p$, a contradiction. This implies that $D(f)$ with f positive homogeneous elements are actually a basis.

Here is another exercise that resembles what we did in the affine case:

Proposition 1.10. 1. Suppose that I is any homogeneous ideal of S contained in S_+ and f is a homogeneous element of positive degree, then f vanishes on $V(I)$ iff $V(I) \subset V(f)$ there is an n such that $f^n \in I$. In particular, suppose that f, g are both homogeneous element of positive degree, just as in the affine case, $D_+(f) \subset D_+(g)$ iff $V(f) \supset V(g)$ iff there is a n such that $f^n \in (g)$ and vise versa.

2. If $Z \subset ProjS$ is any set, we define $I(Z)$ to be the subset in S_+ where it's defined to be the intersection of all the homogeneous primes in Z such that it's contained in S_+. We claim that it's a homogeneous prime contained in S_+ and for two Z_1, Z_2, we have $I(Z_1 \cup Z_2) = I(Z_1) \cap I(Z_2)$.

3. For any subset $Z \subset ProjS$, we have $V(I(Z)) = \overline{Z}$.

Proof. We won't write out all the details. First, the hint is that for a homogeneous ideal, it's radical can be written as the intersection of all the homogeneous primes containing it. First, notice that it definitely contains the intersection of all the primes, but for any prime, we take out all the homogeneous element, it generates a homogeneous ideal, certainly contains I, it's an easy exercise to show this is prime. Therefore, the intersection of all the primes containing I can be written as all the homogeneous primes containing it. The second part is obvious and we won't comment too much. For the last one, the left hand side obviously contains the right hand side, to show the reversed containment, we need to show that suppose that $V(J)$ contains Z, then any $p \in V(J)$, it's in $V(I(Z))$, this omitted. □

The following result is also a little bit cumbersome, but it explains why we call S_+ the irrelevant ideal to some extent.

Proposition 1.11. Suppose that S is a graded ring and $I \subset S_+$ is a homogeneous ideal. Then the following are equivalent:

1. $V(I) = \varnothing$.

2. For any homogeneous set of f_i that generates I, $\bigcup_i D_+(f_i) = ProjS$.

3. $\sqrt{I} \supset S_+$.

Proof. We still only comment on the ideas. 1 to 2 is trivial. 2 to 1 is also easy, as $I = (f_i)$. Then we prove that 1 and 3 are equivalent, notice that each $f \in S_+$ homogeneous, we see that $V(I) \subset V(f_i)$, thus $f_i \in \sqrt{I}$, conversely, $V(I) = V(\sqrt{I})$ for our earlier characterization of the radical using intersection of homogeneous primes, then $V(\sqrt{I}) \subset V(S_+) =$. □

Finally, let's construct $ProjS$ as a scheme by endow it with a structure sheaf. Let's first describe a topological embedding of $Spec(S_{(f)})$ into $D_+(f) \subset ProjS$ that we will use later:

Proposition 1.12. Under our identification of $Spec(S_{(f)})$ with $D_+(f)$, this inclusion is a topological embedding.

Proof. We still won't write out everything, but the goal is to proof that for each homogeneous ideal I not containing S_+, we want to show that under this inclusion, $D_+(f) \cap V(I)$ correspond to $V(I_{(f)})$ in $Spec(S_{(f)})$. The details is omitted here, as long as you still remember how we constructed the identification, it's just a matter of checking definition. □

One immediate corollary is that when we take the subspace topology on $D_+(f)$, the map above is a homeomorphism. Another fact we need later when we glue stuff is:

Proposition 1.13. Suppose f, g are two homogeneous element of positive degree, then $D_+(f) \cap D_+(g)$ has inverse image, under the previous homeomorphism we defined, $D(\frac{g^{degf}}{f^{degg}})$, as topological spaces.

Proof. Again, this seems indeed obscure, but I promise you, don't get scared out, try to write down what this means and remember the identification we have in Prop 1.8, you will figure it out. □

Then, we want to set the structure sheaf on $D_+(f)$ to be $Spec(S_{(f)})$, and then glue them together, but to do so, we need a preparation for the gluing to work out:

Proposition 1.14. Let $f, g \in S_+$ be homogeneous and nonzero, there is a scheme isomorphism between $Spec(S_{(fg)})$ and the distinguished open subset $D(\frac{g^{degf}}{f^{degg}})$ of $Spec(S_{(f)})$.

Proof. This is definitely a pain to check, however, the idea is straight forward, since we are working over affines, we only need to give isomorphisms of rings. Consider the following diagram:

$$\begin{tikzcd} S_{(f)} \arrow[d] \arrow[dr] & \\ S_{(f)}|_{\frac{g^{degf}}{f^{degg}}} \arrow[r, dashed] & S_{(fg)} \end{tikzcd}$$

Use universal property of localization to give a isomorphism from $S_{(f)}$ to $S_{(fg)}$ in a way that compatible with the 0-degree piece. Then use it again to give the horizontal map, it's a pain but straightforward to show this is a bijection. □

Similarly, we can identify $Spec(S_{(fg)})$ with the distinguished open of $D(\frac{f^{degg}}{g^{degf}})$ of $Spec(S_{(g)})$. Now, we begin the most confusing gluing part, here the argument is miserable since we are actually using the homeomorphism $Spec(S_{(f)})$ to $D_+(f)$ to define the sheaf on $D_+(f)$ by the pushforward sheaf. Notice that topologically, as what we proved earlier, the intersection $D_+(fg)$, will be identified topologically as the distinguished open $D(\frac{g^{degf}}{f^{degg}})$, thus has the pushforward sheaf. The above proposition is saying that this restriction sheaf is isomorphic with the sheaf we directly defined on $D_+(fg)$, which is the pushforward sheaf of $Spec(S_{(fg)})$, this will serve as the isomorphism we use to glue sheaves, where the isomorphism is connected by the common third $Spec(S_{(fg)})$. Next, we need to check the cocycle condition is satisfied, the following proposition also serves as the definition of $ProjS$

Proposition 1.15. Under the open cover $D_+(f)$ where f ranges through all positive degree homogeneous elements, the sheaves we defined on these open covers are equipped with isomorphisms satisfies the cocycle condition on triple intersections, thus they glue to a sheaf on $ProjS$ such that on each each $D_+(f)$, the restriction sheaf is isomorphic to the pushforward sheaf of $Spec(S_{(f)})$ under the homeomorphism we defined. Therefore, we will identify the sheaf on $D_+(f)$ as $Spec(S_{(f)})$ as long as we keep track of the correspondence between sets under the homeomorphism. This sheaf is unique by a unique isomorphism under those sheaves on $ProjS$ such that the restriction on $D_+(f)$ is $Spec(S_{(fg)})$ and the gluing property is respected.

Proof. We only need to check the cocycle condition, which in the end, since we are dealing with affines, boils down to checking the following diagram of rings is commutative:

$$\begin{tikzcd} S_{(f)}|_{\frac{(gh)^{degf}}{f^{degh+degg}}} & S_f|_{\frac{g^{degf}}{f^{degg}}} \arrow[l, "loc"'] & S_{(g)}|_{\frac{f^{degg}}{g^{degf}}} \arrow[l, "iso"'] \\ S_f|_{\frac{h^{degf}}{f^{degh}}} \arrow[u, "loc"] & & S_{(h)}|_{\frac{f^{degh}}{h^{degf}}} \arrow[u, "iso"'] \arrow[ll, "iso"] \end{tikzcd}$$

But as you can see, everything is canonically defined, thus you can easily check that they commute. □

Definition 1.16. We define **projective space over a ring** A by:

$$ProjA[x_1, \cdots, x_n]$$

The coordinates x_i are called **projective coordinates**.

It's a useful exercise to see why this definition agrees with our previous construction:

Proposition 1.17. Show that this new definition agree with our previous definition of $ProjA[x_1, \cdots, x_n]$.

Proof. The question is how to establish an isomorphism, of course we have to work locally and then glue things together. The first question we need to address is $D_+(x_i)$ will cover $ProjA[x_1, \cdots, x_n]$ in our new definition. □

Definition 1.18. If S is a finitely generated graded ring over A, we will call $ProjS$ a **projective scheme over** A, or a **projective** A**-scheme**. A **quasiprojective** A**-scheme**, is a quasicompact open subscheme of a projective A-scheme.

A silly example you should know is:

Example 1.19. $\mathbb{P}^0_A = ProjA[T] \cong SpecA$. This is just by definition, $D_+(T)$ will cover $ProjA[T]$ since the irrelevant ideal is just generated by T. But notice that over $D_+(T)$, we know that it's $Spec(A[T]_{(T)}) \cong SpecA$ since $A[T]_{(T)} \cong A$.

Here is a nontrivial example that we will use later, called the **projectivization of a vector space**:

Example 1.20. Let V be a vector space of dimension $n+1$ over a field k, also you can think of this as a ring and a free module. We have the dual of the vector space $V^\vee$, then we form the symmetric algebra $Sym^\bullet V^\vee$. We know that this is a graded ring, then we can define $Proj(Sym^\bullet V^\vee)$, which we will denote as $\mathbb{P}V$, the **projectivization** of V. Notice that if we have a basis of $V^\vee$ associated to a basis $x_1, \cdots, x_n$ in, we have an isomorphism of $Sym^\bullet V^\vee$ with $k[x_1, \cdots, x_n]$. One minor comment is that in the review before, we used V instead of $V^\vee$, but the polynomial ring structure still make sense and it even works better if you think of x_1 to x_n as linear functionals.

The last example we want to give here is how to identify the **closed points** when k is algebraically closed.

Example 1.21. First, our construction of this $Projk[x_0, \cdots, x_n]$ is isomorphic to the first construction we have of just gluing just $n+1$ copies of affine spaces, thus we want to first show that the closed points of this easy version is in bijection with the classical points in the projective spaces.

Now, let's also review how to give morphisms between projective spaces, which will resemble the construction of specs, though it's not an entirely and not a perfectly generailization of the morphisms between affine spaces. As you already know, we have a one to one correspondence between morphisms of affine spaces and rings maps going the opposite direction. The construction here is less nice in the sense that first, not every map of gradeding rings will yield a map of projective schemes and not every morphism of projective schemes comes from a map of graded rings and different maps of graded rings could yield the same morphism of projective schemes. But it's still useful to know this since it can actually helps you to define maps. The following is the key construction:

Definition 1.22. Suppose that $\Phi : S_\bullet \to R_\bullet$ is a **morphism of $\mathbb{Z}^{\geqslant 0}$ graded rings**. This means that this is a map of rings which respects the grading in the following sense: there is a $d > 0$ such that S_n is mapped to R_{dn} for all n where S_n and R_{dn} are the graded pieces of S and R. This makes sense since as you can check that $a \cdot b$ where $a \in S_n$ and $b \in S_m$ is mapped to $R_{d\cdot(m+n)}$.

Proposition 1.23. The above map of graded rings induces a morphism of schemes:

$$Proj(R_\bullet) - V(\Phi(S_+)) \to Proj(S_\bullet)$$

.

Before we prove this, the reviewing of how morphisms of ringed spaces glue is helpful:

Proposition 1.24. Suppose that $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are ringed spaces. Let U_i be an open cover of X and we have morphisms of ringed spaces $\pi_i : U_i \to Y$ that agree on the overlaps, then there is a unique morphism of ringed spaces $\pi : X \to Y$ such that the restriction to U_i is π_i.

Proof. **** later* □

Now let's prove Prop 1.23:

Proof. The idea of the proof is not that complicated, as we usually just define the maps locally and show that they glue together. And we only have a good understand of $Proj$ over $D_+(f)$, which is fortunately affine. Therefore, we will try to define for each $f \in S_+$, a homogeneous element, a morphism $D_+(\Phi(f)) \to D_+(f)$ and then show that they glue together.

Clearly, the map from $D_+(\Phi(f)) \to D(f)$ should be defined by the following map ring map induced by $S \to R$:

$$S_{(f)} \to R_{(\Phi(f))}$$

Then we need to check that they agree on intersections: Suppose that f, g both in S_+ homogeneous, then we have the corresponding maps, we claim that checking that they do agree reduces check that the middle square of the following diagram commute:

$$\begin{tikzcd}
& S_{(f)} \arrow[r] \arrow[dl, "loc"'] & R_{(\Phi(f))} \arrow[dr, "loc"] & \\
S_{(f)|_-} \arrow[rrr, dashed] \arrow[d, leftrightarrow, "iso"'] & & & R_{(\Phi(f))}|_- \arrow[d, leftrightarrow, "iso"] \\
S_{(g)}|_- \arrow[rrr, dotted] & & & R_{(\Phi(g))}|_- \\
& S_{(g)} \arrow[r] \arrow[ul, "loc"] & R_{(\Phi(g))} \arrow[ur, "loc"'] &
\end{tikzcd}$$

It's important to figure out why this commutative diagram is enough rather than checking it, since to be honest, there is a lot annoying details. Fortunately, when things are nice enough, for example when you are over a field and has polynomial rings, it's easy to check this. Therefore, you onle have to figure out why this works, in simple words, the vertical isomorphism is the transition function, and you should use your smooth manifold experience.

It's also important that if both R and S are A-algebras, then the construction clearly shows that the resulted map is a morphism of A-schemes. And notice that if $V(\Phi(S_+))$ is empty, then this morphism has definition on all of $ProjR_\bullet$, this possible locus where it has no definiton is due to the fact that we are only defining this morphism on $\bigcup D(\Phi(f))$ for $f \in S_+$ a homogeneous element. Some special cases where this does happen is where the radical of $\Phi(S_+)$ is the whole irrelevant ideal or $\Phi(S_+)$ itself contains the irrelevant ideal. $\square$

Now, let's look at some examples:

Example 1.25. We will try to scheme theoretic interpret the map of complex projective manifolds $\mathbb{CP}^1 \to \mathbb{CP}^2$ given by $[s,t] \mapsto [s^{20}, s^9t^{11}, t^{20}]$. Notice that this is well defined in the sense that scalar multiples doesn't affect the definition.

We claim that this, algebraically, correspond to a map of graded rings in the opposite direction $\mathbb{C}[x,y,z] \to \mathbb{C}[s,t]$ defined by $x \mapsto s^{20}$, $y \mapsto s^9t^{11}$ and $z \mapsto t^{20}$.

Notice that there is no map of complex manifold $\mathbb{CP}^2 \to \mathbb{CP}^1$ given by $[x,y,z] \mapsto [x,y]$, because the map is not defined when $x = y = 0$. As you can also see, the fact that the ring map $\mathbb{C}[s,t] \to \mathbb{C}[x,y,z]$ given by $s \mapsto x$ and $t \mapsto y$ doesn't satisfy the hypothesis that $V(\Phi(S_+))$ is empty, as you can see easily, (x,y) is a homogeneous prime ideal in $V(\Phi(S_+))$.

