Appendix 4 for Math 632: 2025 Winter Notes

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1 Review on Closed Embeddings

This section is meant to review on closed embeddings, where we start straight out with the definition of a closed embedding:

Definition 1.1. A morphism $\pi : X \to Y$ is a **closed embedding**, or **closed immersion** as we sometimes call it if it's an affine morphism and for every affine open subset $SpecV \subset Y$ with $\pi^{-1}SpecB \cong SpecA$, the map $B \to A$ is surjective. We prefer embedding more than immersion for reasons we will define a lot later. Here, just like the case in open embeddings, we use $\hookrightarrow$ to indicate it. As we will see later, all locally closed embeddings will be indicated by $\hookrightarrow$.

In particular, if X on the level of topological spaces, is a subset of Y(closed), and the above π is just the inclusion map on the level of sets, we will say that X is a closed subscheme of Y. Then with this definition, as we will see later that a closed embedding is just the composition of an isomorphism with a closed subscheme.

Proposition 1.2. Suppose that $\pi : X \to Y$ is a closed embedding, show that the image of π is a closed subset of Y.

Proof. Suppose that we choose an open cover of Y, then we know that $\pi(X)$ is closed in Y iff it's closed in each of these open cover. Let's cover it by affine opens, then we see that $\pi(X) \cap U$, is the image of the $SpecA \to SpecB = U$ where $\pi^{-1}U \cong SpecA$, thus we only have to show that if $SpecA \to SpecB$ is induced by an surjective map of rings:

$$B \to A$$

The image will be closed. But we have:

$$\begin{tikzcd} B \arrow[r] \arrow[d] & A \arrow[dl, "iso"] \\ B/Ker & \end{tikzcd}$$

Thus the image of $SpecA$ will be the same as the image of $SpecB/Ker$ in $SpecB$, which is a closed subset as we showed before. □

Notice that closed subschemes are more than a closed subset, since we have $Speck[x]/(x)$ is not isomorphic to $Speck[x]/(x^2)$, but they can be set theoretically viewed as a closed subset of $Speck[x]$, consisting of just one point.

Proposition 1.3. Closed embedding are finite morphisms, thus of finite type.

Proof. **** later after we reviewed properties of morphisms* □

Proposition 1.4. Compositions of closed embeddings are closed embeddings.

Proof. Work locally, you will see this follows easily. □

Proposition 1.5. Being a closed embedding is affine local on the target.

Proof. This is another application of the affine communication lemma. First, suppose that $SpecB \subset Y$ has inverse image $SpecA$, then it's easy to show that for any distinguished open $D(f) \subset SpecB$, we have the restriction of the diagram to the inverse image of $D(f)$ is $D(\varphi(f)) \subset SpecA$, and the morphism is induced by $B_f \to A_{\varphi(f)}$, clearly surjective if $B \to A$ is.

On the other hand, suppose that $B_{f_i} \to A_{\varphi(f_i)}$ for $1, \cdots, r$ are sujective where $(f_1, \cdots, f_r) = B$, we claim that $B \to A$ is surjective. To see this, notice that the condition implies that for each i, we have:

$$\varphi(b_i f_i^{m_i}) = a\varphi(f_i^{n_i})$$

Notice that $(f_1, \cdots, f_r)$ generates B is equivalent of $D(f_i) \cup \cdots \cup D(f_r)$ is $SpecB$ which is equivalent of $D(f_i^{n_1}) \cup \cdots \cup D(f_r^{n_r})$ covers $SpecB$ which is equivalent of $(f_1^{n_1}, \cdots, f_r^{n_r})$ generates B. Thus we are done. □

Below is an important property, where we will show that closed subschemes correspond to quasicoherent sheaves of ideals:

First, we call a sub-$\mathcal{O}_Y$-module of $\mathcal{O}_Y$ an **ideal sheaf**, or **sheaf of ideals** of $\mathcal{O}_Y$. Notice that if $\pi : X \to Y$ is a closed embedding, it induces a morphism of sheaves of rings: $\mathcal{O}_Y \to \pi_*\mathcal{O}_X$, notice that we already saw this is also morphism of $\mathcal{O}_Y$-modules.

Proposition 1.6. This morphism of $\mathcal{O}_Y$-modules are surjective.

Proof. Notice that on stalks, it can be can be computed by $SpecA \to SpecB$, where we have $B \to A$ is surjective, we consider q a prime in A such that the inverse image is p prime in B, we claim that $B_q \to A_p$ is also surjective. There is no fancy arguments going on here, let's just do it explicitly, you will find this is true. □

Therefore, consider the ideal subsheaf of $\mathcal{O}_Y$, which is defined by the kernel of the morphism above, is a sheaf of ideals. We will denote the kernel by $\mathscr{I}X/Y$, notice that this is the kernel, thus we can directly take the open by open definition. Therefore, on each open subset U, it gives an ideal of $\mathcal{O}_Y(U)$, denoted by $\mathscr{I}_{X/Y}(U)$. Notice that on an affine open, it gives the ideal A determining the ideal of B, denoted by I where $B \to A = B/I$ is the surjection. Next proposition, we will show that this $\mathscr{I}_{X/Y}$ is quasicoherent.

Proposition 1.7. $\mathscr{I}_{X/Y}$ is quasicoherent.

Proof. We will actually prove that $\pi_*\mathcal{O}_X$ is quasicoherent, then use the fact that quasicoherent modules is an abelian Category, thus the kernel is also quasicoherent.

To prove our assertion that $\pi_*\mathcal{O}_X$ is quasicoherent, we use the distinguished criterion we discussed before. This means we need to prove that suppose that $U = SpecB$ an affine open in Y, α is the following diagram is an isomorphism:

$$\begin{tikzcd}
\Gamma(SpecB, \pi_*\mathcal{O}_X) \arrow[r, "\phi"] \arrow[d, "loc"'] & \Gamma(SpecB_f, \pi_*\mathcal{O}_X) \\
\Gamma(SpecB, \pi_*\mathcal{O}_X)_f \arrow[ur, dashed, "\alpha"'] &
\end{tikzcd}$$

But we see that by the definition of $\pi_*\mathcal{O}_X$ and the fact that the inverse image of $SpecB$ is $SpecA$ with a surjective ring map $\varphi : B \to A$. Then we know that $\Gamma(SpecB_f, \pi_*\mathcal{O}_X)$ is $A_{\varphi(f)}$ where $\Gamma(SpecB, \pi_*\mathcal{O}_X)_f$ is A_f where we view A as a B-module. Thus we are trying to show that we have an isomorphism of B_f-modules:

$$A_f \to A_{\varphi(f)}$$

But this map is obviously induced by the universal property by $\frac{a}{f^n} \to \frac{a}{\varphi(f)^n}$, it's an isomorphism since we can find an inverse. □

Via the above result, we see that given a closed embedding $X \hookrightarrow Y$, we have an associated quasicoherent ideal sheaf on $\mathcal{O}_Y$ and a short exact sequence:

$$0 \to \mathscr{I}_{X/Y} \to \mathcal{O}_Y \to \pi_*\mathcal{O}_X$$

This is called the closed subscheme short exact sequence.

Next, we want to show that the converse also holds, in the sense that given a quasicoherent ideal sheaf of $\mathcal{O}_Y$, we produces a closed embedding. We see that if Y is just an affine space $SpecB$ then we see that each of such quasicoherent sheaf comes from some ideal of B, then the closed embedding we have is just $SpecA \to SpecB$ where $A = B/I$. We will see that this works globally:

Proposition 1.8. Given any quasicoherent ideal sheaf of $\mathcal{O}_Y$

Proof. This is again a proposition that can be done by a gluing argument, **** later details.* □

Next, let's discuss a few examples of closed subschemes where we will make the notions of finite **scheme-theoretic union** and **scheme theoretic intersection** of closed subschemes precise:

Let's deal with with the intersection first, and the union will follow similarly. Suppose that $X_1 \hookrightarrow Y$ and $X_2 \hookrightarrow Y$ are two closed subschemes of Y, corresponding to quasicoherent ideal sheaves of $\mathcal{O}_Y$, denoted by $\mathscr{I}_1$ and $\mathscr{I}_2$ respectively. The following proposition is the key ingredient:

Proposition 1.9. Suppose that $\mathscr{I}$ and $\mathscr{J}$ are two quasicoherent ideal sheaves of $\mathcal{O}_Y$. Then there is a presheaf of $\mathcal{O}_Y$, denoted by $\mathscr{I} + \mathscr{J}$, where on each open subset $U \subset Y$, we associate to U $\mathscr{I}(U) + \mathscr{J}(U)$ with the obvious restriction maps, we then sheafify it, denote the associated sheaf still by $\mathscr{I} + \mathscr{J}$. It's still quasicoherent.

