Appendix 9 for Math 632: 2025 Winter Notes

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In this chapter, we will introduce two useful construction called $\mathcal{S}pec$ and $\mathcal{P}roj$, which is the globalization and generalization of the well-known construction $Spec$ and $Proj$ of rings and graded rings. We will use universal property and local description to describe them, very similar to what we did for fibered product and normalizations. The universal property helps us to glue properly without too much pain. The difficulty mainly lies in how to find the right universal property, especially for $\mathcal{P}roj$.

1 The Global Spec Construction

Given an A-algebra R with $A \to R$, we can apply $Spec$ to get an A-scheme $SpecR$, we will now use a universal property to globalize this construction. Consider an arbitrary scheme X and a quasicoherent sheaf of algebras $\mathcal{R}$ on X, we want to define the "Spec" of the sheaf of algebras and we will get an X-scheme that is "affine" over X, in the sense that the structure morphism to X is affine. This will also be a good place to investigate the universal property under Yoneda's lemma, which generalizes the way we used to describe universal properties.

First and most concretely, we know that since $\mathcal{R}$ is a quasicoherent sheaf of algebras, over each affine open $SpecA$, $\mathcal{R}$ gives you an A-algebra R, and $\mathcal{S}pec\mathcal{R}$ locally over $SpecA$ is just $SpecR$. Then we need to figure out how to glue them, and the right way to do it is via the following universal property. To get some motivation, recall that if you have an A-algebra R, then we know that $SpecR \to SpecA$ satisfies the universal property that for any A-scheme $W \to SpecA$, A-scheme morphisms $\mu : W \to SpecR$ are in functorial bijections with the A-algebra morphisms $R \to \Gamma(W, \mathcal{O}_W)$. Since $\mathcal{R}$ and $\mathcal{O}_X$ are quasicoherent, you can think of this as the statement that the following diagram of schemes:

$$\begin{tikzcd} SpecR \arrow[dr, "\beta"'] & & W \arrow[ll] \arrow[dl, "\mu"] \\ & SpecA & \end{tikzcd}$$

corresponds in a one to one fashion to the following diagram of $\mathcal{O}_X$-algebra morphisms:

$$\begin{tikzcd} \mathcal{R} \arrow[rr] & & \mu_*\mathcal{O}_W \\ & \mathcal{O}_X \arrow[ul] \arrow[ur] & \end{tikzcd}$$

where $\mathcal{O}_X \to \mu_*\mathcal{O}_W$ comes from the morphism μ and $\mathcal{O}_X \to \mathcal{R}$ comes from the $\mathcal{O}_X$-algebra structure.

You may have noticed that this universal property is not that similar to what we have used to. Usually a universal property is given by a statement using factorizations within a certain category. But here we work with the category of A-schemes and $\mathcal{O}_X$-algebras, two categories. The correct way to interpret a universal property is via the Yoneda's lemma and universal property always means that an object represents a certain functor. For example, here this functor is:

$$F : A\text{-schemes}^{op} \to Sets \qquad W \mapsto \mathrm{Hom}_{A\text{-algebra}}(R, \Gamma(W, \mathcal{O}_W))$$

And the universal property says that $SpecR$ is the object X in the category of A-schemes, such that we have an isomorphism of functors:

$$\mathrm{Hom}_{A-\mathrm{scheme}}(W, X) \simeq F(W)$$

Then Yoneda's lemma tells you that any other object Y representing this functor is unique up to an isomorphism to $SpecR$. This unique isomorphism comes from the isomorphism $X \simeq Y$ $\mathrm{Hom}_{A-\mathrm{scheme}}(X, X) \simeq \mathrm{Hom}_{A-\mathrm{scheme}}(X, Y)$ and is the image of the identity morphism between X.

Quite a lot of universal property can be interpreted using this point of view. To start with something easy, we reinterpret the universal property of quotients. The first you see this universal property, it's always looks like if you have an A-module M and N is a submodule, then any morphism of A-modules $M \to S$ where N is mapped to 0 factors uniquely through $M/N \to S$. Then I claim that M/N actually represents the following functor from A-modules to Sets:

$$F : A\text{-module} \to \text{Sets} \qquad X \to \{f \in Hom(M, X) : f(N) = 0\}$$

Then you find that M/N represents the functor by $X \mapsto Hom_A(M/N, X)$, you can also use similar point of views to interpret the normalization of an integral scheme and quite a lot of other stuff. Another example is the tensor product of A-modules, I claim that $M \otimes N$ represents the functor:

$$F : A\text{-modules} \to \text{Sets} \qquad X \mapsto \{R\text{-balanced morphisms from } M \oplus N \text{ to } X\}$$

Then you will see that the tensor product $M \otimes_R N$ represents the functor by $X \mapsto Hom_A(M \otimes_R N, X)$.

The old argument we used for the unique morphism satisfying the universal property is elementary, and we can interpret this using Yoneda again. For example, you can think the unique isomorphism in the universal property of quotients. Usually, say $M \to Y$ also satisfies the universal property of M/N which gives you a functor isomorphism $Hom_A(Y, -) \simeq F$ via the following diagram:

$$\begin{tikzcd} M \arrow[r] \arrow[d] & S \\ Y \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Then the unique isomorphism between M/N and Y is formerly given by elementary argument, but we can also think of this as given by the isomorphism of functors:

$$Hom_A(Y, -) \simeq F \simeq Hom(M/N, -)$$

and is given by the image of the identity map on Y to $Hom_A(M/N, Y)$. This is because the identity of Y is mapped by the isomorphism $Hom_A(Y-) \simeq F$ to the morphism $M \to Y$ we started with, which maps N to 0, then this morphism is mapped by the isomorphism $F \simeq Hom_A(M/N, -)$ to the one we got using the elementary argument.

Carry this observation to our definition of $SpecR$ using the universal property, suppose that $Y \to X$ is another object representing the functor F we defined above, then the unique isomorphism between $SpecR$ and Y is given by the image of the identity between $SpecR$, but in order to specify what this isomorphism is concretely, we need to know what is the explicit isomorphism between $Hom(-, Y)$ and F, otherwise we can only abstractly define the isomorphism, which is the image of the identity in $\mathrm{Hom}_{A\text{-algebra}}(R, R)$ under the functor isomorphism $F(SpecR) \simeq Hom(SpecR, Y)$.

Then we will try to globalize this discussion above. The universal property of $\beta : \mathcal{S}pec\mathcal{R} \to X$ for a quasicoherent $\mathcal{O}_X$-algebra $\mathcal{R}$ satisfies is as follows: given any X-scheme $\mu : W \to X$, the X-scheme morphisms $W \to \mathcal{S}pec\mathcal{R}$ is in functorial (in W) bijection with morphisms of $\mathcal{O}_X$-algebras $\mathcal{R} \to \mu_* \mathcal{O}_W$, as shown in the following diagram:

$$\begin{tikzcd} \mathcal{R} \arrow[rr] & & \mu_* \mathcal{O}_X \\ & \mathcal{O}_X \arrow[ul] \arrow[ur] & \end{tikzcd}$$

where the $\mathcal{O}_X$-algebra structure of $\mu_* \mathcal{O}_W$ comes from the morphism $\mu : W \to X$ of schemes. Or under the scheme of our discussion above using Yoneda's lemma, you can say that $\mathcal{S}pec\mathcal{R}$ represents the functor:

$$F : X\text{-schemes}^{op} \to \text{Sets} \qquad W \mapsto Hom_{\mathcal{O}_X\text{-algebras}}(\mathcal{R}, \mu_* \mathcal{O}_W)$$

via some isomorphism of functors $Hom(-, \mathcal{S}pec\mathcal{R}) \simeq F$. Then the usual Yoneda nonsense tells you that if this $\mathcal{S}pec\mathcal{R}$ exists, it's unique up to a unique isomorphism. Therefore, out task is to show that it exists, or in other words, find the representing object $\mathcal{S}pec\mathcal{R}$ and the functor isomorphism $Hom(-, \mathcal{S}pec\mathcal{R}) \simeq F$.

The first observation we should have right now is that if $X = SpecA$, then $SepcR \to SpecA$ induced by $A \to R$ satisfies universal property, which is what we discussed before. Now, just as what we did for fibered products and normalizations, we take a look at what happens on open subschemes:

Proposition 1.1. Suppose that $\beta : \mathcal{S}pec\mathcal{R} \to X$ satisfies the universal property for $(X, \mathcal{R})$ and $i : U \hookrightarrow X$ is an open subscheme, then $\beta|_U : \mathcal{S}pec\mathcal{R} \times_X U \to U$ satisfies the universal property for $(U, \mathcal{R}|_U)$.

Proof. We need to show that the inverse image of U under $\beta : \mathcal{S}pec\mathcal{R} \to X$ represents the functor:

$$F : U\text{-schemes}^{op} \to \text{Sets} \qquad W \mapsto Hom_{\mathcal{O}_U\text{-algebras}}(\mathcal{R}|_U, \mu_* \mathcal{O}_W)$$

or in other words, we need to show that there is a functorial (in W) bijection between:

$$Hom_U(W, \mathcal{S}pec\mathcal{R} \times_X U) \simeq Hom_{\mathcal{O}_U\text{-algebras}}(\mathcal{R}|_U, \mu_* \mathcal{O}_W)$$

Now suppose that we are given a U-scheme morphism $W \to \mathcal{S}pec\mathcal{R} \times_X U$, consider the following diagram:

$$\begin{tikzcd}
W \arrow[dr] \arrow[drr, dashed, bend left] \arrow[ddr, bend right, "f"'] & & \\
& \mathcal{S}pec\mathcal{R} \times_X U \arrow[r, hook] \arrow[d, "\beta|_U"'] & \mathcal{S}pec\mathcal{R} \arrow[d, "\beta"] \\
& U \arrow[r, hook, "i"'] & X
\end{tikzcd}$$

we naturally have a X-scheme morphism $W \to \mathcal{S}pec\mathcal{R}$, which comes from the universal property of fibered product, which is functorial in W. Say $i \circ f = \mu$, then by the universal property $\mathcal{S}pec\mathcal{R}$, it's mapped by a functorial isomorphism to an $\mathcal{O}_X$-algebra morphism:

$$\begin{tikzcd}
\mathcal{R} \arrow[rr] & & \mu_* \mathcal{O}_W \\
& \mathcal{O}_X \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

recall that we have an adjoint pair, which gives you a bifunctorial isomorphism, thus the above morphism of $\mathcal{O}_X$-algebras is mapped to:

$$\begin{tikzcd}
\mu^* \mathcal{R} \arrow[rr] & & \mathcal{O}_W \\
& \mu^* \mathcal{O}_X \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

notice that $\mu^* = f^*(|_U)$, and we use the adjointness of (f^*, f_*) to conclude that we get the following diagram:

$$\begin{tikzcd}
\mathcal{R}|_U \arrow[rr] & & f_* \mathcal{O}_W \\
& \mathcal{O}_U \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

we only have to show this is a functorial (in W) isomorphism but notice that this is a composition of several functorial isomorphisms coming from fibered product and adjointness, we are done. $\square$

Therefore the above proposition shows that if X is an affine scheme and $\mathcal{R}$ is a quasicoherent $\mathcal{O}_X$-algebra on X then, for any open subscheme U, $\mathcal{S}pec\mathcal{R}|_U$ exist. Now we can show the existence of $\mathcal{S}pec\mathcal{R}$ for an arbitrary scheme X and an arbitrary $\mathcal{O}_X$-algebra on X. It's not hard, since we still first construct locally on X for each affine open $SpecA$, and we only have to show that over intersections, these locally defined stuff agree and the result satisfies the universal property. The gluing is easy, once you understand what does this following diagram means, which is the fundamental reason why all the other gluing argument works:

$$\begin{tikzcd}
& \mathrm{Hom}_X(-,Z) \arrow[dr, bend left, "\simeq"] & \\
& F \arrow[u, "\simeq"] \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & \\
\mathrm{Hom}_X(-,Y) \arrow[uur, bend left, "\simeq"] & & \mathrm{Hom}_X(-,W) \arrow[ll, bend left, "\simeq"]
\end{tikzcd}$$

Here Y, Z, W are three objects representing F together with the prescribed isomorphism to F. I will let you figure out what does this mean and why it suffices, but this finishes the construction. We still have to show that the result satisfies the universal property, say you have the following diagram:

$$\begin{tikzcd}
\mathcal{S}pec\mathcal{R} \arrow[dr, "\beta"'] & & W \arrow[ll] \arrow[dl, "\mu"] \\
& X &
\end{tikzcd}$$

and you want to define an $\mathcal{O}_X$-algebra morphism:

$$\begin{tikzcd}
\mathcal{R} \arrow[rr] & & \mu_*\mathcal{O}_W \\
& \mathcal{O}_X \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

we can only define it locally and show that the agree on the intersection, thus gives you a well defined morphism, and we need to show that this is a functiorial isomorphism. Over each affine open $U = SpecA$ of X, universal property locally on an affine open gives you an $\mathcal{O}_U$-morphism:

$$\begin{tikzcd}
\mathcal{R}_U \arrow[rr] & & (\mu|_U)_*\mathcal{O}_{\mu^{-1}U} = (\mu_*\mathcal{O}_W)|_U \\
& \mathcal{O}_U \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

we only have to show that they agree on intersections to use gluing, which still comes from the universal property on the intersection and again is a consequence of the diagram above that you need to figure out yourself. This is an isomorphism since the above construction is invertible, it's inverse is given by restricting the $\mathcal{O}_X$-algebra diagram and define morphisms from W to $\mathcal{S}pec\mathcal{R}$ locally and universal property shows that they agree on intersections.

Remark 1.2. Several quick observations of the $\mathcal{S}pec$ construction is readily available. First, it's computed affine locally so you understand the local behavior quite well.

We also have an isomorphism of $\mathcal{O}_X$-algebras $\phi : \mathcal{R} \simeq \beta_*\mathcal{O}_{\mathcal{S}pec\mathcal{R}}$ which is the image of the identity morphism $\mathcal{S}pec\mathcal{R} \to \mathcal{S}pec\mathcal{R}$. It's an isomorphism since we can check it locally and it reduces to the affine case, and you only need to show that the identity A-morphism $SpecR \to SpecR$ induces an isomorphism $\widetilde{R} \simeq \widetilde{R}$ which is induced by the identity A-algebra map $R \to R$.

Finally the result structure morphism $\mathcal{S}pec\mathcal{R} \to X$ is an affine morphism.

Proposition 1.3. Given a X-morphism γ:

$$\begin{tikzcd}
W \arrow[rr, "\gamma"] \arrow[dr, "\mu"'] & & \mathcal{S}pec\mathcal{R} \arrow[dl, "\beta"] \\
& X &
\end{tikzcd}$$

we know that it gives you an $\mathcal{O}_X$-algebra map $\alpha : \mathcal{R} \to \mu_* \mathcal{O}_W$, then α is the composition:

$$\mathcal{R} \to \beta_* \mathcal{O}_{\mathcal{S}pec\mathcal{R}} \to \beta_* \gamma_* \mathcal{O}_W = \mu_* \mathcal{O}_W$$

Proof. Let's check this locally. By our construction of $\mathcal{S}pec\mathcal{R}$, we know that α locally on $SpecA \subset X$ is given by the following A-algebra morphism $R \to \Gamma(\mu^{-1}SpecA, \mathcal{O}_W)$, then since we know that β is affine, $\beta_* \mathcal{O}_{\mathcal{S}pec\mathcal{R}}$ is still quasicoherent, then on $SpecA$, $\mathcal{R} \to \beta_* \mathcal{O}_{\mathcal{S}pec\mathcal{R}} \to \beta_* \gamma_* \mathcal{O}_W = \mu_* \mathcal{O}_W$ is determined by:

$$R \xrightarrow{id} R \to \Gamma(\mu^{-1}SpecA, \mathcal{O}_W)$$

thus they agree, we are done. $\square$

This global spec construction gives an important way to think about affine morphisms:

Proposition 1.4. If $\mu : Z \to X$ is an affine morphism, then we have a natural isomorphism $Z \simeq \mathcal{S}pec\mu_* \mathcal{O}_Z$.

Proof. First notice that since μ is affine, then the pushforward is indeed a quasicoherent $\mathcal{O}_X$-algebra, thus $\mathcal{S}pec\mu_* \mathcal{O}_Z$ makes sense.

Then notice that via the universal property of $\mathcal{S}pec\mu_* \mathcal{O}_Z$, we obtain a X-morphism from the following diagram:

$$\begin{tikzcd} \mu_* \mathcal{O}_Z \arrow[rr, "id"] & & \mu_* \mathcal{O}_Z \\ & \mathcal{O}_X \arrow[ul] \arrow[ur] & \end{tikzcd}$$

by construction, the induced morphism can be locally described by the A-algebra homomorphism:

$$B \xrightarrow{id} B$$

if we assume $\mu^{-1}SpecA = SpecB$. $\square$

Therefore, a scheme morphism $Z \to X$ is affine iff it's isomorphic as a X-scheme to some $\mathcal{S}pec\mathcal{R} \to X$ for some quasicoherent $\mathcal{O}_X$-algebra $\mathcal{R}$.

Proposition 1.5. Suppose that $\mu : \mathcal{S}pec\mathcal{R} \to X$ the morphism we constructed above, then the category of quasicoherent sheaves on $\mathcal{S}pec\mathcal{R}$ is equivalent to the category of quasicoherent sheaves on X with the structure of $\mathcal{R}$-modules, i.e. you don't think the ringed space structure on X is given by $\mathcal{O}_X$, rather it's given by $\mathcal{R}$.

Moreover the category of finite type quasicoherent sheaves on $\mathcal{S}pec\mathcal{R}$ is equivalent to finite type quasicoherent sheaves on X with $\mathcal{R}$-module structure.

Proof. For the first assertion, I define the functor to be the pushforward:

$$\mu_* : \text{Quasicoherent } \mathcal{O}_{\mathcal{S}pec\mathcal{R}}\text{- sheaves} \to \text{Quasicoherent } \mathcal{R}\text{-sheaves} \qquad \mathcal{F} \mapsto \mu_* \mathcal{F}$$

We need to show this is an equivalence of categories, i.e. it's fully faithful and essentially surjective. First you need to understand what does quasicoherent $\mathcal{R}$-modules on X mean. Say $\mathcal{F}$ is a quasicoherent $\mathcal{R}$-module on X then in the first place it's an $\mathcal{R}$-module, then via the morphism $\mathcal{O}_X \to \mathcal{R}$ is also an $\mathcal{O}_X$-module, then the quasicoherent here means that we think of $\mathcal{F}$ as an $\mathcal{O}_X$-module. Then we need to show that the pushforward $\mu_* \mathcal{F}$ has an $\mathcal{R}$-module structure such that via the induced $\mathcal{O}_X$-module structure, it's quasicoherent.

We already know that as the pushforward, it's naturally a $\mu_* \mathcal{O}_{\mathcal{S}pec\mathcal{R}}$-module, and it viewed as an $\mathcal{O}_X$-module (naturally quasicoherent since μ is affine) via $\mathcal{O}_X \to \mu_* \mathcal{O}_{\mathcal{S}pec\mathcal{R}}$, recall that we have an $\mathcal{O}_X$-algebra isomorphism: $\mathcal{R} \to \mu_* \mathcal{O}_{\mathcal{S}pec\mathcal{R}}$, thus $\mu_* \mathcal{F}$ is naturally equipped with an $\mathcal{R}$-module structure such that the induced $\mathcal{O}_X$-module is quasicoherent on X.

The fact that it's fully faithful is easy, once you realize μ is affine, then we can cover X using $SpecA$ and we denote the inverse images by $SpecR$, then say $\mathcal{F} \to \mathcal{G}$ on $SpecR$ is $F \to G$, then we know that on $SpecA$, the pushforward looks like $\widetilde{F}$ and $\widetilde{G}$ and the morphism is given by $F \to G$ (this module morphism is both an R-module morphism and an A-module morphism). This proves injectivity since agreeing on each

$SpecA$ measn that agreeing on each $SpecR$. To show it's surjective, suppose that you have a morphism of f : $\mathcal{R}$-modules $\mu_*\mathcal{F} \to \mu_*\mathcal{G}$, we want a morphism from $\mathcal{F}$ to $\mathcal{G}$ such that the pushforward is f. We still cover X by $SpecA$, whose inverse image is $SpecR$, then we know that f on $SpecA$ is just an R-module morphism $f_A : M_A \to N_A$, then we define $\mathcal{F} \to \mathcal{G}$ to be the one that on each $SpecR$ is f_A. We only have to show that this definition agrees on intersections to show they glue to a morphism g. After this, it's obvious that the pushforward of g is f.