Recall the **projectivization** of a vector space we discussed before:

Proposition 1.26. Suppose that we have a injective linear map $V \hookrightarrow W$, then it induces a closed embedding $\mathbb{P}V \to \mathbb{P}W$. This is another reason why we used the dual to define the projectivization.

Proof. Notice that if you consider the dual of this map, it's going to be surjective. Then we claim that it induces a surjective ring homomorphism:

$$Sym^\bullet W^\vee \to Sym^\bullet V^\vee$$

This is induced by the universal property of Sym, and it's easy to see that it's surjective. Thus this induces a closed embedding of the proj. $\square$

Definition 1.27. The closed subscheme we had in the proposition above is called a **linear space**. After we review the dimension theory, we will call this $\mathbb{P}V$ a linear space of dimension $dimV - 1 = dim\mathbb{P}V$. More explicitly, notice that we mentioned before when you have V with basis associated with $x_1. \cdots, x_n$, we recover $\mathbb{P}^n$ by considering $\mathbb{P}V$. Then a linear space of dimension dimension n in $\mathbb{P}^N$ is the closed subscheme cut out by $N - n$ k-linearly independent homogeneous linear polynomials in $x_1, \cdots, x_N$. In particular, a linear space of dimension 1 is called a **line**.

Remark 1.28. Let's make a brief remark on how these two notions of the vanishing scheme cut out by $N-n$ linear homogeneous polynomials and the notion of $\mathbb{P}V \hookrightarrow \mathbb{P}W$ related to each other.

Now, as a practice of the above notion of linear spaces, we look at a very special case of **Bezout's Theorem**:

Example 1.29. This is a very special case of Bezout's theorem we will deal with later. Suppose that X is a degree d hyper surface of $\mathbb{P}_k^n$ cut out by a degree d homogeneous ideal $f = 0$. Suppose that l is a line not contained in X, then show that X and l meet with multiplicity d. **** more to say on this after we finish bezout theorem*

Now let's discuss a particular nice case of projective spaces where the graded ring is finitely generated in degree 1:

Proposition 1.30. Suppose that S is a A-graded ring and it's finitely generated as an A-algebra in degree 1 by $n+1$ elements $x_0, x_1, \cdots, x_n$, then $ProjS$ can be described as a closed subscheme of $\mathbb{P}_A^n$.

Proof. The proof is not involved. Consider the free A-module $A^{\oplus n+1}$ with $n+1$ generators associated to $x_0, \cdots, x_n$, then we know that the symmetric algebra $Sym^\bullet(A^{\oplus n+1}) \cong A[t_0, \cdots, t_n]$. The universal property of symmetric algebra tells us that we have a surjection:

$$Sym^\bullet(A^{\oplus n+1}) = A[x_0, \cdots, x_n] \to S$$

Then this induces a closed embedding $ProjS \to \mathbb{P}_A^n$. Then consider the fact we talked about, we always know that if S is finitely generated, we can always assume that it's finitely generated in degree 1, thus this is a general construction. ☐

Remark 1.31. This is like that if we know that if S is finitely generated A-algebra, then $SpecS$ is a closed subscheme of $\mathbb{A}_A^n$, by choosing coordinates. Here, we have the projective analogous statement where we can also view $ProjS$ as a closed subscheme of the projective space, by choosing projective coordinates(one more coordinate).

There is also an important example called the **Veronese embedding**, where it's good to introduce a related notion called the veronese subring:

Suppose that $S_\bullet$ is a finitely generated graded ring, define the n-th **Veronese subring** of $S_\bullet$ by $S_{n\bullet} = \oplus_{j=0}^{\infty} S_{nj}$, where we are only taking out the pieces whose degree is a multiple of n. To simplify the notation, we will just use S_n in this section to denote $S_{n\bullet}$, but when it's not clear what is S_n later in other parts of the notes, will explicitly write out the $\bullet$ to indicate the grading. Below are a few easy properties of Veronese subrings:

Proposition 1.32. The inclusion of rings, which is also a map of graded rings induces an isomorphism $ProjS \to ProjS_n$.

Proof. This proposition actually has two parts. First, we need to show that it induces an global morphism and we need to show that it's an isomorphism. We will denote the ring inclusion by i

First, we claim that $Rad(i(S_{n+})) = S_+$, which implies that we have a global morphism. To see this, notice that suppose that $f \in S_+$, we want to show that there is a natural number n such that $f^n \in S_{n+}$, we may asssume that f is homogeneous, then this is trivial. Suppose that $f \in Rad(i(S_{n+}))$, then write out it's homogeneous pieces, we only have to show that it doesn't contains degree 0 part. We already proved that all positive degree homogeneous elements is in the radical, thus if we have a degree 0 element, it's also in the radical, but there is an easy contradiction from this.

Another way to see this, which is also helpful for us to prove the isomorphism is that: suppose that $f \in S_+$, then we see that $f^n \in S_{n+}$, and also, we see that $D_+(f^n) = D_+(f)$ in $projS$ since you can just think of it as the intersection of $D_+(f)$ with itself for n times. Thus, we can cover $ProjS$ by $D_+(f)$ where f is in S_{n+}, homogeneous.

Then the isomorphism is easy to see, we only have to prove that for each $f \in S_{n+}$, homogeneous, we have a natural isomorphism of rings induced by localization:

$$S_{n(f)} \to S_{(f)}$$

First, this is injective since localization is exact, it's surjective since if $\frac{a}{f^r}$ is in $S_{(f)}$, then we see that the degree of a is $rdegf$ where $f = an$ for some a positive natural number, thus $a \in S_{+n}$, thus surjective. $\square$

Proposition 1.33. If S is generated in degree 1, then S_n is also generated in degree 1.

Proof. Suppose that $f \in S_n$, which is not in degree 0, we only have to consider those homogeneous elements, notice that it's also in S, thus, since S is generated in degree 1, we can assume that it's a polynomial of elements in degree 1 with coefficents in degree 0. Suppose that $deff = mn$, we know that each terms in the polynomial contains exactly mn degree 1 elements, thus can be rearranged to form a polynomial of coefficients in degree 0 and with variables homogeneous positive degree elements of S_{n+}. $\square$

We have some rather enlightening propositions:

Proposition 1.34. Suppose that S and R are both finitely generated over a base ring A such that if we write $S = \oplus S_n$ and $R = \oplus R_n$, there is a positive natural number N such that we have a ring homomorphism $S \to R$ where it's restriction to each graded piece is an ismorphism of abelian groups passing N, then $ProjS \cong ProjR$.

Proof. Notice that if we consider S_N and R_N and consider the restriction of the ring homomorphism, it's an isomorphism. $\square$

The following result shows that there is little harm when assume that a finitely generated graded ring is generated in degree 1.

Proposition 1.35. Suppose that S is generated by $f_1, \cdots, f_n$. There is a d such that S_d is finitely generated in the new degree 1 (old degree d).

Proof. Suppose that these generators are of degree $d_1, \cdots, d_n$, we claim that $d = nd_1 \cdots d_n$. To see this, suppose that we have an element x in this new ring. We can assume that it's homogeneous and of positive degree. Then we can assume that it's a monomial $f_1^{a_1} \cdots f_n^{a_n}$, then we know that at least one of a_i has to be larger than $\prod_j d_j/d_i$ otherwise the degree can not sum up to $d = nd_1 \cdots d_n$ and it's generated by all possible combination of $\prod f_i^{a_i}$ with degree $nd_1 \cdots d_n$(only finite of them). Suppose that we have any homogeneous elements $g \in S_{d+}$, notice that if it's in the new degree 1, it's trivial that we can write it in the form of the desired generators. Then let's do induction, suppose that we have this for degree less than $k-1$, consider the degree k, notice that we can factor out a $f_i^{\prod d_j/d_i}$ for some i and after we factor out it, the degree is still larger than $nd_1 \cdots d_n$, we do this n times the stuff we factored out is of degree $nd_1 \cdots d_n$, thus we are back to the hypothesis, we are done. $\square$

Remark 1.36. This above result, plus the fact that the proj of Verosene subring is isomorphic to the original proj, tells us that we can always assume that a finitely generated graded ring is generated in degree 1 if we only care about the proj.

Now let's start the construction of Veronese embedding: We begin with an easy example, where if we have $S = k[x, y]$, we have $ProjS = \mathbb{P}^1$. Then the 2-th Veronese subring is $k[x^2, xy, y^2]$, it's easy to see that it's isomorphic to $k[u, v, w]/(uw - v^2)$, thus we have a new graded ring which is generated by degree 1 elements, thus we can think of it sitting as $\mathbb{P}^2$ as a conic by taking the closed embedding we mentioned above for graded rings generated in degree 1. This observation plus the following algebra exercise gives an interesting result:

Lemma 1.37. Suppose that k is an algebraic closed field whose characteristic is not 2. Then we can diagonalize any quadric forms in n-variables by a change of variable to be the sum of at most n squares where the linear forms appearing in the squares in these squares are independent.

Proof. Let's do this by induction of the number of variables n. When $n = 1, 0$, it's trivial. Suppose that we are now in the case of $n = 2$, then since we can always complete the square, we can always write it as up to 2 squares, and and if the linear terms are dependent, then you can just absorb it into another term. Then let's do the general construction, suppose that this holds up to $n - 1$, for n, consider we have a quadric of the form:

$$Q(x_1, \cdots, x_n) = ax_1^2 + x_1 B(x_2, \cdots, x_n) + R(x_2, \cdots, x_n)$$

where B is a linear term, and R is still a quadric form, then if we make the change of variables $y_1 = (x_1 + \frac{B}{2})$, then we can form $ay_1^2 + R'(x_2, \cdots, x_n)$, then we can apply the hypothesis to R', since the linear terms in the result we obtained from R' only contains $x_2, \cdots x_n$ are linearly independent and doesn't contain y_1, we are done.

Actually, we can prove even a little bit more, we can show that the number of squares we need to express this quadric only depends on the quadric itself. This is called the **rank of this quadric form**. If the number we need is equal to the number of the variables, we will say that we have full rank. One easy way to see this is by writing this quadric form as $(x_1, \cdots, x_n)M(x_1, \cdots, x_n)^t$, and we see that a change of variable correspond to $(y_1, \cdots, y_n)TMT^T(y_1, \cdots, y_n)^t$ where T is the transition matrix between the two basis, then since the rank of M doesn't change under multiplying invertible elements, we are done.

One special case we will use is that $uv - w^2 = ((u+v)/2)^2 + (i(u-v)/w)^2 + (iv)^2$, here i is still the square root of -1. $\square$

The above result, show that the quadrics (conics) in $\mathbb{P}^2$ only comes in three forms, which are sums of three or two squares or just one simple square. Thus, consider the example $uv - w^2$ we mentioned, notice that any plan conic that can be written as the sum of three squares can be written as $uv - w^2 = ((u+v)/2)^2 + (i(u-v)/w)^2 + (iv)^2$ by a change of coordinate (make this precise by giving a ring isomorphism), thus it's isomorpic to $\mathbb{P}^1$.

Corollary 1.38. The above argument shows that if k is an algebraically closed field whose characteristic is not 2. Then any conic in $\mathbb{P}_k^2$ which can be written as the sum of three square of linear terms that are independent is isomorphic to $\mathbb{P}_k^1$.

Proof. The previous paragraph is the proof. $\square$

Now we soup up the example:

Example 1.39. We still take $S = k[x, y]$, we will show that $ProjS_d$ is given by the equations that all the 2×2 minors of the following matrix vanish:

$$\begin{bmatrix} y_0 & \cdots & y_{d-2} & y_{d-1} \\ y_1 & \cdots & y_{d-1} & y_d \end{bmatrix}$$

To see this, we see that we have a natural ring homomorphism:

$$k[y_0, \cdots, y_d] \to k[x^d, x^{d-1}y, \cdots, xy^{d-1}, y^d]$$

given by $y_0 \mapsto x^d$, $y_1 \mapsto x^{d-1}y$ and etc. Then it's easy to see that all the 2×2 minors are in the kernel, thus we have a well defined morphism from the quotient to $k[x^d, x^{d-1}y, \cdots, xy^{d-1}, y^d]$. We claim that is has an inverse given by mapping x^d to $\overline{y_0}$ and the others similarly maps to the class of what we difine the map in the reversed direction. We only have to verify that this is well defined . The way we do it is by look at each degree, where degree $0, 1$ is trivial. Suppose that we have it for degree $k-1$, let's consider k. **** more to say on this*

More generally, we give the definition of **Veronese embedding**:

Definition 1.40. Suppose that $S = k[x_0, \cdots, x_n]$, we define the closed embedding $v_d : ProjS_d \to \mathbb{P}^{N-1}$ where N is the number of homogeneous polynomials of degree d in variables $x_0, \cdots, x_n$. This closed embedding is defined by mapping the projective coordinate in $\mathbb{P}^{N-1}$ to the corresponding homogeneous polynomial of degree d, it's obviously a surjective morphism, thus induces a closed embedding.

Remark 1.41. The number N is $\binom{n+d}{d}$, which you can think through if you are interested. And since we know that the proj of the Veronese subring is isomorphic to the proj of the original ring, we can think that we are embedding lower dimensional projective spaces into higher dimensional projective spaces.

The next important notion is affine and projective cones, which we will use repreated in the review of dimension theory: we begin straight up with the definition:

Definition 1.42. Suppose that S is a finitely generated graded ring, then the **affine cone** of $ProjS$ is defined to be $SpecS$. At this point, we can only give such a not ideal definition since this $SpecS$ depends on the ring S but not on $ProjS$.

The following exercise is useful in terms helping us picture the affine cone:

Proposition 1.43. Let $ProjS$ be a projective scheme over a field k, then there is a natural morphism $SpecS - V(S_+) \to ProjS$ where $V(S_+)$ is just a point and should be intuitively thought of as the origin. And more generally, if S is a finitely generated graded ring over A, then we have a natural morphism from $SpecS - V(S_+)$ to $ProjS$.

Proof. The map is still defined locally and you need to check that they glue together, by the obvious inclusion of $S_{(f)}$ into S_f. One interesting point is that when we are over a field k, $V(S_+)$ is indeed on point. To see this, we see that $S \cong k[x_0, \cdots, x_d]/I$ where I is a homogeneous ideal. Then we see that S_+ will be the ideal generated by the class of $x_0, \cdots, x_d$, we know this is a maximal ideal. □

Based on this notion, you can view the affine cone minus the origin as projected to $ProjS$.