Proof. This is a local property, thus we might as well assume that X is $SpecA$ and we have $\widetilde{I}$ and $\widetilde{J}$ two ideal sheaves of $\mathcal{O}_X = \widetilde{A}$. We claim that $\widetilde{I} + \widetilde{J} \cong \widetilde{I + J}$. To show this, consider how we define a morphism from $\widetilde{I} + \widetilde{J}$, you first consider the presheaf, then you show this is a isomorphism by considering the stalks, we leave the details for you. But you need to figure out how to view $\widetilde{I}$ and $\widetilde{J}$ as a subscheme of $\widetilde{A}$ in order to do the summation operation. □

Based on the above result, we can define the closed subscheme defined by $\mathscr{I}_1 + \mathscr{I}_2$ to be the scheme theoretic intersection of X_1, X_2. Similarly, we can define arbitrary intersections this way, the only thing we need to make sure of is $\sum_i \mathscr{I}_i$ is still quasicoherent when i is not necessarily finite, but this is just a mild generalization of the previous. Here, we consider the intersection of closed subschemes, later, after we review the fiber product, you can define intersections of arbitrary schemes, not necessarily closed subschemes.

For unions, the picture is clearer, we still have the above $\mathscr{I}$ and $\mathscr{J}$, we form the intersection subsheaf, but in this case, we see that the gluing of the pieces on individual open subsets indeed lies in the intersection, thus we can just define the union of two closed subschemes to be the one defined by the intersection sub-ideal sheaf.

Here are two emamples:

Example 1.10. Consider the scheme theoretic intersection of $V(y - x^2)$ and $V(y)$ in $\mathbb{A}^2$ and their scheme theoretic union.

After this easy example, consider the scheme-theoretic intersection of $V(y^2 - x^2)$ and $V(y)$, use these two examples above to give an example where under the scheme theoretic operation $(X \cap Z) \cup (Y \cap Z) \neq (X \cup Y) \cap Z$. This says that the set theoretic point of view doesn't always carry over to schemes.

The next exercise shows that closed embeddings are monomorphisms just like open embeddings:

Proposition 1.11. Closed embeddings are monomorphisms.

Proof. Consider:

$$X \to Y \to Z$$

where $f : Y \to Z$ is a closed embedding, suppose that g, h are morphisms from $X \to Y$. We first show that on the level of topological spaces, the morphism is injective, thus it's a monomorphism in the sense of topoplogical spaces. Then notice that two morphisms will agree if they agree on the preimage of a basis of the target. But when we consider affine covers, we see that the corresponding ring maps is surjective, try to use it to show the assertion. □

Below are more emamples of closed subschems that is useful:

Example 1.12. Suppose that Y is a scheme and $s \in \Gamma(Y, \mathcal{O}_Y)$, we want to define the closed subscheme cut out by s, which will be called the **vanishing scheme** $V(s)$ of s. Clearly, we want to use the correspondence between closed subschems and quasicoherent ideal sheaves, but how to find such a ideal sheaf. On each affine open, we would like it be be like $SpecB/(s_B)$ where s_B is the restriction of s to $SpecB$, thus we would like to define this ideal sheaf locally by: $\widetilde{(s_B)}$. But it's important to understand what this means. We know that $Y|_U$ is isomorphic to $SpecA$ for some affine scheme, this is a good place to think about what quaiscoherent means on a space that is only isomorphic to an affine space $SpecA$, after you figure this out, try to use this definition to piece these to a global sheaf on M, and try to show that it's a subsheaf of $\mathcal{O}_Y$ where you could possibly use the fact that pushforward by isomorphisms preserves subsheaf structure since it's left exact, and more over is an ideal sheaf. Finally, quasicoherent will be easy form how you constructed on opens.

Under this definition, it's easy to show that if u is an invertible function in the global section, $V(s) = V(us)$ as closed subschemes. You only have to check that they agree on each open subset. The reason is that as subschemes of $\mathcal{O}_Y$, if they agree on an open cover, they are necessarily equal, this is also easy to show.

More generally, if S is a set of functions we can define $V(S)$ to be the closed subscheme cut out by the ideal sheaf such that on each affine open, we have $\widetilde{(S|_U)}$ where $S|_U$ stands for the image of S under the restriction. **** more comments later*

Our next topic is locally closed embeddings and locally closed subschemes, which is also going to be useful in the review of fiber products.

Definition 1.13. We say that a morphism $\pi : X \to Y$ is a **locally closed embedding** or **locally closed immersion** if it can be factored into:

$$X \xhookrightarrow{\rho} Z \xhookrightarrow{\tau} Y$$

where ρ is a closed embedding and τ is an open embedding.

One quick fact is that since we know that open and closed embeddings are monomorphisms and being monomorphisms is preserved under composition, thus a locally closed embedding is also a monomorphism.

Here is a handy example of locally closed embedding:

Example 1.14. Show that $Speck[t, t^{-1}] \to Speck[x, y]$ given by $x \mapsto t$ and $y \mapsto 0$ is a locally closed embedding.

The idea here is that we first notice that the image of $Speck[t, t^{-1}]$ is in $D(x)$, and $D(x) \hookrightarrow Speck[x, y]$ is an open embedding, thus we only have to show that the first is a closed embedding. The details are omitted here.

Again, the warning of the terminology here is that we will avoid using immersion since the differential geometry version of immersion is more closely related to the notion of being **ramified** which we will introduce later.

Proposition 1.15. Locally closed embeddings are locally of finite type.

Proof. **** later* □

There is one useful criterion of showing that a locally closed embedding is in fact a closed embedding, which is purely topological:

Proposition 1.16. Suppose that $f : X \to Y$ is a locally closed embedding whose image is closed in Y, then f is a closed embedding.

Proof. Suppose that f can be factored into:

$$X \hookrightarrow U \hookrightarrow Y$$

where the first one is a closed embedding and the later one is an open embedding. Notice that being a closed embedding can be checked affine local on the target. Suppose that $f(X)$ is the image of f in Y, then since it's closed, we know that $Y - f(X)$ and U will cover Y. Notice that we can choose an affine open cover that only consist opens that is either in U or $Y - f(X)$, if it's in U, then we know that it satisfies the closed embedding property. If it's in $Y - f(X)$, we know that the preimage is the zero ring, thus surjective, we are done. □

Now, we begin to solve an interesting question. Clearly, an open subscheme U of a closed subscheme V can be viewed as a closed subscheme of an open subscheme. Notice that U is the intersection of some open subset U' with V, then we can take $V' = V \cap U'$ which is the same of U and we can embed it into U' as an closed subscheme, then followed by $U' \hookrightarrow X$ an open embedding. To see that $V' \to U'$ is a closed embedding, choose an affine open cover of U', then it's clear that the inverse image is an affine open in V that is contained in U, since $V \to X$ is closed embedding, this is surjective, thus a closed embedding.

However, if we have V', a closed subscheme of an open subscheme U', can we express it as an open subscheme of an closed subscheme? In the category of topological spaces, we will take V''s closure, then V' is an open subset of $\overline{V'}$, which is an closed subset of X. In the case of schemes, we can only define constructions like this in good schemes, but to give an motivation, let's consider the following:

Proposition 1.17. Suppose that $V \to X$ is a morphism of schemes, consider the following three conditions:

1. V is the "intersection" of an open closed subscheme and a closed subscheme.
2. V is an open subscheme of a closed subscheme of X.
3. V is a closed subscheme of an open subscheme of X, i.e. $V \to X$ is a closed embedding.

where we use a preliminary definition of intersection here by defining it to be the fibered product. We claim that 1 and 2 are equivalent and both imply 3. Even though 3 doesn't always imply $1, 2$.

Proof. This proposition is about the following diagram:

$$\begin{tikzcd}
V \arrow[r, hook, "open"] \arrow[d, hook, "closed"'] & K \arrow[d, hook, "closed"] \\
U \arrow[r, hook, "open"'] & X
\end{tikzcd}$$

We only make some comments on how 2 implies 1 since other implications are not hard. Notice if we assume 2, we only have the upper right half of the above diagram, we need to recover the lower left half. First, what is the U, we just take the U such that $U \cap K = V$ and the morphism form V to U will be the restriction of $V \to K \to X$ restricted to U. Then it's not hard to show that this diagram commutes. The point is that why it's Cartesian, you need to verify the universal property, but this is just a tedious work is you remember the universal property of the open embedding. Then everything else is just tautological checking. $\square$

One of the consequences of the above result is the we can show that composition of locally closed embedding is still locally closed embedding:

Proposition 1.18. Composition preserves locally closed embedding.