To show that they agree on intersections, notice that the intersection $SpecR_1 \cap SpecR_2$ is the inverse image of $SpecA_1 \cap SpecA_2$, the good news is that we can cover $SpecA_1 \cap SpecA_2$ using distinguished opens in both $SpecA_1$ and $SpecA_2$, thus the inverse image of these distinguished opens cover $SpecR_1 \cap SpecR_2$, we only have to show that they agree on these distinguished opens, but this can be checked on X, thus we are done.

Finally, we need to show that this is essentially surjective. Say you have an $\mathcal{R}$-module $\mathcal{F}$ on X such that with the induced $\mathcal{O}_X$-module structure, it's quasicoherent. We need to show that there is a quasicoherent module $\mathcal{G}$ on $\mathcal{S}pec\mathcal{R}$ such that the pushforward is isomorphic to this module $\mathcal{F}$ as an $\mathcal{R}$-module. The strategy is still to define $\mathcal{G}$ locally on the inverse image $SpecR$ of $SpecA$ for each affine open $SpecA$ in X and show that they glue.

The above gluing argument is kind of annoying but the advantage is it's elementary, an alternative way to do this is to define a quasi-inverse to F, recall that we have an isomorphism $\mathcal{R} \to \mu_*\mathcal{O}_{\mathcal{S}pec\mathcal{R}}$ of $\mathcal{O}_X$-algebras, then by the adjunction property, we know that we have a morphism $\mu^*\mathcal{R} \to \mathcal{O}_{\mathcal{S}pec\mathcal{R}}$, this is still a morphism of sheaves of rings, then we define the quasi-inverse to be:

$$H : \text{Quasicoherent } \mathcal{R}\text{-sheaves} \to \text{Quasicoherent } \mathcal{O}_{\mathcal{S}pec\mathcal{R}}\text{- sheaves} \qquad \mathcal{F} \mapsto \mu^*\mathcal{F} \otimes_{\mu^*\mathcal{R}} \mathcal{O}_{\mathcal{S}pec\mathcal{R}}$$

Then we need to show they are indeed inverses of each other. Recall that we have functorial morphisms of $\mathcal{O}_{\mathcal{S}pec\mathcal{R}}$-modules coming from adjunction:

$$\mu^*\mu_*\mathcal{F} \to \mathcal{F}$$

this induces a functorial morphism $H(F(\mathcal{F})) \to \mathcal{F}$, we only need to show this is an isomorphism for quasicoherent $\mathcal{F}$. We only have to check this on affine opens, say on $SpecR$, inverse images of $SpecA$. Notice that this is induced by the natural morphism:

$$(F \otimes_A R) \otimes_{(R \otimes_A R)} R \to F$$

we only have to show this is an isomorphism of R-modules. First, you need to understand how $F \otimes_A R$ is an $R \otimes_A R$-module, after this you can define an inverse $F \to (F \otimes_A R) \otimes_{(R\otimes_A R)} R$ pretty easily. Therefore this is an isomorphism of R-modules, then via $A \to R$, can be viewed as an isomorphism of A-modules, thus the converse is also proved, we are done. $\square$

Remark 1.6. The second approach given in the above proof requires more understanding of the adjunction of (μ^*, μ_*) on affine schemes for quasicoherent sheaves.

Say you have $f : SpecR \to SpecA$ is a scheme morphism induced by $A \to R$, then you already know that pushforward of a quasicoherent sheaf on $SpecR$, say $\widetilde{F}$ is just $\widetilde{F}$ but viewed as an A-module and pullback of $\widetilde{G}$ is $\widetilde{G \otimes_A R}$ which is proved using the tensor hom adjunction $Hom_R(G \times_A R, F) \simeq Hom_A(G, F)$ (recall we defined the pullback to be the left adjoint of the pushforward), we also know that the definition of the adjunction map is compatible with restrictions therefore for the adjunction map between quasicoherent sheaves, we understand how it looks like locally quite well and it's just the tensor hom adjunction.

Using this interpretation, you know that on $SpecR$, the canonical morphism $\mu^*\mathcal{R} \to \mathcal{O}_{\mathcal{S}pec\mathcal{R}}$ is just the adjunction of the identity $R \to R$, which is $R \otimes_A R \to R$.

There are of course more that you need to know, for example, we didn't talk about tensor product like the one below $\mu^*\mathcal{F} \otimes_{\mu^*\mathcal{R}} \mathcal{O}_{\mathcal{S}pec\mathcal{R}}$ since it's tensor over something not the structure sheaf, but fortunately $\mu^*\mathcal{R}$ is a quasicoherent $\mathcal{O}_{\mathcal{S}pec\mathcal{R}}$-module, thus if you remember how we proved $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is quasicoherent for quasicoherent $\mathcal{F}, \mathcal{G}$, the same method applies here to show $\mu^*\mathcal{F} \otimes_{\mu^*\mathcal{R}} \mathcal{O}_{\mathcal{S}pec\mathcal{R}}$ is still quasicoherent and over each affine it's just the tensor you expected.

Another thing you need to know is how $\mu^*\mathcal{F}$ and $\mu^*\mathcal{R}$-module. We know that $\mu^*\mathcal{F}$ is an $\mathcal{O}_{\mathcal{S}pec\mathcal{R}}$-module, and we have a morphism of sheaves of rings $\mu^*\mathcal{R} \to \mathcal{O}_{\mathcal{S}pec\mathcal{R}}$, thus $\mu^*\mathcal{F}$ has a $\mu^*\mathcal{R}$-module structure.

We have to admit there are quite a lot of stuff to check, but these are all something that when you want to check, it can be checked.

Remark 1.7. The equivalence of categories above is useful when X is easy to understand while $\mathcal{S}pec\mathcal{R}$ is complicated, for example, we will use this later when $X = \mathbb{P}^1$. Also, it will appear in the proof of Serre duality.

Proposition 1.8. Suppose that $\mu : Z \to X$ is any morphism of schemes, and $\mathcal{R}$ is a quasicoherent sheaf of $\mathcal{O}_X$-algebras on X, then there is a natural isomorphism $Z \times_X \mathcal{S}pec\mathcal{R} \simeq \mathcal{S}pec\mu^*\mathcal{R}$.

Proof. Recall that we have already showed this for an open embedding $i : U \hookrightarrow X$, and the proposition generalizes this special case we saw.

Consider the following diagram:

$$\begin{tikzcd}
W \arrow[dr, "p"] \arrow[ddr, bend right, "g"'] & & \\
& Z \times_X \mathcal{S}pec\mathcal{R} \arrow[r] \arrow[d] & \mathcal{S}pec\mathcal{R} \arrow[d, "\beta"] \\
& Z \arrow[r, "\mu"] & X
\end{tikzcd}$$

given a Z-scheme W with structure morphism g, let's denote $f = \mu \circ g$, then we know that by the universal property of fibered product, we have a functorial isomorphism between p and $q : W \to \mathcal{S}pec\mathcal{R}$, which by the universal property, gives you a functorial isomorphism to $\mathcal{O}_X$-algebra morphisms $\mathcal{R} \to f_*\mathcal{O}_W$, then the adjunction property give you another functorial isomorphism and it's mapped to $f^*\mathcal{R} \to \mathcal{R} = g^*\mu^*\mathcal{R} \to \mathcal{O}_W$, then again by the adjunction property we have $\mu^*\mathcal{R} \to g_*\mathcal{O}_W$, therefore we know that $Z \times_X \mathcal{S}pec\mathcal{R}$ is the representing object, thus it's isomorphic via a unique isomorphism to $\mathcal{S}pec\mu^*\mathcal{R}$. $\square$

Now enough of these abstract constructions, we know look at an important exampe of $\mathcal{S}pec$, which we will use a lot later:

Definition 1.9. Suppose that X is a scheme and $\mathcal{F}$ is a finite rank locally free sheaf on X, the total space of $\mathcal{F}$ is defined to be $\mathcal{S}pec(Sym^\bullet\mathcal{F}^\vee)$.

Proposition 1.10. Suppose that $\mathcal{F}$ is locally free of rank n, then the total space of $\mathcal{F}$ is a rank n vector bundle, i.e. given any point $p \in X$, there is an open neighborhood $p \in U \subset X$ such that the inverse image via the morphism $\mathcal{S}pec(Sym^\bullet\mathcal{F}^\vee) \to X$ is isomorphic to $\mathbb{A}^n_U$.

Proof. First recall that $\mathbb{A}^n_X$ is independent of the base scheme since we have:

$$\begin{tikzcd}
\mathbb{A}^n_X \arrow[r] \arrow[d] & \mathbb{A}^n_B \arrow[r] \arrow[d] & \mathbb{A}^n_{\mathbb{Z}} \arrow[d] \\
X \arrow[r] & B \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

thus we can just define it over any base ring. I can first prove that if $\mathcal{F}$ is free of rank n, i.e. $\mathcal{F} \simeq \mathcal{O}^n_X$, then the total space is $\mathbb{A}^n_X$. We will show that $\mathbb{A}^n_X \to X$ represents the function that the total space represent. Let's define $\mathbb{A}^n_X$ over the base $\mathbb{Z}$, then we know that by the universal property, suppose $W \to X$ is also a X-scheme, then it correspond to a morphism $\mu : W \to \mathbb{A}^n_{\mathbb{Z}}$ such that the composition to $\mathbb{Z}$ agrees, but this is the same as an $\mathbb{Z}$-algebra homomorphism:

$$\begin{tikzcd}
\Gamma(W, \mathcal{O}_W) & & \mathbb{Z}[x_1, \cdots, x_n] \arrow[ll] \\
& \mathbb{Z} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

notice this is isomorphic to an $\mathcal{O}_X$-module homomorphism $\mathcal{O}^n_X \to \mu_*\mathcal{O}_W$ where the $\mathbb{Z}$-algebra homomorphism that maps x_i to f_i is mapped to the $\mathcal{O}_X$-module morphism that maps e_i to f_i and the inverse is obvious. But by the universal property of $Sym^\bullet(\mathcal{O}^n_X)$, this is again mapped to a $\mathcal{O}_X$-algebra morphism $Sym^\bullet(\mathcal{O}^n_X) \to \mu_*\mathcal{O}_W$, therefore, we have functorial isomorphisms:

$$Hom_{\mathcal{O}_X\text{-algebra}}(Sym^\bullet(\mathcal{O}^n_X), \mu_*\mathcal{O}_W) \simeq Hom_{\mathcal{O}_X\text{-module}}(\mathcal{O}^n_X, \mu_*\mathcal{O}_W) \simeq Hom_X(W, \mathbb{A}^n_X)$$

therefore, there is a unique X-scheme isomorphism induced by this isomorphism:

$$\begin{tikzcd} \mathbb{A}^n_X \arrow[rr, "\simeq"] \arrow[dr] & & \mathcal{S}pec Sym^\bullet \mathcal{O}^n_X \arrow[dl] \\ & X & \end{tikzcd}$$

Then recall that the inverse image of U is just $\mathcal{S}pec(Sym^\bullet \mathcal{F}^\vee|_U)$, with the $\mathcal{O}_U$-algebra structure induced from the $\mathcal{O}_X$-algebra structure. Then the above proof suffice to show that $\mathcal{S}pec(Sym^\bullet \mathcal{F}^\vee|_U) \simeq \mathbb{A}^n_U$ since $(Sym^\bullet \mathcal{F}^\vee)|_U \simeq Sym^\bullet(\mathcal{F}^\vee|_U) \simeq Sym^\bullet(\mathcal{O}^n_U)$ and there is a unique X-scheme morphism such that $\mathcal{S}pec(Sym^\bullet \mathcal{F}^\vee|_U) \simeq \mathcal{S}pec(Sym^\bullet \mathcal{O}^n_U)$. But we can also prove it using a more elementary way as follows:

Recall that we proved before that the symmetric algebra construction of a quasicoherent sheaf is also quasicoherent, and we can assume that $U = SpecA$ is a trivializing neighborhood of $\mathcal{F}^\vee$ where it's isomorphic to $\mathcal{O}^n_U$, then I claim that $Sym^\bullet \mathcal{F}^\vee|_U \simeq \widetilde{A[x_1, \cdots, x_n]}$. To see this, we consider the following sequence of isomorphisms:

$$(Sym^\bullet \mathcal{F}^\vee)|_U \simeq Sym^\bullet(\mathcal{F}^\vee|_U) \simeq Sym^\bullet(\widetilde{A}^n) \simeq \widetilde{Sym^\bullet(A^n)} \simeq \widetilde{A[x_1, \cdots, x_n]}$$

where the second to last isomorphism follows from the fact that he symmetric algebra is quasicoherent, thus on affine open, we only have to show the global sections are isomorphic and the last isomorphism comes from the isomorphism of A-algebra:

$$Sym^\bullet(A^n) \simeq A[x_1, \cdots, x_n]$$

By the universal property of $Sym^\bullet(A^n)$, it comes from an A-module morphism $A^n \to A[x_1, \cdots, x_n]$ where e_i is mapped to x_i, the inverse is given by x_i is mapped to e_i using the universal property of $A[x_1, \cdots, x_n]$, thus they are isomorphic, we are done.

Therefore, we know that over $SpecA$, the total space is $SpecA[x_1, \cdots, x_n]$, we are done. $\square$

Remark 1.11. For later reference, we state here: in particular, if $\mathcal{F} = \mathcal{O}^n_X$, then $\mathcal{S}pec Sym^\bullet(\mathcal{O}^n_X) \simeq \mathbb{A}^n_X$.

We probably will wonder why we use the dual of $\mathcal{F}$ instead of $\mathcal{F}$ to define the total space, the main reason is we that want the following assertion to be true:

Proposition 1.12. Still $\mathcal{F}$ is a locally free sheaf of finite rank on X, say of rank n, but this doesn't really matter in this proposition, then $\mathcal{F}$ is isomorphic to the sheaf of sections of the total space $\mathcal{S}pec(Sym^\bullet \mathcal{F}^\bullet)$. For this reason the total space is called the **vector bundle associated to a locally free sheaf** $\mathcal{F}$.

Proof. First, let's take a look at how this sheaf of sections is an $\mathcal{O}_X$-module.

Say $U \subset X$ is an open subscheme, then we can consider the following diagram:

$$\begin{tikzcd} U \arrow[r] \arrow[dr, "is"'] & \mathcal{S}pec(Sym^\bullet \mathcal{F}^\vee)|_U \arrow[r, hook] \arrow[d] & \mathcal{S}pec(Sym^\bullet \mathcal{F}^\vee) \arrow[d] \\ & U \arrow[r, hook] & X \end{tikzcd}$$

then by the universal property, we know that the section $\mathcal{S}$ correspond to $\mathcal{O}_U$-algebra homomorphism:

$$Hom_{\mathcal{O}_U}(Sym^\bullet \mathcal{F}^\vee|_U, \mathcal{O}_U)$$

this is obviously a $\Gamma(U, \mathcal{O}_U)$-module, and this holds for each open subscheme $U \subset X$. Therefore, we have the following sequence of isomorphisms:

$$(Sym^\bullet \mathcal{F}^\vee)^\vee(U) \simeq \mathrm{Hom}_{\mathcal{O}_U}(Sym^\bullet \mathcal{F}^\vee|_U, \mathcal{O}_U) \simeq \mathcal{S}(U)$$

and motivated by this, you probably want to show that $\mathcal{S} \simeq (Sym^\bullet \mathcal{F}^\vee)^\vee$, but this is not true since the Hom sets here is not module morphisms, but algebra homomorphism, thus if you replace module homomorphisms by algebra homomorphisms, this is indeed true (you need to think a little bit about why the algebra homomorphisms is still a sheaf and a sheaf of $\mathcal{O}_X$-modules) and for simplicity let's use the superscript a to indicate algebra homomorphisms and differentiate from module homomorphisms, i.e. $\mathrm{Hom}^a_{\mathcal{O}_U}(Sym^\bullet\mathcal{F}^\vee|_U, \mathcal{O}_U)$ and $\mathcal{H}\mathrm{om}^a_{\mathcal{O}_U}(Sym^\bullet\mathcal{F}^\vee|_U, \mathcal{O}_U)$, but to prove this, we need a different point of view. Recall that if we denote $i : U \hookrightarrow X$ by the open embedding, we have a natural isomorphism of $\mathcal{O}_X$-modules:

$$\mathrm{Hom}^a_{\mathcal{O}_X}(Sym^\bullet\mathcal{F}^\vee, i_*\mathcal{O}_U) \simeq \mathrm{Hom}^a_{\mathcal{O}_U}(Sym^\bullet\mathcal{F}^\vee|_U, \mathcal{O}_U)$$

Then by the functoriality of the universal property, since we know that the open embedding $j : V \hookrightarrow X$ factors through $i : U \hookrightarrow X$ for $V \subset U$, we know the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}^a_{\mathcal{O}_U}(Sym^\bullet\mathcal{F}^\vee|_U, \mathcal{O}_U) \arrow[d, "res"'] & \mathrm{Hom}^a_{\mathcal{O}_X}(Sym^\bullet\mathcal{F}^\vee, i_*\mathcal{O}_X) \arrow[l, "\simeq"'] \arrow[d] & \mathcal{S}(U) \arrow[l, "\simeq"'] \arrow[d, dashed, "res"] \\
\mathrm{Hom}^a_{\mathcal{O}_V}(Sym^\bullet\mathcal{F}^\vee|_V, \mathcal{O}_V) & \mathrm{Hom}^a_{\mathcal{O}_X}(Sym^\bullet\mathcal{F}^\vee, j_*\mathcal{O}_X) \arrow[l, "\simeq"'] & S(V) \arrow[l, "\simeq"']
\end{tikzcd}$$

where the vertical arrow in the middle is induced by composition with the natural morphism $i_*\mathcal{O}_X \to j_*\mathcal{O}_X$, then you get the desired isomorphism $\mathcal{S} \simeq (Sym^\bullet\mathcal{F}^\vee)^\vee$.

To prove this is isomorphic to $\mathcal{F}$, we use the transition function argument. We pick an affine cover $SpecA_i$ of X, each of which is a trivializing neighborhood of $\mathcal{F}$, with transition functions T_{ij}. We need to show that $(Sym^\bullet\mathcal{F}^\vee)^\vee$ also have transition function T_{ij}. To see how to get the transition function, consider the following diagram:

$$\mathcal{H}\mathrm{om}^a_{\mathcal{O}_U}(Sym^\bullet\mathcal{F}^\vee|_U, \mathcal{O}_U) \xrightarrow{\simeq} \mathcal{H}\mathrm{om}^a_{\mathcal{O}_U}(Sym^\bullet\mathcal{O}^n_U, \mathcal{O}_U) \xrightarrow{\simeq} \mathcal{O}^n_U$$

where the last isomorphism comes from the universal property of the symmetric algebra: it corresponds to $\mathcal{O}_X$-module homomorphisms from $\mathcal{O}^n_W \to \mathcal{O}_W$, i.e. $(\mathcal{O}^n_W)^\vee \simeq \mathcal{O}^n_W$, then on intersections $W = U \cap V$ where $U = SpecA_i$ and $V = SpecA_j$, we have:

$$\begin{tikzcd}
\mathcal{H}\mathrm{om}^a_{\mathcal{O}_W}(Sym^\bullet\mathcal{F}^\vee|_W, \mathcal{O}_W) \arrow[r, "\simeq"] & \mathcal{H}\mathrm{om}^a_{\mathcal{O}_W}(Sym^\bullet\mathcal{O}^n_W, \mathcal{O}_W) \arrow[r, "\simeq"] \arrow[d, dashed, "\simeq"] & \mathcal{H}\mathrm{om}^a_{\mathcal{O}_W}(\mathcal{O}^n_W, \mathcal{O}_W) \arrow[r, dashed, "\simeq"] \arrow[d, dashed, "\simeq"] & \mathcal{O}^n_W \arrow[d, dashed, "\simeq"] \\
\mathcal{H}\mathrm{om}^a_{\mathcal{O}_W}(Sym^\bullet\mathcal{F}^\vee|_W, \mathcal{O}_W) \arrow[r, "\simeq"] & \mathcal{H}\mathrm{om}^a_{\mathcal{O}_W}(Sym^\bullet\mathcal{O}^n_W, \mathcal{O}_W) \arrow[r, "\simeq"] & \mathcal{H}\mathrm{om}^a_{\mathcal{O}_W}(\mathcal{O}^n_W, \mathcal{O}_W) \arrow[r, dashed, "\simeq"] & \mathcal{O}^n_W
\end{tikzcd}$$

where the second vertical isomorphism is induced by the $\mathcal{O}_W$-algebra isomorphism $Sym^\bullet\mathcal{O}^n_W \to Sym^\bullet\mathcal{O}^n_W$ given by the isomorphism $(T^{-1}_{ij})^t : \mathcal{O}^n_W \to \mathcal{O}^n_W$, thus by functoriality of the universal property the third vertical isomorphism is given by precomposing $(T^{-1}_{ij})^t$ we only have to show that the isomorphism corresponding to the last vertical arrow is T_{ij}. But you know that we are just taking the dual again, thus we get the transition function is $(((T^{-1}_{ij})^t)^{-1})^t = T_{ij}$, we are done. $\square$

Remark 1.13. You may find the following remark conceptually useful.