Definition 1.44. In the same set up, we say that the **projective cone** of $ProjS$ is $ProjS[T]$ where S is a new variable of degree 1. For example, the projective cone of the conic $Projk[x, y, z]/(z^2 - x^2 - y^2)$ is $Projk[x, , y, z, T]/(z^2 - x^2 - y^2)$.

This is called the projective cone because we have the following:

Proposition 1.45. The $ProjS[T]$ has a closed subscheme which is isomorphic to $ProjS$, moreover, it's complement is isomorphic to the affine cone $SpecS$.

Proof. We define the closed subscheme to be $V(T)$, informally, it's correspond to the locus where T vanishes. There are several ways to see that this closed subscheme is isomorphic to $ProjS$, one way where you can take advantage of the gluing work you did before is to consider the morphism induced by:

$$S_{(f)} \to S[T]_{(f)} \to S[T]_{(f)}/T$$

which induces a morphism. This is an isomorphism. The complement is trivially isomorphic to S since it's just get rid of T on $D_+(T)$. □

Remark 1.46. This construction $ProjS[T]$ can be viewed as the union of the affine cone plus the points at infinity which correspond to $ProjS$. This is the algebric version of the classical picture where we view:

$$\mathbb{P}^{n+1} = \mathbb{R}^{n+1} \coprod \mathbb{P}^n$$

The last definition we have about constructions of these flavour is the **weighted projective space** $\mathbb{P}(d_0, d_1, \cdots, d_n)$ where we take the *proj* of the graded ring $k[x_0, \cdots, x_n]$, but we give each variable a new degree, where x_i has degree d_i. As some practice:

Proposition 1.47. $\mathbb{P}(m, n)$ is isomorphic to $\mathbb{P}^1$.

Proof. Notice that we can't really apply the usual way of constructing maps between projs by giving a map of graded rings, since it's not really clear how we are gong to map x, y. However, we still can use the fact that $\mathbb{P}(m, n)$ is covered by $D_+(x)$ and $D_+(y)$. Only notice that the ring we have here is $Speck[\frac{x^{m/d}}{y^{n/d}}]$ where d is the gcd of m, n. Then the isomorphism and they agree on intersections are not hard to check. □

Another worth doing example is:

Example 1.48. We claim that $\mathbb{P}(1, 1, 2)$ is isomorphic to $Projk[u, v, w, z]/(uw - v^2)$, which is the projective cone over a conic in $\mathbb{P}^2$.

We do this by the following trick, notice that the 2-th Veronese subring is $k[x_0^2, x_1^2, x_0x_1, x_2]$, and you take the proj, it's isomorphic to the proj of the original ring. Then notice that we have an obvious isomorphsm of $k[x_0^2, x_1^2, x_0x_1, x_2]$ with $k[u, v, w, z]/(uw - v^2)$.

Before we end this section, we will also review the quasicoherent sheaf construction on a projective space, which is important afterwords in the cohomology of prjective spaces, this is also covered in the notes. We will first follow the treatment of Hartshorne and then review using Vakil's book:

Definition 1.49. Suppose that S is a graded ring and M is a graded S-module. We will define the sheaf **associated to** M **on** $PorjS$ as follows. For each homogeneous prime ideal in $ProjS$, let $M_{(p)}$ be the degree 0 piece of M_p, the **homogeneous localization** of M at the homogeneous prime ideal p. To be more precise, we localize M at the multiplicative set T consisting of all the homogeneous elements in $S - p$. As you know that localization of a graded module at a multiplicative set consisting of all homogeneous elements gives us another $T^{-1}S$-module with the obvious grading.

Then for any open subset $U \subset ProjS$, we define $\widetilde{M}(U)$ to be the functions $s : U \to \prod_{p \in U} M_{(p)}$ which are locally fractions. This means that for every $p \in U$, there is a neighborhood V of p in U and homogeneous element $m \in M$ and homogeneous element $f \in S$ of the same degree, such that for each $q \in V$, we have $f \notin q$ and $s(q) = \frac{m}{f} \in M_{(q)}$. The restriction are obvious restriction of functions.

Notice that the above definition make $\widetilde{M}$ a presheaf for sure. It also satisfies the identity axiom obviously. Notice that this is a local definition, in the sense that it resembles the definition of locally constant function, thus we should expect it to satisfy the gluability condition, which you can check really handy.

The following several propositions should look familiar:

Proposition 1.50. Let S be a graded ring and M is a graded algebra of S, let $X = ProjS$, then:

1. For any $p \in X$, the stalk $\widetilde{M}_p = M_{(p)}$. Or in other words, the is an obvious and natural isomorphism between the stalk and $M_{(p)}$.

2. For any homogeneous $f \in S_+$, we have $\widetilde{M}|_{D_+(f)} \cong \widetilde{M}_{(f)}$ via the isomorphism of $D_+(f)$ with $SpecS_{(f)}$. The first tilde is the projective version and the second one is the affine version.

3. $\widetilde{M}$ is a quasicoherent $\mathscr{O}_X$-module and if S is noetherian and M is finitely generated, then $\widetilde{M}$ should be coherent.

Proof. The first assertion is easy to prove as you can view shrinking open subsets containing one point and then restricting the function is like approaching the value of the function at that point, which should range all possible values in $M_{(p)}$ when we range all the possible functions.

For the second assertion, just as how we proved the isomorphism of $D_+(f)$ and $SpecS_{(f)}$, the most important relation we need to establish is, how to relate: $M_{(p)}$ and $(M_{(f)})_{\varphi(p)}$ where $\varphi(p)$ is the map we defined in showing the isomorphism of $D_+(f)$ and $SpecS_{(f)}$. Notice that one of them is $(S_{(f)})_{\varphi(p)}$-module, another one is a $S_{(p)}$-module. Then there is a natural question that if we have two isomorphic rings: $f : A \to B$, and two modules M, N as A, B-module respectively. Is there any way to say that they are isomorphic? Well, they are not over the same ring, but the good news is that the two rings are isomorphic, thus restriction of scalars are not really a bit deal. Then we can restrict scalars such that N is now a B-module, then we can ask whether M and N are isomorphic as B-modules.

This is exactly what is happening here, what we saw is that we have an isomorphism of scheme where $\varphi : D_+(f) \cong SpecS_{(f)}$, and we have $\widetilde{M}|_{D_+(f)}$ on $D_+(f)$, we push it forward to $SpecS_{(f)}$, we want to show that the pushforward is isomorphic to $\widetilde{M_{(f)}}$. What is the isomorphism of sheaves, we need to find a natural map that connects: functions of the form $s : \varphi U \subset SpecS_{(f)} \to \prod M_{(f)}|_{\varphi(p)}$ and functions of the form $s : U \to \prod M_{(p)}$ such that this is an isomorphism of sheaves, in the sense that on each open U, the map need to be an isomorphism.

This is not hard to check if you know what the isomorphism between $S_{(p)}$ and $S_{(f)}|_{\varphi(p)}$ is. You should work to find this isomorphism, while the hint is try to define the morphism directly going out of $S_{(f)}|_{\varphi(p)}$. Essentially, this can be reproduced by exactly the same formula on mudules then you can similarly show that $M_{(p)} \cong M_{(f)}|_{\varphi(p)}$ after the restriction of scalars. Then we define $s : U \subset SpecS_{(f)} \to \prod M_{(f)}|_{\varphi(p)}$ to the function such that we compose s with the isomorphism at each $\varphi(p) \in \varphi(U)$. Then we are done. This in particular shows that $\widetilde{M}$ is a quasicoherent module.

The last assertion is also easy to show, if S is noetherian, and M is finitely generated, we know that $S_{(f)}$ is noetherian and we only have to show that $M_{(f)}$ is finitely generated over $S_{(f)}$. This is easy to show if we can assume that S is generated by degree 1 elements. While the general case will be proved later. ☐

Here comes the crucial definition:

Definition 1.51. Let S be a graded ring and let M be a graded S-module, then we will denote $M(l)$ by another graded module defined out of M where the underlying set is still M, while the grading is shifted by l. We know assume that $M(l)_d$, the d graded piece is M_{d+l}. It's called the **twisted module by** l. Then suppose that $X = ProjS$, we define ${}_X(n) = \widetilde{S(n)}$, where we view S as an graded module over itself. We will call $\mathscr{O}_X(1)$ the twisting sheaf of Serre. For any sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$, we denoted by $\mathscr{F}(n)$ the **twisted sheaf** $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(n)$.

The following proposition contans some main properties we will use:

Proposition 1.52. Let S be a graded ring and $X = ProjS$. Assume that S is generated over S_1 as an S_0-algebra. Then:

1. $\mathscr{O}_X(n)$ is a Line bundle over X.
2. For any graded module M, we have a natural isomorphism of sheaves of $\mathscr{O}_X$-modules:

$$\widetilde{M}(n) \cong \widetilde{M(n)}$$

   i.e. first twisting and then take the associated sheaf or first take the associated sheaf then twisting doesn't matter.

3. Let T be another graded ring generated by T_1 over T_0. Let $\varphi : S \to T$ be a map of graded rings of degree 1. If we denote $f : ProjT - V(\varphi(S_+)) \to ProjS$ be the morphism induced by a map of graded rings. Then $f^*(\mathscr{O}_X(n)) = \mathscr{O}_Y(n)|_U$ and $f_*(\mathscr{O}_Y(n)|_U) = (f_*\mathscr{O}_U)(n)$.

Proof. When we assume S is generated over S_0 by S_1, we only have to look at $f \in S_1$, homogeneous elements whose $D_+(f)$ will cover X. The first assertion is just showing that $S(n)_{(f)}$ is free of rank 1 as a $S_{(f)}$-module. Notice that since we twisted S by n in $S(n)$, the degree 0 elements in $S(n)$ is now S_n, the degree n piece in the original grading, thus we see that $S(n)_{(f)}$ are all the elements of degree n in S_f. This is because that when we localize at f, we are still viewing f as an degree 1 element in S, before we twist it. Therefore we have an isomorphism of $S_{(f)}$-modules:

$$S_{(f)} \to S(n)_{(f)}$$

where we map $\frac{u}{f^n} \to u$. This is a well defined map, since we only have to care where 1 goes, where we map it to f^n. This makes sense even if n is less than 0 since no matter what n we pick, f^n will be in $S(n)_{(f)}$.

The let's consider the second assertion. Notice that by definition:

$$\widetilde{M}(n) = \widetilde{M} \otimes_{\mathscr{O}_X} \mathscr{O}_X(n)$$

We claim that for any graded S-modules M, N, we have $\widetilde{M \otimes_S N} \cong \widetilde{M} \otimes \tilde{N}$ if S is generated over S_0 by S_1. Indeed, we need to check this on $D_+(f)$, where $f \in S_1$ homogeneous. Notice that on $D_+(f)$, it's isomorphic to $(\widetilde{M \otimes_S N})_{(f)}$, then notice that $(M \otimes_S N)_{(f)} \cong M_{(f)} \otimes_{S_{(f)}} N_{(f)}$, **** also need to check gluing conditions and stuff, which we left to our later review using vakil*. In particular, this applies to $S(n)$. Therefore, we only have to show that $M(n)_{(f)} \cong M_{(f)} \otimes_{S(f)} S(n)_{(f)}$. This isomorphism is easy to establish.

**** we wait for vakil's treatment of the last argument* □

Now we look at the somewhat inverse procedure where we want to obtain a graded S-module from any sheaf of $\mathscr{O}_X$-modules on $X = ProjS$.

Definition 1.53. Let S be a graded ring let $X = ProjS$ and let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules. We define the graded S-module associated to $\mathscr{F}$ as a group, to be $\Gamma_*(\mathscr{F}) = \oplus_{n\in\mathbb{Z}}\Gamma(X, \mathscr{F}(n))$. We give it a structure of graded S-module as follows:

Notice that if $s \in S_d$, we claim that it determines a global section $s \in \Gamma(X, \mathscr{O}_X(d))$. To see this, notice that global sections of $\mathscr{O}_X(d)$ are functions s that maps p in $ProjX$ to $\prod S(d)_{(p)}$, which is the function that maps every p in $ProjX$ to $\frac{s}{1}$. This makes sense since we are considering the ring twisted by d. Then for any $t \in \Gamma(X, \mathscr{F}(n))$, we define $s \cdot t \in \Gamma(X, \mathscr{F}(n+d))$ by the isomorphism:

$$\mathscr{F}(n) \otimes \mathscr{O}_X(d) \cong \mathscr{F}(n+d)$$

And the corresponding section on the global section determined by the tensor product $s \otimes t$ in the presheaf. Then we extend this action linearly. It's easy to verify this is a well defined S-action.

As a practice of the definition above, we consider:

Proposition 1.54. Let A be a ring and $S = A[x_0, \cdots, x_r]$ for $r \geqslant 1$. Then if $X = ProjS$, we have $\Gamma_*(\mathscr{O}_X) = S$.

Proof. Let's first consider the pieces of $\Gamma_*(\mathscr{O}_X)$. For each n, the degree n piece is $\Gamma(X, \mathscr{O}_X(n))$. Notice the global sections of $\Gamma(X, \mathscr{O}_X(n))$ can be identified with an $r+1$-tuple of t_i such that each t_i is a section on $D_+(x_i)$ such that they agree on $D_+(x_i x_j)$. Notice that we already know that sections of $\Gamma(X, \mathscr{O}_X(n))$ on $D_+(x)$ are degree n homogeneous elements in S_{x_i} such that they agree on $S_{x_i x_j}$, therefore after you sum all the n's you can identify elements in the $\Gamma_X(\mathscr{O}_X)$ by the set of $r+1$-tuples of o=polynomials in S_{x_i} such that they agree on $S_{x_i x_j}$, notice that since x_i's are not zerodivisors, thus we can view the localizations as embedding in a larger ring. Since every localization can be embedded in a compatible way into $S_{x_0 x_1 \cdots x_r}$, then we are just taking the intersection of all S_{x_i} in $S_{x_0 x_1 \cdots x_r}$. Notice that we know that S is definitely contained in the intersection, thus we want to show that any stuff in the intersection is in S. We can just consider the homogeneous part, thus if all homogeneous are in S then their sum will be in S, this is an easy exercise left. ☐

Remark 1.55. If S is not the polynomial ring, the above may not hold that the global section of $\mathscr{O}_X$ is still S.