Proof. Consider the following diagram:

$$\begin{tikzcd}
V_1 \arrow[r, hook, "closed"] & U_1 \arrow[r, hook, "open"] \arrow[dr, hook, "closed"'] & V_2 \arrow[r, hook, "closed"] & U_2 \arrow[r, hook, "open"] & X \\
 & & T \arrow[ur, hook, "open"'] & &
\end{tikzcd}$$

where T comes from applying the result above. $\square$

These are the general discussion of locally closed embeddings and closed embeddings, later, we would like to apply them to get some interesting results about projective geometry. Let's first look at an explicit example of closed embedding in projective space:

Example 1.19. Consider the projective space $\mathbb{P}^n_A$ where we have $f \in A[x_0, \cdots, x_n]$ a homogeneous polynomial. We claim that any homogeneous polynomial defines a subscheme, notice that f maynot be a global function, thus this is not quite the same as the vanishing scheme we defined earlier. On each of the opens $U_i = D_+(x_i)$, we define it to be the closed subscheme cut out by $f(x_{0/i}, \cdots, x_{n/i})$ where we omit the $x_{i/i}$ coordinate of interprete it as 1. Privately, we think of it as the closed subscheme cut out by $f(x_0, \cdots, f_n)/x_i^{degf}$.

Before we globalize this construction idea, we will use the following straightforward results: Suppose that X is any topological space and $\mathcal{F}$ is a sheaf on X, then if we we have $\mathcal{G}_U$ subsheaves of $\mathcal{F}|_U$ on an open cover U such that they are equal to each other on intersections, then there is a global subsheaf $\mathcal{G}$ of F such that the restriction of $\mathcal{G}$ on U is equal to $\mathcal{G}_U$, and if they are all $\mathcal{O}_X$-modules, the resulting global sheaf is also an $\mathcal{O}_X$-module. This can be viewed as a special case of gluing sheaves together via identity maps on intersections.

**** details later.*

The above construction gives an important definition in projective geometry:

Definition 1.20. As usual, let k be any field, a closed subscheme of $\mathbb{P}_k^n$ cut out by a single nonzero homogeneous equation f is called a **hypersurface** in $\mathbb{P}_k^n$, obviously, this is not the same notion we defined earlier where we have a closed subscheme cut out by a global function, since here as we proved in a previous review, the global sections of $\mathbb{P}_k^n$ is just k, i,e. there is no nonconstant global functions on $\mathbb{P}_k^n$. The **degree** of a hypersurface is the degree of the polynomial f. A hypersurface of degree 1 is called a **hyperplane**, similarly we have **quadric, cubic, quartic, quintic, sextic, septic** and **octic**, $\cdots$ hypersurafaces.

We also have different names for them if we are in $\mathbb{P}_k^2$, where hypersurfaces will be called **curves**. A degree 1 curve is called a **line** and a degree 2 curve is called a **conic**. If we are in $\mathbb{P}_k^3$, hypersurfaces are called **surfaces**.

As a small exercise, consider the following example:

Example 1.21. Suppose that we work now in $\mathbb{P}_k^3$. We will try to figure out why the following three equations defines a irreducible closed subscheme in $\mathbb{P}_k^3$, which we will call a **twisted cubic** and often denoted by C. We will also show that it's isomorphic to $\mathbb{P}_k^1$.

Consider $wz = xy, x^2 = wy$ and $y^2 = xz$. The coordinate ring is $k[x, y, z, w]$. Notice that each of these quadic equations, as what we discussed earlier, defines a quadric hypersurface in $\mathbb{P}_k^3$. We will take their scheme theoretic intersections, and we will prove that this is an irreducible closed subscheme which is isomorphic to $\mathbb{P}_k^1$. The following argument will be given in a very hands on way, we will list all the steps you need to do, but the details are left, which should be easy to fill in.

The first claim is that it's covered by $D_+(z)$ and $D_+(w)$. To see this, suppose that now we are outside the union of $D_+(z)$ and $D_+(w)$, it's left as an exercise that locally the closed subscheme in $D_+(x)$ and $D_+(y)$ are just a point which is contained respectively in $D_+(x) \cap D_+(w)$ and $D_+(y) \cap D_+(z)$. Since we assume that we are outside $D_+(z)$ and $D_+(w)$, then we get empty scheme.

Moreover, we claim that the closed subscheme that is not inside $D_+(z)$ is only one point, which should be easy to show if you compute what is the closed subscheme inside $D_+(w)$ and then take out the part in $D_+(w) \cap D_+(z)$, where you can think of showing that the intersection of two closed subset is just one point ($V(\cdots) \cap V(\frac{w}{z}) = V(\frac{x}{z}, \frac{y}{z}, \frac{w}{z})$ and you should figure out what is the $\cdots$), we see this also shows that it's open inside $D_+(x)$. Now we are in good shape to show that this is irreducible. First, you should know that if we have a polynomial ring $k[x, y, z]$ and an ideal $(xy - z, x^2 - yz, y^2 - x)$, this ideal is prime, which means that the closed subscheme is irreducible. This is because the quotient is isomorphic to $k[y]$, this is left as an exercise. We claim that the twisted cubic in $D_+(z)$ is isomorphic to this and since we know that the closure in $\mathbb{P}^3$ of the twisted cubic inside $D_+(z)$ can't be itself since we know that it's open, thus it's must contains the twisted cubic itself since it's closed, and they differ by only one point, thus the closure is the twisted cubic, then the closure of an irreducible subset in an open subset is irreducible, we are done.

Finally, let's show that it's isomorphic to $\mathbb{P}^1$. The above discussion is still useful where we have the isomorphism $k[x, y, z]/(xy - z, x^2 - yz, x - y^2) \cong k[y]$, this is the isomorphism of $C \cap D_+(z)$ to $k[y]$. Similarly, you have $C \cap D_+(w)$ is isomorphic to $k[y]$, where you should think of $\mathbb{P}^1$ is covered by two $\mathbb{A}^1$, then it's left as an exercise that these two morphisms agree on intersections, thus we glue them to a global isomorphism.

Following the last example, recall that we showed before that a morphism of graded rings induce a morphism of projective spaces on an open subset, not necessarily a globally defined morphism. But a special case is that when the map of graded rings is surjective:

Proposition 1.22. Suppose that $S_\bullet \to R_\bullet$ is a surjection of graded rings, then it induces a morphism from $ProjR$ to $ProjS$.

Proof. We only have to show that $V(\Phi(S_+))$ is empty. Suppose that we have a element $e \in R_+$, a homogeneous element, since the ring maps is surjective, then we have an element $f \in S$ such that $\Phi(f) = e$, we claim that f is in S_+. Suppose that we we write f in homogeneous pieces, since this is a map of graded rings, suppose that the degree of the map is $d > 0$, not the zero map. Then we see that f must have positive degree. ☐

Remark 1.23. We will show later after we develop enough tools that actually all closed embeddings $X \hookrightarrow ProjS$ into a projective A-scheme (S is a finitely generated graded A-algebra) is of the form $Proj(S/I)$ where I is a homogeneous ideal.

But we should interpret this correctly, since this is only saying that we have such a commutative diagram:

$$\begin{tikzcd} X \arrow[rr, hook] \arrow[dr, leftrightarrow, "iso"'] & & ProjS \\ & ProjS/I \arrow[ur, hook] & \end{tikzcd}$$

where $ProjS/I \hookrightarrow ProjS$ is induced by $S \to S/I$, but we are not saying that is we have a closed embedding $ProjR \hookrightarrow ProjS$, R is of the form S/I and it's induced by $S \to S/I$. This is again since the imperfect generalization of the morphism of proj induced by maps of graded rings.

Below is an example of what we mean by the above remark, still using the twisted cubic:

Example 1.24. We will show that the map of graded rings $k[x, y, z, w] \to k[s, t]$ given by $(x, y, z, w) \mapsto (s^3, s^2t, st^2, t^3)$ induces an closed embedding $\mathbb{P}_1^k \hookrightarrow \mathbb{P}_3^k$, which yields an isomorphism between $\mathbb{P}_1^k$ and the twisted cubic C we defined earlier in Example 1.21.

Notice that this is not a surjective map of rings for obvious reasons, but we can easily see that the the radical of (s^3, s^2t, st^2, t^3) is the whole irrelevant ideal (s, t), thus by a previous result, we see that this actually defines a morphism from all of $\mathbb{P}_1^k$ to $\mathbb{P}_3^k$. This is what we meant to convey in the last remark, since you see that this is not a surjective map, but we can actually show that we have a commutative diagram:

$$\begin{tikzcd} \mathbb{P}_1 \arrow[rr, hook] \arrow[dr, leftrightarrow, "iso"'] & & \mathbb{P}_3 \\ & C = Proj(k[x, y, z, w]/(\cdots)) \arrow[ur, hook] & \end{tikzcd}$$

Notice that the isomorphism is **not** induced by a map of graded rings.

The reason what $\mathbb{P}_1 \hookrightarrow \mathbb{P}_3$ will induce an isomorphism to C is still checked locally, which is basically what we did in Example 1.21. Another map is also an isomorphism to C is also checked locally on $D_+(x)$, $D_+(y)$, $D_+(z)$ and $D_+(w)$. The idea is that this closed embedding resembles how we defined C locally, thus you will find naturally that this is an isomorphism.

We also have something to say about this projective space such as Veronese embedding, affine projective cones which help us understand the algebraic picture of the classic picture we have of projective spaces, but we include it in the sections of the review of proj constructions, you can find the stuff you need to review there.