Recall that when we motivated the importance of the study of locally free sheaves, we discussed the equivalence of categories between locally free sheaf of rank n and rank n vector bundles over a manifold. Because we work with manifold, with have an $\mathbb{R}^n$ structure thus we can discuss transition functions of a vector bundle, and you saw that the transition function of the vector bundle is the same as the transition functions of the sheaf of sections. Therefore, you can say that in some sense, the vector bundle is associated to the sheaf of sections.

We want to tell a similar story in the algebraic setting, i.e, given a locally free sheaf of finite rank $\mathcal{F}$, we want to associate it with a vector bundle such that the sheaf of sections is isomorphic to $\mathcal{F}$. However, for the scheme total space, we don't have any transition functions like we used to have for vector bundles over manifolds, thus we can't produce the similar story using the elementary method we had for manifold and we have more difficulties than we expected, for example, how we are able to interpret the sheaf of sections as an $\mathcal{O}_X$-module in the first place as for a general scheme morphism $Y \to X$ there is no natural structure on the sheaf of sections, which is just a sheaf of sets or sheaf of groups under the compositions operation.

The $\mathcal{S}pec$ construction is precisely the tool we use to overcome the difficulties stated above. We have a locally free sheaf $\mathcal{F}$ of rank n on X, we associated it with the total space of $\mathcal{F}$, which is a vector bundle over X, then you find the sheaf of sections is isomorphic to $\mathcal{F}$, only with one caveat that the total space is defined using the symmetric algebra $\mathcal{F}^\vee$. You may want to say that we still have an equivalence of category between locally free sheaves and "vector bundles" over X, thus fully recovering the story for manifolds and justify our usage of vector bundles for locally free sheaves. But some questions arise here, for example, how do we define a vector bundle over X. With the right definition, our expectation is indeed true, but since we will not use it later, we omit it here.

As an example of the above vector bundle construction, we construct the **tautological line bundle** on $\mathbb{P}^n_k$ for a field k. For the setup, we consider the fibered product $\mathbb{A}^{n+1}_k \times \mathbb{P}^n_k$, and notice that we have a line bundle $\mathcal{O}_{\mathbb{A}^{n+1}_k} \boxtimes \mathcal{O}(1)$ consider the global sections $x_i y_j - x_j y_i$ if we assume the coordinate on $\mathbb{A}^{n+1}_k$ is $x_0, \cdots, x_n$ and the coordinate on $\mathbb{P}^n_k$ are $y_0, \cdots, y_n$, then the global sections $x_i y_j - x_j y_i$ are just the tensor of the pullback x_i, y_j and x_j, y_i. Recall that we already saw that a single global sections of a line bundle cuts out a closed subscheme, the same argument works for finitely many global sections of a line bundle, thus these sections $x_i y_j - x_j y_i$ cuts out a closed subscheme $X \hookrightarrow \mathbb{A}^{n+1}_k \times \mathbb{P}^n_k$, then we can compose with the projection to $\mathbb{P}^n_k$ and get $\pi : X \to \mathbb{P}^n_k$.

For some geometric intuition about what is this closed subscheme X, we still can describe what X looks like locally, say over the standard affine cover $\mathbb{A}^{n+1} \times \mathbb{A}^n$. Notice that on this affine open where $y_k \neq 0$, the restriction of the global sections are just functions $x_i \frac{y_j}{y_k} - x_j \frac{y_i}{y_k}$, thus it's just cut out in $Speck[x_0, \cdots, x_n, \frac{y_0}{y_k}, \cdots, \frac{y_n}{y_k}]$ by the ideal $(x_i \frac{y_j}{y_k} - x_j \frac{y_i}{y_k})$. Then if you consider say the k-valued point $[y_0, \cdots, y_n]$ where $y_i \in k$ and $y_k \neq 0$, then the computation of the fiber of this point is in the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r, hook] \arrow[d] & \cdots \arrow[r] \arrow[d, hook] & k \arrow[d, hook] \\
\pi^{-1}(D_+(y_k)) \arrow[r, hook] \arrow[d] & \mathbb{A}^{n+1} \times \mathbb{A}^n \arrow[r] \arrow[d] & \mathbb{A}^n \arrow[d] \\
X \arrow[r, hook] & \mathbb{A}^{n+1} \times \mathbb{P}^n \arrow[r] & \mathbb{P}^n
\end{tikzcd}$$

the fiber is the stuff on the left top corner. Where the stuffs on the second row are given by:

$$Speck[x_0, \cdots, x_n, \tfrac{y_0}{y_k}, \cdots, \tfrac{y_n}{y_k}]/I \longrightarrow Speck[x_0, \cdots, x_n, \tfrac{y_0}{y_k}, \cdots, \tfrac{y_n}{y_k}] \longrightarrow Speck[\tfrac{y_0}{y_k}, \cdots, \tfrac{y_n}{y_k}]$$

with obvious ring morphisms. Then I will let you check that if we denote $r_i = \frac{y_i}{y_k}$, then the fiber is just $Speck[x_0, \cdots, x_n]/(x_i r_j - x_j r_i)$, then you see that the k-valued points in $\mathbb{A}^{n+1}_k$ that are in the fiber must satisfies $x_i y_i - x_j y_i = 0$, in other words, there is a nonzero scalar multiple $\lambda \in k^\times$ such that:

$$\lambda[x_0, \cdots, x_n] = [y_0 : \cdots : y_n]$$

Therefore if you only want to consider the k-valued points.

Proposition 1.14. The above morphism $\pi : X \to \mathbb{P}^n_k$ is the vector bundle associated to $\mathcal{O}(-1)$. (The total space associated to $\mathcal{O}(-1)$, since we saw that the fiber of a k-valued point is the line corresponding to the homogeneous coordinates of that point, $\mathcal{O}(-1)$ is always called the tautological line bundle on $\mathbb{P}^n_k$.)

Proof. To see the assertion, let's first try to understand the total space of the line bundle $\mathcal{O}(-1)$:

$$\mathcal{S}pec\mathcal{S}ym^\bullet\mathcal{O}(1) \to \mathbb{P}^n_k$$

We don't use any fancy complicated arguments, instead we just try to understand this locally on affine trivializing neighborhoods and see how they are glued together. This requires us to understand the situation on affine opens quite well. Say you work over $SpecA = X$ and you have two quasicoherent $\mathcal{O}_X$-algebra that are isomorphic via an $\mathcal{O}_X$-algebra isomorphism $R \to S$, then we know that the unique isomorphism coming from the universal property that makes $SpecR$ and $SpecS$ isomorphic as an A-module is just $R \to S$.

Therefore the lesson we learn about $\mathcal{S}pec$ construction on affine schemes is that we know the functorial isomorphism coming from universal property well, thus basically we can do anything without too much pain. Therefore apply this to $D_+(y_k) = U_k$, we get the following diagram:

$$\begin{tikzcd}
Speck[x, \frac{y_0}{y_k}, \cdots, \frac{y_n}{y_k}] \arrow[r, "\simeq"] \arrow[dr] & Speck[x, \widetilde{\frac{y_0}{y_k}, \cdots}, \frac{y_n}{y_k}] \arrow[r, "\simeq"] \arrow[d] & \mathcal{S}pecSym^\bullet\mathcal{O}(-1)|_U \arrow[d] \\
& Speck[\frac{y_0}{y_k}, \cdots, \frac{y_n}{y_k}] \arrow[r, "\simeq"] & U_k
\end{tikzcd}$$

thus we can identity the restriction to $D_+(y_k)$ to the the morphism $Speck[x, \frac{y_0}{y_k}, \cdots, \frac{y_n}{y_k}] \to Speck[\frac{y_0}{y_k}, \cdots, \frac{y_n}{y_k}]$, then since we know that restricting further correspond to taking fibered product, we know that we have the following diagram:

$$\begin{tikzcd}
Speck[x, \frac{y_0}{y_k}, \cdots, \frac{y_n}{y_k}, \frac{y_0}{y_l}, \cdots, \frac{y_n}{y_l}] \arrow[r, hook] \arrow[d] & Speck[x, \frac{y_0}{y_k}, \cdots, \frac{y_n}{y_k}] \arrow[d] \\
Speck[\frac{y_0}{y_k}, \cdots, \frac{y_n}{y_k}, \frac{y_0}{y_l}, \cdots, \frac{y_n}{y_l}] \arrow[r, hook] & Speck[\frac{y_0}{y_k}, \cdots, \frac{y_n}{y_k}]
\end{tikzcd}$$

on the intersection $D_+(y_k) \cap D_+(y_l)$. You can apply the same argument on $D_+(y_l)$ then restrict to $D_+(y_l)$, then you know that we have the diagram below:

$$\begin{tikzcd}
Speck[x, \frac{y_0}{y_k}, \cdots, \frac{y_n}{y_k}, \frac{y_0}{y_l}, \cdots, \frac{y_n}{y_l}] \arrow[r, dashed, "\simeq"] \arrow[dr] & \mathcal{S}pec(\Gamma(U_k \cap U_l, Sym^\bullet\mathcal{O}(-1))) \arrow[d] & Speck[x, \frac{y_0}{y_k}, \cdots, \frac{y_n}{y_k}, \frac{y_0}{y_l}, \cdots, \frac{y_n}{y_l}] \arrow[l, dashed, "\simeq"'] \arrow[dl] \\
& Speck[\frac{y_0}{y_k}, \cdots, \frac{y_n}{y_k}, \frac{y_0}{y_l}, \cdots, \frac{y_n}{y_l}] &
\end{tikzcd}$$

You should think through what doesn't this diagram means and see that the two local identification is connected via multiplication by $\frac{y_k}{y_l}$.

Then back to the morphism $X \to \mathbb{P}^n$, we know that over each $D_+(x_i)$, it looks like:

$$\begin{tikzcd}
Speck[x, \frac{y_0}{y_k}, \cdots, \frac{y_n}{y_k}] \arrow[r, dashed, "\simeq"] \arrow[dr] & Speck[x_0, \cdots, x_n, \frac{y_0}{y_k}, \cdots, \frac{y_n}{y_k}]/(x_i\frac{y_j}{y_k} - x_j\frac{y_i}{y_k}) \arrow[d] \\
& Speck[\frac{y_0}{y_k}, \cdots, \frac{y_n}{y_k}]
\end{tikzcd}$$

where the vertical isomorphism is given by $x_k \mapsto x, x_i \mapsto \frac{y_i}{y_l}$, then on $D_+(y_k) \cap D_+(y_l)$ we can identify π using:

$$\begin{tikzcd}
Speck[x, \frac{y_0}{y_k}, \cdots, \frac{y_n}{y_k}, \frac{y_0}{y_l}, \cdots, \frac{y_n}{y_l}] \arrow[r] \arrow[d] & Speck[x, \frac{y_0}{y_k}, \cdots, \frac{y_n}{y_k}] \arrow[d] \\
Speck[x_0, \cdots, x_n, \frac{y_0}{y_k}, \cdots, \frac{y_n}{y_k}, \frac{y_0}{y_l}, \cdots, \frac{y_n}{y_l}]/(x_i\frac{y_j}{y_k} - x_j\frac{y_i}{y_k}) \arrow[r] \arrow[d] & Speck[x_0, \cdots, x_n, \frac{y_0}{y_k}, \cdots, \frac{y_n}{y_k}]/(x_i\frac{y_j}{y_k} - x_j\frac{y_i}{y_k}) \arrow[d] \\
Speck[\frac{y_0}{y_k}, \cdots, \frac{y_n}{y_k}, \frac{y_0}{y_l}, \cdots, \frac{y_n}{y_l}] \arrow[r, hook] & Speck[\frac{y_0}{y_k}, \cdots, \frac{y_n}{y_k}]
\end{tikzcd}$$

You need to understand the gluing of $\mathbb{A}^{n+1} \times \mathbb{P}^n$ very well which allows you to see how the glue works on the closed subscheme, but the result is till given by multiplication by $\frac{y_k}{y_l}$, thus we are done. $\square$

2 Relative Proj of a quasicoherent sheaf of graded algebras

In parallel with the $\mathcal{S}pec$ construction, we define a relative version of Proj, denoted $\mathcal{P}roj$ called relative proj, curly proj or sheaf proj. It's defined for a quasicoherent graded sheaf of algebra on a scheme X. When the scheme X is affine, this is just the usual proj construction we defined before. Since graded rings and modules makes everything confusing, even we can state the universal property and glue, we won't go in this way since the universal property should make your life easier, not harder. Instead, we take a more pedestrian approach.

We first need some preparations:

Proposition 2.1. Suppose that $A \to B$ is a map of ring and $S_\bullet$ is a $\mathbb{Z}^{\geqslant}$-graded ring over A, then there is a canonical isomorphism:

$$\alpha : Proj_B(S_\bullet \otimes_A B) \to Proj_A S_\bullet \times_A SpecB$$

Here the subscript is just to remind you what base ring we are using.

Proof. First, notice that $S_\bullet = \oplus_{n\geqslant 0} S_n$, with component of each degree an A-module, the tensor product gives you another direct sum decomposition:

$$S_\bullet \otimes_A B \simeq \oplus_{n\geqslant 0}(S_n \otimes_A B)$$

Recall that the ring structure on $S_\bullet \otimes_A B$ is given by $a \otimes b \cdot c \otimes d = ac \otimes bd$, then since the degree 0 piece is $A \otimes_A B = B$, we know that multiplication on $S_\bullet \otimes_A B$ indeed makes it a graded ring wrt the decomposition we stated above.

Recall that by the universal property of fibered product, to give a morphism to $Proj_A(S_\bullet) \times_A SpecB$ is just to give the following diagram:

$$\begin{tikzcd}
Proj_B(S_\bullet \otimes_A B) \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& Proj_A(S_\bullet) \times_A SpecB \arrow[r] \arrow[d] & SpecB \arrow[d] \\
& Proj_A(S_\bullet) \arrow[r] & A
\end{tikzcd}$$

A natural choice of $Proj_B(S_\bullet \otimes_A B) \to SpecB$ is the structure morphism we use such that it's a B-scheme. To be more precise, if $S_\bullet$ is a graded module over A, we define a morphism $ProjS_\bullet \to SpecA$ by on each $Spec(S_f)_0$ we define it to be the morphism given by the natural ring map $A \to (S_f)_0$, it's obvious that they agree on intersections:

$$\begin{tikzcd}
& (S_f)_0 \arrow[dr] & \\
A \arrow[ur] \arrow[dr] & & (S_{fg})_0 \\
& (S_g)_0 \arrow[ur] &
\end{tikzcd}$$

Here we apply this to $Proj_B(S_\bullet \otimes_A B) \to SpecB$.

The natural choice for $Proj_B(S_\bullet \otimes_A B) \to Proj_A(S_\bullet)$ is given by the morphism of graded rings:

$$i : S_\bullet \to S_\bullet \otimes_A B$$

Notice that the homogeneous elements in $S_\bullet \otimes_A B$ od degree d are of the form $\sum f_i \otimes b_i$ where $f_i \in S_d$, then we know that they are $i(f_i) \cdot b_i$, thus the induced scheme morphism is well defined on all of $Proj_B(S_\bullet \otimes_A B)$.

The final step is to show this induces an isomorphism. Notice that on $Spec(S_\bullet \otimes_A B_{f\otimes 1})_0 \to Spec(S_f)_0 \times_A SpecB$, it's given by the following diagram:

$$\begin{tikzcd}
Spec(S_\bullet \otimes_A B_{f\otimes 1})_0 \arrow[dr, dashed] \arrow[drr] \arrow[ddr] & & \\
& Spec(S_f)_0 \times_A SpecB \arrow[r] \arrow[d] & SpecB \arrow[d] \\
& Spec(S_f)_0 \arrow[r] & SpecA
\end{tikzcd}$$

i.e. it's given by the ring map:

$$(S_f)_0 \otimes_A B \to (S_\bullet \otimes_A B_{f\otimes 1})_0$$

where $\frac{x}{f^n} \otimes b \mapsto \frac{x \otimes b}{f^n \otimes 1}$. We only have to show this is a ring isomorphism. To show this, first consider the following ring map:

$$S_f \otimes_A B \to (S_\bullet \otimes_A B)_{f\otimes 1}$$

This is an isomorphism of graded rings, since it's inverse is induced by the following diagram:

$$\begin{tikzcd} S_\bullet \otimes_A B \arrow[r] \arrow[d] & S_f \otimes_A B \\ (S_\bullet \otimes_A B)_{f\otimes 1} \arrow[ur, dashed] & \end{tikzcd}$$

Then you only have to check the composition is the identity.

Then we take the 0-th degree of $S_f \otimes_A B \to (S_\bullet \otimes_A B)_{f\otimes 1}$, which gives you an isomorphism:

$$(S_f \otimes_A B)_0 \to (S_\bullet \otimes_A B_{f\otimes 1})_0$$

However, we know that $(S_f \otimes_A B) \simeq (S_f)_0 \otimes_A B$ via the isomorphism:

$$(S_f)_0 \otimes_A B \to (S_f \otimes_A B)_0 \qquad \frac{x}{f^n} \otimes b \mapsto \frac{x}{f^n} \otimes b$$

This is an isomorphism since tensor commutes with direct sums and you see the ring map above is exactly the inclusion of the 0-th degree component. $\square$

Proposition 2.2. Suppose that X is a projective A-scheme, then $X \times_A SpecB$ is a projective B-scheme.

Proof. This is essentially just a consequence of the above proposition.

Recall the definition of a projective A-scheme: X is a projective A-scheme if $X \simeq ProjS_\bullet$ where $S_\bullet$ is a finitely generated graded ring over A. Then we proved that:

$$X \times_A SpecB \simeq Proj_A S_\bullet \otimes_A SpecB \simeq Proj_B(S_\bullet \otimes_A B)$$

Therefore, we only need to prove $S_\bullet \otimes_A B$ is finitely generated over B, which is obvious. $\square$

Proposition 2.3. Recall that if $S_\bullet$ is generated in degree 1, then $\mathcal{O}_{ProjS_\bullet}(1)$ is an invertible sheaf. Clearly, in this case, we know $S_\bullet \otimes_A B$ is also generated in degree 1 as a B-algebra. Then there is an isomorphism:

$$\mathcal{O}_{Proj_B(S_\bullet \otimes_A B)}(1) \simeq \alpha^* \gamma^* \mathcal{O}_{S^\bullet}(1)$$

where γ is the morphism $Proj_B(S_\bullet \otimes_A B) \to Proj_A(S_\bullet)$ in the fibered product.

Proof. We will use transition functions to prove this. Recall that if $D_+(f)$ covers $Proj_A(S_\bullet)$ if f runs over all homogeneous elements of degree 1. Then we know that $D_+(f \otimes 1)$ covers $Proj_B(S_\bullet \otimes_A B)$ if f runs over all homogeneous elements of degree 1.