Our main goal is to prove the following result:

Proposition 1.56. Suppose that S is a graded ring finitely generated by S_1 as an S_0-algebra. Let $X = ProjS$, then for any quasicoherent sheaf $\mathscr{F}$ on X, there is a natural isomorphism $\beta : \widetilde{\Gamma_*(\mathscr{F})} \to \mathscr{F}$

To prove this, we need two preparation results, one of them is of it's own interest:

Proposition 1.57. Let $X = SpecA$ be an affine scheme. The functor $\tilde{-}$ and the global sections functor are adjoint in the sense that: for any A-module M and any sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$, there is a natural isomorphism:

$$Hom_A(M, \Gamma(X, \mathscr{F})) \to Hom_{\mathscr{O}_X}(\widetilde{M}, \mathscr{F})$$

Proof. Given any M and a map of A-modules $M \to \Gamma(X, \mathscr{F})$, by the general morphism of sheaves theory, it gives a morphism of sheaves on the basis, which determines a unique morphism of sheaves. Conversely, given a morphism of sheaves, by taking the global sections, we have a map of A-modules. We claim that these are inverses of each other, which are still general sheaf theory.

This isomorphism is also natrual in the sense that it gives an adjointness, this is easy to check, we omit it here. ☐

Lemma 1.58. Suppose that X is any scheme and $\mathscr{L}$ is a line bundle on X. Let X_f be the open subset of points in X such that $f_x \notin m_x \mathscr{L}_x$ and let $\mathscr{F}$ be a quasicoherent sheaf on X, then:

1. Suppose that X is quasicompact, let $s \in \Gamma(X, \mathscr{F})$ be a global section of $\mathscr{F}$ whose restriction to X_f is 0. Then for some $n > 0$, we have $f^n s = 0$ where $f^n s$ is considered to be a global section of $\mathscr{F} \otimes \mathscr{L}^n$.

2. Suppose that further more X has a finite affine cover U_i such that $\mathscr{L}$ is trivial on U_i and such that $U_i \cap U_j$ are quasicompact for all i, j. Then given a section $t \in \Gamma(X_f, \mathscr{F})$, then for some $n > 0$, there is a section $f^n t \in \Gamma(X_f, \mathscr{F} \otimes \mathscr{L}^n)$ such that it extends to a global section in $\mathscr{F} \otimes \mathscr{L}^n$.

Proof. This is a generlization of the result we proved before when we tried to prove the affineness condition, where we only consider $\mathscr{F} = \mathscr{L} = \mathscr{O}_X$. Here we want to be more general.

We will first prove part 1, which can be used to prove part b. Notice that since we have a finite cover, we can convert this question into a local question, where we can prove that the restriction of f^n and s restricted to U_i will be mapped to zero in $\mathscr{F} \otimes \mathscr{L}^n(U)$, then the global result holds because of identity axiom. Then, suppose that over $U = SpecA$, we have $\mathscr{L}|_U \cong \mathscr{O}_U$ via some isomorphism ψ. Then we know that the restriction of f to $\mathscr{L}|_U$ again correspond to $f \in A$. Similarly, since $\mathscr{F}$ is quasicoherent, then on U we know that it's $\widetilde{M}$ Then we see that $X_f \cap U = D(f)$, then we know that **** details later, clear but too much to write down*

Then for the second assertion, the condition that we want the intersection to be quasicomapct is we want to apply the part 1 result here. The ideal is simple, it's the technical details that are confusing. We only state the ideas here while the details verifications are left to you. You need to figure out what kind of technical details you need such that the idea work, which are all very simple parts.

The idea here is that on U_i, everything is affine, then s restrict to $U \cap X_f$ is an element in a localized module, then by the definition of localization, there will be $f_i^{n_i} t|_{U_i \cap X_f}$ such that it comes from an element in the section on U_i, take n to be the maximum of all n_i, then show that they agree on $U_i \cap U_j$, then apply the above part 1, take maximum again, we are done. $\square$

Remark 1.59. The above hypothesis is satisfied if either if X is noetherian or X is qcqs.

Then we are ready to prove the isomorphism we promised:

Proof. We first define the natural morphism $\widetilde{\Gamma_*(\mathscr{F})} \to \mathscr{F}$. Notice that we still can only do it locally, where we consider $D_+(f)$. Notice that $\widetilde{\Gamma_*(\mathscr{F})}$ is quasicoherent, thus we only have to define a module map from the section of $\widetilde{\Gamma_*(\mathscr{F})}$ over $D_+(f)$ to the section of $\mathscr{F}$ over $D_+(f)$. Notice that such a morphism can be defined on homogeneous pieces of the section of $\widetilde{\Gamma_*(\mathscr{F})}$ over $D_+(f)$. Suppose that one of such a section is:

$$\frac{m}{f^d} \qquad m \in \Gamma(X, \mathscr{F}(d))$$

We want to associate this an element in the section of $\mathscr{F}$ over $D_+(f)$. Notice that if we view f^{-d} as an element of $\mathscr{O}_X(-d)$ over $D_+(f)$, we can consider the tensor product $\mathscr{F}(d) \otimes \mathscr{O}_X(-d) \cong \mathscr{F}$, we can obtain a section of $\mathscr{F}$ on $D_+(f)$ by first restricting m then taking tensor product, this defines the desired β.

Now we still have to show that this is an isomorphism, which is still done locally. We will use the lemma above, consider the line bundle $\mathscr{O}_X(1)$. Notice that we have showed before that $\mathscr{O}_X(1)$ is trivial over $D_+(f_i)$ where f_i's are the finitely many homogeneous element in S_1 generating S as an S_0 algebra. And we can also view f as a global section of $\mathscr{O}_X(1)$, to see this, notice that if f is homogeneous of degree 1. Then over each $D_+(f_i)$, it's an element since we shifted degree by 1. Obviously, they agree on the intersections, thus giving a global section such that the restriction to each $D_+(f_i)$ is f. Thus the hypothesis of the above lemma is satisfied. Now notice that given any section in $\mathscr{F}(D_+(f))$, we know that the X_f in the lemma is $D_+(f)$ in our case. This requires some proof but at least you should expect this happen as you can see many evidence suggesting it. Then, we know that by the above lemma, there will be f^d such that the section s satisfies that sf^d comes from the restriction of global section in $\mathscr{F}(d)$, this is exactly the inverse process of the map we defined above, thus we are done. $\square$

We end this section by a few stray remarks I don't really know where to put, but are still useful sometimes when you deal with the local situations and helpful at least psychologically:

Proposition 1.60. $S_{(p)}$ is indeed a local ring for any homogeneous prime ideals p. Moreover, in this case, it's maximal ideal consist of all the elements of the form $\{\frac{f}{g}\}$ where $f \in p$ homogeneous and the degree of $\frac{f}{g}$ is zero in the homogeneous localization of S at p, it's also the zero degree piece of the extension of p in the homogeneous localization at p.

Proof. We use a characterization of local ring by the fact that all non invertible elements forms an ideal.

Let's first determine the noninvertible elements in $S_{(g)}$, suppose that $\frac{f}{g}$ where $f \in p$, then this is noninvertible, otherwise a contradiction to p is prime. Conversely, if $\frac{f}{g}$ is noninvertible then $f \in p$, otherwise it's obviously invertible.

This clearly forms and ideal since p is an ideal, thus we are done. □

Proposition 1.61. Suppose that $N \hookrightarrow M$ is a sub-gradedmodule of a graded module M over a graded ring S, then suppose that p in S is a homogeneous prime, then if we consider the homogeneous localizations at p, we have $(M/N)_{(p)} \cong M_{(p)}/N_{(p)}$ as a module of $S_{(p)}$. That is taking zero degree piece commute with quotients that preserves degrees.

Proof. Consider:

$$0 \to N \hookrightarrow M \to M/N \to 0$$

we know that taking 0 sections and homogeneous localization are both exact. Apply homogeneous localization and taking 0 degree, we still have a exact sequence, we are done. □

2 Review on Curly Proj Construction and Below up

3 Review on Separated and Proper Morphisms

We begin to describe the notion of separatedness in this section. This is a fundamental notion that requires a lot prerequisites we developed before to make sense of. This is the analog of Hausdorffness in the world of schemes. This definition in the beginning looks a little bit wired in the beginning, but we will develop more after we give the definition. Gradually, you will find this definition natural.

We start this section by reviewing Hausdorffness in the world of manifolds to see why we want it and what we can expect in the world of schemes. First, we include the notion of Hausdorffness to exclude some wild and not natural examples such as the line with a doubled origin. Here, you will see that separatedness is finally the notion that allows us to define variety, which is a lot like the manifolds. Also, since we see our affine space and projective space to be natural counterparts of manifolds, we would expect then to be separated, also since a subspace of Hausdorff space is automatically Hausdorff, we would like any subschemes of a separated scheme to be separated. Therefore, we know that if X is a subset of a manifold Y that satisfies all the technical conditions of a manifold except hausdorffness, it's automatically a manifold. Similarly, a locally closed embedding into something separated results in separarted stuff.

As an unexpected consequence, you will see that under some separated condition, intersection of affine schemes will be affine. This can be thought of as the counterpart of the fact that in $\mathbb{R}^n$, the intersection of convex sets is convex. The reason for this metaphor lies in the fact that from the point of view of cohomology, affine schemes and convex subsets in $\mathbb{R}^n$ don't have any nontrivial cohomology.

The last comment we would like to give before we proceed to the real content, besides that the convention that we should define any property of schemes in terms of properties of morphisms and the scheme have that property if the morphism to the final object have that property, is that the definition of separatedness is given in terms of the **diagonal morphism**, by which we can derive many pleasant properties. This is also important in other fields of geometry, our first goal is to understand it, which also makes the notion of quasiseparatedness easier to understand:

Definition 3.1. Given any morphism of schemes $f : X \to Y$, we can study the diagram below:

$$\begin{tikzcd}
X \arrow[dr, dashed] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "id"'] & & \\
& X \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

the dotted morphism $\delta_f : X \to X \times_Y X$ induced by the universal property of fibered product is called **the diagonal morphism**.

The first proposition we would like to prove is to finish our promise a while ago about the nice class of morphism satisfies the cancellation property. The setup is easy, we still assume that P is a property satisfied by some morphisms, however, we now define an auxiliary class of morphism, these are the morphisms $\pi : X \to Y$ whose diagonal morphism δ_π satisfies P. We will call this class of morphism $P\delta$.

Theorem 3.2. Let P be a property of morphisms that is preserved by composition and base change, (we still assume that it's enjoyed by isomorphisms to exclude stupid examples) thus every reasonable class of morphisms satisfies this. Then for the commutative diagram below:

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "\tau"'] & & Y \arrow[dl, "\rho"] \\ & Z & \end{tikzcd}$$

if π is in P and ρ is in $P\delta$, then π is in P.

Before we prove it, here are some examples where this cancellation theorem holds in the categroy of sets, which may help you understand how this proof below works: Suppose that the property P is injective, then if τ is injective then π is injective. This can be easily proved and since in the category of sets, the diagonal morphism is always injective, this can be viewed as the application of the cancellation theorem. Similarly, if we now take P to be surjective τ is surjective while ρ is injective, then π is surjective. Notice that since ρ is injective, then the morphism $Y \to Y \times_Z Y$ will be surjective (do you see why?), thus still an application of the cacellation theorem. After you see the proof of the general theorem below, you will find the the same ideal applies to this case as well.

Now we prove Thm 3.2:

Proof. The first key component is the **diagonal-base-change** diagram, which is of great importance itself. Suppose that we are given morphisms $X_1, X_2 \to Y$ and morphisms $Y \to Z$ and suppose that all the fibered product exists, then we will have the following Cartesian square:

$$\begin{tikzcd} X_1 \times_Y X_2 \arrow[r] \arrow[d] & X_1 \times_Z X_2 \arrow[d] \\ Y \arrow[r] & Y \times_Z Y \end{tikzcd}$$

it's called diagonal-base-change since we are trying to perform base change using the diagonal morphism. Let's first figure out what are the morphisms connecting each of them, since this such a general statement, we don't have a lot to try, so we can only see if there is any canonical morphisms we can get from $X_1, X_2 \to Y$ and $Y \to Z$. I will let you figure how to get them, for example if you want a morphism from $X_1 \times_Z X_2$ to $Y \times_Z Y$, you can only use the universal property where you need to give morphisms $X_1, X_2 \to Y$ such that they agree after you compose with $Y \to Z$, you only have one natural choice...

Now we prove this is Cartesian, suppose that W has morphisms into Y and $X_1 \times_Z X_2$ such that they agree after we go to $Y \times_Z Y$, this first gives a morphism from W to $X_1 \times_Y X_2$. The picture is hard to draw here since it involves too many terms, but the process is $W \to X_1 \times_Z X_2 \to X_i \to Y = W \to X_1 \times_Z X_2 \to Y \times_Z Y \to Y = W \to Y \to Y \times_Z Y \to Y = W \to Y$ which is the morphism we started with and in particular equal to each other, thus if we take $W \to X_1 \times_Z X_2 \to X_i$ as the morphism from W to X_1, X_2, they induce a unique morphism π to $X_1 \times_Y X_2$ by the universal property, we claim that this morpihsm make the below diagram commutes and it is the unique morphism such that it commutes.

$$\begin{tikzcd} W \arrow[dr, dashed, "\pi"] \arrow[drr, "f"] \arrow[ddr, "g"'] & & \\ & X_1 \times_Y X_2 \arrow[r, dashed, "\tau"] \arrow[d] & X_1 \times_Z X_2 \arrow[d] \\ & Y \arrow[r] & Y \times_Z Y \end{tikzcd}$$

The fact that the left lower triangle commutes comes from the argument in the above paragraph, the other upper right triangle commutes is the content of the below diagram:

$$\begin{tikzcd}
W \arrow[dr, dashed, "\pi"] \arrow[drr, dashed, "f"] \arrow[drrr, bend left] \arrow[ddrr, bend right] & & & \\
& X_1 \times_Y X_2 \arrow[r, dashed, "\tau"] \arrow[drr] \arrow[dr] & & \\
& & X_1 \times_Z X_2 \arrow[r] \arrow[d] & X_2 \arrow[d] \\
& & X_1 \arrow[r] & Y \arrow[r] & Z
\end{tikzcd}$$

the composition of the two short dotted morphism is the long dotted morphism by the universal property of fibered product (uniqueness).

Finally we prove the uniqueness. This is easier than the argument above, suppose that besides π, you have a π' such that the desired diagram commutes. Consider the diagram above we just drew, you will see that if you replace π by π', we still have the desired commutativity, thus it must equal to π.