We move on to a new topic where we use the closed subscheme structure to define a few notions which are all of form of the smallest closed subscheme such that something is true. We start with the scheme theoretic image:

To give some kind of motivation, we first see that the set theoretic version of the image behave badly in the scheme setting. For example, consider the morphism $\mathbb{A}^2 \to \mathbb{A}^2$ where $(x, y) \mapsto (x, xy)$, you will later find that it's hard to put a reasonable structure on the set theoretic image of this map, if that is even possible. Therefore, the most we can expect is to approximate the set theoretic image using closed subschemes, which is the notion we will introduce:

Definition 1.25. Suppose that $i : Z \to Y$ is a closed subscheme, giving an exact sequence $0 \to \mathscr{I}_{Z/Y} \to \mathcal{O}_Y \to i_*\mathcal{O}_Z \to 0$. We will say that the image of the morphism $\pi : X \to Y$ lies in Z if the composition

$\mathscr{I}_{Z/Y} \to \mathcal{O}_Y \to \pi_* \mathcal{O}_X$ is zero. Informally speaking, we think of $\mathcal{I}$ as those functions that vanish on Z, then the notion of X has it's image in Z is saying that the functions that vanish on X should also vanish on X, which is a reasonable notion of X has it's image in Z.

Notice that if π has it's image inside $Z_j \hookrightarrow Y$ where Z_j's are a collection of closed subschemes, then notice that the intersections are defined by the sum of $\mathscr{I}_j$, then notice that though we have to take the sheafification, since before we sheafify, all the sections in the presheaf goes to 0, thus after sheafify, still all sections go to 0, we see that X is still contained in the scheme theoretic intersection of all Z_j. Therefore, we define the **scheme-theoretic image** of π to be a closed subscheme of Y, which is the smallest closed subscheme such that π has it's image inside the closed subscheme, or equivalently, it's the intersection of all the closed subschemes which contains the image of π.

Let's try to understand this explicitly when Y is affine:

Proposition 1.26. When $Y = SpecA$ is affine, the scheme theoretic image of $X \to Y$ is the closed subscheme of Y cut out by the global functions of Y where it's pullback to 0 in X.

Proof. Consider the morphism of rings $A \to \pi_* \mathcal{O}_X(X)$, then consider the kernel of this map, which is an ideal of A, we claim that scheme theoretic image of π is $SpecA/I$. First, notice that if we denote the kernel by K, then the closed subscheme if cut out by $\widetilde{K}$, then it's easy to see that the following composition is indeed zero:

$$0 \to \widetilde{K} \to \mathcal{O}_Y \to \pi_* \mathcal{O}_X \to 0$$

Then we have to prove that this is the smallest one. If $\widetilde{K'}$ is another quasicoherent ideal sheaf defining another closed subscheme such that the image of π lies inside it. Then we have:

$$0 \to \widetilde{K'} \to \mathcal{O}_Y \to \pi_* \mathcal{O}_X \to 0$$

Then take the global section, we see that $K' \subset K$ since this is the definition of kernel. Then since taking the associated quasicoherent module $\widetilde{\ }$ is exact, thus we know that $\widetilde{K'}$ is a subsheaf of $\widetilde{K}$. We are done. $\square$

Based on the above result, we can give a few examples:

Example 1.27. First, consider the image of $Speck[\epsilon]/(\epsilon^2) \to Speck[x]$ given by $x \mapsto \epsilon$. Then we see that the kernel is (x^2), thus the scheme theoretic image is $Speck[x]/(x^2)$, the image of the fuzzy point still has fuzz.

The second example is still consider a morphism $Speck[\epsilon]/(\epsilon^2) \to Speck[x]$, but this time it's given by $x \mapsto 0$. Then notice that the kernel is (x), thus the scheme theoretic image is $Speck[x]/(x) = Sepck$, thus reduced.

The third one is the morphism $Speck[t, t^{-1}] = \mathbb{A}^1 - \{0\} \hookrightarrow \mathbb{A}^1 = Speck[u]$ given by $u \mapsto t$. Notice that the kernel is 0, thus the scheme theoretic image is $\mathbb{A}^1$, which is not the set theoretic image. But this as you can see, is not so crazy since we know that the scheme theoretic image is a closed subscheme, and since the closure of $\mathbb{A}^1 - \{0\}$ is just $\mathbb{A}^1$, we are not surprised. Therefore, we may assume that the underlying set of scheme theoretic image is always the closure of the set-theoretic image. This is true in reasonable situations, but still not true in general just like in the following pathological situation.

The last example is a counter example to the hypothesis we had in the last paragraph: Consider the scheme $X = \coprod Speck[\epsilon_n]/(\epsilon_n^n)$ and $Y = Speck[x]$, and the morphism is given by form each $Speck[\epsilon_n]/(\epsilon_n^n)$, we define a morphism by $x \mapsto \epsilon_n$. Then we see that if a function pulls back to 0, then since it's a polynomial, we can consider it's Taylor expansion, which must be zero at each degree, thus the function is 0, thus the kernel is 0, thus the image is the scheme itself. However, the set theoretic image is easy to see just the origin.

Next, let's see how to get rid of this pathology and how to compute the image in nice situations. The general idea is that, if Y is an affine shceme, we already know how to deal with it, thus it's natural to ask can we compute this image affine locally. Notice that on an affine open subset $SpecB \subset Y$, we see that if we define the ideal $I(B) \subset B$ to be the kernel of the map $B \to \Gamma(SpecB, \pi_* \mathcal{O}_X)$. Then we want to see that if we can glue this together to get a global quasicoherent ideal sheaf $\mathscr{I}$, this sheaf is the one defining the image. As you can see, it is one of those containing the image of X, and suppose that you have another one, then locally to the inverse image of $SpecB$, you can prove that it injects locally into $\mathscr{I}$, thus it's a subsheaf of $\mathscr{I}$. Therefore, we are just trying to figure out under what conditions can this local definition of $I(B)$ glue together. Recall the characterization of quasicoherent modules we had before using localizations **** more to say one this after we finish the review of retreatment of quasocoherent ideals.* You only have to prove the natural map $\Phi : I(B)_g \to I(B_g)$ is an isomorphism.

Let's first check the injectivity of this map, where suppose that we have $\frac{f}{g^n}$ which is mapped to 0. This is a question that is a little bit delicate, notice that under Φ, the $\frac{f}{g^n}$ is mapped to the element that is represented by $\frac{f}{g^n}$, thus if this mapped it well defined, we see that since these two elements are zero under the same conditions, we are done, but this map is defined by the universal proeprty, thus we are done.

We only have to consider the surjectivity, which is equivalent of asking if $\frac{r}{g^n}$ is pulled back to 0 on $\pi^{-1}(SpecB_g)$, is there an element in $I(B)_g$ that is mapped to it. Notice that if we have a natural number m, such that rg^m is pullback to 0 on $\pi^{-1}(SpecB)$, then we know that rg^m/g^{m+n} is the target we are looking for. If $\pi^{-1}(SpecB)$ is reduced, we are good since we claim that rg is the element we are looking for. First, we see that since $\frac{r}{g^n}$ is pullback to 0 on $\pi^{-1}(SpecB_g)$, but since g are invertible on B_g, thus r is pullback to 0 on $SpecB_g$, then again, since g vanishes on $V(g)$, we see that rg will vanish on $\pi^{-1}D(g) \cup \pi^{-1}V(g) = \pi^{-1}SpecB$.

We are also in nice situations if $\pi^{-1}SpecB$ is affine since if we write $\pi(r) = r'$ and $g = \pi(g)$ in $SpecA = \pi^{-1}SpecB$, then we get r' vanishes on $D(g')$, thus there will be m such that $r'g'^m = 0$, thus we are done.

Finally, if $\pi^{-1}SpecB$ is quasicompact, we are also good since we can cover it by finitely many affine opens, for each open, we have rg^{m_i} when pulled back to this open set, then we take m to be the maximum. In conclusion, we proved the following theorem:

Theorem 1.28. Suppose that $\pi : X \to Y$ is a morphism of schemes, if X is reduced or π is quasicomapct, then the scheme theoretic image of π can be computed affine locally: on each $SpecA \subset Y$, it's cutout by functions of A that pulls back to 0 on $\pi^{-1}SpecA$.

This theorem has a corollary where we can say that in nice situations as in the above theorem, the closure of the set theoretic image is the underlying set of the closed subscheme:

Corollary 1.29. In the above setup as in the above theorem, the closure of the set theoretic image is the set underlying the scheme-theoretic image. In particular, if the set theoretic image is closed, then they agree.

Proof. First it's easy to see that the set theoretic image is contained in the underlying set of the scheme-theoretic image, since it can be checked affine locally, moreover, we can also assume that the source is also an affine by choosing an affine open of the point we are interested in. Then it's just an easy algebra exercise to see this is true.