Now if you still remember how we defined $\mathcal{O}_{Proj_B(S_\bullet \otimes_A B)}(1)$, it's defined to be the associated quasicoherent sheaf of $(S_\bullet \otimes_A B)(1)$. Generally speaking, we know the transition function of $\mathcal{O}_{ProjS_\bullet}(1)$ from $D_+(f)$ to $D_+(g)$ is multiply by $\frac{f}{g}$, thus the transition function of $\mathcal{O}_{Proj_B(S_\bullet \otimes_A B)}(1)$ from $D_+(f \otimes 1)$ to $D_+(g \otimes 1)$ is just multiply by $\frac{f1}{g\otimes 1}$.

Then we compute the transition function of $\alpha^* \gamma^* \mathcal{O}_{S^\bullet}(1)$. Recall that the inverse image of $D_+(f)$ under γ is $D_+(f) \times_A SpecB$, whose inverse image under α is $D_+(f \otimes 1)$, then the inverse image of $D_+(f) \cap D_+(g)$ under $\gamma \circ \alpha$ is $D_+(f \otimes 1) \cap D_+(g \otimes 1)$. Then notice that via the ring morphism:

$$(S_{fg})_0 \to (S_{fg})_0 \otimes_A B \to ((S_\bullet \otimes_A B)_{fg\otimes 1})_0$$

$\frac{f}{g}$ is mapped to $\frac{f\otimes 1}{g\otimes 1}$ (if you are really careful, you may want to say $\frac{f^2}{fg}$ is mapped to $\frac{f^2 \otimes 1}{fg\otimes 1}$). Then recall how the transition function via pullback works, we are done. $\square$

Remark 2.4. The above argument can also be used to show the pullback of $\mathcal{O}(n)$ is $\mathcal{O}(n)$ since the transition functions can be computed similarly.

After all these preparations, we are ready to introduce a general way to define a scheme over X, if we know what it looks like over every affine open and how they behave under open embeddings of affine open set into each other (not necessarily distinguished!).

Proposition 2.5. Suppose that we are given a scheme X and the following data:

1. For each affine open $U \subset X$, we are given some morphism $\pi_U : Z_U \to U$, i.e. a scheme over U.
2. For each open embedding of affine open $V \subset U \subset X$, we are given an open embedding $\rho_V^U : Z_V \hookrightarrow Z_U$.

Assume that the data satisfies:

1. For each $V \subset U \subset X$, ${}_V^U$ induces an isomorphism $Z_V \to \pi_U^{-1}V$ of schemes over V and,
2. Whenever $W \subset V \subset U \subset X$ are three nested affine open subsets, $\rho_W^U = \rho_V^U \circ \rho_W^U$.

Then there exists an X-scheme $\pi : Z \to X$ and isomorphisms $i_U : \pi^{-1}U \simeq Z_U$ over each affine open such that for each nested affine open subsets $V \subset U$, ρ_V^U agrees with the composition:

$$Z_V \xrightarrow{i_V^{-1}} \pi^{-1}V \hookrightarrow \pi^{-1}U \xrightarrow{i_U} Z_U$$

Proof. We will first define Z as a set, then a topological space finally as a scheme, so the morphism $\pi : Z \to Y$ will be maps of sets, topological spaces and finally schemes. You probably will think of the argument about gluing schemes along isomorphic open subschemes. This is indeed similar to what we will do here and you will see that the principles we use are the same. However, they are not the same, so you can not use gluing schemes to prove this.

Also, before we prove it, we want to state here that in the following proof, the fact that we are considering only affine opens are not special, the proof works for any base of the topology on a ringed space.

To start with, let's first define Z as a topological space. We will define Z to be the disjoint union of all Z_U quotient out the equivalence condition induced by open embedding, i.e. $(x, Z_U) \sim (y, Z_V)$ iff there is another affine open $W \subset U \cap V$ such that via the open embeddings $Z_W \hookrightarrow Z_U$ and $Z_W \hookrightarrow Z_V$ there is a $z \in Z_W$ such that it's mapped to $x \in Z_U$ and $y \in Z_V$. Under our hypothesis, this is indeed an equivalence relation, reflexive and symmetric is easy, to prove it's transitive, if $(x, Z_U) \sim (y, Z_V)$ via the common affine open (a, Z_S) and $(y, Z_V) \sim (z, Z_W)$ via the common affine open (b, Z_T), we can find an affine open of $T \cap S$, say K, then once we find an element (c, Z_K) such that via the open embedding $Z_K \hookrightarrow Z_S$ and $Z_K \hookrightarrow Z_T$, c is mapped to a, b, we are done since we assumed compositions of open embeddings in nested affine open agree. To show there is indeed such a (c, Z_K), we consider the image of a and b in S and T, since we assumed that the open embedding induces isomorphisms on the inverse images, we can assume the image of a, b are the same and are in $T \cap S$, we just pick an affine open K in $T \cap S$ which contains the image of a, b then we know again from the open embedding induces isomorphisms of inverse images that there is indeed exactly one such a (c, Z_K), we are done.

This describes Z as a topological space together with open embeddings $i_U : Z_U \hookrightarrow Z$ with the image $i_U(Z_U)$ such that the following diagram commutes for any $V \subset U$:

$$\begin{tikzcd} Z_U \arrow[r, "i_U"] & i_U(Z_U) \\ Z_V \arrow[u, hook, "\rho_V^U"] \arrow[r, "i_V"'] & i_V(Z_V) \arrow[u, hook, "\text{inclusion}"'] \end{tikzcd}$$

Then we describe Z as a scheme together with a universal property that tells you that there is a unique scheme morphism $\pi : Z \to X$ satisfying the hypothesis we imposed.

You probably want to define the sheaf of rings on Z by define it to be the pushforward of the structure sheaf on Z_u on each $i_U(Z_U)$ and then use the gluing argument to glue them. But the problem here is that two affine intersections are not necessarily affine thus it's hard to define the gluing isomorphism on the intersection $i_U(Z_U) \cap i_V(Z_V)$. Therefore, instead, we retreat a little bit and if you still remember the method we used to constructed the glued sheaf, the same ideal applies here to give this construction we want here. The result will be a sheaf of rings denoted $\mathcal{O}_Z$ on Z together with open embeddings $i_U : Z_U \hookrightarrow Z$ such that it induces isomorphism with Z_U and $i_U(Z_U)$ such that following diagram commutes:

$$\begin{tikzcd} Z_U \arrow[r, "i_U"] & i_U(Z_U) \\ Z_V \arrow[u, hook, "\rho_V^U"] \arrow[r, "i_V"'] & i_V(Z_V) \arrow[u, hook, "\text{inclusion}"'] \end{tikzcd}$$

This is exactly the diagram we used when constructing the topological space, but here we think of everything as schemes. Or you can check it is equivalent to the commutativity of the following diagram:

$$\begin{tikzcd} Z_U \arrow[rr, hook, "i_U"] & & Z \\ & Z_V \arrow[ul, hook, "\rho_V^U"] \arrow[ur, hook, "i_V"'] & \end{tikzcd}$$

To be precise, we define the sheaf $\mathcal{O}_Z$ on Z directly where for each open subset W in Z, we define:

$$\mathcal{O}_Z(W) = \{(s_U)_U : s_U \in \Gamma(U \cap i_U^{-1}(W), \mathcal{O}_{Z_U}) \text{ such that for any } V \subset U \; s_v|_{(V \cap W)} = \rho_V^U(s_U)\}$$

You can directly check this satisfies the sheaf axioms with the obvious restriction essentially because $\mathcal{O}_{Z_U}$ satisfies the sheaf axioms.

Then we need to show there exists such open embeddings $i_U : Z_U \to Z$. We define the map of topological space to be i_U we defined for the topological spaces, and for the morphism of sheaf of rings from $\mathcal{O}_Z \to i_U^* \mathcal{O}_{Z_U}$, we just define it by for each $W \subset X$, we have the following map:

$$\mathcal{O}_Z(W) \to \mathcal{O}_{Z_U}(i_U^{-1}W) \qquad (s_U)_U \mapsto s_U$$

It is obviously compatible with restrictions and is a morphism of sheaves of rings. We have to show it's an open embedding to $i_U(Z_U)$ and the diagram above commutes.

Notice that when $W \subset U$, the map $\mathcal{O}_Z(W) \to \mathcal{O}_{Z_U}(i_U^{-1}W)$ we defined above is an isomorphism. I will let you check it's injective, and the fact that it' surjective comes from the compatibility condition in the definition of $\mathcal{O}_Z$, say we have some $s_U \in \Gamma(i_U^{-1}W, \mathcal{O}_{Z_U})$, we can define for other $V \subset U$ a $s_V = \rho_V^U(s_U)$, then the compatibility condition in the definition is automatically satisfied, thus it's surjective as well. This fact plus $i_U : Z_U \to Z$ induces open embedding to $i_U(Z_U)$ proves i_U as a morphism of schemes is indeed an open embedding. Finally the fact that the diagram commutes comes from the fact that it commutes as a map of topological spaces and the definition of the structure sheaf on Z, I will also let you check it.

The construction Z satisfies a certain universal property, i.e. to give a morphism $f : Z \to Y$, it suffices to give morphism $f_U : Z_U \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} Z_U \arrow[rr, "f_U"] & & Y \\ & Z_V \arrow[ul, hook, "\rho_V^U"] \arrow[ur, "f_V"'] & \end{tikzcd}$$

for each pair of U, V such that $V \subset U$. It's easy to give such a diagram from a morphism $f : Z \to Y$, since we can just compose the open embedding, it's also easy to give a morphism f from such a diagram, if you recall that morphisms of scheme glue.

Using this universal property, with our hypothesis, it's easy to give a morphism $\pi : Z \to X$ from the following diagram:

$$\begin{tikzcd} Z_U \arrow[rr, "\pi_U"] & & X \\ & Z_V \arrow[ul, hook, "\rho_V^U"] \arrow[ur, "\pi_V"'] & \end{tikzcd}$$

this diagram indeed commutes since we assumed that ρ_V^U induces an isomorphism with $\pi_U^{-1}(V)$, which means that ρ_V^U factors through an isomorphism with $\pi^{-1}V$ and the open embedding $\pi^{-1}V \hookrightarrow Z_U$.

The last assertion that ρ_V^U agrees with the composition:

$$Z_V \xrightarrow{i_V^{-1}} \pi^{-1}V \hookrightarrow \pi^{-1}U \xrightarrow{i_U} Z_U$$

follows from the way we define f and the commutativity diagram. Therefore we are done, and as we mentioned in the beginning, this argument works for ringed spaces and any base of the topology. ☐

Remark 2.6. If you go over the above argument, you will find that we don't even have to assume we have a base of the topology. In fact, what we need is the following data: an open cover of the space X, such that for any two open U, V in this cover, the intersection can be covered again by opens in this open cover. Then you find that the argument above can be carried out verbatim in this situation and we have a scheme Z together with a structure morphism $Z \to X$.

Moreover, suppose that you have a base of X together with the data in the above proposition, and then you pick out a subcollection of the base such that it satisfies the assumptions in the preceding paragraph, then you can construct two X-schemes Z_1, Z_2 using the base and the open cover, and you will find they are naturally isomorphic. The reason is also simple, the universal property of Z_2 is given by any morphism out of Z_2 is equivalent to a bunch of morphisms out of Z_U such that it satisfies the compatibility condition for $V \hookrightarrow U$ an open embedding. Then since the data defining Z_2 us just part of the data defining Z_1 and both satisfies the compatibility condition, we naturally have a morphism $Z_2 \to Z_1$ such that it's an isomorphism on each Z_U, thus a global isomorphism. The same argument shows that you can pick out a different open cover satisfying the assumptions in the previous paragraph and construct $Z_3 \to X$, and it will be isomorphic as a X-scheme to Z_2 since Z_2, Z_3 are both isomorphic to Z_1. This is also an advantage of doing the construction using all the affines.

Using the above construction, we can construct the relative proj, for the set up, let X be any scheme and $\mathcal{I}_\bullet = \oplus_{n\geqslant 0}\mathcal{I}_n$ (a clean way to state this condition is that we have an injective morphism $\mathcal{I}_n \hookrightarrow \mathcal{I}_\bullet$ for each n, which induces an isomorphism $\oplus_{n\geqslant n}\mathcal{I}_n \simeq \mathcal{I}_\bullet$ via the universal property of direct sums, we will also require $\mathcal{I}_n$ is also quasicoherent as a $\mathcal{O}_X$-module for each n) be any quasicoherent sheaf of graded algebra on X, then over each affine open $SpecA \simeq U \subset X$, $\mathcal{I}_\bullet(U)$ is an A-algebra which is graded (can you figure out what is the grading? The key here is that even if infinite direct sums of sheaves need to be sheafified, quasicoherent condition helps to drop the sheafification step on affine opens, i.e. just as the way we proved tensor product of quasicoherent sheaves don't have to be sheafified on affine opens, we can prove the same conclusion for direct sums of quasicoherent modules which essentially is the statement that arbitrary direct sums commutes with localizations) and we can define an U-scheme by: $\beta_U : Z_U = Proj_A\mathcal{I}_\bullet(U) \to U$.

Proposition 2.7. Using the above construction of X-scheme, the local construction $Proj_A\mathcal{I}_\bullet(U) \to U$ glues to a X-scheme, called $\mathcal{P}roj_X\mathcal{I}_\bullet$, together with a structure morphism $\beta : \mathcal{P}roj_X\mathcal{I}_\bullet \to X$. (The structure morphism is part of the data in the relative proj.)

Proof. Before we prove this, a comment is that we can't not just considering affine opens and distinguished affine inclusions, since in the above proposition, we need to use all affine opens, which is a base of the topology. However, a version of affine open and distinguished affine open inclusion of the above proposition can be given and indeed will make this proposition easier in the sense that we don't have to use the fact that the usual proj commutes with affine base change which we proved in the beginning of this section. However, considering only the distinguished affine inclusions makes the above construction considerably harder than using all affine opens, thus we won't go that way.

Now suppose that $V = SpecB \subset U \simeq SpecA$, then we know that over V, the V-scheme structure is given by:

$$\beta_V : Z_V = Proj_B(\mathcal{I}_\bullet(V)) \to V$$

To use the construction, we need two things: an open embedding $\rho_V^U : Proj_B(\mathcal{I}_\bullet(V)) \hookrightarrow Proj_A(\mathcal{I}_\bullet(U))$ and it should induces an isomorphism to $\beta_U^{-1}(V)$.

First, consider the following fibered product of open embeddings of affine opens:

$$\begin{tikzcd}
Proj_B(\mathcal{I}_\bullet(U) \otimes_A B) \simeq Proj_A(\mathcal{I}_\bullet(U)) \times_A SpecB \arrow[r, hook] \arrow[d] & Proj_A(\mathcal{I}_\bullet(U)) \arrow[d, "\beta_U"] \\
V = SpecB \arrow[r, hook] & SpecA = U
\end{tikzcd}$$

which follows from Proj construction commutes with affine base change.

Now, notice that $\mathcal{I}_\bullet(V)$ is just $\mathcal{I}_\bullet(U) \otimes_A B$, since this is the global section of the restriction of $\mathcal{I}_\bullet$ on U, since we are working with affine schemes and open embeddings, restriction is just pullback using the open embedding, thus it's just tensor with B. To be more precise, there is a canonical isomorphism of B-modules induced by the adjoint property:

$$\mathcal{I}_\bullet(V) \simeq \mathcal{I}_\bullet(U) \otimes_A B$$

and if you think through how this isomorphism is induced (think thourg the adjoint property that let's you conclude restriction is the same as pullback then the image of the identity is the isomorphism) it's just the one induced by $\mathcal{I}_\bullet(U) \to \mathcal{I}_\bullet(V)$ and $B \to \mathcal{I}_\bullet(V)$, which happens to be a B-algebra isomorphism! Therefore, we have the following diagram:

$$\begin{tikzcd}
Proj_A(\mathcal{I}_\bullet(U)) \arrow[dr] & & Proj_B(\mathcal{I}_\bullet(U) \otimes_A B) \arrow[ll, hook'] \arrow[dr] & & Proj_B(\mathcal{I}_\bullet(V)) \arrow[ll, "\simeq"'] \arrow[dl] \\
& U = SpecA & & V = SpecB \arrow[ll, hook']
\end{tikzcd}$$

this tells you what is ρ_V^U and why it induces isomorphsims to the inverse image.

Finally, we need to show that the composition $\rho_W^U = \rho_V^U \circ \rho_W^V$. Recall that we already saw the morphism $Proj_B(\mathcal{I}_\bullet(U) \otimes_A B) \hookrightarrow Proj_A(\mathcal{I}_\bullet(U))$ is induced by the B-algebra homomorphism $\mathcal{I}_\bullet(U) \to \mathcal{I}_\bullet(U) \otimes_A B$ and the isomorphism $Proj_B(\mathcal{I}_\bullet(U) \otimes_A B) \simeq Proj_B(\mathcal{I}_\bullet(V))$ is induced by the B-algebra homomophism $\mathcal{I}_\bullet(U) \otimes_A B \to \mathcal{I}_\bullet(V)$, therefore ρ_V^U is induced by the B-algebra homomorphisms $\mathcal{I}_\bullet(B) \to \mathcal{I}_\bullet(V)$ and you find it's justt the restriction!

Therefore, since composition of restriction maps agree, we are done. $\square$

Remark 2.8. As we mentioned before, the data on every affine open together with every inclusion is redundant in the sense that we can construct an isomorphic X-scheme $\mathcal{P}roj\mathcal{I}_\bullet$ only using only a part of the affine opens

Remark 2.9. Essentially the same ideal can be used to construct X^{red}, normalization and $\mathcal{S}pec\mathcal{R}$ by using affine opens and glue use the proposition above. However, there will be a price to be paid when you do the glue argument by losing universal property, for example, here we have to use the fact that Proj commutes with affine base change and a little bit more than that.

Recall that the usual proj construction is most useful when applied to a graded A-algebra $S_\bullet$ with some reasonable assumptions, for example, finitely generated as an A-algebra in degree 1. For the same reason, for the rest of the notes, we will also make some reasonable assumptions on $\mathcal{I}_\bullet$ and will keep them fixed.

Definition 2.10. For the rest of the notes, we will require $\mathcal{I}_\bullet$ to be a quasicoherent sheaf of graded $\mathcal{O}_X$-algebras **finitely generated in degree** 1, i.e. each component $\mathcal{I}_n$ is a quasicoherent sheaf of $\mathcal{O}_X$-modules, $\mathcal{I}_0 = \mathcal{O}_X$, $\mathcal{I}_1$ is a finite type quasicoherent sheaf on X and $\mathcal{I}_\bullet$ is generated in degree 1.

For the last condition, the cleanest way to state it is to require a surjection:

$$Sym^\bullet_{\mathcal{O}_X}\mathcal{I}_1 \to \mathcal{I}_\bullet$$

which is induced by the universal property of $Sym^\bullet_{\mathcal{O}_X}$ and it's given by the usual inclusion $\mathcal{I}_1 \hookrightarrow \mathcal{I}_\bullet$.

Recall that the $Sym^\bullet_{\mathcal{O}_X}$ construction for a quasicoherent module $\mathcal{F}$ is also quasicoherent, and on each affine open A, it's sections is just $Sym^\bullet_A\mathcal{F}(SpecA)$, then we can just check the surjection affine locally, thus generation in degree 1 is something we can check on each affine.

The motivation for this hypothesis is because we want the following construction:

Proposition 2.11. Let $\mathcal{I}_\bullet$ to be finitely generated in degree 1, then there is an invertible sheaf $\mathcal{O}_{\mathcal{P}roj_X\mathcal{I}_\bullet}(1)$ on $\mathcal{P}roj_X\mathcal{I}_\bullet$ such that it restricts to $\mathcal{O}_{Proj_A\mathcal{I}_\bullet(SpecA)}(1)$ over each affine open subset $SpecA \subset X$.

Proof. We first show that similar to gluing sheaves on an open cover via isomorphisms on the intersection, we can produce a sheaf using similar method on an cover of X via an open cover of X say $\{U\}$ such that for each U, V in the cover there is some W in the cover such that $W \subset U \cap V$ and such W's cover $U \cap V$.