Given this important result, we can prove the cancellation theorem by considering the second main ingredient, which is the below diagram:

$$\begin{tikzcd}
X \arrow[r, leftrightarrow, "iso"] \arrow[rrrr, bend left, "\pi"] & X \times_Y Y \arrow[r] \arrow[dl] & X \times_Z Y \arrow[r] \arrow[dl] \arrow[dr] & Z \times_Z Y \arrow[r, leftrightarrow, "iso"] \arrow[dr] & Y \\
Y \arrow[r, "\delta"] & Y \times_Z Y & & X \arrow[r, "\tau"] & Z
\end{tikzcd}$$

Notice that the left square is Cartesian by our diagonal base change diagram above. Also, the right square is also Cartesian since you can consider:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & Z \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d] \\
X \arrow[r] & Z \arrow[r] & Z
\end{tikzcd}$$

Then since $Y \to Z$ is in $P\delta$, δ in the diagram is in P, and since P is stable under base change, $X \times_Y Y \to X \times_Z Y$ is in P, also $X \times_Z Y \to Z \times_Z Y$ are in P. Now since isomorphisms are in P and it's stable under composition, if we can show that the composition of the top four morphisms is just π, we are done.

Let's first deal with the two middle maps, I will let you see that the two maps are induced by the following diagram:

$$\begin{tikzcd}
X \times_Y Y \arrow[dr, dashed] \arrow[drr, "p"] \arrow[ddr, "q"'] & & \\
& X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d, "id"] \\
& X \arrow[r, "\pi"'] & Y \arrow[r, "\rho"'] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times_Z Y \arrow[d, dashed] \arrow[rrr, "p"] \arrow[dd, bend right, "q"'] & & & \\
\cdots \arrow[rr] \arrow[d] & & Z \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d, "\rho"] \\
X \arrow[rr, "\tau"] & & Z \arrow[r, "id"] & Z
\end{tikzcd}$$

Therefore, now you can use it to conclude that the composition of the middle two morphism with $Z \times_Z Y \to Y$ is just $p : X \times_Y Y \to Y$, then precompose it with the isomorphism $X \to X \times_Y Y$, we get the composition is π, thus we are done. $\square$

Plus the above Cancellation theorem we showed, we know have 5 properties enjoyed by nice class of morphisms: they are closed under composition, base change, they are local on the target, closed under products and they satisfies this cancellation property. There is another property of nice class of morphisms we need to show:

Theorem 3.3. Suppose that P is a nice class of morphisms, then $P\delta$ is also a nice class.

Proof. Let's verify that $P\delta$ satisfies the axioms for nice class of morphisms. First, we need to show that it's closed under composition, which means that if $f : X \to Y$ and $g : Y \to Z$ are in $P\delta$, we want to show that $g \circ f$ is in P, to do this, we need to show that $X \to X \times_Z X$ is in P. Consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "\in P"] \arrow[rr, bend left, "\in P"] & X \times_Y X \arrow[r] \arrow[d] & X \times_Z X \arrow[d] \\
& Y \arrow[r, "\in P"'] & Y \times_Z Y
\end{tikzcd}$$

Notice that the right square is Cartesian because it's the base-change Cartesian square, then we see that the above two morphisms $X \to X \times_Y X$ and $X \times_Y X \to X \times_Z X$ are both in P, thus if we can prove that the composition is $X \to X \times_Z X$ which is induced by $g \circ f$, we are done. Now, recall how we get the morphism $X \times_Y X \to X \times_Z X$, we put them together, and get the following diagram:

$$\begin{tikzcd}
X \times_Y X \arrow[dr, dashed] \arrow[drr] \arrow[ddr] & & & \\
& X \times_Z X \arrow[r] \arrow[d] & X \arrow[d] & \\
& X \arrow[r] & Y \arrow[r] & Z
\end{tikzcd}$$

Then after you precompose $X \to X \times_Y X$, notice that this map satisfies the universal property, thus is the one $X \to X \times_Z X$ we desire.

Then we need to show that it's preserved by base change, which means that if $f : X \to Y$ is in $P\delta$, then we want to show that any base change of f, which is the map $g : V \to W$ in the left Cartesian diagram below is also in $P\delta$, but you can consider the right diagram below:

$$\begin{tikzcd}
V \arrow[r, "g"] \arrow[d] & W \arrow[d] \\
X \arrow[r, "f"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r] \arrow[d] & V \times_W V \arrow[d] \\
X \arrow[r] & X \times_Y X
\end{tikzcd}$$

Notice that if we can show that the right diagram is Cartesian, then we are done. There are probably many ways you can proceed but some of them is extremely ugly and confusing, we present here a not so complicated way of doing it. First, consider the following diagram:

$$\begin{tikzcd}
X \times_Y X & X \arrow[l] \arrow[d] & V \arrow[l] \arrow[d] & V \times_W V \arrow[l] \arrow[d] & V \arrow[l] \\
& Y & W \arrow[l] & V \arrow[l] &
\end{tikzcd}$$

since stacks of Cartesian is Cartesian, we get $V \times_W V \cong X \times_Y V$, also the top morphism also shows that the the diagram that we are trying to show is Cartesian is really commutative. We want to reduce what we need to show to the following diagram:

$$\begin{tikzcd}
V \times_X X \arrow[r] \arrow[d] & V \times_Y X \arrow[d] \\
X \arrow[r] & X \times_Y X
\end{tikzcd}$$

which is just the base-change diagram. But notice that $V \times_X X \cong V$ and $V \times_Y X \cong V \times_W V$, this is promising, the only thing left to do is to check certain commutativity. It easy to show that suppose that we have A, B, C, D below is a Cartesian diagram, and we have isomorphism $A \cong E$ and $B \cong F$, the outer square is Cartesian iff the inner square is Cartesian:

$$\begin{tikzcd}
E \arrow[r, leftrightarrow, "i"'] \arrow[dr, "f\circ i^{-1}"'] & A \arrow[r, "u"] \arrow[d, "f"'] & B \arrow[r, leftrightarrow, "j"'] \arrow[d, "g"] & F \arrow[dl, "g\circ j^{-1}"] \\
& C \arrow[r, "v"'] & D &
\end{tikzcd}$$

In our case, we only have to show that in the below diagram, the three morphisms on the top composes to the diagonal morphism $V \to V \times_W V$.

$$\begin{tikzcd}
V \arrow[r] \arrow[dr] & V \times_X X \arrow[r] \arrow[d] & V \times_Y X \arrow[r] \arrow[d] & V \times_W V \arrow[dl] \\
& X \arrow[r] & X \times_Y X &
\end{tikzcd}$$

This is the content of the following diagram:

$$\begin{tikzcd}
V \times_X X \arrow[dd, leftrightarrow, "iso"] \arrow[dr, dashed] \arrow[ddrrrr, bend left] \arrow[ddrrrr, bend left=10] & & & & \\
& V \times_Y X \arrow[ddr] \arrow[dr, dashed, leftrightarrow] \arrow[drrr] & & & \\
V \arrow[drr, leftrightarrow, "id"] & & V \times_W V \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddl, leftrightarrow, bend left, "id"] \\
& & V \arrow[r] \arrow[dr] & W \arrow[r] & Y \\
& & & X &
\end{tikzcd}$$

I will let you figure out why this diagram suffice.

Finally, let's prove it's local on the target. Suppose there is an open cover U_i of Y and there is $f : X \to Y$ such that we have the following diagram:

$$\begin{tikzcd}
V_i \arrow[r] \arrow[d, hook] & U_i \arrow[d, hook] \\
X \arrow[r] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
V_i \arrow[r] \arrow[d, hook] & V_i \times_{U_i} V_i \arrow[d, hook] \\
X \arrow[r] & X \times_Y X
\end{tikzcd}$$

where V_i are the inverse images of U_i. Since we assume that $V_i \to U_i$ is in $P\delta$, then we have $V_i \to V_i \times_{U_i} V_i$ is in P. Notice that the right side diagram is just a special case of the Cartesian diagram we had in the precious case, thus it's also Cartesian. Then if we can show that $V_i \times_{U_i} V_i$ covers $X \times_Y X$, we are done since P is local on the target. To show this, the key is to observe that U_i covers Y and you only have to prove that under the canonical projection map $X \times_Y X \to X$, the inverse image of V_i is $V_i \times_{U_i} V_i$, to see this, consider the following diagram:

$$\begin{tikzcd}
V_i \times_Y V_i \arrow[r, hook] \arrow[d, hook] & X \times_Y V_i \arrow[r] \arrow[d, hook] & V_i \arrow[d, hook] \\
V_i \times_Y X \arrow[r, hook] \arrow[d] & X \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\
V_i \arrow[r, hook] & X \arrow[r] & Y
\end{tikzcd}$$

One important fact you should notice is that since the two canonical maps are the same $X \times_Y \to X$, then we know that the image of $X \times_Y V_i$ and $V_i \times_Y X$ in $X \times_Y X$ are the same, thus their intersection is just itself, thus we proved that $V \times_Y V_i$ covers $X \times_Y V_i$, but notice that we already showed that since the image of V_i is contained in U_i, the morphism: $V_i \times_Y V_i \to X \times_Y X$ is canonically isomorphic to the morphism: $V_i \times_{U_i} V_i \to X \times_Y X$, in the sense that a desired diagram commutes, any way, this completes the proof. ☐

We can still continue our list of useful properties of nice class of morphisms:

Proposition 3.4. Suppose that P is a nice class of morphisms, and $\rho : X \to Y$, $\sigma : X \to Z$ are morphisms of S-schemes in class P, if we also assume that $X \to S$, the structure morphism is in $P\delta$, then the morphism $(\rho, \sigma) : X \to Y \times_S Z$ is in P as well.

Proof. This is still a proof by diagram, and the right one you should consider is the following:

$$\begin{tikzcd}
X \arrow[dr, "P"] \arrow[drrr, bend left, "id"] \arrow[ddr, bend right, "id"'] & & & \\
& X \times_S X \arrow[r, "P"'] \arrow[d, "P"'] & Y \times_S X \arrow[r] \arrow[d, "P"'] & X \arrow[d, "P"'] \\
& X \times_S Z \arrow[r, "P"] \arrow[d] & Y \times_S Z \arrow[r] \arrow[d] & Z \arrow[d] \\
& X \arrow[r, "P"'] & Y \arrow[r] & S
\end{tikzcd}$$

First, this is Cartesian since stacks of Cartesian diagram is Cartesian, the only confusing issue is the left bottom square and the right upper square, we we have the fibered product $X \times_S Z$ and $Y \times_S X$, this is because that we are considering the larger bottom rectangle consisting of two small squares and similarly, another fibered product is what we wrote in the diagram because we are considering the larger rectangle on the right consisting of two small squares. ☐

For applications of the propositions we proved above:

Proposition 3.5. A morphism $\pi : X \to Y$ is a monomorphism iff it's diagonal morphism $\delta_\pi : X \to X \times_Y X$ is an isomorphism. Then, since we know that isomorphisms form a nice class, then monomorphisms form a reasonable class.

Proof. This is actually the content of the following assertion: A morphism $\pi : X \to Y$ is a monomorphism if and only if the fibered product exists and the induced diagonal morphism $\delta_\pi : X \to X \times_Y X$ is an isomorphism.

Let's first assume that $\pi : X \to Y$ is a monomorphism, then consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "id"] \arrow[d, "id"'] & X \arrow[d, "\pi"] \\
X \arrow[r, "\pi"'] & Y
\end{tikzcd}$$

We claim that it's Cartesian, which will suffice. But this is just the fact that π is a monomorphism, and I will let you think about it.

Conversely, suppose that we have the following diagram:

$$\begin{tikzcd}
X \arrow[dr, dashed, "iso"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "id"'] & & \\
& X \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\
& X \arrow[r, "\pi"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
W \arrow[dr, dashed] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& X \arrow[r, dashed, "id"] \arrow[d, dashed, "id"] & X \arrow[d] \\
& X \arrow[r] & Y
\end{tikzcd}$$

I will let figure out why this suffice. ☐

As what we promised earlier, we now give a new definition of a morphism is quasiseparated, which helps us prove certain properties easier:

Definition 3.6. We say that a morphism of schemes $\pi : X \to Y$ is quasiseparated of the diagonal morphism is $\delta_\pi : X \to X \times_Y X$ is quasicompact.

Proposition 3.7. This definition above agrees with our definition earlier.

Proof. First, suppose that $X \to X \times_Y X$ is quasicompact. Then, suppose that V is an affine open in X, we want to show that it's inverse image is quasiseparated, in the sense that the intersection of two affine open is quasicomapct. To show this, consider the following diagram:

$$\begin{tikzcd}
X \arrow[dr, dashed] \arrow[drr, "id"] \arrow[ddr, "id"'] & & \\
& X \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\
& X \arrow[r] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
U \times_V W \cong U \times_Y W \arrow[r, hook] \arrow[d, hook] & X \times_Y W \arrow[r] \arrow[d, hook] & W \arrow[d, hook] \\
U \times_Y X \arrow[r, hook] \arrow[d] & X \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\
U \arrow[r, hook] & X \arrow[r, "\pi"'] & Y
\end{tikzcd}$$

From the left one, we see that the intersection of U, W, which are affine opens in the inverse image of V, is the inverse image of the intersection in $X \times_Y X$. The right morphism says that the intersection in $X \times_Y X$ is just $U \times_V W$, which is affine, which is in particular quasicompact, thus since we know that $X \to X \times_Y X$ is quasicompact, then the inverse image in X is quasicompact, we are done. □

Now via, this we can prove another characterization of quasiseparated:

Proposition 3.8. A morphism of schemes $\pi : X \to Y$ is quasiseparated iff for any affine open $SpecA$ of Y and two affine opens U, V in the inverse image of $SpecA$, we have $U \cap V$ is a finite union of affines.

Proof. Suppose that $\pi : X \to Y$ is affine, then the conclusion is contained in the proof of the previous proposition.

Conversely, this is just the definition if you still remember that a a scheme is quasicomapct iff it's the union of finitely many affine opens. □

Remark 3.9. Using the above characterization of quasiseparated, we can give a good interpretation of a morphism being qcqs.

Notice that if $\pi : X \to Y$ is quasicompact, we know that the inverse image of an affine open is quasicompact, which means that it's the finite union of affines. If further more, this is a quasiseparated morphism, then we know that for this finite covering, any intersection of two of them is still a finite union of affine opens. This turns out to be one of the most useful characterizations of qcqs, and you will see an application of this later.

Remark 3.10. Also, since we know that quasicompact morphism is a nice class of morphisms, using the characterization of quasiseparated morphisms by the diagonal being quasicompact, we know that quasiseparated morphisms are a nice class of morphisms.

We pause this general Category discussion here for a while and look at how this diagonal morphism is related to our geometry:

We start by a proposition showing that for any morphism $\pi : X \to Y$, the diagonal morphism $\delta_\pi : X \to X \times_Y X$ is not just any arbitrary kind of morphism:

Proposition 3.11. Let $\pi : X \to Y$ be a morphism of schemes the the following holds:

1. If X and Y are affine, then the diagonal morphism $\delta_\pi : X \times_Y X$ is a closed embedding.

2. In general, we have $X \to X \times_Y X$ is a locally closed embedding.

Proof. The first is left to you which is just an algebraic translation.