Therefore the closure of the set-theoretic image is contained in the underlying set of the scheme-theoretic image. Let U be the complement of the closure of the set theoretic image, thus $\pi^{-1}U = \varnothing$. Since we know that the scheme-theoretic image can be computed affine locally, we know that on U, the scheme-theoretic image is empty. □

Also, we can show that if $\pi : X \to Y$ is a morphism from a reduced scheme X, then we claim that the scheme theoretic image of π is also reduced. To see this, notice that this is a local property (and since X is reduced, by the above theorem we can compute everything affine locally), on $SpecA \subset Y$, the scheme-theoretic image is cut out by $ker(\Phi : A \to \Gamma(SpecA, \pi_*\mathcal{O}_X)) = \Gamma(\pi^{-1}SpecA, X)$, thus we know that if $a^r \in Ker$, then $\Phi(a)^r = 0$, then since X is reduced, this is a reduced ring, thus $\Phi(a) = 0$, thus $a \in Ker$, thus reduced.

Definition 1.30. We define the **scheme-theoretic closure** of a locally closed embedding $\pi : X \to Y$ to be the scheme-theoretic image of π.

Recall that when we discussed locally closed embedding before, where we provided three ways of thinking of a locally closed embedding: where we know that the intersection of a closed and an open embedding is equivalent to the notion of a open subscheme of a closed subcheme, which both implies that it's a locally closed subscheme, while a locally closed subscheme will not in the most general case imply the above two notions. However:

Proposition 1.31. Suppose that $V \hookrightarrow X$ is a locally closed embedding, if V is reduced or this morphism is quasicompact, then this implies the other two notions we discussed in the previous paragraph.

Proof. One key setp in this is **** more to say on this* ☐

Then, let's consider the reduced subscheme construction, we begin by the general case, where we have there different definitions, which are all equivalent:

Definition 1.32. Suppose that X^{set} is a closed set of a scheme Y, then we can define a canonical scheme structure X on X^{set} such that it's reduced and a closed subscheme of Y. The following definitions are equivalent, which we will show later:

1. It's the closed subscheme cut out by functions on each open subset U such that they vanish on $X^{set} \cap U$.

2. We can also define it to be a scheme theoretic image. Suppose that for each point p in X^{set}, we associate it with a scheme which is just a ring since it's a point, thus we associate it with *Spec* of the residue field of $\mathcal{O}_{Y,p}$. Notice that this is reduced since it's the spec of a field. Then, we take the disjoint union of these schemes for each point in X, we obtain a canonical map where each $SpecK_y$ is mapped to $p \in Y$, with an isomorphism of the residue field. Then the scheme structure on X is the scheme theoretic image of this morphism.

3. Or we can define it to be the smallest closed subscheme whose underlying set contains X.

Proposition 1.33. All the above three definitions are equivalent.

Proof. Before we start, the first definition needs a little bit of justification. We define a subsheaf $\mathscr{I}$ of $\mathcal{O}_X$ where on each open subset U, we define:

$$\mathscr{I}(U) = \{f \in \mathcal{O}_X(U) : f(t) = 0, \text{all } t \in X^{red} \cap U\ \}$$

Notice that if f has value 0 in U at t, then in any smaller open subset, it's the same, thus with the obvious restriction, it's presheaf. However, this is a sheaf, since this definition is local in nature. You can check that if f_i's are functions on U_i such that their value is 0 at $X^{set} \cap U_i$, and they agree on intersections, we see that by the gluability of $\mathcal{O}_X$, we have a function on U such that it restrict to f_i, then it's obviously that the value is 0 at any $U \cap X^{set}$.

We only have to show that this is a quasicoherent sheaf. By the characterization we had before, suppose that $U = SpecR$, then it's an exercise that $\mathscr{I}(U) = I$ for a unique radical ideal of R, then on any distinguished open $D(g)$, this $\mathscr{I}(D(g))$ is all $r \in R_g$ such that r is zero at $D(g) \cap X^{red}$, we claim that this is isomorphic to I_g, which shows that this is a quasicoherent sheaf. To see this, notice that if we can show that $V(I) \cap D(g) = V(I_g)$, we are done. Notice that it's a general algebra fact that taking radical commutes with localization, thus I_g is still a radical. Then since the $\mathscr{I}(D(g))$ will correspond to a radical ideal in R_g, which is the intersction of all primes in $V(I) \cap D(g)$. If we can show that $V(I) \cap D(g) = V(I_g)$, we are done. The is an algebra exercise, which is omitted here.

Then, let's prove that definition 1 and 2 are equivalent, which means that we only have to show that they are defined by the same quasicoherent ideal sheaf. Notice that the disjoint sheaf is reduced, then we can compute the image affine locally, then suppose that we are in $SpecA \subset Y$, then we only have to consider the kernel of the above morphism we defined, which is an easy exercise and you see that they agree.

Finally, let's see why they are equivalent to definition 3. To be more precise on what we mean by the smallest, we mean that it's the intersection of all closed subschemes containing X^{set}. Notice that we see that the above two definition gives a subscheme that contains X^{set}, this is easy to see by the second definition since the source is reduced, thus we know that the closure of the set theoretic image is the underlying set, which is X^{set} since it's closed in the first place. Then, we only need to show that it's the smallest. Suppose that Z is another closed subscheme of Y which contains X^{set} set theoretically. Then choose any $x \in X^{set} \subset Z$, there is an affine open U in X, which satisfies $U \cap Z$ is isomorphic to $Spec(A/I)$ where I is an ideal and $SpecA$, an easy algebra exercise shows that since Z is closed, then $Z \cap U$ is closed in U, thus we know that $Z \cap U$ correspond to another unique radical ideal denoted by I', then $I' \subset I$ since $V(I') \supset V(I)$, thus we see that the ideal sheaf defining Z is contained $\mathscr{I}$, thus we see that the intersection of the ideal sheaves defineing then is $\mathscr{I}$, thus this is equivalent of saying that this is the smallest. ☐

Remark 1.34. The above constructions are called the induced reduced subscheme structure on the closed subset X^{set}. In some other literature, it's also defined to be the unique closed subscheme of Y, whose underlying set is X^{set} and is reduced. We already see from the above constructions that the induced structure is indeed reduced and it has the underlying set equal to X^{set}. The technique we used to show the last property shows that if we have another reduced subscheme of the same property, it must has the defining ideal sheaf the same as that of X, thus we are done. We will use this as the **official definition**, while using properties we proved above.

One special case is that X^{set} is the whole space Y, by this we obtain a reduced subscheme of Y, called the reduction of Y. By the construction we did before, we see that this is indeed the closed subscheme defined by the sheaf of nilradicals $\mathfrak{N}$, which has several equivalent definitions (one of which being a special case of what we define above) **** more to say on this after finish reviewing quasicoherent sheaves.*

The last topic we would like to mention in this section is **effective Cartier divisor**s and how to do "**slicing**" using effective Cartier divisors. After that we briefly introduce regular embeddings which is useful in later dissussion of dimension:

We start by a definition:

Definition 1.35. A closed subscheme is called **locally principal** if on each open subset of a small enough open cover, it's cut out by a single equation, or in other words, by a principal ideal, hence the terminology. More precisely, a locally principal closed subscheme is a closed subschem $\pi : X \hookrightarrow Y$ such that there is an open cover $\{U_i\}$ of Y for which for each i, $\pi^{-1}(U_i) \hookrightarrow U_i$ is the closed subscheme $V(s_i)$ of U_i where $s_i \in \Gamma(U_i, \mathcal{O}_X)$.

Remark 1.36. Notice that in the definition above, the open cover may not be affine. We are only saying that the closed embedding restricted to $\pi : \pi^{-1}U \to U$ is defined by a global section on $\mathscr{O}_X(U)$.

Recall what we mean by this, if a closed embedding $X \hookrightarrow Y$ is said to be defined by a global section of Y, we actually mean that we use this global section to create a quasicoherent ideal sheaf by defining it on every affine open by the ideal generated by the restricted global section. Thus, this construction is naturally suited to restricted to affine opens, which means that if $X \hookrightarrow Y$ is a locally principal closed embedding, we can always choose this open cover where we discuss whether if principal or not to be an affine open cover.

We have already saw an example of this kind of constructions, where a hypersurface in $\mathbb{P}^n_k$ is locally principal since we know that on each the $n+1$ affine open covers, it's defined by a single equation.

If the ideal sheaf is locally generated near every point by a nonzero divisor, we will say that the closed subscheme is an **effective Cartier Divisor**. More precisely, if $\pi : X \hookrightarrow Y$ is the closed embedding we are interested, there is an affine open of Y, consisting of $SpecA_i$ and the ideal sheaf generating the closed subscheme in $SpecA_i$ is generated by a $t_i \in A_i$ which is not a zero divisor. In this case, we will say that X is an **effective Cartier divisor on** Y, later we will also use this phrase **slicing by an effective Cartier Divisor**, which basically means restrict to an effective Cartier Divisor, which is something we will make precise later.

Generally speaking, the notion of an effective Cartier is central by the notion of a locally principal closed subscheme is less so.