Say on X you have such an open cover $\{U\}$ and on each U, there is a sheaf $\mathcal{F}_U$ together with isomorphisms $\varphi_V^U : \mathcal{F}_V \simeq \mathcal{F}_U|_V$ such that for any nested opens in the cover $W \subset V \subset U$, we have $\varphi_W^U = \varphi_V^U|_W \circ \varphi_W^V$, then there is a sheaf $\mathcal{F}$ on X together with isomorphisms $\psi_U : \mathcal{F}|_U \simeq \mathcal{F}_U$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}|_V \arrow[r, "\psi_U|_V"] & \mathcal{F}_U|_V \\
\mathcal{F}|_V \arrow[u, "id"] \arrow[r, "\psi_V"'] & \mathcal{F}_V \arrow[u, "\varphi_V^U"']
\end{tikzcd}$$

The way to prove it is still to define $\mathcal{F}$ by on each $W \subset X$, we set:

$$\mathcal{F}(W) = \{(s_U)_U : s_U \in \Gamma(W \cap U, \mathcal{F}_U), \text{ such that for each } V \subset U \ s_U|_{V\cap W} = \varphi_V^U(s_V)\}$$

Again, it's a sheaf with obvious restrictions since each $\mathcal{F}_U$ is a sheaf, and the isomorphism ψ_U comes from the the natural map where for each $W \subset U$, we define $(s_U)_U \mapsto s_U \in \Gamma(W, \mathcal{F}_U)$. This is injective since $W \subset U$, it's surjective since for any $s \in \Gamma(W, \mathcal{F}_U)$ we can define a compatible collection. To be precise, for each V in the open cover, we consider all the other opens S in the cover such that S is in the intersection $U \cap V$, and we define $s_V \in \Gamma(W \cap V, \mathcal{F}_V)$ to be the one that restricts to $\varphi_S^V \circ (\varphi_S^U)^{-1}(s) \in \Gamma(W \cap V \cap S, \mathcal{F}_V)$, we notice that for we want to show they agree on intersections, such that it indeed gives a section on $W \cap V$, well suppose that we have the section $\varphi_T^V \circ (\varphi_T^U)^{-1}(s) \in \Gamma(W \cap V \cap T, \mathcal{F}_V)$ and we want to show they agree on $W \cap V \cap S \cap T$. The way we check this is via the fact that we can still cover $S \cap T$ with opens in the cover, say one of them is K. Then we restrict to $W \cap V \cap K$ and if they agree on every $W \cap V \cap K$, we are done.

This essentially follows from the compatibility conditions on φ's, and I will let you check it. Then once we check this satisfies the condition in the definition of $\mathcal{F}$ it's indeed surjective. Suppose that $V' \subset V$, we need to show that $s_V|_{V'\cap W} = \varphi_{V'}^V(s_{V'})$ then we only have to check this on all T where T in in $V' \cap U$, but remember how we defined s_V and $s_{V'}$ on T, I will let you check this suffice to show the compatibility condition. The fact that the certain diagram commutes is built in the definition.

Then, let's use this to construct $\mathcal{O}_{\mathcal{P}roj\mathcal{I}_\bullet}(1)$. To begin with we give a general argument. Say $\mathcal{L}, \mathcal{M}, \mathcal{N}$ are three locally free sheaf of rank n on X, say you cover X by some opens U, on each U, all three of them are trivial and the transition functions are the same. Under this assumption, we know we have three isomorphisms:

$$\psi_L^M : \mathcal{L} \simeq \mathcal{M} \quad \psi_M^N : \mathcal{M} \simeq \mathcal{N} \quad \psi_L^N : \mathcal{L} \simeq \mathcal{N}$$

induced by the transition functions. Then I claim $\psi_N^L = \psi_M^N \circ \psi_L^M$.

The reason is because if you consider what is the isomorphism ψ_L^M, it's given by the universal property since $\mathcal{L}$ represents the sheaf glued together the local free sheaves, then you have a functorial isomorphisms:

$$Hom(\mathcal{L}, -) \simeq Hom(\mathcal{M}, -) \quad Hom(\mathcal{M}, -) \simeq Hom(\mathcal{N}, -) \quad Hom(\mathcal{L}, -) \simeq Hom(\mathcal{N}, -)$$

such that the composition of the first two is the third, then Yoneda's lemma tells you the conclusion. Or you can just do this show that the isomorphism between $\mathcal{L}$ and $\mathcal{M}$ is just defined locally by:

$$\mathcal{L}|_U \simeq \mathcal{O}_U^n \simeq \mathcal{M}|_U$$

and it naturally agrees on intersections thus glue to a global isomorphism. We will use this in the proof below.

Now by construction of $\mathcal{P}roj\mathcal{I}_\bullet$, we have the following commutative diagram:

$$\begin{tikzcd}
Proj_C(\mathcal{I}_\bullet(W)) \arrow[r, hook] \arrow[d, dashed, "\simeq"] & Proj_C(\mathcal{I}_\bullet(V)) \arrow[r, hook] \arrow[d, dashed, "\simeq"] & Proj_C(\mathcal{I}_\bullet(V)) \arrow[d, dashed, "\simeq"] \\
\beta^{-1}W \arrow[r, hook] \arrow[d, "\beta"] & \beta^{-1}V \arrow[r, hook] \arrow[d, "\beta"] & \beta^{-1}U \arrow[d, "\beta"] \\
W = SpecC \arrow[r, hook] & V = SpecB \arrow[r, hook] & U = SpecA
\end{tikzcd}$$

where composition of arrows in each column is the structure morphism for the usual proj. Then I define a sheaf on $\beta^{-1}U$ by $\mathcal{O}_{Proj_A(\mathcal{I}_\bullet(U))}(1)$. Then to show these assignment extends to a global invertible sheaf on X, we need to show that we have the isomorphism φ_V^U satisfying compatibility conditions. Let's first define these φ_V^U's for each pair of U, V.

Recall that we have the following diagram:

$$\begin{tikzcd}
 & Proj_B(\mathcal{I}_\bullet(U) \otimes_A B) \arrow[dr, hook] & \\
Proj_B\mathcal{I}_\bullet(V) \arrow[ur, "\simeq"] \arrow[rr, hook] & & Proj_B\mathcal{I}_\bullet(U)
\end{tikzcd}$$

and we already saw the pullback of $\mathcal{O}_{Proj_A}(1)$ is isomorphic to $\mathcal{O}_{Proj_B\otimes}(1)$, then it's again pullback to something isomorphic to $\mathcal{O}_{ProjB}(1)$, and the isomorphism is defined via transition functions. We will take this to be the isomorphism φ_V^U.

Let's denote the open embeddings by $i : Proj_B\mathcal{I}_\bullet(V) \hookrightarrow Proj_B\mathcal{I}_\bullet(U)$ and $j : Proj_B\mathcal{I}_\bullet(W) \hookrightarrow Proj_B\mathcal{I}_\bullet(V)$. We know that $\varphi_W^U : j_*i_*\mathcal{O}_A(1) \simeq \mathcal{O}_C(1)$ where the subscript is used to indicate it's over $SpecA$ or $SpecC$. Then to show the compatibility, we only need to prove that the following diagram commutes:

$$\begin{tikzcd}
j_*i_*\mathcal{O}_A(1) \arrow[rr, "\simeq"] \arrow[dr, "\simeq"'] & & \mathcal{O}_C(1) \\
 & j_*\mathcal{O}_B(1) \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

where the isomorphism $j_*i_*\mathcal{O}_A(1) \simeq j_*\mathcal{O}_B(1)$ is the isomorphism we obtained by apply j_* to the isomorphism $i_*\mathcal{O}_A(1) \simeq \mathcal{O}_B(1)$. Notice the isomorphisms $j_*\mathcal{O}_B(1) \simeq \mathcal{O}_C(1)$ and $j_*i_*\mathcal{O}_A(1) \simeq \mathcal{O}_C(1)$ are induced by the transition functions. Therefore if we can prove $i_*\mathcal{O}_A(1) \simeq \mathcal{O}_B(1)$ is also induced by the transition functions, we are done. But I will let you check that this is generally true for any scheme morphisms $f : X \to Y$ because if you have a locally free sheaf $\mathcal{L}$ of finite rank on Y with trivializing neighborhoods U, then the pullback $f^*\mathcal{L}$ is trivial on $f^{-1}U$ and with respect to this trivialization, the transition functions are pullbacks of the transitions functions. We are done. $\square$

Now we can also prove the relative proj construction behaves well wrt base change:

Proposition 2.12. Suppose that $\mathcal{I}_\bullet$ is a quasicoherent sheaf of graded algebra finitely generated in degree 1 on X, let $\rho : Z \to X$ be any morphism of schemes, then there is a natural isomorphism:

$$(\mathcal{P}roj\rho^*\mathcal{I}_\bullet, \mathcal{O}_{\mathcal{P}roj\rho^*\mathcal{I}_\bullet}(1)) \simeq (Z \times_X \mathcal{P}roj\mathcal{I}_\bullet, \psi^*\mathcal{O}_{\mathcal{P}roj\mathcal{I}_\bullet}(1))$$

where ψ is the morphism in the following fibered product diagram:

$$\begin{tikzcd}
Z \times_X \mathcal{P}roj\mathcal{I}_\bullet \arrow[r, "\psi"] \arrow[d] & \mathcal{P}roj\mathcal{I}_\bullet \arrow[d] \\
Z \arrow[r] & X
\end{tikzcd}$$

Informally speaking, this proposition is saying that fibered product of relative proj is also a relative proj.

Proof. Consider the canonical morphism induced by the following diagram:

$$\begin{tikzcd}
\mathcal{P}roj\rho^*\mathcal{I}_\bullet \arrow[rrd, bend left] \arrow[rd, dashed] \arrow[rdd, bend right] & & \\
& Z \times_X \mathcal{P}roj\mathcal{I}_\bullet \arrow[r, "\psi"] \arrow[d] & \mathcal{P}roj\mathcal{I}_\bullet \arrow[d] \\
& Z \arrow[r] & X
\end{tikzcd}$$

where $\mathcal{P}roj\rho^*\mathcal{I}_\bullet \to Z$ is the structure morphism. To figure out what is the morphism $\mathcal{P}roj\rho^*\mathcal{I}_\bullet \to \mathcal{P}roj\mathcal{I}_\bullet$, we describe $\mathcal{P}roj\rho^*\mathcal{I}_\bullet$ not using all the affine opens and inclusions of Z, but a subcollection of that.

To be precise, consider the morphism $\rho : Z \to X$, for any affine open U of X, we consider the inverse image $\rho^{-1}U$, we choose all the affine opens of $\rho^{-1}U$. We know that we can cover X using affine opens, and we can consider the inverse image, thus we only consider those affine opens in Z whose image lies in an affine open of X. Then they indeed cover Z and suppose U, V are such two affine opens whose image lies in $SpecA$ and $SpecB$ in X, then we know that for each point $p \in U \cap V$, $\rho(p)$ is in $SpecA \cap SpecB$, then we know that all the affine opens in in $U \cap V$ that contains p is an affine open whose image is in an affine open. Therefore, this is indeed a subcollection of all the affine opens of Z, and by a previous remark, we can construct $\mathcal{P}roj\rho^*\mathcal{I}_\bullet$ using this collection and obtain isomorphic object.

Using this point of view, it's easy to describe $\mathcal{P}roj\rho^*\mathcal{I}_\bullet \to \mathcal{P}roj\mathcal{I}_\bullet$. To be precise, by the universal property of $\mathcal{P}roj\rho^*\mathcal{I}_\bullet$ we only have to describe a morphism over each $SpecB \subset Z$ whose image is in some affine open $SpecR$ in X such that the compatibility condition is satisfied. Notice that over $SpecA$, we know $\mathcal{P}roj\rho^*\mathcal{I}_\bullet$ is just:

$$Proj\rho^*\mathcal{I}(SpecA) \to SpecA$$

However, we know that $\rho^*\mathcal{I}(SpecA) \simeq \mathcal{I}(SpecR) \otimes_R A$, then we know we only have to give a morphism $Proj\mathcal{I}(SpecR) \otimes_R A \to Proj\mathcal{I}(SpecR)$ which is obvious. Then we only have to show this is compatible in the following diagram:

$$\begin{tikzcd}
Proj\mathcal{I}(SpecR) \otimes_R B \arrow[r, hook] \arrow[rr, bend left] \arrow[d, "\simeq"'] & Proj\mathcal{I}(SpecR) \otimes_R A \arrow[r] \arrow[d, "\simeq"] & Proj\mathcal{I}(SpecR) \\
Proj\rho^*\mathcal{I}(SpecB) \arrow[r, hook] \arrow[d] & Proj\rho^*\mathcal{I}(SpecA) \arrow[d] & \\
SpecB \arrow[r, hook] & SpecA &
\end{tikzcd}$$

where the top morphism is induced by the graded ring map $\mathcal{I}(SpecR) \otimes_R A \to \mathcal{I}(SpecR) \otimes_R B$. (There is an interesting question: why the B-module isomorphism $Proj\mathcal{I}(SpecR) \otimes_R A \simeq \rho^*\mathcal{I}(SpecB)$ is also a B-algebra isomorphism. The question is not difficult to answer once you know this isomorphism can be explicitly written as $x \otimes a \mapsto \rho^{-1}x \otimes a$, the important step is why this isomorphism looks like this, this requires you to know that say for any sheaf $\mathcal{F}$ on Y and morphisms of scheme $f : X \to Y$, where does a global section of $\mathcal{F}$ under the morphism $\mathcal{F} \to f_* f^{-1}\mathcal{F}$ induced by the adjunction property and the identity morphism $f^{-1}\mathcal{F} \to f^{-1}\mathcal{F}$, I will let you review this. **** Another point which is technical and annoying is that we need to make sure that if $SpecA$ is mapped to both $SpecR$ and $SpecT$, what we stated above if well defined, this is hard to check if we don't assume anything on the morphism ρ or on the base scheme X. For example, if X is separated, then intersection of affine is affine again, then we can use the intersection to show it's well defined, or if the morphism is affine we can also avoid the confusion*)

To show this diagram indeed commutes, we only have to show the top square commutes, i.e. the following diagram commutes:

$$\begin{tikzcd}
\mathcal{I}(SpecR) \otimes_R A \arrow[r] \arrow[d, dashed, "\simeq"] & \mathcal{I}(SpecR) \otimes_R B \arrow[d, dashed, "\simeq"] \\
\rho^*\mathcal{I}(SpecA) \arrow[r, "res"] & \rho^*\mathcal{I}(SpecB)
\end{tikzcd}$$

Recall that if $SpecB \hookrightarrow SpecA$ is an open embedding and $\widetilde{M}$ is a quasicoherent module on $SpecA$ associated to the A-module M, then the isomorphism between $\widetilde{M}(SpecB) \simeq M \otimes_A B$ is induced by the restriction map $M \to \widetilde{M}(SpecB)$, then to show the above diagram commutes, it's equivalent to show the following diagram commutes:

$$\begin{tikzcd}
& \mathcal{I}(SpecR) \otimes_R B = \mathcal{I}(SpecR) \otimes_R A \otimes_A B \arrow[d, "\simeq"] \\
\mathcal{I}(SpecR) \otimes_R A \arrow[ur] \arrow[r, "res"] \arrow[d, "\simeq"] & \Gamma(\widetilde{\mathcal{I}(SpecR) \otimes_R A}, SpecB) \arrow[d, "\simeq"] \\
\rho^*\mathcal{I}(SpecA) \arrow[r, "res"] & \rho^*\mathcal{I}(SpecB)
\end{tikzcd}$$

notice the square commutes because we have an isomorphism of sheaves, the top triangle commutes because we know the isomorphism $\mathcal{I}(SpecR) \otimes_R A \otimes_A B \simeq \mathcal{I}(SpecR) \otimes_R B = \mathcal{I}(SpecR) \otimes_R A \otimes_A B$ is induced by the restriction map.

This defines a morphism from $\mathcal{P}roj\rho^*\mathcal{I}_\bullet$ to $\mathcal{P}roj\mathcal{I}_\bullet$, and it's easy to show that it makes the fibered product diagram commutes: you only have to check this locally, therefore, we can first assume X is affine, say it's $SpecR$, then you can assume $\rho^{-1}SpecR$ is also affine say $SpecA$, then you notice that you just need to check the following diagram commutes:

$$\begin{tikzcd}
Proj\mathcal{I}(SpecR) \otimes_R A \arrow[d, "\simeq"'] \arrow[dr] & \\
Proj\rho^*\mathcal{I}(SpecA) \arrow[r] \arrow[d] & Proj\mathcal{I}(SpecR) \arrow[d] \\
SpecA \arrow[r] & SpecR
\end{tikzcd}$$

which is something we already proved: usual proj commutes with affine base change.

Then the universal property of fibered product gives you a morphism from $\mathcal{P}roj\rho^*\mathcal{I}_\bullet$ to $Z \times_X \mathcal{P}roj\mathcal{I}_\bullet$ this is an isomorphism since we can still check this locally, notice that $SpecA \times_R Proj\mathcal{I}(SpecR)$ covers the fibered product, then locally over $SpeA$ and $SpecR$, we already saw the fibered product looks like:

$$\begin{tikzcd}
Proj(\mathcal{I}(SpecR) \otimes_R A) \arrow[r] \arrow[d] & Proj\mathcal{I}(SpecR) \arrow[d] \\
SpecA \arrow[r] & SpecR
\end{tikzcd}$$

then it's clear why this is locally an isomorphism.

We also need to show this isomorphism is compatible with $\mathcal{O}(1)$, i.e. the pullback of $\mathcal{O}(1)$ on $\mathcal{P}roj\mathcal{I}_\bullet$ is $\mathcal{O}(1)$ on $\mathcal{P}roj\rho^*\mathcal{I}_\bullet$. I claim it suffice to prove $\mathcal{O}(1)$ on $Proj\mathcal{I}(SpecR)$ pulls back to $\mathcal{O}(1)$ on $Proj(\mathcal{I}(SpecR)\otimes_R A)$, which is true since we already proved this using transition functions.

This is because if this holds, then the data that defines $\mathcal{O}(1)$ on $\mathcal{P}roj\rho^*\mathcal{I}_\bullet$ is isomorphic to the data that defines the pullback. To be precise, we can view the pullback to be the sheaf constructed by sheaves on each $SpecA$ and open inclusions $SpecB \hookrightarrow SpecA$ whose image is in an affine open $SpecR$ in X. This makes the description of the pullback of $\mathcal{O}(1)$ easier since $\square$

The global proj construction allows us to generalize the notion of relatively very ample:

Definition 2.13. Suppose $\pi : X \to Y$ is proper and $\mathcal{L}$ is an invertible sheaf on X, we say $\mathcal{L}$ is very ample with respect to π, or relatively very ample or π-very ample if we can write $X = \mathcal{P}roj_Y\mathcal{I}_\bullet$ with $\mathcal{L} \simeq \mathcal{O}(1)$ where $\mathcal{I}_\bullet$ is a quasicoherent sheaf of graded algebra finitely generated in degree 1.

Proposition 2.14. Suppose that $\mathcal{I}_\bullet$ is finitely generated in degree 1, then there is a map of graded quasi-coherent sheaves $\psi : \mathcal{I}_\bullet \to \oplus_{n\geqslant 0}\beta_*\mathcal{O}(n)$ where β is the structure morphism and $\mathcal{O}(n)$ is just the n-th tensor power of $\mathcal{O}(1)$.

Proof. First notice that just like $\mathcal{O}(1)$ on $\mathcal{P}roj\mathcal{I}_\bullet$, we can also define $\mathcal{O}(n)$ on $\mathcal{P}roj\mathcal{I}_\bullet$ by gluing the usual $\mathcal{O}(n)$ on the usual proj. Moreover $\mathcal{O}(n)$ obtained by gluing or by tensoring $\mathcal{O}(1)$ are isomorphic, the reason is the tensor product definition can also be realized using the gluing data. Suppose that $\mathcal{L}$ and $\mathcal{M}$ are two line bundles on X with the same transition function, which gives you an isomorphism $f : \mathcal{L} \simeq \mathcal{M}$, then we know that the transition functions of $\mathcal{L}^{\otimes n}$ and $\mathcal{M}^{\otimes n}$ are the same and the corresponding isomorphism is given by $f^{\otimes n}$. Then try to apply this to $\mathcal{O}(1)$ and $\mathcal{O}(n)$ here.

Then, recall that to describe a morphism of sheaves on X, we can only describe a morphism of sheaves on the base, and we can use all the affine opens as a base on X.