To show the second assertion, we will show that there is an open subscheme U of $X \times_Y X$, which is a union of smaller open subschemes, where we have $X \hookrightarrow U$ is a closed embedding and the morphism $X \to X \times_Y X$ factors as $X \hookrightarrow U \hookrightarrow X$. You have seen all the ideas we are using below in the previous propositions, thus we won't spend long in explaining them.

Suppose that Y is covered with affine opens V_i and we will cover the inverse image in X by affine opens U_{ij}. Then we can consider $U_{ij} \times_{V_i} U_{ij}$, which are open subschemes of $X \times_Y X$, as what we proved earlier. Also, the inverse image of $U_{ij} \times_{V_i} U_{ij}$ in X under δ_π is just U_{ij}, then we see that locally on $U_{ij} \times_{V_i} U_{ij}$, δ_π is a closed embedding and the fact that U_{ij}'s will cover X means that $X \to \cup(U_{ij} \times_{V_i} U_{ij})$ is a closed embedding and we have the factorization:

$$X \hookrightarrow \cup(U_{ij} \times_{V_i} U_{ij}) \hookrightarrow X \times_Y X$$

This suffices to show our assertion. $\square$

Remark 3.12. One of the key point you should be aware of is that these $U_{ij} \times_{V_i} U_{ij}$'s may not cover $X \times_Y X$, which is very different from the case of $U_i \times_{V_i} U_i$ if we take U_i to be the inverse image of V_i and we also used it in a previous proposition.

Sometimes, they do cover, but these are just very special cases. You can get a geometric intuition by think the case in the category of topological spaces, and you don't even need a concrete example. I will let you do this, try to convince yourself you actually got it.

As one application of one proposition we proved before, we can also prove that:

Proposition 3.13. If $X \to Y$ and $X \to Z$ are all locally closed embedding, then $X \to Y \times Z$ is a locally closed embedding.

Proof. Notice that locally closed embedding is a nice class of morphisms, let denote it by P, and notice that we showed that the diagonal morphism is always a locally closed embedding, that $X \to Spec\mathbb{Z}$ is in δP, thus we use Prop 3.4 to conclude that $X \to Y \times Z$ is in P, thus a closed embedding. $\square$

Now, here comes the important definition of sepratedness:

Definition 3.14. A morphism is separated if the diagonal morphism is a closed embedding. Also, an A-scheme X is called separated if the structure morphism $X \to SpecA$ is separated.

Notice that we have proved that if the diagonal morphism has a closed image in pure topological sense, this locally closed embedding is actually a closed embedding, thus a morphism being separated is a purely topological condition on the diagonal, reminiscent of the definition of Hasudorffness, as we promised. However, separatedness only looks like Hausdorffness while not implying it since in the category of schemes, the underlying space $X \times_Y X$ is usually not the product space.

Proposition 3.15. Separated morphisms form a nice class of morphisms.

Proof. Notice that the class of closed embeddings is a nice class of morphisms, thus we are done. $\square$

Proposition 3.16. Monomorphisms are separated, and in particular, locally closed embeddings are separated.

Proof. Recall that $X \to Y$ is a monomorphism iff the diagonal morphism $X \to X \times_Y X$ is an isomorphism, in particular a closed embedding. $\square$

Proposition 3.17. Separated morphisms are quasiseparated.

Proof. Notice that we define quasiseparated by the diagonal morphism is quasicompact, but when it's separated, the diagonal morphism is even a closed embedding, thus of course quasicompact. $\square$

The first example of separated morphisms are the affine morphisms:

Proposition 3.18. If $X \to Y$ is affine, then it's separated.

Proof. Being separated is affine on the target, then we pick an affine open cover of Y, we only have to show the restriction is separated for each of them. However, we see that morphisms between affine schemes are always separated. $\square$

The following results uses the Cancellation Thm 3.2:

Proposition 3.19. An A-scheme X is separated iff it's separated as an $\mathbb{Z}$-scheme.

Proof. Consider the following diagram:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & SpecA \arrow[dl] \\ & Spec\mathbb{Z} & \end{tikzcd}$$

Notice that if it's separated as an A-scheme, then the top morphism is separated, and $SpecA \to Spec\mathbb{Z}$ is also separated since it's a morphism between affine schemes. Then we can do composition, and since separated morphisms are closed under compositions, we are done.

Then suppose that it's an separated $\mathbb{Z}$-scheme, then notice that since $SpecA \to Spec\mathbb{Z}$ is also separated, thus the diagonal morphism is a closed embedding, thus also separated, thus by the cancellation theorem, we know the top morphism $X \to SpecA$ is also separated. $\square$

Proposition 3.20. Suppose that we have morphisms of schemes $X \xrightarrow{\pi} Y \xrightarrow{\rho} Z$, then:

1. If $\rho \circ \pi$ is a locally closed embedding or locally of finite type or separated then so is π.
2. If $\rho \circ \pi$ is quasicompact and Y is Noetherian, then π is quasicompact.
3. If $\rho \circ \pi$ is quasiseparated, then π is quasiseparated.

Proof. To prove assertion 1, let's consider locally closed embeddings first, notice that since the diagonal morphism is always a locally closed embedding, then we can use the Cancellation Thm. If we are dealing with locally of finite type morphisms, again, we already proved that locally closed embeddings are locally of finite type, then use the fact that locally of finite type is nice class and cancellation thm. Sepratedness is proved similarlly, we are done.

To show assertion 2, we know that any morphis from a Noetherian space is quasicompact, then use cancellation thm.

Finally, **** I didn't find out how to prove this via cancellation thm, here I give a different proof:*, To prove that $\pi : X \to Y$ is quasiseparated, we can use the local on target property of being quasiseparated, then we can assume that Z is affine, say $SpecA$. Then consider the following daigram:

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & X \times_Y X \arrow[d] \\ & X \times_Z X \end{tikzcd}$$

It's left to you to figure what the maps should be an why this is commutative. We know by assumption that $X \times X \times_X$ is quasicompact, we are trying to show that $X \rightharpoonup X \times_Y X$ is quasicompact. To do this, suppose that you start with an affine open cover of Y, say V_i, then take the inverse image in X, and cover them by affine opens, then we know that for affine opens U, V we had in X, the inverse image of $U \times_V W$ under $X \to X \times_Z X$ is known to be $U \cap W$, which is quasicompact since Z is affine, then $U \times_Z W$ is affine. But we claim that the inverse image of $U \times_Z W$ under $X \times_Y X \to X \times_Z X$ is $U \times_V W$, which is still affine, and we know that as U, V, W runs over all possible choices, we cover $X \times_Y X$, and the inverse image of $U \times_V W$ under $X \to X \times_Y X$ is still $U \cap W$, which is quasicompact, we are done. To show our claim, it fastest to think of it as the intersection of two inverse images, and you need to figure this out. $\square$

Proposition 3.21. Suppose that $\mu : Z \to X$ is a morphism and σ is a section of μ. Then, we have σ is a locally closed embedding and if μ is separated, σ is a closed embedding, if μ is quasiseparted, then σ is quasicompact.

Proof. The picture is this:

$$\begin{tikzcd} Z \arrow[r, "\mu"] & X \arrow[l, bend right, "\sigma"'] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "\sigma"] \arrow[dr, "id"'] & & Z \arrow[dl, "\mu"] \\ & X & \end{tikzcd}$$

Now, notice that $\mu \circ \sigma$ is the identity on X, which is in particular a locally closed embedding then σ is a locally closed embedding.

Now, suppose that μ is separated, we can use the cancellation thm since the diagonal is closed embedding and id is also a closed embedding.

Finally, suppose that μ is quasiseparated, we still can use cancellation. □

Below is another important definition:

Definition 3.22. Suppose that $\pi : X \to Y$ is a morphism of Z-schemes, the morphism $\gamma_\pi : X \times_Z Y$ given by (id, π) is called the **graph morphism**. Notice that π factor as $pr_2 \circ \gamma_\pi$, where pr_2 is the second projection morphism.

Notice that we have $pr_1 \circ \gamma_\pi$ is the identity, thus we know that γ_π is a section of pr_2. Apply the previous problem to pr_2, we obtain the following result:

Proposition 3.23. The graph morphism γ_π is always a locally closed embedding and if Y is a separated Z-scheme, γ_π is a closed embedding and is Y is a quasiseaprated Z-scheme, γ_π is quasicompact.

Proof. Use the fact that separated and quasiseaprated morphisms are closed under base change and then apply the previous proposition. □

Since $\gamma_\pi : X \to X \times_Z Y$ is always a locally closed embedding, we know will call the image of γ_π the **graph** of π,

Proposition 3.24. The line with the double origin is not separated, in particular you can show it by showing that the image of the diagonal morphism is not closed.

Proof. **** later* □

Next, we will show a result that gives us the first example of a separated morphism that is not affine:

Proposition 3.25. The morphism $\mathbb{P}^n_A \to SpecA$ is separated.

Proof. We will give two proofs, the first one is just direct computation, and the second one is by using the Veronese and Segre embedding we constructed in other review sections.

Let's do the direct computation first, we cover $\mathbb{P}^n_A$ by $U_0, \cdots, U_n$ the standard $n+1$ affine opens. The the standard fact is that $U_i \times_A U_j$ covers $\mathbb{P}^n_A \times_A \mathbb{P}^n_A$, now we are trying to prove:

$$\mathbb{P}^n_A \to \mathbb{P}^n_A \times_A \mathbb{P}^n_A$$

is a closed embedding, we use affine locality, we only have to show this for $U_i \times_A U_j$, notice that when $i = j$, we already saw that this is:

$$U_i \to U_i \times_A U_i$$

which is always a closed embedding, thus we only have to look at the case when $i \neq j$.

Recall how we defined the morphism $\mathbb{P}^n_A \to \mathbb{P}^n_A \times_A \mathbb{P}^n_A$, it defined by gluing local morphisms on affines, the notice that the intersection of U_i and U_j under pr_1 and pr_2 in $\mathbb{P}^n_A \times_A \mathbb{P}^n_A$ is $U_i \times_A U_j$, then the intersection in $\mathbb{P}^n_A$ is $U_i \cap U_j$. Then the morphism from $U_i \times U_j$ to U_i and U_j is defined by: $A[x_{0/i}, \cdots, x_{n/i}]/(x_{i/i} - 1) \to A[x_{0/i}, \cdots, x_{n/i}, x_{j/i}^{-1}]/(x_{i/i} - 1)$ where $x_{k/i} \mapsto x_{k/i}$ and $A[y_{0/j}, \cdots, y_{n/j}]/(y_{j/j} - 1) \to A[x_{0/i}, \cdots, x_{n/i}, x_{j/i}^{-1}]/(x_{i/i} - 1)$ where $y_{k/j} \mapsto \frac{x_{k/i}}{x_{i/j}}$, by this you can see that the morphism that goes from $A[x_{0/i}, \cdots, x_{n/i}]/(x_{i/i} - 1) \otimes_A A[y_{0/j}, \cdots, y_{n/j}]/(y_{j/j} - 1) \to A[x_{0/i}, \cdots, x_{n/i}, x_{j/i}^{-1}]/(x_{i/i} - 1)$ is surjective, we are done.

Now let's see the second approach: **** later* □

Now we can finally give the definition in the classic algebraic geometry, varieties:

Definition 3.26. A **variety** over a field k, or a k-**variety**, is a reduced separated scheme over k.

For example, a reduced, finite type affine k-scheme is a variety. Here the reducedness is in the assumption, it's separated because it's an affine scheme over $Speck$, thus the structure morphism is always separated. Recall that in the classic algebraic geometry, we define an **affine variety** over k to be a affine scheme that is reduced and finite type over k. Also, we also can define (**quasi**)**projective** varieties over k to be a reduced (quasi)projective k-scheme. Remember that quasiprojective here means a quasicompact open subscheme of $ProjS$ where S is a finitely generated graded ring over A.

Remark 3.27. Sometimes, in different literature, we may also assume that a variety is irreducible or the field is algebraically closed.

It's also easy to see that k-varieties form a category with morphisms being morphisms of k-schemes.

We can also define a **subvariety** of a k-variety X, it's just a reduced locally closed subscheme of X. It's by definition reduced it's separated becasue that composition of separated morphism is still separated. Therefore, any **open subvariety** of X is an open subvariety of X. Also, a **closed subvariety** of X is just a reduced closed subscheme of X.

Now, we begin to look at products of varieties:

Proposition 3.28.
1. If k is a perfect field the product of k-varieties, defined to be the fibered product over k, is still a k-variety.
2. If k is algebraically closed, then the product of irreducible varieties is irreducible.
3. If k is algebraically closed, then the product of connected varieties is connected.

Proof. Let's deal with the first assertion. Suppose that X, Y are k-varieties, then we have:

$$\begin{tikzcd} X \times_k Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Speck \end{tikzcd}$$

Use the fact that finite type and separatedness is preserved by base change, we only have to prove the reducedness part. Notice that a scheme is reduced iff there is an open cover of it which is reduced, thus we can reduced to the affine case, where $X = SpecA$ $Y = SpecB$ both A, B are reduced k-algebra, then we need to prove $A \otimes_k B$ is still reduced. **** fill in the missing part after we are done with the review.*

Then for the second assertion, **** fill in the details after we are done with the review.* ☐

We go back to the general properties of separatedness, where we prove an surprising fact is that intersection of affine opens is affine in a separated scheme:

Proposition 3.29. Suppose that $X \to SpecA$ is a separated morphism, then if U, V are affine opens of X, $U \cap V$ is also an affine open of X. In particular, intersection of any two affine opens of $SpecA$ is affine, which is surprising since we already know that in general we have little handle on general open subsets of $SpecA$.

The proof of the previous statement uses the following result:

Proposition 3.30. Suppose that U, V are open subsets of an A-scheme X, then we have a canonical isomorphism: $\Delta \cap (U \times_A V) \cong U \cap V$.