The following is an easy easy algebra exercise we will use later:

Proposition 1.37. Suppose that $t \in A$ is a nonzero divisor iff t is not an effective divisor in A_p for every prime $p \subset A$.

Proof. The direction where t is not a zero divisor implies that it's not a zero divisor in A_p is easy. Conversely, suppose that $ta = 0$, then we see that $\frac{t}{1}\frac{a}{1} = 0$ in A_p, therefore we know that $\frac{a}{1} = 0$ in any A_p, then consider the natural map:

$$(a) \hookrightarrow A$$

We localize at each prime p to get 0, thus the map is zero, thus a is 0. ∎

Remark 1.38. It's easy to think that if D is an effective Cartier of $SpecA$, then $D = V(t)$ for some $t \in A$ nonzerodivisor. However, this is not the case. If you look at the definition carefully enough, you will find that this definition of locally principal closed subschemes only said there is an affine cover where we can verify the condition, but this condition **may not** hold on an arbitrary affine open cover. Therefore, this condition is not an affine open condition in this obvious sense.

If we work with locally Noetherian schemes, since the notion of associated points are of good property in this setting, we have the following result of checking something is a effective Cartier divisor using associated points:

Proposition 1.39. Suppose that X is a locally Noetherian scheme if $t \in \Gamma(X, \mathscr{O}_X)$ is a function on X, show that t defines an effective Cartier divisor iff t doesn't vanish on any associated point of X. Then we can show that a locally principal closed subscheme of X is an effective Cartier divisor iff it doesn't contain any associated points of X.

Proof. Let's first suppose that $Y \hookrightarrow X$ where Y is defined by t is an effective Cartier divisor and $p \in X$ is an associated point, then by definition, for any affine open around p, say $SpecA$ where A is Noetherian, the prime corresponding to p in $SpecA$ is an associated prime of A, then p contains only zero divisors, thus if t vanishes at p, then $t|_{SpecA}$ is going to be a zero divisor. Then since $V(t)$ is assumed to be an effective Cartier divisor, then around p, there is a $SpecA$ such that it's defined by $t|_{SpecA}$ which is not a zero-divisor, a contradiction. Then converse is proved similarly.

Now, say $Y \hookrightarrow X$ is only a locally principal divisor, then if it's actually an effective Cartier divisor, then suppose that $p \in X$ is an associated point, and there is $p \in SpecA$ where Y is defined by $t \in \mathscr{O}_X(SpecA)$, which is not a zero divisor, then we only have to show that $V(t)$ in $SpecA$ doesn't contain any associated point of A, but this is because t is not a zero divisor. The converse is similarly proved. ☐

Proposition 1.40. Suppose that $V(t) = V(t') \hookrightarrow SpecA$ are both effective Cartier divisors with t, t' non-zerodivisors in A, then t and t' differ by a invertible element in A.

Proof. We notice that this implies that $(t) = (t')$ since the closed embedding are the same implies that the ideal defining them are the same. Since the ideal defining $V(t)$ and $V(t')$ are (t) and (t'), then we know that $t = t'a$ and $t' = bt$, therefore $t = abt$ since t is not a zerodivisor, then $ab = 1$, thus we are done. ☐

The idea of effective Cartier divisors naturally lead to the notion of regular embeddings, we first introduce some algebra notion which is due to Serre, called **regular sequence**:

The geometric idea of regular sequence comes from iterating out effective Cartier divisor construction: we locally take an effective Cartier divisor and then on this effective Cartier divisor, we take another effective Cartier divisor of this effective Cartier divisor, and we keep going for a finite number of times. of course, there are some technical issues you need to pay attention to, for example, is the finally result we obtain depend on the order we choose the equations we imposed and so on. We will make this clear later, but let's introduce the formal definition first:

Definition 1.41. Let M be an A-module, a sequence $x_1, \cdots, x_r \in A$ is called an M**-regular sequence** (or **a regular sequence for** M) if the following two conditions are satisfied:

1. For each i, x_i is not a zerodivisor for $M/(x_1, \cdots, x_{i-1})M$ and when $i = 1$, we should interpret this as x_1 is not a zerodivisor for M.
2. We have a proper inclusion $(x_1, \cdots, x_r)M \hookrightarrow M$, or in other words, $M/(x_1, \cdots, x_{i-1})$ is not zero for all $1 \leqslant i \leqslant r + 1$.

The case which is mostly relevant to us is the case where $M = A$ itself. We will say that r is the length of the regular sequence $x_1, \cdots, x_r \in A$ and a A-regular sequence is just called a **regular sequence** for simplicity. This definition is also related to something called complete intersection, which we will define after we finish what we want to say about regular embeddings:

Proposition 1.42. Let M be an A-module, then an M-regular sequence continues to satisfy the requirement 1 in Def 1.41 upon any localization.

Proof. Let's try to make this precise first, we we mean is that suppose that S is a multiplicative system and we localize at S to get $S^{-1}M$ is a $S^{-1}A$-module, then the images of $x_1, \cdots, x_r$ in $S^{-1}A$ also satisfies requirement 1 in Def 1.41.

To see this, let's first consider if $\frac{x_1}{1}$ is a zerodivisor for $S^{-1}M$, suppose that it is, then by our assumption, there will be a nonzero $\frac{m}{s}$ such that $\frac{x_1 m}{s} = 0$, then notice that this implies that $x_1 s' x = 0$, notice that since x_1 is not a zerodivisor in M, then $s'x = 0$, therefore $\frac{m}{s} = 0$, a contradiction.

Then consider $S^{-1}M/(\frac{x_1}{1}, \cdots, \frac{x_{i-1}}{1})S^{-1}M$, we want to show that $\frac{x_i}{1}$ is not a zerodivisor, again suppose that $\frac{x_i}{1} \cdot \frac{m}{s} \in (\frac{x_1}{1}, \cdots, \frac{x_{i-1}}{1})S^{-1}M$, then I will let you to show that why this is equivalent to the fact that there is an $s' \in S$ such that $x_i s' m \in (x_1, \cdots, x_{i-1})M$, then we know that $s'm \in (x_1, \cdots, x_{i-1})M$, which means that $\frac{m}{s} \in (\frac{x_1}{1}, \cdots, \frac{x_{i-1}}{1})S^{-1}M$. You will actually see later after we discuss the notion of flatness that requirement 1 is preserved by any flat base change. □

Proposition 1.43. Suppose that x, y is an M-regular sequence, then x^N, y is also an M-regular sequence for any $N \in \mathbb{Z}_{>0}$.

Proof. Notice that if x is not a zerodivisor, then x^N is not for any N since if it is, then $x^N m = 0$ for some m, then $x^{N-1}m = 0$, then you keep going, finally you get $xm = 0$, thus m is 0. Also, since $(x^N, y) \subset (x, y)$, then the requirement 2 of Def 1.41 is satisfied. The only thing we need to check now is y is not a zerodivisor in $M/(x^N)M$. This is done by induction on N, suppose that $ym \in (x^N)M$, then we know that $ym = x^N k$ for some $k \in M$, then notice that $ym \in (x)M$, then $m = xk'$, therefore we have $yxk' = x^N k$, then since x is not a zero divisor, we know that $yk' = x^{N-1}k$, thus by our induction hypothesis, we know that $k' \in (x^{N-1})M$, then since $m = xk'$, then $m \in (x^N)M$, we are done. □

Remark 1.44. Notice that the above argument can be easily modified to show that if $x_1, \cdots, x_r$ is an M-regular sequence, then $x_1^{a_1}, \cdots, x_r^{a_r}$ is also a M-regular sequence. This is based on doing induction on the sum of a_i's, which starts from r.

As we mentioned earlier, the order of the sequence matters since you may encounter special cases where if you switch the order $x_1, \cdots, x_r$, it fails to be regular anymore, just as the following exercise, which also motivates an explanation of when the ordering problem may disappear:

Example 1.45. Let $A = k[x, y, z]/(x-1)z$, geometrically thought of as the union of the $z = 0$ plane and the $x = 1$ plane. We claim that the sequence $x, (x-1)y$ is a regular sequence while $(x-1)y, x$ is not a regular sequence.

This is understood most efficiently via geometric interpretation. Let $X = SpecA$, which as mentioned above is the union of the $z = 0$ plane and the $x = 1$ plane. Let's first see what does it mean for some thing to be a nonzerodivisor geometrically in our case, notice that this ring $A = k[x, y, z]/(x-1)z$ is reduced, then there is no embedded associated primes, and we can prove that the minimal primes are just $(x-1)$ and z, then we know that an element is not a zero divisor iff it's not in these two primes, and you can think of these as it completely vanishes on the $x = 1$ or $z = 0$ plane. Therefore x is not, since you can either algebraically think of it as it's not in any of the two minimal primes or you can geometrically think of it as missing $x = 1$ and not completely vanish on $z = 0$ plane. Then you consider $A/(x) \cong k[x, y, z]/(x, z)$, we see that it's still reduced and we know that an element is an associated prime iff it's in (x, z), thus we know that $(x-1)y$ is not for the same reason.