Say $U = SpecA$ is an affine open of X, we only recall that we need to describe an A-module morphism $\mathcal{I}_\bullet(SpecA) \to \oplus_n \beta_*\mathcal{O}(n)(SpecA)$.

Recall that for any graded algebra $S_\bullet$ finitely generated in degree 1, we have the natural saturation map $M_\bullet \to \Gamma_\bullet(\widetilde{M_\bullet}) = \oplus_n(ProjS_\bullet, \widetilde{M_\bullet}(n))$ for any graded module $M_\bullet$, then take $M_\bullet$ to be $S_\bullet$, we get a natural morphism $S_\bullet \to \oplus_n\Gamma(ProjS_\bullet, \mathcal{O}(n))$.

Then notice that for each n, we know $\mathcal{O}(n)$ is quasicoherent on $\mathcal{P}roj\mathcal{I}_\bullet$, and the structure morphism β is obviously qcqs since both qc and qs can be checked affine locally on the target and we know that for a graded A-algebra $S_\bullet$ finitely generated in degree 1, the structure morphism $ProjS_\bullet \to SpecA$ is qcqs, thus we know that the pushforward by β is also quasicoherent on X for each n, therefore since direct sum of quasicoherent sheaves on an affine open don't need to be sheafified, we know that we only need a morphism:

$$\mathcal{I}_\bullet(SpecA) \to \oplus_n\mathcal{O}(n)(\beta^{-1}SpecA)$$

If you remember the definition of $\mathcal{O}(n)$, then notice that on $\beta^{-1}SpecA$ it's just $\mathcal{O}(n)$ on $Proj_A\mathcal{I}_\bullet(SpecA)$, then we can just take this to be the natural saturation map described above, the only thing left is to verify this is compatible with restrictions to smaller affine opens.

Say $V = SpecB \subset SpecA = U$ are two affine opens of X, then we need the following diagram to commute:

$$\begin{tikzcd}
\mathcal{I}_\bullet(SpecA) \arrow[r] \arrow[d, dashed, "res"] & \oplus_n(Proj_A\mathcal{I}(SpecA), \mathcal{O}(n)) \arrow[d, dashed, "res"] \\
\mathcal{I}_\bullet(SpecB) \arrow[r] & \oplus_n(Proj_B\mathcal{I}(SpecB), \mathcal{O}(n))
\end{tikzcd}$$

We need to first figure out what is the restriction on the right hand side. We only have to do this for each n since the restriction respects the direct sum, and for each n, recall that $\mathcal{O}(n)$ is defined by relative gluing and for such $SpecBSpecA$, the relative gluing is given by the Cartesian diagram:

$$\begin{tikzcd}
Proj_B\mathcal{I}(SpecB) \arrow[r, hook] \arrow[d] & Proj_A\mathcal{I}(SpecA) \arrow[d] \\
SpecB \arrow[r, hook] & SpecA
\end{tikzcd}$$

we described before. Therefore, the restriction is defined by the pullback morphism. To be more precise, suppose that you have an open embedding $i : X \hookrightarrow Y$, then for any sheaf $\mathcal{F}$ on Y, the pullback by i, $i^*\mathcal{F}$ will be isomorphic to the restriction to the image of i and the pullback morphism is just the restriction. This operation is compatible with restriction in the sense that if $U \subset Y$ is an open subset with inverse image V, then in the open embedding Cartesian diagram, restriction and pullback commutes. Therefore for $Proj_A\mathcal{I}(SpecA)$, say $f_1, \cdots, f_n$ in $\mathcal{I}(SpecA)$ generates $\mathcal{I}$, we know that their restriction in $\mathcal{I}(SpecB)$ generates $\mathcal{I}(SpecB)$, then I claim for a element in $\Gamma(Proj_A\mathcal{I}(SpecA), \mathcal{O}(n))$ that corresponds to x_n in each $D_+(f_n)$ will be mapped to the element in $\Gamma(Proj_B\mathcal{I}(SpecB), \mathcal{O}(n))$ that correspond to the pullback of x_n by the morphism of affine schemes:

$$D_+(f_n|_{res}) \to D_+(f_n)$$

The reason is simple, just recall how the isomorphism determined by transition functions is defined, then we are done with the restriction map, but with this description of the restriction, the commutativity is easy and I will let you do this. $\square$

Proposition 2.15. Suppose that $\mathcal{L}$ is an invertible sheaf on X and $\mathcal{I}_\bullet$ is a quasicoherent sheaf of graded algebras finitely generated in degree 1, we can define a new quasicoherent sheaf of algebras finitely generated in degree 1 using the above two data: we define $\mathcal{I}'_\bullet = \oplus_n \mathcal{I}_n \otimes \mathcal{L}^{\otimes n}$, this has a natural algebra structure inherited from $\mathcal{I}_\bullet$, then there is a natural isomorphism of X-schemes with line bundles:

$$(\mathcal{P}roj\mathcal{I}'_\bullet, \mathcal{O}(1)) \simeq (\mathcal{P}roj\mathcal{I}_\bullet, \mathcal{O}(1) \otimes \beta^*\mathcal{L})$$

Informally speaking, you can twist the graded sheaf of algebra by a line bundle $\mathcal{L}$ and get isomorphic X-scheme, but the $\mathcal{O}(1)$ is twisted by the pullback of the line bundle $\mathcal{L}$.

Proof. Since we have a direct sum decomposition of $\mathcal{I}'$, to define the algebra structure, we only have to describe how to multiply elements in different degrees that respects the new grading. To be precise, we need morphisms:

$$\mathcal{I}_n \otimes \mathcal{L}^{\otimes n} \times \mathcal{I}_m \otimes \mathcal{L}^{\otimes m} \to \mathcal{I}_{m+n} \otimes \mathcal{L}^{\otimes m+n}$$

that satisfies a few ring multiplication axioms. This is equivaelant by the universal property of tensor products to a multilinear morphism of $\mathcal{O}_X$-modules:

$$\mathcal{I}_n \times \mathcal{I}_m \times \prod^{m+n} \mathcal{L} \to \mathcal{I}_{m+n} \otimes \mathcal{L}^{\otimes m+n}$$

then it's clear how to define it and the fact it's satisfies the ring multiplication axioms is easy to check.

With this structure, $\mathcal{I}'$ is finitely generated in degree 1. It's obvious the 0 degree part is $\mathcal{O}_X$ and since $\mathcal{I}_1$ is finite type and $\mathcal{L}$ is locally free, the tensor is quasicoherent and finite type. Finally, we need to show:

$$Sym^\bullet \mathcal{I}'_1 \to \mathcal{I}'_\bullet$$

is surjective. Recall this can be cheked affine locally and it just reduces to if $Sym^\bullet I_1(SpecA) \to \mathcal{I}_\bullet(SpecA)$ is surjective, then $Sym^\bullet(I_1(SpecA) \otimes \mathcal{L}(SpecA)) \to \mathcal{I}'_\bullet(SpecA)$ is surjective. I will let you prove this.

Then we need to define a natural morphism and show it's an isomorphism. By the functoriality of the relative gluing, we only need to give a morphism of the gluing data. To do so, we notice that the affine trivializing neighborhoods of $\mathcal{L}$ forms a basis of X and we use this as the relative gluing data. For each $SpecA$ in X, we equip it with a trivializing isomorphism determined by an ismorphism $\mathcal{L}(SpecA) \simeq A$, then we need a morphism of A-schemes:

$$Proj_A(\mathcal{I}'(SpecA)) \to Proj_A(\mathcal{I}(SpecA))$$

that is compatible with the gluing morphisms. Let's first define this morphism and then show the compatibility. It's defined by a morphism of graded algebras:

$$\mathcal{I}(SpecA) \to \mathcal{I}'(SpecA)$$

recall $\mathcal{I}'(SpecA)$ is just $\oplus_n \mathcal{I}_n(SpecA) \otimes \mathcal{L}(SpecA)^{\otimes n}$, then using the isomorphism $\mathcal{L}(SpecA) \simeq A$, say y is mapped 1, then we define this morphism to be $x \in \mathcal{I}_n$ is mapped to $x \otimes y^{\otimes n}$, then it's easy to check this is a morphism of algebra. Notice that if $f_1, \cdots, f_n$ in $I_1(SpecA)$ generates $\mathcal{I}(SpecA)$, then we know that $f_1 \otimes y, \cdots, f_n \otimes y$ generates $\mathcal{I}'(SpecA)$, therefore, this induces a well defined morphism of *proj* that is also an A-scheme morphism. This is actually an isomorphism of A-algebra since the inverse is given by $x \otimes y_1 \otimes \cdots \otimes y_n$ is mapped to $x \cdot a_1 \cdots a_n$ where a_n is the image of y_n in A via the isomorphism $\mathcal{L}(SpecA) \to A$.

We need to show the respects the relative gluing morphism, i.e. the following diagram commutes:

$$\begin{tikzcd}
Proj_B\mathcal{I}(SpecB) \arrow[r, hook] & Proj_A\mathcal{I}(SpecA) \\
Proj_B\mathcal{I}'(SpecB) \arrow[r, hook] \arrow[u] & Proj_A\mathcal{I}'(SpecA) \arrow[u]
\end{tikzcd}$$

we only need to show the corresponding diagram of algebras commute modding out possible difference by multiplying an invertible element of the ring (recall that two ring morphism $S_\bullet \to S_\bullet$ that is finitely generated in degree 1 that differs by the multiplication of an invertible element on degree 1 defines the same morphism after we apply proj construction.) This gives the desired isomorphism:

$$\begin{tikzcd} \mathcal{P}roj\mathcal{I}' \arrow[rr, "\alpha"] \arrow[dr, "\beta'"'] & & \mathcal{P}roj\mathcal{I} \arrow[dl, "\beta"] \\ & X & \end{tikzcd}$$

Finally, we need to show the isomorphism between line bundles. To see why we need to twist by the pullback of $\mathcal{L}$, we need to see why $\mathcal{O}(1)$ is not isomorphic to $\alpha^*\mathcal{O}(1)$ and this also tells you why you need to twist. Recall that we can also view $\alpha^*\mathcal{O}(1)$ as a sheaf obtained using relative gluing, then if we want to show $\mathcal{O}(1)$ is isomorphic to $\alpha^*\mathcal{O}(1)$, we again use the functoriality of relative gluing and try to give a morphism of the gluing data. Notice that over $SpecA \subset X$, the local isomorphis is just between two usual proj, then we can consider the following general statament: Suppose that $S_\bullet \to T_\bullet$ is an isomorphism of two graded A-algebra finitely generated in degree one, then via the induced isomorphism:

$$ProjT_\bullet \to ProjS_\bullet$$

the two $\mathcal{O}(1)$ are isomorphic with the isomorphism defined by transition functions. Then we also try to define the morphism between $\mathcal{O}(1)$ and $\alpha\mathcal{O}(1)$ using the above idea, which is natural, however, this isomorphism isn't compatible with restrictions of the gluing data. The reason is the following: in the diagram below:

$$\begin{tikzcd} Proj_B\mathcal{I}(SpecB) \arrow[r, hook] & Proj_A\mathcal{I}(SpecA) \\ Proj_B\mathcal{I}'(SpecB) \arrow[r, hook] \arrow[u] & Proj_A\mathcal{I}'(SpecA) \arrow[u] \end{tikzcd}$$

the isomorphism of $\mathcal{O}(1)$ on the right hand side is induced by the ring morphism:

$$\mathcal{I}(SpecA) \to \mathcal{I}'(SpecA) \qquad x \mapsto x \otimes y_1^{\otimes n}$$

but the isomorphism of $\mathcal{O}(1)$ on the left hand side is induced by the ring morphism:

$$\mathcal{I}(SpecB) \to \mathcal{I}'(SpecB) \qquad x \mapsto x \otimes y_2^{\otimes n}$$

where y_1 restricted to $SpecB$ differs from y_2 by an invertible element of B. Therefore, if you restrict the isomorphism on the top to the stuff on the bottom, you don't get the same isomorphism, and they differ by the transition function of $\mathcal{L}$, pulled back to the inverse image of $SpecB$, this the the reason we need to twist $\alpha^*\mathcal{O}(1)$ by the pullback of $\mathcal{L}$. $\square$

Now let's dicsuss the projectivization of vector bundles:

Definition 2.16. If $\mathcal{F}$ is a finite rank locally free sheaf on X, then we can form $Sym^\bullet\mathcal{F}^\vee$, this is a graded sheaf of algebra, and since $\mathcal{F}$ is locally free of finite rank, it's finitely generated in degree 1, thus we can talk about $\mathcal{P}roj(Sym^\bullet\mathcal{F}^\vee)$, we will say this X-scheme is the projectivization of $\mathcal{F}$ and is denoted by $\mathbb{P}\mathcal{F}$.

If $\mathcal{F}$ is locally free of rank $n+1$, we say $\mathbb{P}\mathcal{F}$ is a **projective bundle** or **$\mathbb{P}^n$-bundle** over X.

Remark 2.17. We redefine $\mathbb{P}^n_X$ by $\mathbb{P}(\mathcal{O}_X^{\oplus n+1})$, then you can check $\mathbb{P}_{SpecA}$ in this definition agrees with our earlier definition of $\mathbb{P}^n_A$. This is simple since over an affine scheme $SpecA$, the relative proj is just the usual proj we define earlier. And notice that $Sym^\bullet \widetilde{A^{n+1}}^\bullet$ is just $A[\widetilde{x_0, \cdots}, x_n]$.

If $\mathcal{F}$ is locally free of finite rank, then we have a canonical isomorphism $\mathbb{P}\mathcal{F} \simeq \mathbb{P}(\mathcal{L}\otimes\mathcal{F})$ for a line bundle $\mathcal{L}$. This only holds for line bundles and fails for other vector bundles and it is because we know that $Sym^\bullet(\mathcal{L}\otimes\mathcal{F})^\vee = \oplus_n Sym^n(\mathcal{L}\otimes\mathcal{F})^\vee = \oplus_n(Sym^n(\mathcal{F}^\vee)\otimes\mathcal{L}^{\otimes -n})$. This holds more generally: suppose that $\mathcal{L}$ is a line bundle and $\mathcal{F}$ is a locally free sheaf of finite rank, then:

$$Sym^n(\mathcal{L}\otimes\mathcal{F}) \simeq Sym^n\mathcal{F}\otimes\mathcal{L}^{\otimes n}$$

To see this holds, recall that via the universal property of Sym^n, we have a natrual morphism from left hand side to right hand side, defined using:

$$\mathcal{L} \times \mathcal{F} \times \cdots \times \mathcal{L} \times \mathcal{F} \to Sym^n \mathcal{F} \otimes \mathcal{L}^{\otimes n}$$

that is multilinear and for symmetric in each pair of $\mathcal{L}, \mathcal{F}$. Then recall that on the open subset U where $\mathcal{L}$ is $\mathcal{O}_U$, this morphism is obvious an isomorphism and you also see why this fails if $\mathcal{L}$ is not rank 1.

This gives you the desired isomorphism between $\mathbb{P}(\mathcal{F})$ and $\mathcal{F} \otimes \mathcal{L}$ and we will use this later.

Remark 2.18. There is a serious disagreement on whether $\mathbb{P}\mathcal{F}$ should be defined using $Sym^\bullet(\mathcal{F}^\vee)$ or $Sym^\bullet(\mathcal{F})$, therefore we will try to avoid the over use of the notation $\mathbb{P}(\mathcal{F})$, we only use it when $\mathcal{F}$ is locally free of finite rank and define it using the dual.

Example 2.19. We can use this relative proj construction to define the notion of ruled surfaces.

Let C be a regular curve and $\mathcal{F}$ is locally free of rank 2, then $\mathbb{P}(\mathcal{F})$ is called a ruled surface over C, and if C is further isomorphic to $\mathbb{P}^1$, $\mathbb{P}(\mathcal{F})$ is called a Hirzebruch surface. We will see later that all vector bundles on $\mathbb{P}^1$ splits as a direct sum of line bundles, and since the line bundles on $\mathbb{P}^n$ are just $\mathcal{O}(n)$, we know that $\mathbb{P}(\mathcal{F})$ is $\mathbb{P}(\mathcal{O}(n_1) \oplus \mathcal{O}(n_2))$, since we can twist $\mathcal{O}(n_1) \oplus \mathcal{O}(n_2)$ using a line bundle and get isomorphic space, we know that the Hirzebruch surface only depends on $n_2 - n_1$. Therefore, the Hirzebruch surface $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n))$ for $n \geqslant 0$ is always denoted by $\mathbb{F}_n$, we will discuss Hirzebruch surfaces in greater length later, in particular you will see that for different n, $\mathbb{F}_n$ are all distinct.

Proposition 2.20. Suppose $\mathcal{I}_\bullet$ is a graded sheaf of algebra finitely generated in degree 1 on X, then there is a canonical closed embedding i of X-schemes:

$$\begin{tikzcd} \mathcal{P}roj\mathcal{I}_\bullet \arrow[rr, hook, "i"] \arrow[dr] & & \mathcal{P}roj(Sym^\bullet \mathcal{I}_1) \arrow[dl] \\ & X & \end{tikzcd}$$

such that we have an isomorphism of line bundles $\mathcal{O}(1) \simeq i^*\mathcal{O}(1)$.

In particular, if $\mathcal{I}_1$ is locally free, we have embedded $\mathcal{P}roj\mathcal{I}_\bullet$ in a projective bundle of X.

Proof. We define the morphism similarly on the inverse image of each $SpecA$ and show that it's compatible with the restriction of gluing data, finally show the isomorphsim of line budnles.

Notice that over $SpecA$, you have:

$$\begin{tikzcd} Proj\mathcal{I}_\bullet(SpecA) \arrow[rr, hook, "i_A"] \arrow[dr] & & Proj(Sym^\bullet \mathcal{I}_1(SpecA)) \arrow[dl] \\ & SpecA & \end{tikzcd}$$

where the closed embedding is induced by the surjective graded ring morphism:

$$Sym^\bullet \mathcal{I}_1(SpecA) \to \mathcal{I}_\bullet(SpecA)$$

Then we need to show the following diagram commutes:

$$\begin{tikzcd} Proj\mathcal{I}_\bullet(SpecA) \arrow[r, hook, "i_A"] & Proj(Sym^\bullet \mathcal{I}_1(SpecA)) \\ Proj\mathcal{I}_\bullet(SpecB) \arrow[r, hook, "i_B"] \arrow[u, hook] & Proj(Sym^\bullet \mathcal{I}_1(SpecB)) \arrow[u, hook] \end{tikzcd}$$

which translates to show the commutativity of ring morphisms, which is easy. This defines a well defined morphism and it's a closed embedding since you can check it locally since closed embedding is local on the target.

As for the isomorphism of line bundles this again comes from the functoriality of relative gluing and it's similar to what we saw before and it reduces to the fact that isomorphism of line bundles induced by transition functions is compatible with composition. $\square$

Proposition 2.21. Suppose that $\mathcal{F}$ is a locally free sheaf of rank $n+1$ on X, then there is a bijection between the set of sections of $s : X \to \mathbb{P}\mathcal{F}$ of $\mathbb{P}\mathcal{F} \to X$ and the set of surjective homomorphisms $\mathcal{F}^\vee \to \mathcal{L} \to 0$ of $\mathcal{F}^\vee$ onto invertible sheaves on X.

This is a special case of a functorial description of $\mathbb{P}(\mathcal{F})$ as a X-scheme, but we won't use this.