Proof. Consider the following diagram:

$$\begin{tikzcd} \cdots \arrow[r] \arrow[d] & U \times_A V \arrow[d, hook] \\ X \arrow[r, "\delta"'] & X \times_A X \end{tikzcd}$$

with $\cdots$ representing the fibered product, we are trying to figure out what is $\cdots$. Using the base change diagram, and the morphis $X \to SpecA$, we can conclude that $\cdot$ is just $U \times_X V$, which is canonically isomorphic to $U \cap V$ since this is already the first result we had for fibered products of open embeddings. ☐

Now, let's start the proof of Prop 3.29:

Proof. Notice that $U \times_A V$ is affine, then notice that $\Delta \cap (U \times_A V)$ is a closed subscheme of an affine open, thus affine which is isomorphic to $U \cap V$, thus the intersection is affine. $\square$

Below is a result that generalize the classical definition of quasiprojective varieties:

Proposition 3.31. Every quasiprojective A-scheme is separated over A, then in particular

Proof. Suppose that X is a quasiprojective A-scheme, then by definition, this structure morphism is decomposed into X is an locally closed subschem of $\mathscr{P}^n$ (make sure you know why!) followed by $\mathbb{P}^n_A \to SpecA$, which are all separated, then use separatedness if closed under composition. $\square$

Now we begin to deal with an important theme. Recall that we defined what is a rational map before, and we promised back then that, **in good circumstances**, we have a largest domain of definition for rational maps. We will make this precise in the following discussion:

Proposition 3.32. Suppose that $\pi : X \to Y$ and $\pi' : X \to Y$ are two morphisms over some scheme Z:

$$\begin{tikzcd} X \arrow[rr, shift left, "\pi'"] \arrow[rr, shift right, "\pi"'] \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

Suppose that $\mu : W \to X$ is any morphism of schemes, we say that π and π' agrees on μ if $\pi \circ \mu = \pi' \circ \mu$. Then there is a locally closed embedding $i : V \hookrightarrow X$ on which π and π' agree such that this V is the largest such locally closed subschemes in the sense that any morphism $\mu : W \to X$ on which π and π' agree factors uniquely through i, or you can think of the following diagram:

$$\begin{tikzcd} V \arrow[dr, hook, "i"] & & & \\ & X \arrow[rr, shift left, "\pi'"] \arrow[rr, shift right, "\pi"'] \arrow[dr] & & Y \arrow[dl] \\ W \arrow[uu, "\exists ! j"] \arrow[ur, "\mu"] & & Z & \end{tikzcd}$$

Moreover, if Y is separated over Z, then this $i : V \to X$ is a closed embedding.

Proof. This is the first example where you see that if the target is separated then there is some good properties for the morphism to enjoy.

Now to define this $i : V \to X$, which is supposed to be a locally closed embedding and has a certain mapping property that has maps into it, we guess that V is probably some fibered product, and this $i : V \hookrightarrow X$ is probably some pullback of another locally closed embedding and you want it to be closed when $Y \to Z$ is separated. What maybe a good candidate? The only natural locally closed embedding that is closed when $Y \to Z$ is separated is $Y \to Y \times_Z Y$, then you can consider the following daigram:

$$\begin{tikzcd} V \arrow[d] & Y \arrow[d] \\ X & Y \times_Z Y \end{tikzcd}$$

You only need to figure out what is the morphism $X \to Y \times_Z Y$, and what could it be that's related to both π, π, it can only naturally be the morphism induced by the universal property of $Y \times_Z Y$ and π, π'. This is why you get:

$$\begin{tikzcd} V \arrow[r] \arrow[d] & Y \arrow[d, "\delta"] \\ X \arrow[r, "(\pi,\pi')"] & Y \times_Z Y \end{tikzcd}$$

Then you only have to check that V satisfies the mapping property we mentioned above, I will let you think about it. $\square$

The $i : V \to X$ in the above proposition is called the **largest locus where π, π' agree**.

Below are some useful remarks you can use to think of this $i : V \hookrightarrow X$:

Remark 3.33. It's not a surprise that V can be nonreduced. Consider the two maps from $\mathbb{A}^1_k$ to itself defined by $k[x] \to k[x]$ where $x \mapsto 0$ and $x \mapsto x^2$. Notice that in terms of the points, they agree on $x = 0$, the origin, however, the closed embedding $i : V \hookrightarrow X$ is defined to be the fibered product, and you can see what it is: $Speck[t]/(t^2)$, and I will let you compute it, since everything here is affine.

Also, the above construction is the justification of the universal property definition of $i : V \to W$, and after you finished the construction, for theoretical purposes, you should think of it using the universal property.

Also, even two morphisms agrees at the level of the map of sets, these two morphisms might still be different as morphisms of schemes. To see an example, consider the two morphism $Spec\mathbb{C} \to Spec\mathbb{C}$ defined by the identity and the complex conjugation. As morphisms over $SpecR$, it's obvious that as a map of topological spaces, they both map (0) to (0). However, this is not the same as morphisms of schemes, and you can see this since they have different maps on the global sections of the sturcture sheaf.

Finally, we can define the set-theoretic locus where π, π agree by: π, π' agrees as morphisms of schemes if they agree as maps of sets $\pi(p) = \pi'(p)$ and the induced morphism on the residue fields are the same. This is a reasonable definition, since what this definition is capturing is the set-theoretic image of the map $i : V \hookrightarrow X$. If p is in this image, then we see that since $i : V \to X$ is a locally closed embedding, then we have the following diagram:

$$\begin{tikzcd} V \arrow[r, hook, "closed"'] & U \arrow[r, hook, "open"'] & X \arrow[r, shift left, "\pi'"] \arrow[r, shift right, "\pi"'] & Y \end{tikzcd}$$

Notice that we can compute the induced morphism on residue field by choosing affine opens, then since p is in the image of i, thus it's also in the image of $V \hookrightarrow U$, choose affines in Y and in U such that we the induced morphims on residue fields is:

$$B_p/pB_p \to A_q/qA_q$$

But notice that you can also consider the morphism on residue fields induced by $i \hookrightarrow X \to Y$, which is going to be:

$$B_p/pB_p \to A_q/qA_q \to (A_q/I_q)/(qA_q/I_q) \cong A_q/qA_q$$

It's left for you to think about why this suffice.

Conversely, if $\pi(p) = \pi'(p)$ and we have the map on residue fields agree, then you can use the universal property to show that p lies in the image of i.

The following result is important in terms of our understanding of k-varieties over a field k and also gives you some practice in terms of the notion of locus where two morphisms agree:

Proposition 3.34. Suppose that $\pi, \pi' : X \to Y$ are two morphisms of $k = \overline{k}$-varieties that agree on closed points not in the sense we defined above but only in terms of $\pi(p) = \pi'(p)$ for all closed points. Then we have $\pi = \pi'$ even in the sense of morphisms of schemes. Finally, this is also true even when k is not algebraically closed.

Proof. Let's first do this for algebraically closed fields, then generalize it appropriately.

You should notice that for a closed point, the Nullstellensatz said that the residue field is just a finite extension of k, but in this case this is just itself since k is algebraically closed, thus the induced morphism of residue fields is just a map between k. It's obviously not the zero map since we assumed that k is not the zero field. Then we consider the following diagram:

$$\begin{tikzcd} k \arrow[rr, "iso"] & & k \\ & k \arrow[ul, "id"] \arrow[ur, "id"'] & \end{tikzcd}$$

you can figure why this suffice to show that the maps on residue fields by both the morphisms are just the identity.

This proves that each closed points in in the image of $i : V \hookrightarrow X$, then since we assumed that the variety is separated, then the i is a closed embedding, thus the image is closed, containing all closed points, which is dense, thus the image of i is the whole variety.

Finally, we need to think why this implies the result for non algebraically closed fields k. Since we know that every algebraically closed fields are prefect, Consider the following diagram:

$$\begin{tikzcd} X \arrow[r] & Y \\ X \times_k \overline{k} \arrow[u] \arrow[r] & Y \times_k \overline{k} \arrow[u] \end{tikzcd}$$

Being perfect means that the bottom two stuff are still varieties, and since closed points are mapped to closed points, we know that the induced map $X \times_k \overline{k} \to Y \times_k \overline{k}$ agree on closed points, thus we consider:

$$\begin{tikzcd} V \arrow[r, "i"'] & X \arrow[r] & Y \\ V' \arrow[u, dashed] \arrow[r, "i'"'] & X \times_k \overline{k} \arrow[u, "surjective"'] \arrow[r] & Y \times_k \overline{k} \arrow[u] \end{tikzcd}$$

This implies that $i : V \hookrightarrow X$ has image the whole X, thus i is an isomorphism, we are done. $\square$

Remark 3.35. We want to make a special comments here about our assertion in the above prove that morphisms of varieties maps closed points to closed points.

This actually holds for morphisms between schemes locally of finite type over k. Say $f : X \to Y$ is a morphism of schemes locally of finite type over k and $x \in X$ is a closed point, we want to prove that $f(x)$ is finite type, take two affine open neighborhoods of $x, f(x)$, say $SpecA$ and $SpecB$, such that A, B are both finitely generated k-algebras.

To proceed further, we need the statement of Hilbert's Nullstellensatz. Suppose that k is a field then for any maximal ideal of $k[x_1, \cdots, x_n]$, the residue field is a finite extension of k. Using this theorem, plus the fact that if A is an integral domain which is a finite k-algebra, then it's necessarily a field. Then say that p is a prime ideal of $k[x_1, \cdots, x_n]$ such that the residue field is a finite extension of k, we claim that p is a maximal prime. This is equivalent to prove that $k[x_1, \cdots, x_n]/p$ is a field, but notice that it's a integral domain, and we have a natural inclusion:

$$A/p \hookrightarrow (A/p)_p \cong A_p/pA_p$$

thus A/p is a finite dimensional k-vector space, thus A/p is a field by the above discussion, we are done.

Now, consider this field extension:

$$k(f(x)) \hookrightarrow k(x)$$

induced by the morphism of schemes notice that this fits into this diagram:

$$\begin{tikzcd} k(f(x)) \arrow[rr, hook] & & k(x) \\ & k \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

Notice that $k(x)$ is a finite dimensional k-vector space and $k(f(x))$ is now a subspace, thus still finite dimensional, thus by the discussion above this corresponds to a maximal ideal in $SpecA$, now we only have to show that it's a closed point globally. Suppose that now we cover X by $SpecA_i$'s, we only have to show that in each $SpecA_i$, it's closed, but notice that the above discussion shows that p is closed in each $SpecA_i$ that contains it, we are done.

Now, we begin to prove the central result we promised earlier, the reduced to separated theorem, which makes sense of the domain of definition for rational maps:

Theorem 3.36. Suppose that two S-morphisms $\pi : U \to Z$ and $\pi' : U \to Z$ is agrees on a dense open of U, then if U is reduced and Z is separated over S, π and π' agree on all of U.

Proof. Let V be the locus where π, π' agree, it's a closed subscheme of U since we assumed that Z is separated over S. Also, since we assumed that the points where π, π' agree is a dense open in U, V contains all of U. However, notice that since U is reduced, V as a closed subscheme of U whose underlying topological space is all of U is U itself.

The last assertion holds because say the closed subscheme is defined by the ideal sheaf $\mathscr{I}$, then on any affine open $SpecA$ of U, the inverse image is $SpecA/I$ for $I = \mathscr{I}(SpecA)$, however, since we assume that $SpecA/I$ is via the morphism $SpecA/I \to SpecA$ homeomorphic to $SpecA$, this $I \subset \mathfrak{N}(A)$, but we assumed that the ring is reduced, thus $I = 0$, thus $\mathscr{I}$ is 0, we are done. □

Example 3.37. This following example shows that if we give up the assumption on the reducedness of the source or the separatedness of the target, the theorem fails.

For the failure when the source is separated, just consider the case where we have morphisms from $Speck[x, y]/(y^2, xy)$ to $Speck[t]$ defined by $t \mapsto x$ and $t \mapsto x + y$. Then, notice that the target is definitely separated over k since the structure morphism to k is just an affine morphism. Also, you should notice that they agree on $D(x)$ which is a dense open since $\mathbb{A}^2$ is integral, but you can easily see that they are two different morphisms since the ring homomorphisms are different.

Then, when you consider the case when the separatedness of the target is removed, say, you take the affine line with the double origin, then you can just see that set-theoretically, the agreeing locus can't be agreeing at the origin.

Now, we see some consequences of this reduced to separatedness theorem:

Remark 3.38. If X, Y are such as in Thm 3.36, then suppose that U, V are open subschemes of X, which are automatically reduced. Then if you consider two morphisms $\pi : U \to Y$ and $\pi' : V \to Y$ such that they agree on a dense open of the intersection, then they necessarily agree on $U \cap V$.

This gives us the right of saying the domain of definition of a rational map. Suppose that we are still in the same set up as above, the discussion tells us that any two representatives of a rational map automatically agrees on the intersection. Thus the union of all the domains of every representative of a rational map is the largest domain of definition.

For example, we will ask what is the rational map $\mathbb{A}^2 \dashrightarrow \mathbb{P}^1$ defined by the map:

$$(x, y) \mapsto [x : y]$$

has the domain of definition $\mathbb{A}^2_k - \{0\}$ **** later check this!*.

The complement of the domain of definition is called the **indeterminacy locus** of the rational map and it's points are called the **fundamental points** of the rational.

Now, we begin to describe the graph of a rational map and see an example how can we use this it to extend a rational even away from the domain of definition:

**** later after we review the rational maps*

Finally, we want to see that some variations of the proof we gave before to Thm 3.36, and something more can be said in this direction:

Proposition 3.39. Suppose that X is a Z-scheme, which is not necessarily reduced and Y is a separated Z-scheme

Proof. **** later after we review Cartier divisors* □

Proposition 3.40. Two S-morphisms $\pi, \pi' : U \to Z$ from a locally Noetherian scheme to a separated S-scheme Z agreeing on an open subset cintaining the associated points of U are the same.

Proof. **** later after we review the associated points* □

Finally before we end this section, we introduce the notion of proper morphisms, which we will have more to say in other sections of review:

We first start by some topological motivations: Recall that a continuous map is said to be proper if the preimage of any compact set is compact. The notion of proper morphisms in the category of schemes is a similar property: For example, a $\mathbb{C}$-variety is proper if it's "compact" in the classical topology which is good to know as a geometric intuition, thus you can probably guess that $\mathbb{A}^n_{\mathbb{C}}$ is not going to be proper over $\mathbb{C}$ while $\mathbb{P}^n_{\mathbb{C}}$ is going to be proper.

Now, we start defining what is a proper morphism: Recall that a morphism of topological spaces is called **closed** if image of closed subsets is closed and a morphism of schemes is called closed if the underlying topological morphism is closed, however, this is a property that is not preserved by base change, even though it's easy to see that it's preserved under composition and local on the target(you should make sure you know how to show it). To resolve this property, we define the following notion:

Definition 3.41. A morphism of schemes $\pi : X \to Y$ is called **universally closed** if for every morphism $Z \to Y$, the induced morphism $Z \times_Y X \to Z$ is closed, and this is exactly the condition where being closed is preserved by base change.

Thus, we automatically have the following result:

Proposition 3.42. The class of universally closed morphisms is a nice class of morphisms.