However, if you use the order $(x-1)y, x$, we are in trouble since $(x-1)y$ completely vanishes on $x = 1$ plane.

So, if you ask why this ordering problem happens, the reason is simple if you think of it as slicing by effective Cartier divisors. In the first order $x, (x-1)z$, we first slice by $x = 0$, which completely wipes out $x = 1$ components and then if you slice by $(x-1)y$, there is nothing wrong happens. However, if you first use $(x-1)y$, without using $x = 0$ to clear the $x-1$ component, the problem appears. Therefore, in some sense, this is a problem of we are noting being local, since you you are only doing the operation around $x = y = z = 0$, such problems won't happen, which indicates that we might have some sort of good property of the order if we work in a local ring.

Theorem 1.46. Suppose that (A, m) is a Noetherian local ring, and M is a finitely generated A-module, then any M-regular sequence $(x_1, \cdots, x_r)$ remains regular upon any reordering.

Before we treat the general case, let's first prove the first nontrivial case which is $r = 2$ and we want to show that switching the order of each other doesn't change regularity, which happens to be a baby case of the Koszul Complex:

Since we are working in a local ring, if (x, y) is a M-regular sequence, then x, y should both belong to the maximal ideal, otherwise they will be invertible elements, thus won't be regular. Then this just means that x is a nonzero divisor of M and y is a nonzero divisor of M/xM. This is a good exercise to get a small practice problem of the use of **spectral sequences**:

Let's consider the following small double complex:

$$\begin{tikzcd} M \arrow[r, "\times(-x)"] & M \\ M \arrow[u, "\times y"] \arrow[r, "\times x"] & M \arrow[u, "\times y"'] \end{tikzcd}$$

The bottom left one is considered to be in position $(0, 0)$. Now we compute the cohomology of the total complex using a very simple **spectral sequence**: **** later after we introduce the notion of spectral sequences, pages and so on, then you will find this is really easy*

Now we can prove Thm 1.46:

Proof. **** later* □

With the above discussion in hand, we can now define what is a regular embedding:

Definition 1.47. Suppose that $\pi : X \hookrightarrow Y$ is a locally closed embedding, we say that π is a **regular embedding of codimension** r **at a point** $p \in X$ if in the local ring $\mathscr{O}_{Y,p}$ the ideal of X is generated by a regular sequence of length r. We say that π is **a regular embedding of codimension** r if it's a regular embedding of codimension r at each $p \in X$.

Remark 1.48. Let's first make a comment here about the definition of regular embeddings of codimension r. Notice that we start by a locally closed embedding $\pi : X \hookrightarrow Y$ which means that we have a decomposition:

$$X \xhookrightarrow{closed} U \xhookrightarrow{open} Y$$

Then if we treat X as a closed embedding of U, then it has a uniquely defined ideal sheaf of $\mathscr{O}_U$, say $\mathscr{I}$. Then suppose that $p \in X$, we know that $\mathscr{O}_{Y,p} = \mathscr{O}_{X,p}$ since the stalk can be computed at any open around that point. Then it makes sense to talk about the ideal of X in $\mathscr{O}_{Y,p}$, it's just take the stalk at p of the injective morphism:

$$\mathscr{I} \hookrightarrow \mathscr{O}_U$$

Remark 1.49. Another remark here is about the codimension r, at this point, you might still be confused by why this codimension appears in the definition, but it will be clear after you see the generalized Krull's Theorem later.

One important property of a regular embeddings is that the condition for it being regular is open:

Proposition 1.50. Suppose that $\pi : X \hookrightarrow Y$ is a locally closed embedding of locally Noetherian schemes which is regular at $p \in X$, then is a regular embedding at some open neighborhood of p in X.

Proof. Notice that we can just assume that π is a closed embedding instead of a locally closed embedding since if whether it's regular at a point only depends on U if $\pi : X \hookrightarrow U \subset Y$ is a closed embedding.

Then notice that we can also assume that Y is affine since $\mathscr{I}_p \to \mathscr{O}_{U,p}$ can be determined at any affine open of p. Thus we are in the case of:

$$SpecA/I \hookrightarrow SpecA$$

where A is Noetherian, and we assume that $I_p \hookrightarrow A_p$ is defined by a regular sequence of length r. Then notice that we can assume that the r elements in I_p are images of $f_1, \cdots, f_r \in I$ (notice that we can certainly assume thet generate the ideal, the fact that after we clear the denominators they are still a regular sequence follows form the regular sequence is insensitive to multiplication by invertible elements), what we are trying to prove is that there is an open neighborhood V of p such that for every $q \in V$, the images of $f_1, \cdots, f_r$ generate I_q and they form a regular sequence.

We need the following algebra fact: suppose that I, J are ideals of a Noetherian ring A and p is prime in A such that $I_p = J_p$, then there is an $a \in A - p$ such that $I_a = J_a$ in A_a. To show this, we first claim that we can reduce the problem to the case where $I \subset J$. To see this, notice that we have a canonical isomorphism:

$$S^{-1}(I \cap J) \cong S^{-1}I \cap S^{-1}J$$

This isomorphism can be interpreted as an equality in $S^{-1}A$ since you can see that this map maps an element to itself is we think of everything in $S^{-1}A$. Then, notice that we can consider $I \cap J$, since localization commutes with finite intersections, then $(I \cap J)_p = I_p \cap J_p = I_p$, then if we can show that for some $a \in A - p$ such $(I \cap J)_a = I_a$, we know that $I_a \cap J_a = I_a$, then J_a contains I_a. Conversely, we also have $I_p \cap J_p = J_p$, then if we can show that for some $b \in A - p$ such that $(I \cap J)_b = J_b$, we know that I_b contains J_b. Now, notice that localization is exact, we know that $(J_a)_{\frac{b}{1}} \subset (I_a)_{\frac{b}{1}}$ and $(J_b)_{\frac{a}{1}} \subset (I_b)_{\frac{a}{1}}$. Finally, we have canonical isomorphisms for each A-modules M, N, which is a submodule K:

$$\begin{tikzcd}
N_a \arrow[r, "loc"] & (N_a)_{\frac{b}{1}} \arrow[r, "iso"] & N_{ab} \\
M_a \arrow[u, hook] \arrow[r, "loc"'] & (M_a)_{\frac{b}{1}} \arrow[u, hook] \arrow[r, "iso"'] & M_{ab} \arrow[u, hook]
\end{tikzcd}$$

where the morphism in the first row is the restriction of the identity $K_a \to K_a$, then you can see that the last one should be the restriction of the identity $K_{ab} \to K_{ab}$, then you can do the same stuff for a, b replace, you will get the restriction of the identity $K_{ab} \to K_{ab}$ to $N_{ab} \hookrightarrow M_{ab}$, thus $N_{ab} = M_{ab}$ as submodule of K_{ab}, we are done, then you can apply this to the case of ideals.

To prove this for $I \subset J$, we consider $K = J/I$ which is a finitely generated A-module since A is Noetherian, then we want to prove that if $K_p = 0$ then there is some $a \in A - p$ such that $K_a = 0$. Notice that suppose that K is generated by $f_1, \cdots, f_r$, then $K_a = 0$ iff there are a_i's such that $f_i a^{n_i} = 0$ for all i. But notice that $K_p = 0$, then notice that there are some $a_1 \in A - p$ such that $f_i a_i^{n_i} = 0$, then take a to be the product of all a, we are done. To make what we are doing more clear about what we have in the beginning and what we are trying to prove, consider the following SES:

$$0 \longrightarrow I \longrightarrow J \longrightarrow J/I \longrightarrow 0$$

then we can localize at p, which is:

$$0 \longrightarrow I_p \longrightarrow J_p \longrightarrow (J/I)_p \longrightarrow 0$$

notice that this $I_p \hookrightarrow J_p$ is the inclusion in A_p, then we assumed that $(J/I)_p = 0$. If we can prove from $(J/I)_p = 0$ that there is some $a \in A - p$ such that $(I/J)_a$ is 0, then if you consider the following SES:

$$0 \longrightarrow I_a \longrightarrow J_a \longrightarrow (J/I)_a \longrightarrow 0$$

you will get what we want.

Now back to our original question, we are trying to prove regularity is a open condition. Notice that if we consider the ideal generated by $(f_1, \cdots, f_r)$ and I, since they agree after localizing at p, then by our discussion above, there is an $a \in A - p$ such that $(f_1, \cdots, f_n)_a = I_a$, then notice that for all $q \in D(a)$ which means that $a \notin q$, we have localizing at a then at the image of $A - q$ is canonically isomorphic to directly localizing at q(similar to the diagram above we drew). Also, we need to show that $f_1, \cdots, f_n$'s image in A_q is a regular sequence.