Proof. To see how this bijection is built, we first need to notice that there is a canonical surjection:

$$\beta^*\mathcal{F}^\vee \to \mathcal{O}(1)$$

where β is the structure morphism $\mathbb{P}\mathcal{F} \to X$. To see what is this surjective morphism, recall we have a canonical morphism:

$$Sym^\bullet(\mathcal{F}^\vee) \to \oplus_n \beta_* \mathcal{O}(n)$$

and on degree 1, it's just:

$$\mathcal{F}^\vee \to \beta_* \mathcal{O}(1)$$

then via the adjoint property, we get a morphism $\beta^*\mathcal{F}^\vee \to \mathcal{O}(1)$ and I claim that this is surjective. To check this, we need a general statement that if $S_\bullet$ is a graded A-algebra finitely generated in degree 1 by $f_1, \cdots, f_n$, then $\mathcal{O}(1)$ is also globally generated $f_1, \cdots, f_n$. Then notice that over the inverse image of $SpecA$, if the morphism $\beta^*\mathcal{F}^\vee \to \mathcal{O}(1)$ gives a map on the global section such that $\mathcal{F}^\vee(SpecA)$ is in the image, then it's surjective, and since surjectivity can be checked locally, we are done. Consider $\mathcal{F}^\vee \to \beta^*\mathcal{O}(1)$ on $SpecA$, the corresponding map on the global section has an image that contains $\mathcal{F}^\vee$, then recall how the adjoint property is defined, we first use the adjointness of β^{-1} and β_* to get $\beta^{-1}\mathcal{F} \to \mathcal{O}(1)$ which then gives a morphism $\beta^{-1}\mathcal{F} \otimes_{\beta^{-1}\mathcal{O}_X} \mathcal{O}_{\mathbb{P}\mathcal{F}} \to \mathcal{O}(1)$. Then I will let you review the construction of the adjoint between β^{-1}, β_* and see that the image of $\beta^{-1}\mathcal{F} \to \mathcal{O}(1)$ on the global section contains $\mathcal{F}(SpecA)$, then we are done. Then we consider the following diagram, which represents a section of $\mathbb{P}\mathcal{F} \to X$:

$$\begin{tikzcd} X \arrow[rr, "s"] \arrow[dr, "id"'] & & \mathbb{P}\mathcal{F} \arrow[dl, "\beta"] \\ & X & \end{tikzcd}$$

we pullback the surjection $\beta^*\mathcal{F}^\vee \to \mathcal{O}(1)$ along s to get $\mathcal{F}^\vee \to s^*\mathcal{O}(1) = \mathcal{L}$, since pullback is right exact, it preserves surjection, we get a surjective morphism $\mathcal{F}^\vee \to \mathcal{L}$ on X.

Then for the converse, we first need to prove that if $\mathcal{L}$ is a line bundle on X, then $\mathbb{P}\mathcal{L}^\vee \to X$ is an isomorphism. To see why, notice that this morphism over $SpecA$ where A is a trivializing neighborhood of $\mathcal{L}$, it looks like $ProjSym^\bullet\mathcal{L}(SpecA) \to SpecA$, but notice that since $SpecA$ is trivializing, there is a commutative diagram:

$$\begin{tikzcd} ProjSym^\bullet\mathcal{L}^\vee(SpecA) \arrow[rr, "\simeq"] \arrow[dr] & & ProjA[x] \simeq SpecA \arrow[dl] \\ & SpecA & \end{tikzcd}$$

therefore we are done. Moreover, via this isomorphism of schemes, we have an isomorphism of line bundles $\mathcal{L}{\simeq}\mathcal{O}(1)$. To see why, notice that over the inverse image of $SpecA$, $\mathcal{O}(1)$ is trivial and suppose that the transition function of $\mathcal{L}$ over $SpecA$ and $SpecB$ are given, then the transition function of $\mathcal{O}(1)$ between the inverse image of $SpecA$ and $SpecB$ is the same. There is some identification issues going on but if you work out the details, you will find what I said is correct.

Then suppose we have a surjective morphism $\mathcal{F}^\vee \to \mathcal{L}$, we can take the symmetric algebra construction and get a surjective morphism of graded sheaf of algebra finitely generated in degree 1:

$$Sym^\bullet(\mathcal{F}^\vee) \to Sym^\bullet(\mathcal{L})$$

Then we apply the relative proj construction to both of them and get:

$$\begin{tikzcd} \mathbb{P}\mathcal{L} \simeq X \arrow[rr] \arrow[dr, "id"'] & & \mathbb{P}\mathcal{F} \arrow[dl, "\beta"] \\ & X & \end{tikzcd}$$

We need to check these two are inverses of each other. This follows basically from the local version of the statement. We have seen before that $\mathbb{P}^n_A \to SpecA = X$ is characterized by the fact that morphisms of A-schemes to $\mathbb{P}^n_A$ is equivalent to a surjective morphism $\mathcal{O}_X^{\oplus n+1} \to \mathcal{L}$ to some line bundle $\mathcal{L}$ on X. Then we notice that to check two morphisms are the same, we can check it locally on the target, which lets us use the local result. $\square$

Remark 2.22. This argument above can be modified without too much pain to show that when $\mathcal{F}$ is locally free of finite rank, then we have a functorial description of $\mathbb{P}\mathcal{F}$ as an X-morphism, say Y is an X-scheme with structure morphism $f : Y \to X$, then the X-scheme morphism from Y to $\mathbb{P}\mathcal{F}$ correspond to surjective morphism $f^*\mathcal{F} \to \mathcal{L}$ of sheaves on Y where $\mathcal{L}$ is a line bundle on Y.

We have to admit that the way we choose to go about when defining the relative proj is the more elementary but more accessible approach. The sacrifice is that we didn't discuss some quite important properties of the relative proj, for example, the universal property of $\mathcal{P}roj$. We make a comment here about what are they but postpone the proof. You will only realize the important of this remark and want to see the proof (which is quite complicated) after you used the universal property enough and see that it is not easily deduced using our elementary gluing construction of $\mathcal{P}roj$.

Remark 2.23. Suppose that X is again a scheme and $\mathcal{I}_\bullet$ is a quasicoherent sheaf of algebras over X. We will show that the old $\mathcal{P}roj\mathcal{I}_\bullet$ actually represent a functor which gives you a universal property that lets you read the next section easier.

We will define a functor F, which is contravariant going from the category of X-schemes to the category of sets. We define $F(T)$ for a X-scheme $f : T \to X$ to be the set of pair $(\mathcal{L}, \psi)$ quotient out an equivalence relation. Here $\mathcal{L}$ is an invertible sheaf over T and ψ is a morphism of graded $\mathcal{O}_T$ algebras $\psi : f^*\mathcal{I}_\bullet \to \oplus_{n\geqslant 0}\mathcal{L}^{\otimes n}$ such that on the degree 1 component, you have a surjective morphism of $\mathcal{O}_T$-modules $f^*\mathcal{I}_1 \to \mathcal{L}$.

The equivalence relation here is defined by $(\mathcal{L}, \psi)$ is equivalent to $(\mathcal{L}', \psi')$ if there is an isomorphism $\beta : \mathcal{L} \to \mathcal{L}'$ such that $\beta \circ \psi = \psi'$ as an graded algebra morphism $f^*\mathcal{I}_\bullet \to \oplus_{n\geqslant 0}(\mathcal{L}')^{\otimes n}$. You can easily see this is indeed an equivalence relation. We also need to consider if $h : T' \to T$ is a morphism of schemes, where does this morphism go under F, it's just given by pullback, in the sense that a triple $(\mathcal{L}, \psi)$ is mapped to $(h^*\mathcal{L}, h^*\psi)$ and you can also see this respect the equivalence relation.

Then the claim is F is represented by a particular X-scheme, and that scheme is exactly $\mathcal{P}roj\mathcal{I}_\bullet \to X$ we constructed before. We won't prove it here sine the proof need a quite a lot of new notions we don't need. But out of this universal property, we can deduce that X-morphisms $T \to \mathcal{P}roj\mathcal{I}_\bullet$ is equivalent to a morphism of graded $\mathcal{O}_T$-algebras $f^*\mathcal{I}_\bullet \to \oplus_{n\geqslant 0}\mathcal{L}^{\otimes n}$ which is surjective on degree 1. This is quite useful since

3 Projective Morphisms

One of the reason we introduce the relative proj construction is that we want to define projective morphisms in general over any scheme X, not just over an affine scheme $SpecA$.

We already redefined affine morphisms using the relative spec contruction, and since projective morphism is such a classic notion, you probably will think there is some obvious definition of a projective morphism. But this is unfortunately not the case, and all the definitions of projective morphisms are imperfect in some sense, including the one we will accept below.

Definition 3.1. A morphism $\pi : X \to Y$ is **projective** if there is an isomorphism:

$$\begin{tikzcd} X \arrow[rr, "\simeq"] \arrow[dr, "\pi"'] & & \mathcal{P}roj\mathcal{I}_\bullet \arrow[dl] \\ & Y & \end{tikzcd}$$

for some quasicoherent sheaf of algebras $\mathcal{I}_\bullet$ on Y finitely generated in degree 1.

If the above condition holds, we will say X is a **projective Y-scheme**, or X is **projective over Y**.

Remark 3.2. Recall that in our earlier definition of projective A-schemes, it's of the form $ProjS_\bullet$ for some finitely generated graded algebra over A. We didn't require $S_\bullet$ to be something generated in degree 1, but this is not a problem. Since we know that if $S_\bullet$ is generated over S_0 by $f_1, \cdots, f_n$ each of which is homogeneous of degree d_i, then there is a d such that $S_{d\bullet}$ is generated in degree 1, and we have an isomorphism:

$$\begin{tikzcd} ProjS_\bullet \arrow[rr, "\simeq"] \arrow[dr] & & ProjS_{d\bullet} \arrow[dl] \\ & SpecA & \end{tikzcd}$$

Therefore, our new definition generalize the old definition of a projective A-scheme.

We now show a few useful characterizations of projective morphisms, but we will need the following lemma:

Lemma 3.3. Let Y be a scheme and $\mathcal{S}_\bullet$ a quasicoherent sheaf of algebras on Y finitely generated in degree 1, then suppose that we have a closed embedding of Y-schemes:

$$i : X \hookrightarrow \mathcal{P}roj\mathcal{S}_\bullet$$

then X is isomorphic as a Y-scheme to $\mathcal{P}roj(\mathcal{S}_\bullet/\mathcal{I}_\bullet)$ for some homogeneous quasicoherent ideal sheaf $\mathcal{I}_\bullet$ of $S_\bullet$ and the closed embedding is induced by applying $\mathcal{P}roj$ to the surjective morphism $S_\bullet \to S_\bullet/\mathcal{I}_\bullet$.

Proof. Recall that if X is affine say $SpecA$, this is just the statement that any closed embedding of the proj comes from an ideal of the graded ring.

I claim that this ideal sheaf $\mathcal{I}_\bullet$ is the kernel of the canonical morphism of graded sheaf of algebras:

$$S_\bullet \to \oplus_{n\geqslant 0}\pi_*\mathcal{O}(n) \to \oplus_{n\geqslant 0}\pi_* i_*\mathcal{O}_Z(n)$$

where the second morphism comes from the fact that we have a natural morphism $\mathcal{O} \to i_*\mathcal{O}_X$, then we twist by $\mathcal{O}(n)$ and the apply p_*.

To see this ideal sheaf defines X, we check it locally, notice that over $SpecA$ since everythings here is quasicoherent (closed embeddings are qcqs and when $S_\bullet$ is finitely generated in degree 1, the structure morphism is also qcqs, thus pushforward transforms quaiscoherent stuff to quasicoherent stuff.) the canonical morphism is determined by the global section:

$$S_\bullet(SpecA) \to \oplus_n\Gamma(ProjS_\bullet(SpecA), \mathcal{O}(n)) \to \Gamma(ProjS_\bullet(SpecA), i_*\mathcal{O}_X(n))$$

where the first morphism is the saturation morphism and the second morphism is induced by the saturation functor, therefore, by our understanding before the kernel indeed determines X, thus we are done.

Be careful that we are taking advantage of the fact that if you have two subsheaves of a sheaf, then to prove the two subsheaves are equal is equivalent to prove that they are equal on any open cover, we don't have to worry about the compatibility problem since everything is identified using the identity! In other words, we proved that the quasicoherent ideal sheaf determined by the closed embedding of X is the same as that determined by the closed embedding determined by the homogeneous ideal sheaf. $\square$

Proposition 3.4. Suppose that $\pi : X \to Y$ is a morphism, then π is projective if and only if there is a finite type quasicoherent sheaf $\mathcal{I}_1$ on Y and a closed embedding $i : X \hookrightarrow \mathcal{P}rojSym^\bullet\mathcal{I}_1$ of Y-schemes (notice that since $\mathcal{I}_1$ may not be locally free, we avoid using the $\mathbb{P}$ notation.)

Proof. Suppose that X is porjective, say $X = \mathcal{P}roj\mathcal{I}_\bullet$, then this closed embedding is easy to construct since we have a natural surjection induced by the universal property of symmetric algebra:

$$Sym^\bullet\mathcal{I}_1 \to \mathcal{I}_\bullet$$

Conversely, suppose that you have a closed embedding of Y-schemes $i : X \hookrightarrow \mathcal{P}rojSym^\bullet\mathcal{I}_1$, we know from the previous lemma that X is the relative proj of a quotient of $Sym^\bullet\mathcal{I}_1$ by a homogeneous ideal sheaf, thus X is projective. $\square$

Remark 3.5. Sometimes when we want to say $\pi : X \to Y$ is projective with some line bundles $\mathcal{L}$ on X, we want to ask via the isomorphism of Y-schemes $X \simeq \mathcal{P}roj\mathcal{I}_\bullet$, is $\mathcal{L}$ isomorphic to $\mathcal{O}(1)$. We can adapt the proposition above to include this situation. Recall that if you have a closed of Y-schemes:

$$\mathcal{P}roj\mathcal{I}_\bullet \to \mathcal{P}roj\mathcal{S}_\bullet$$

induced by a surjective morphism of sheave of algebras $S_\bullet \to I_\bullet$, we know that the pullback of $\mathcal{O}(1)$ is still $\mathcal{O}(1)$, thus we know that π is projective with $\mathcal{L} \simeq \mathcal{O}(1)$ iff there is a finite type quasicoherent sheaf $\mathcal{I}_1$ on Y with a closed embedding of Y-schemes $i : X \hookrightarrow \mathcal{P}rojSym^\bullet\mathcal{I}_1$ such that $i^*\mathcal{O}(1)_{\mathcal{P}rojSym^\bullet\mathcal{I}_1} \simeq \mathcal{L}$.

There are other possible definitions of projective morphisms you will see in the literature, for example, you will see some authors define $\pi : X \to Y$ to be projective if there is a closed embedding of Y-schemes for some n:

$$\begin{tikzcd}
X \arrow[rr, hook, "i"] \arrow[dr, "\pi"'] & & \mathbb{P}^n_Y \arrow[dl] \\
& Y &
\end{tikzcd}$$

Here $\mathbb{P}^n_Y$ is defined to be $\mathcal{P}roj\mathcal{O}_Y[x_0, \cdots, x_n]$. Our earlier definition of $\mathbb{P}^n_Y$ is defined to be the fibered product of $\mathbb{P}^n_{\mathbb{Z}}$ with Y over $\mathbb{Z}$, these two are isomorphic since you can consider the following diagram of fibered product:

$$\begin{tikzcd}
\mathbb{P}^n_B \arrow[r] \arrow[d] & \mathbb{P}^n_A \arrow[r] \arrow[d] & \mathbb{P}^n_Y \arrow[r] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d] \\
SpecB \arrow[r, hook] & SpecA \arrow[r, hook] & Y \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

and think of a fibered product as obtained by relative gluing over Y, then it's easy to show the two of them are isomorphic. Thus this new definition agrees with the old one.

This definition of projective morphism agrees with the definition we give in the beginning under some assumptions, but in general, these two are not the same. The details are given in the following proposition:

Proposition 3.6. Suppose that Y is a scheme that admits an ample line bundle in the absolute setting for example when Y is a projective A-scheme, or an affine scheme or a quasiprojective A-scheme. Then $\pi : X \to Y$ is projective iff there is a closed embedding of Y-schemes $i : X \to \mathbb{P}^n_Y$.

Proof. Recall that if we have such a closed embedding, no matter what kind of Y we have, $\pi : X \to Y$ is projective by the characterization we had before.

The converse is where we need ampleness. Since we didn't really discuss ampleness in the absolute setting in any serious manner, we will only use a property of absolute amplenss, which is quite simialr to relative ampleness that we know. Say $\mathcal{L}$ on Y is ample, then first Y is quasicompact and if Y is also quasiseparated, then for any finite type quaiscoherent sheaf $\mathcal{F}$ on Y, there is an n_0 such that for $n \geqslant n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated. Then if we assume $\pi : X \to Y$ is projective in our definition together with a ample line bundle $\mathcal{L}$ on Y, we know that in this case for the finite type quasicoherent sheaf $\mathcal{I}_1$ on Y, there is a large enough n such that $\mathcal{I}_1 \otimes \mathcal{L}^{\otimes N}$ is finitely globally generated. Or in other words, there is a surjection:

$$\mathcal{O}_Y^{\oplus n+1} \to \mathcal{I}_1 \otimes \mathcal{L}^{\otimes N} = \mathcal{I}_1 \otimes \mathcal{M}$$

where $\mathcal{M} = \mathcal{L}^{\otimes N}$ still a line bundle. Then apply the symmetric algebra construction this gives you a surjection of quasicoherent sheaves of graded algebras finitely generated in degree 1:

$$\mathcal{O}_Y[x_0, \cdots, x_n] \to Sym^\bullet(\mathcal{I}_1 \otimes \mathcal{M})$$

this gives a closed embedding:

$$\mathcal{P}rojSym^\bullet(\mathcal{I}_1 \otimes \mathcal{M}) \hookrightarrow \mathbb{P}^n_Y$$

Recall that $\mathcal{P}rojSym^\bullet(\mathcal{I}_1 \otimes \mathcal{M}) \simeq \mathcal{P}rojSym^\bullet\mathcal{I}_1$ since we already know that twisting by line bundles gives you isomorphic stuff. Moreover $\pi : X \to Y$ is projective means we have the following diagram:

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[drr, "\pi"'] & \mathcal{P}rojSym^\bullet\mathcal{I}_1 \arrow[r, "\simeq"] \arrow[dr] & \mathcal{P}rojSym^\bullet(\mathcal{I}_1 \otimes \mathcal{M}) \arrow[r, hook] \arrow[d] & \mathbb{P}^n_Y \arrow[dl] \\
& & Y &
\end{tikzcd}$$

we are done. $\square$

Now let's investigate properties of projective morphisms:

Remark 3.7. Notice that if $\pi : X \to Y$ is projective, then for any morphism of schemes $\rho : Z \to Y$, the base change morphism $Z \times_X Y \to Z$ is also projective. This is because relative proj construction behaves well under base change, and you can consider the following diagram:

$$\begin{tikzcd} \mathcal{P}roj\rho^*\mathcal{I}_\bullet \arrow[r] \arrow[d] & \mathcal{P}roj\mathcal{I}_\bullet \arrow[r, "\simeq"] \arrow[d] & X \arrow[dl, "\pi"] \\ Z \arrow[r, "\rho"'] & Y & \end{tikzcd}$$

the square is Cartesian.

Also, you can see that if $\pi : X \to Y$ is projective, it's proper since proper is local on the target, thus we only have to show that projective A-schemes are proper over A, which is something we know before. In particular, projective morphisms are separated, finite type and universally closed.