Proof. Clear from our discussion above. □

Notice that this definition also makes sense in the category of topological spaces, and as a motivation of the following definition of proper morphisms of schemes, one can prove that in the category of topological spaces, a morphism between locally compact, hausdorff spaces with a countable bais is proper iff it's universally closed, which leads to:

Definition 3.43. A morphism $\pi : X \to Y$ of schemes is called **proper** if it's separated, finite type and universally closed. If we say that a scheme is proper, then what we mean is that some implicit structure morphism is proper. For example, we say that an A-scheme is proper if the structure morphism $X \to SpecA$ is proper.

In the above definition, the separatedness is imitating the Hausdorffness, the finite type is trying to imitating the countable condition and the locally compactness. And we don't have to mention that since each property of the morphism is preserved by composition, base change and local on the target, proper morphisms are a nice class of morphisms.

Let's try this idea in practice:

Example 3.44. As we promised in the motivation part, we expect that $\mathbb{A}^n_{\mathbb{C}} \to Spec\mathbb{C}$ is not proper. Notice that this morphism is obviously finite type, separated, and closed for trivial reasons. Therefore, the only thing that can go wrong is the universal closed part.

One possible approach is to consider using $\mathbb{P}^1_{\mathbb{C}} \to Spec\mathbb{C}$, and consider the following diagram:

$$\begin{tikzcd} \mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}} \arrow[r] \arrow[d] & \mathbb{P}^1_{\mathbb{C}} \arrow[d] \\ \mathbb{A}^1_{\mathbb{C}} \arrow[r] & Spec\mathbb{C} \end{tikzcd}$$

Consider the diagram, let's first consider what is the closed subset in $\mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$ that we will use. First, notice that $\mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$ is covered by two $\mathbb{A}^2_{\mathbb{C}}$, then if we can find a closed subset in the intersection that is not mapped to a closed subset, we are done. Notice that the intersection is just the inverse image of the intersection of the two $\mathbb{A}^1_{\mathbb{C}}$ in $\mathbb{P}^1_{\mathbb{C}}$, which is characterized by:

$$\begin{tikzcd} Spec\mathbb{C}[t, \frac{x}{y}, \frac{y}{x}] \arrow[r] \arrow[d] & Spec\mathbb{C}[\frac{x}{y}, \frac{x}{y}] \arrow[d] \\ \mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}} \arrow[r] \arrow[d] & \mathbb{P}^1_{\mathbb{C}} \arrow[d] \\ \mathbb{A}^1_{\mathbb{C}} \arrow[r] & Spec\mathbb{C} \end{tikzcd}$$

and it's easy to see that the morphism is determined by $\mathbb{C}[\frac{x}{y}, \frac{y}{x}] \hookrightarrow \mathbb{C}[t, \frac{x}{y}, \frac{y}{x}]$, thus notice that if we consider the closed subscheme of this open subscheme defined by modding out the ideal t, which is $V(t)$, then we are left with the identity morphism $Spec\mathbb{C}[\frac{x}{y}, \frac{y}{x}] \to Spec\mathbb{C}[\frac{x}{y}, \frac{y}{x}]$, thus the image is obvious the open subset $D_+(x, y)$ in $\mathbb{P}^1_{\mathbb{C}}$.

Obviously, this argument can be carried out for any $\mathbb{A}^n_{\mathbb{C}} \to Spec\mathbb{C}$ via the base change by $\mathbb{P}^n_{\mathbb{C}} \to Spec\mathbb{C}$. Therefore, what we really got here is that any $\mathbb{A}^n_{\mathbb{C}}$ is not a proper scheme over $\mathbb{C}$.

Now, we can see two easy examples of proper morphisms:

Example 3.45. First, it's easy to see that closed embeddings are proper. This is because that closed embedding are clearly finite type separated, we only have to show that it's universally closed. But notice that base change of closed embeddings are still closed embeddings, thus we are done.

Following the same line of thinking, we can easily prove that finite morphisms are proper. First, it's finite type and separated since it's an affine morphisms. Also, we know that finite morphisms are closed, and base change of finite morphism is still finite, we are done. We are not including integral morphisms here since an integral morphism maynot be finite type.

Since we already showed that proper morphisms are a nice class of morphisms, we know that it satisfies all the properties that are enjoyed by nice class of morphisms. We won't write them out in detail, even though we do want to write down one of them:

Proposition 3.46. Suppose that we have a commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "\tau"'] & & Y \arrow[dl, "\rho"] \\ & Z & \end{tikzcd}$$

with τ being proper and ρ being separated, then π is proper.

Proof. By Cancellation Theorem, we only have to prove $Y \to Y \times_Z Y$ is proper, but since ρ is separated, the diagonal is a closed embedding, thus proper. □

We can also show that the scheme theoretic image of a morphism coming out of proper scheme under mild assumptions is going to be proper. The reason why we want to prove this is because that we can introduce something we didn't do when we discussed scheme theoretic images before. We start by a useful factorization result which you probably expected before:

Proposition 3.47. Suppose that $f : X \to Y$ is a morphism of schemes, and $Z \hookrightarrow Y$ is a closed subscheme defined by the quasicoherent ideal sheaf $\mathscr{I}$, then if the following composition is zero:

$$\mathscr{I} \hookrightarrow \mathscr{O}_Y \to f_*\mathscr{O}_X$$

you have a factorization:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, dashed, "f'"'] & & Y \\ & Z \arrow[ur, hook] & \end{tikzcd}$$

Proof. Let's first show that we have this factorization in terms of topological spaces, which is exactly saying that the set-theoretic image of f is contained in Z. Now, suppose that $x \in X$, you want to show that $f(x) \in Z$, what you do is simple where you choose affines around X and $f(x)$ such that you have an restricted morphism:

$$SpecA \to SpecB$$

induced by a ring morphism, then notice that the closed subscheme is defined by $SpecB/I \hookrightarrow SpecB$, then since we have the following composition is 0, we have:

$$I \to B \to A$$

is zero, then you can use universal property of quotient to prove the rest.

This result plus basic topology argument gives you f' uniquely. Now, you are supposed to find out the morphism of sheaves, to obtain this, we can use the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{I} \arrow[r] & \mathscr{O}_Y \arrow[r] \arrow[dr] & i_*\mathscr{O}_Z \arrow[r] \arrow[d, dashed] & 0 \\ & & & f_*\mathscr{O}_X & \end{tikzcd}$$

where the morphism from $i_*\mathscr{O}_Z$ to $f_*\mathscr{O}_X$ is induced by the universal property. Then if we want to give $\mathscr{O}_Z \to f'_*\mathscr{O}_X$, we are supposed to give for each $V \subset Z$ open, a morphism $\mathscr{O}_Z(V) \to \mathscr{O}_X((f')^{-1}V)$. I claim that this is just the induced map given above. Also, you have to show that this is a map of locally ringed spaces, but this follows from that we know that if you consider the commutative diagram:

$$\begin{tikzcd} \mathscr{O}_Y \arrow[rr] \arrow[dr] & & i_*\mathscr{O}_Z \arrow[dl, dashed] \\ & f_*\mathscr{O}_Y & \end{tikzcd}$$

For $p \in Z$, the induced map is just the same as:

$$\begin{tikzcd} A_p \arrow[rr] \arrow[dr] & & A_p/I_p \arrow[dl, dashed] \\ & (f_*\mathscr{O}_Y)_p & \end{tikzcd}$$

then you can easily verify this induced morphism maps the maximal ideal into maximal ideal. $\square$

Proposition 3.48. We still consider the diagram as in the previous proposition, we assume that τ is proper, ρ is separated and finite type, then the scheme-theoretic image of π is going to be proper over Z.

Proof. Let's use the property we just proved. Consider the following diagram:

$$\begin{tikzcd} & W \arrow[dr, hook] & \\ X \arrow[rr, "\pi"] \arrow[dr, "\tau"'] & & Y \arrow[dl, "\rho"] \\ & Z & \end{tikzcd}$$

We first know that π is separated, thus the image in Y is a closed subset, and being proper implies that it's quasicompact, thus we know that the image of $W \hookrightarrow Y$ is the same as the image of X in Y. Notice that $W \hookrightarrow Y \to Z$ is separated finite type, thus we only have to prove it's universally closed. Now consider the following diagram where the induced morphism of schemes is what we proved in the previous result:

$$\begin{tikzcd} & W \arrow[dr, hook] & \\ X \arrow[ur, dashed] \arrow[rr, "\pi"] \arrow[dr, "\tau"'] & & Y \arrow[dl, "\rho"] \\ & Z & \end{tikzcd}$$

which is surjective. Then the question now becomes that if we have a surjective morphism $f : X \to Y$ over a scheme Z with $X \to Z$ universally closed, then we want to show that $Y \to Z$ is also universally closed, the picture is as below:

$$\begin{tikzcd} X \arrow[rr, two heads, "\pi"] \arrow[dr, "\tau"'] & & Y \arrow[dl, "\rho"] \\ & Z & \end{tikzcd}$$

To show our assertion, consider the following diagram:

$$\begin{tikzcd} X \times_Z T \arrow[r, two heads] \arrow[d] & Y \times_Z T \arrow[r] \arrow[d] & T \arrow[d] \\ X \arrow[r, two heads] & Y \arrow[r] & Z \end{tikzcd}$$

Notice that since surjectivity is preserved by base change, thus $X \times_Z T \to Y \times_Z T$ is surjective, then say C is closed in $Y \times_Z T$, the inverse image is closed, and also surjective to C via $X \times_Z T \to Y \times_Z T$, now notice that $X \to Z$ is universally closed, thus the top composition is closed, thus we are done. ☐

The last one and most important example of proper morphism is the following:

Theorem 3.49. Projective A-schemes are proper over A, and since we know that finite morphism is projective, this can be used to give the second proof of why finite morphisms are proper.

Proof. We will use a result we haven't proved yet which is the Fundamental Theorem of Elimination Theory. Let's first assume it, and we will do the proof later in other review sections.

First, the structure morphism of any projective A-schemes $X \to SpecA$ can be factored as $X \hookrightarrow \mathbb{P}^n_A \to SpecA$, since we know that closed embeddings are proper and compositions of proper morphisms is proper, we only have to show that $\mathbb{P}^n_A \to SpecA$ is proper. We already know that it's finite type and separated, thus we only have to show that it's universally closed.

Consider the following diagram:

$$\begin{tikzcd} \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} X \arrow[r] \arrow[d] & \mathbb{P}^n_A \arrow[r] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d] \\ X \arrow[r] & SpecA \arrow[r] & Spec\mathbb{Z} \end{tikzcd}$$

I will let you figure out why two small squares are Cartesian, thus you only have to prove that $\mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} X \to X$ is closed. Notice that being closed is local on the target, thus you can assume that X is affine $SpecB$, thus you only have to show that $\mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} SpecB \cong \mathbb{P}^n_B \to SpecB$ is closed for any ring B, which is the Fundamental Theorem of Elimination Theory. ☐

Remark 3.50. We just saw that projective morphisms are proper, but it's hard to give an example of a proper morphism that is not projective. First, how to characterize projective morphisms in a good way such that you can determine if a morphism is projective or not easier is a problem we will discuss later. And you will also see that indeed a lot proper morphisms is indeed projective.

For examples of proper morphisms that are not projective, we also need to learn further.

As one application of the notion of properness, we can revisit a result we had before and prove a upgraded result below. Recall that if k is a field, then we already saw that the global functions on $\mathbb{P}^n_k$ for any field is k, if we assume that $k = \overline{k}$, algebraically closed, we have:

Proposition 3.51. Suppose that X is a connected reduced proper scheme over k, then the global functions are just k.

Proof. Suppose that $f \in \Gamma(X, \mathscr{O}_X)$, a global function, this is the same as giving a morphism of k-schemes $X \to \mathbb{A}^1_k$, now we consider the open embedding $\mathbb{A}^1_k \to \mathbb{P}^1_k$, now since $\mathbb{P}^1_k \to Speck$ is separated we know that $X \to \mathbb{P}^1_k$ is proper, thus in particular the set-theoretic image is a closed subset lying in $\mathbb{A}^1_k \subset \mathbb{P}^1_k$ and it should also be connected since we assumed that X is connected in the first place. Now we claim this implies that the image is either a single closed point or the entire space. This is because that we know well about what is the closed subsets in $\mathbb{P}^1_k$ when k is algebraically closed. It's just the union of finitely many closed points or the entire space. (Make sure you understand this!)

But we also know that the image is in $\mathbb{A}^1$, thus the image is just a single closed point $p \in \mathbb{A}^1$. Notice that this map $X \to \mathbb{A}^1_k$ is with a reduced scheme, thus the scheme theoretic image is also supported in one single closed point. Also, since X is reduced, we already proved that the scheme theoretic image is reduced, thus it's just $Speck \to \mathbb{A}^1_k$ where the point of $Speck$ is mapped to the closed point p since we know that the reduced scheme structure is the only one closed subscheme with the same under lying topological space that is closed. Then consider the following diagram:

$$\begin{tikzcd} X \arrow[rr, "x \mapsto f"] \arrow[dr, "1 \mapsto 1"'] & & \mathbb{A}^1_k \\ & Speck \arrow[ur, "x \mapsto m"'] & \end{tikzcd}$$

you see that $m \in k$ is mapped to $f \in \Gamma(X, \mathscr{O}_X)$ where $X \to Speck$ is induced by the property we proved for the scheme theoretic image. This should be convincing enough for us to believe that the elements in $\Gamma(X, \mathbb{O}_X)$ can be realized by an elements in k, but the formal arguments is the following: notice that the morphism $X \to Speck$ is the structrue morphism of X being a k-scheme. $\square$

Remark 3.52. The conditions in the above result is still two strong, it's natural to ask what are the conditions that can be relaxed. The fact that k is algebraically closed can't be ignored since if you consider $\mathbb{R}$, $Spec\mathbb{C}$ is a nice $\mathbb{R}$-scheme that is reduced, proper(it's finite) and connected, but notice that the global functions of $\mathbb{C}$ is $\mathbb{C}$. But we can now show an example where the scheme is not reduced but the global section is reduced. This is also an example showing that the converse of the fact that a reduced scheme will have reduced global sections is not true.

Consider the closed subscheme in $\mathbb{P}^2_k$ for any field k cut out by $x^2 = 0$. Notice that this implies that in $D_+(x)$, the closed subscheme is empty. Also in $D_+(y)$ and $D_+(z)$, the closed subscheme are just $Speck[\frac{x}{y}, \frac{z}{y}]/(\frac{x}{y}^2 = 0)$ and $Speck[\frac{x}{z}, \frac{y}{z}]/(\frac{x}{z}^2 = 0)$, you can see that this scheme is not reduced. I will let you to do the rest of the computation using the sheaf axioms to see that the global section is just k. (Notice that it's only covered by two affine opens, not 3, thus the computation should be a little bit easier.)

References