Here comes one of the most important property of projective morphisms: finite morphisms are projective:

Proposition 3.8. Suppose $Z \to X$ is finite so that $Z \simeq \mathcal{S}pec\mathcal{B}$ for some finite type quasicoherent sheaf of algebras on X. Then there is a sheaf of graded algebras $\mathcal{I}_\bullet$ on X where $\mathcal{I}_0 = \mathcal{O}_X$ and $\mathcal{I}_n = \mathcal{B}$ for all $n \geqslant 1$, together with an X-scheme isomorphism:

$$Z \simeq \mathcal{P}roj\mathcal{I}_\bullet$$

Proof. Let's first describe the graded algebra structure again we only have to describe it for the direct sum presheaf and it suffice to describe how the multiplication of homogeneous elements works. Informally speaking, the multiplication is still the multiplication in $\mathcal{B}$, just with a label to indicate the degree of the elements. Precisely, we set $x \cdot y$ for x coming from degree n and y coming from degree m to be xy but now it's in degree $m + n$. Then we can consider the following two objects:

$$\begin{tikzcd} \mathcal{S}pec\mathcal{B} \arrow[dr] & & \mathcal{P}roj\mathcal{I}_\bullet \arrow[ll, dashed] \arrow[dl] \\ & X & \end{tikzcd}$$

our question is that is there a natural morphism indicated in the diagram such that it's an isomorphism. The answer is yes and this morphism comes from the universal property of $\mathcal{S}pec\mathcal{B}$. Recall that a X-morphism to $\mathcal{S}pec\mathcal{B}$ correspond to a morphism of $\mathcal{O}_X$-algebra:

$$\begin{tikzcd} & \mathcal{O}_X \arrow[dl] \arrow[dr] & \\ \mathcal{B} \arrow[rr] & & \beta_*\mathcal{O} \end{tikzcd}$$

where $\beta : \mathcal{P}roj\mathcal{I}_\bullet \to X$ is the structure morphism. Then notice that $\mathcal{O}(1)$ on $\mathcal{P}roj\mathcal{I}_\bullet$ is isomorphic to the structure sheaf. This is because first over the inverse image of each $SpecA$, $\mathcal{O}(1)$ is equal to $\mathcal{O}$ by the way we define $\mathcal{I}_\bullet$, and you can think of the structure sheaf and $\mathcal{O}(1)$ obtained from the relative gluing using the same date or in other words use the funtoriality of the relative gluing to obtain the conclusion. Therefore $\mathcal{B} \to \beta_*\mathcal{O}$ is the same as $\mathcal{B} \to \beta_*\mathcal{O}(1)$, which is the canonical morphism, therefore we indeed got a morphism to $\mathcal{S}pec\mathcal{B}$, then to check this is an isomorphism, we can work locally and assume X is affine, then we know that say $\mathcal{B}(SpecA) = B$, the assertion the morphism is an isomorphism is just the fact that in the following diagram the top morphism is an isomorphism:

$$\begin{tikzcd} SpecB \arrow[rr] \arrow[dr] & & ProjI_B \arrow[dl] \\ & SpecA & \end{tikzcd}$$

where I_B is the graded ring where the zero degree is A and all the other degree are B, and the morphism is induced by the identity $\Gamma(ProjI_B, \mathcal{O}) = B \to B$, then it's easy to see this is an isomorphism. $\square$

Remark 3.9. Therefore, since we know that closed embeddings are finite, then closed embeddings are projective, and the following sequence of implication is obvious:

$$\text{closed embedding} \Rightarrow \text{finite morphism} \Rightarrow \text{projective morphism} \Rightarrow \text{proper morphism}$$

We know finite morphism is projective and has finite fiber, later we will see an inverse which states that if a morphism is projective and finite fiber then it's finite.

Remark 3.10. Projective morphism we just defined are not a nice class of morphisms in the sense we talked about before, for example, projective morphism is not local on the target. You will see an example later and we will prove that with reasonable assumptions, projective morphisms is local on the target. Another point you need to pay attention is that composition of projective morphisms is not always projective, we still need some reasonable assumptions for this to hold.

But many properties enjoyed by nice class of morphisms still holds for projective morphisms, as what we will describe later.

Let's first show that projective morphisms still satisfies the cancellation property, we will need a few preparations listed below:

Proposition 3.11. Suppose that $\pi : X \to Y$ is a closed embedding and $\rho : Y \to Z$ is projective, then $\rho \circ \pi$ is projective.

Proof. This is just the following diagram:

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[drr] & Y \arrow[rrr, hook] \arrow[dr] & & & \mathcal{P}rojSym^{\bullet}\mathcal{I}_1 \arrow[dll] \\
& & Z & &
\end{tikzcd}$$

then we only need to use the fact that compositions of closed embeddings is still a closed embedding. $\square$

Proposition 3.12. A morphism $\rho : Y \to Z$ has projective diagonal iff ρ is separated.

Proof. Recall that if ρ is separated, then the diagonal is a closed embedding, thus projective.

Conversely, if the diagonal $Y \to Y \times_Z Y$ is projective, then it's proper, then universally closed, thus closed, thus it's a locally closed embedding with closed image, thus it's a closed embedding, we are done. $\square$

Proposition 3.13. Suppose that $\pi : X \to Y$ is a morphism, $\rho : Y \to Z$ is a separated morphism and the composition $\rho \circ \pi$ is projective, this implies that π is projective.

Proof. Just use the diagram where we proved the cancellation theorem for nice class of morphisms and it applies here with a mild modification where you need to use the fact that closed embedding composed with a projective morphism is projective. $\square$

Using the above cancellation result, we can prove:

Proposition 3.14. A k-scheme morphism from a projective k-scheme to a separated k-scheme is always projective.

Proof. A direct apllication of the cancellation theorem. $\square$

Proposition 3.15. Suppose that $\pi : X \to Y$ and $\pi' : X' \to Y'$ are projective morphisms of S-schemes, then the induced morphism $\pi \times \pi' : X \times_S X' \to Y \times_X Y'$ is also projective.

Proof. Recall that if a class of morphisms is preserved by composition and base change, we have already seen that it's also preserved by product. However, for projective morphisms it's only preserved by base change, therefore, we can not just quote the previous result.

Instead, to prove this, since π, π' are projective, we can just assume $\pi : X \to Y$ is $\pi : \mathcal{P}roj\mathcal{I}_\bullet \to Y$ and π' is $\pi' : \mathcal{P}roj\mathcal{J}_\bullet \to Y'$, then we can observe the product is just $\mathcal{P}roj\mathcal{I}_\bullet \times_S \mathcal{P}roj\mathcal{J}_\bullet \to Y \times_S Y'$, then if we can prove that $\mathcal{P}roj\mathcal{I}_\bullet \times_S \mathcal{P}roj\mathcal{J}_\bullet$ is the relative proj of some quaisocherent sheaf of graded algebra on $Y \times_S Y'$, we are done. To do this, we first consider the following Cartesian diagram where each small square is Cartesian:

$$\begin{tikzcd}
\mathcal{P}roj\mathcal{I}_\bullet \times_S \mathcal{P}roj\mathcal{J}_\bullet \arrow[r] \arrow[d] & \mathcal{P}rojp_2^*\mathcal{J}_\bullet \arrow[r] \arrow[d] & \mathcal{P}roj\mathcal{J}_\bullet \arrow[d] \\
\mathcal{P}rojp_1^*\mathcal{I}_\bullet \arrow[r] \arrow[d] & Y \times_S Y' \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y' \arrow[d] \\
\mathcal{P}roj\mathcal{I}_\bullet \arrow[r] & Y \arrow[r] & S
\end{tikzcd}$$

notice that the scheme on the left top corner is also $\mathcal{P}rojp_1^*\mathcal{I}_\bullet \times_{Y\times_S Y'} \mathcal{P}rojp_2^*\mathcal{J}_\bullet$. Therefore, if we can prove $\mathcal{P}rojp_1^*\mathcal{I}_\bullet \times_{Y\times_S Y'} \mathcal{P}rojp_2^*\mathcal{J}_\bullet$ is the relative proj, we are done.

To do this, we claim it's isomorphic as a $Y \times_S Y'$-scheme to $\mathcal{P}roj \oplus_{n\geqslant 0} (p_1^*\mathcal{I}_n \otimes p_2\mathcal{J}_n)$. This actually holds for more general cases where we can prove that for any scheme Y and qusicoherent sheaf of algebras $\mathcal{I}_\bullet, \mathcal{J}_\bullet$ finitely generated in degree 1 over Y, we have a canonical isomoprphism of Y-schemes:

$$\mathcal{P}roj\mathcal{I}_\bullet \times_Y \mathcal{P}roj\mathcal{J}_\bullet \simeq \mathcal{P}roj(\oplus_{n\geqslant 0}\mathcal{I}_n \otimes \mathcal{J}_n)$$

This is not hard to prove since we have natural morphism from $\mathcal{P}roj(\oplus_{n\geqslant 0}\mathcal{I}_n \otimes \mathcal{J}_n)$ to $\mathcal{P}roj\mathcal{I}_\bullet$ and to $\mathcal{P}roj\mathcal{J}_\bullet$ using the functoriality of relative proj. To prove this, we first notice that suppose that $S_\bullet$ and $T_\bullet$ are two graded A-algebra with $S_0 = T_0 = A$ finitely generated in degree 1, then we have an isomorphism:

$$ProjS_\bullet \times_A ProjT_\bullet \simeq Proj(\oplus_{n\geqslant 0}S_n \otimes T_n)$$

This isomorphism is given locally and then glued together. For example, on $D_+(f) \times_A D_+(g)$ where f, g are two homogeneous degree 1 elements in $S_\bullet$ and $T_\bullet$, then the morphism is given by the ring isomorphism:

$$S_f^0 \otimes T_g^0 \to R_{f\otimes g}^0$$

where the graded ring R is $\oplus_{n\geqslant 0}S_n \otimes T_n$. This ring morphism is given by $\frac{a}{f^n} \otimes \frac{b}{g^m} \to \frac{ag^m \otimes bf^n}{f^{m+n}\otimes g^{m+n}}$ and it's indeed an isomorphism since we can define $\frac{a\otimes b}{f^n \otimes g^n}$ is mapped to $\frac{a}{f^n} \otimes \frac{b}{g^n}$, this is well defined since it can be viewed as $R_{f\otimes g}$ mapped to $S_f \otimes T_g$ defined by $\frac{a\otimes b}{f^n\otimes g^n} \mapsto \frac{a}{f^n} \otimes \frac{b}{g^n}$ restricted to the zero degree and the image is in $S_f^0 \otimes T_g^0$, we are done. Then we need to show that this morphism is well defined in the sense that they agree on intersections. Say we have another pair of degree 1 homogeneosu element x, y, then the morphism is defined by the ring morphism:

$$S_x^0 \otimes T_y^0 \to R_{x\otimes y}^0$$

then they agree on the intersection can be viewed as the commutativity of the following diagram:

$$\begin{tikzcd}
S_f^0 \otimes T_g^0 \arrow[r] \arrow[d] & R_{f\otimes g}^0 \arrow[d] \\
S_{fx}^0 \otimes T_{gy}^0 \arrow[r] & R_{fx\otimes gy}^0 \\
S_x^0 \otimes T_y^0 \arrow[r] \arrow[u] & S_x^0 \otimes T_y^0 \arrow[u]
\end{tikzcd}$$

where the morphism in the middle row is defined similar to what we did above.

Then to prove $ProjS_\bullet \times_A ProjT_\bullet \simeq Proj(\oplus_{n\geqslant 0}S_n \otimes T_n)$, as what we promised, we use the functoriality of relative gluing. First, notice that the fibered product can also be viewed as obtained using relative gluing where on $SpecA$ it's just $ProjS(SpecA) \times_A ProjT(SpecA)$ and for open embeddings $SpecB \hookrightarrow SpecA$, the open embedding in the data of the relative gluing is:

$$ProjS(SpecB) \times_B ProjS(SpecB) \hookrightarrow ProjS(SpecA) \times_A ProjS(SpecA)$$

the stuff on the left hand side can be thought of as a product over A since $SpecB \hookrightarrow SpecA$ is an open embedding and we have seen in the general property of fibered product that in this case we can use $SpecA$ instead of $SpecB$ as the base, then since open embeddings are preserved under product, we are done. Then we need to define a morphism of the relative gluing data in the following diagram:

$$\begin{tikzcd}
ProjS(SpecA) \times_A ProjS(SpecA) \arrow[r, "\simeq"] & Proj(\oplus_{n\geqslant 0} S_n(SpecA) \otimes_A T_n(SpecA)) \\
ProjS(SpecB) \times_B ProjS(SpecB) \arrow[r, "\simeq"] \arrow[u, hook] & Proj(\oplus_{n\geqslant 0} S_n(SpecB) \otimes_B T_n(SpecB)) \arrow[u, hook]
\end{tikzcd}$$

where the isomorphism of the usual proj is the one we defined above. If we can show this diagram is commutative, then we are done. Recall the vertical morphims are induced by the restriction, then we can check the commutativity locally, which just reduces to a commutativity of ring moirphisms. $\square$

Before we end this section, we will talk about global generation and (very)-ampleness in the relative setting, which helps us to answer a lot questions and this notion is also a notion that appears in the research literature a lot.

Definition 3.16. Suppose that $\pi : X \to Y$ is a qcqs morphism and $\mathcal{F}$ is a quasicoherent sheaf on X, we say that $\mathcal{F}$ is relatively globally generated or globally generated with respect to π if the natural morphism of quasicoherent sheaves $\pi^*\pi_*\mathcal{F} \to \mathcal{F}$ is surjective.

Qcqs assumption here is used to ensure $\pi^*\pi_*\mathcal{F}$ is still quasicoherent but this hypothesis is not that restrictive. As you will see later, this notion is mostly useful in the quasicompact setting and most people won't be bothered by the quasiseparated hypothesis.

Remark 3.17. Thanks to our qcqs hypothesis, the morphism $\pi^*\pi_*\mathcal{F} \to \mathcal{F}$ is a morphism of quasicoherent morphism, thus to check it's surjective we can work with affine opens and check the morphism gives you a surjective morphism of modules.

Definition 3.18. Suppose $\mathcal{L}$ is a locally free sheaf on X and $\pi : X \to Y$ is a qcqs morphism, then we say $\mathcal{L}$ is relatively base point free or base point free with respect to π if it's relatively globally generated.

Proposition 3.19. Suppose that $\mathcal{L}$ is a relatively base point free line bundle on X and $\pi : X \to Y$ is a qcqs morphism. Assume further that $\pi_*\mathcal{L}$ is finite type (After we will show that if π is proper and Y is Noetherian then this holds using Grothendieck's Coherence Theorem, but let's just assume it now).

Then there is a canonical morphism $\psi : X \to \mathcal{P}roj\mathcal{S}ym^\bullet(\pi_*\mathcal{L})$.

Proof. Recall that the universal property of relative proj tells you that a Y-scheme morphism from X to $\mathcal{P}roj\mathcal{S}ym^\bullet\pi_*\mathcal{L}$ is equivalent to a surjective morphism from $\pi^*\pi_*\mathcal{L}$ to a line bundle on X, thus we can just take the line bundle to be $\mathcal{L}$ and use the surjective morphism coming from the relative global generation.

The essence of this universal property is not that mysterious. Recall that we have proved that an A-scheme morphism f from X to $\mathbb{P}^n_A = ProjA[x_0, \cdots, x_0] = ProjS_\bullet$ is equivalent to $n+1$ global sections not vanishing simultaneously. However, the essence of this is actually a surjective morphism $f^*\widetilde{S_1} \to \mathcal{L}$ where $f : X \to SpecA$ is the structure morphism, in our case, since $\widetilde{S_1}$ is free of rank $n+1$ on $SpecA$, this is the same as a surjective morphism $\mathcal{O}_X^{\oplus n+1} \to \mathcal{L}$. This description is equivalent to our description of the bijection before since if you have a morphism $\pi : X \to \mathbb{P}^n_A$, our previous description of the surjctive morphism is:

$$\mathcal{O}_X^{\oplus n+1} \to f^*\mathcal{O}(1)$$

where e_i is mapped to f^*x_i. With this new description, we can translate this to a surjective morphism $f^*\widetilde{S_1} \to f^*\mathcal{O}(1)$ by pulling back the natural surjection $\beta^*\widetilde{S_1} \to \mathcal{O}(1)$ where β is the structure morphism $\beta : \mathbb{P}^n_A \to SpecA$. This natural morphism comes from a natural morphism $\widetilde{S_1} \to \beta_*\mathcal{O}(1)$ coming form the saturation map and since x_i goes to x_i then in the pullback $\beta^* x_i$ goes to x_i, therefore you pullback again by π we know that f^*x_i goes to $\pi^* x_i$, and f^*x_i is just e_i.

We can generalize this argument to $Y = ProjSym^\bullet I = ProjR_\bullet$ where I is a finitely generated A-module. An A-scheme morphism $\pi : X \to Y$ correspond to a surjective morphism $f^*\widetilde{I} \to \mathcal{L}$ for a line bundle on X and $f : X \to SpecA$ is the structure morphism, modding out isomorphisms if there is a commutative diagram like below:

$$\begin{tikzcd} f^*\widetilde{I} \arrow[rr, two heads] \arrow[dr, two heads] & & \mathcal{L} \arrow[dl, "\simeq"] \\ & \mathcal{M} & \end{tikzcd}$$

Say $x_0, \cdots, x_n$ generates I as an A-module, and if a surjective morphism $f^*\widetilde{I} \to \mathcal{L}$ maps f^*x_i to y_i, we define a morphism from X to Y by on the locus where y_i doesn't vanish we define:

$$D(y_i) \to D_+(x_i)$$

by the ring morphism $R^0_{x_i} \to \Gamma(D(y_i), \mathcal{O}_X)$ by first consider the morphism $R \to \Gamma(D(y_i), \mathcal{O}_X)$ where $I \to \Gamma(D(y_i), \mathcal{O}_X)$ is defined by the natural morphism $\widetilde{I} \to f_*\mathcal{L}$ which is a morphism of A-modules $A \to \Gamma(X, \mathcal{L})$, then we can just restrict to $\Gamma(D(y_i), \mathcal{L}) \simeq \Gamma(D(y_i), \mathcal{O}_X)$ via a trivialization. Then we have an induced morphism $R^0_{x_i} \mapsto \Gamma(D(y_i), \mathcal{O}_X)$ where x_i is mapped to the rational function corresponding to y_i. We need to check this definition is well defined in the sense that it argees on intersections, but this is similar to the well definedness we proved to $\mathbb{P}^n_A$ since different trivializations differ by an invertible element on the intersection and by dividing x_i or powers of x_i, we eliminate this difference. Conversely, given a morphism $\pi : X \to Y$, we pullback the surjective morphism $\beta^*\widetilde{I} \to \mathcal{O}(1)$ obtained from the saturation map $I \to \Gamma(ProjR_\bullet, \mathcal{O}(1))$ along π where $\beta : ProjR_\bullet \to SpecA$ is the structure morphism. To see why these two are inverses of each other it's very similar to the argument we used in $\mathbb{P}^n_A$.

This local universal property can be used to prove the global universal property of $\mathcal{P}roj\mathcal{S}ym^\bullet\mathcal{I}$ for a finite type quaiscoherent sheaf $\mathcal{I}$ on Y. We claim that Y-morphisms to $\mathcal{P}roj\mathcal{S}ym^\bullet\mathcal{I}$ correspond to surjective morphisms $f^*\mathcal{I} \to \mathcal{L}$ for a line bundle $\mathcal{L}$ on X and $f : X \to Y$ is the structure morphism modding out isomorphisms. To prove this universal property, suppose that we are given such a surjective morphism, then restrict to the inverse image of $SpecA$, the local universal property gives you a morphism. We can do this for each affine open and we only have to show that they agree on the intersection. But we can cover the intersection by inverse image of smaller affine opens, thus the universal property tells you on the intersection is must be the unique morphism. Therefore, a surjection gives a morphism of Y-schemes. Conversely, we know how to get a surjective morphism from a Y-scheme morphism. These are inverses since we can check it locally and surjective morphisms implies that isomorphisms restricted to inverse images of smaller affine opens must agree, thus we are done. □

We can also define relatively ampleness:

Definition 3.20. Say $\mathcal{L}$ is a line bundle on X and $\pi : X \to Y$ is a qcqs morphism of schemes, we say that $\mathcal{L}$ is relatively ample or π-ample or relatively ample with respect to π if for every affine open $SpecB$ of Y $\mathcal{L}|_{\pi^{-1}SpecB}$ is ample over $SpecB$ in the relative setting or equivalently since $\pi^{-1}SpecB$ is quasicompact, it's equivalent to $\mathcal{L}|_{\pi^{-1}SpecB}$ is absolutely ample on $\pi^{-1}SpecB$.

Many of the properties of relatively ampleness in the affine setting applies here without change and in practice we always require π to be proper since when we discussed relatively ampleness over $SpecA$, we required properness. We can also say more form the knowledge of absolute ampleness we know, we won't state all of them, but the following two statements are worth remembering.

Proposition 3.21. Suppose that $\pi : X \to Y$ is proper and Y is locally Noetherian. Let $\mathcal{L}$ be an invertible sheaf on X, assume further $\pi_*\mathcal{L}$ is finite type, then $\mathcal{L}$ is π relatively very ample iff $\mathcal{L}$ is relatively base point free and the canonical morphism $\psi : X \to \mathcal{P}roj\mathcal{S}ym^\bullet(\pi_*\mathcal{L})$ is a closed embedding.

In particular since relatively base point free and closed embedding is affine local on Y, in this case relatively very ample is affine local on Y.

Proposition 3.22. Suppose that $\pi : X \to Y$ and $\rho : Y \to Z$ are projective morphisms and Z is affine or Noetherain, then $\rho \circ \pi$ is projective.

This tells you that composition of projective morphisms is usually projective.

References