This is a little bit tricky, notice that the image of $f_1, \cdots, f_r$ is a regular sequence at A_q, iff the multiplication map:

$$A_q/(f_1, \cdots, f_{i-1})_q \to A_q/(f_1, \cdots, f_{i-1})_q$$

by f_i is an injective map, or in other words, the kernel is 0. But what is the kernel, we can consider the following diagram:

$$0 \longrightarrow Ker \longrightarrow A/(f_1, \cdots, f_{i-1}) \xrightarrow{\times f_i} A/(f_1, \cdots, f_{i-1}) \longrightarrow Coker \longrightarrow 0$$

notice that if you localize this sequence at q, you get:

$$0 \longrightarrow Ker_q \longrightarrow A_q/(f_1, \cdots, f_{i-1})_q \xrightarrow{\times \frac{f_i}{1}} A_q/(f_1, \cdots, f_{i-1})_q \longrightarrow Coker_q \longrightarrow 0$$

moreover, the sequence is still going to be exact, thus Ker_q is the kernel we are looking for. Now notice that when you localize at p, $ker_p = 0$ for any $(f_1, \cdots, f_{i-1})$. Also, if $M_p = 0$ for a finitely generated module, then there is an $a \notin p$ such that for all $q \in D(a)$, $M_q = 0$, then since we are working with the Noetherian case, everything if finitely generated, then we can apply it to find $D(a), D(a_i)$ for $1 \leqslant i \leqslant r+1$ such that for every q in the intersection $(f_1, \cdots, f_r)_q = I_q$ and $(\frac{f_1}{1}, \cdots, \frac{f_r}{1})$ is a regular sequence in A_q, since each of them contains p, the intersection is still a nonempty open, we are done. □

Corollary 1.51. Suppose we are still in the above setup and X is now a locally finite type scheme over k, then to check $X \hookrightarrow Y$ is a regular embedding, it suffices to check it for closed points.

Proof. We know that for schemes which are locally of finite type over a field, the closed points are dense, thus in particular, it at least have one closed point. Then, since being a regular embedding of codimension r is an open property, if it holds at all p closed, then notice that there are going to be open neighborhoods around each closed point such that the embedding is regular of codimension r. Notice that the union of all these opens is the whole space, otherwise the complement is closed, and if you take the reduced scheme structure on it, it's also locally finite type over k, thus in particular has closed points, then a contradiction. □

The last result of regular embeddings of codimension r we want to prove now is that:

Proposition 1.52. Suppose that $\pi : X \hookrightarrow Y$ is a closed embedding of locally Noetherian schemes. Then it's a regular embedding of codimension 1 iff it's an effective Cartier divisor.

Proof. First let's assume that it's a regular embedding of codimension 1, we want to show that the ideal of X is locally generated by a nonzero divisor. We can assume that Y is affine, thus we are in the case where $SpecA/I \hookrightarrow SpecA$ for a Noetherian ring A.

Notice that we know that I_p for any $p \in SpecA/I$ is generated by a single non-zerodivisor in A_p, we want to show that there is an affine open such that I is generated by one element. But notice that we already proved that if we assume that I_p is generated by the image of f in A, then there is an $a \in A - p$ such that $I_a = (f)_a$. Notice that in the affine open $D(a)$, the ideal of X is just I_a, we are done.

Then suppose that $X \hookrightarrow Y$ is an effective Cartier divisor, then we automatically have the ideal of X is locally generated by a nonzero divisor, then you use the fact that being a nonzero divisor is preserved under localization. □

We end our review of closed embeddings right now by a definition we will use later:

Definition 1.53. A codimension r complete intersection in a scheme Y is a closed subscheme X that can be written as the scheme theoretic intersection of r effective Cartier divisors $D_1, \cdots, D_r$ such that for every point $p \in X$ the equations corresponding to $D_1, \cdots, D_r$ for a regular sequence in the local ring $\mathscr{O}_{Y,p}$. If we just say complete intersection, that means that codimension r complete intersection for some r.

2 More Review on (quasi)Coherent Sheaves

We already gave an earlier review on general algebraic properties of (quasi)coherent sheaves. In this section we do a geometrical review on how these notions generalize vector bundles:

we begin with a definition:

Definition 2.1. A free sheaf on a ringed space X is an $\mathscr{O}_X$-module isomorphic to $\mathscr{O}_X^{\oplus I}$ for some index set I. The rank of the free module is the cardinality of I. A locally free sheaf on X is an $\mathscr{O}_X$-module that is locally isomorphic to a free sheaf.

Remark 2.2. Notice that quasichoerent sheaves form an abelian category containing those locally free sheaves and quasicoherent sheaves generalizes locally free sheaves just like general modules generalize free modules. As what we have mentioned, if we use the right definition, coherent sheaves also form an abelian category that contains that finite rank locally free sheaves.

The first result we can do now is that, we will show later that vector bundles are equivalent to the notion of locally free sheaves. You should know from differential topology that a vector bundle over a manifold M gives you the sheaf of sections, which is locally free. Also any locally free sheaf $\mathscr{F}$ on M will give you a vector bundle by gluing a bunch of stuff together. It's left to you to think about why these are inverse operations, which implies that vector bundles and locally free sheaf on a manifold M are equivalent notions.

Now, let's talk about locally free sheaf of rank n on a scheme more generally. By definition, this is an $\mathscr{O}_X$-module over a scheme X, together with an open cover $\{U_i\}$ of X such that on each U_i, we have $\mathscr{F}|_{U_i} \cong \mathscr{O}_{U_i}^{\oplus n}$. Notice that these determine the transitions functions which are $T_{ij} \in GL_n(\mathscr{O}(U_i \cap U_j))$ which satisfies the cocycle conditions, this is because that an isomorphism from $\mathscr{O}_{U_i \cap U_j}^{\oplus n} \to \mathscr{O}_{U_i \cap U_j}^{\oplus n}$ is determined by the map on global sections.

Also, given these data, an open cover $\{U_i\}$ and transition functions, we get a unique up to unique isomorphism locally free sheaf, whose sections on any open U is determined by sections on $U \cap U_i$ that agrees on over laps. More formally, they are $s_i \in \Gamma(U \cap U_i, \mathscr{O}_X)^{\oplus n}$'s such that:

$$T_{ij} s_i = s_j$$

on $U \cap U_i \cap U_j$.

In conclusion, you should think of locally free sheaf of rank n as vector bundles, even though they are not the same, just equivalent notions. Also, sometimes it's also useful to let the rank vary on different connected components, or to let the rank to be infinite. If it's not locally free, just free, you should think of this to be a trivial vector bundle.

Now, we will present our first result about pullback vector bundles, before that, let's review the general pullback functor:

Suppose that $\pi : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of ringed spaces:

Proposition 2.3. $(X, \pi^{-1}\mathscr{O}_Y)$ is a ringed space and the morphism of ringed space π factors into: $(X, \mathscr{O}_X) \to (X, \pi^{-1}\mathscr{O}_Y) \to (Y, \mathscr{O}_Y)$.

Proof. The first assertion is equivalent to $\pi^{-1}\mathscr{O}_Y$ is a sheaf of rings. To show this, we know that before we sheafify, the presheaf is defined to be on each open, it's a colimit of rings, which is still a ring, as you can easily check, then if we have a presheaf of rings, the sheafified version is obviously a sheaf of rings with obvious multiplications since this is just defined to be locally constant functions on stalks. The canonical morphism from the presheaf to the sheaf is a ring homomorphism on each open.

The second assertion regrading the factorization requires us to use the explicit description of the adjunction, by compatible diagrams. First, we need to specify what are the two morphisms in this factorization:

$$(X, \mathscr{O}_X) \to (X, \pi^{-1}\mathscr{O}_Y) \to (Y, \mathscr{O}_Y)$$

We claim that the the first morphism is the one defined by the adjointness of (π^{-1}, π_*) and $\mathscr{O}_Y \to \pi_*\mathscr{O}_X$, the second is (π, Φ), where we need to specify what is Φ as a morphism $\Phi : \mathscr{O}_Y \to \pi_*\pi^{-1}\mathscr{O}_Y$, it's the morphism defined by the identity $\pi^{-1}\mathscr{O}_Y \to \pi^{-1}\mathscr{O}_Y$ via the adjointness (of course you should think of this as compatible diagrams). Then, it's left to you to check that this is indeed the moprhism $\mathscr{O}_Y \to \pi_*\mathscr{O}_X$ we started with. (Hint, the first morphism Φ is defined by diagrams that goes from $\mathscr{O}_Y$ to $\pi^{-1}\mathscr{O}_Y$, but how can you define this, you will find that you have no choice the restriction diagram, the second diagram is easy to find). □

Proposition 2.4. Suppose that $\mathscr{G}$ is an $\mathscr{O}_Y$-module, then $\pi^{-1}\mathscr{G}$ is an $\pi^{-1}\mathscr{O}_Y$-module on X.

Proof. This use another easy property, we say that if we have a presheaf of rings $\mathscr{R}^{pre}$ on a topological X and another presehaf of abelian groups $\mathscr{F}^{pre}$ on X such it has a module structure over each open such that it's compatible with restrictions. Then we take sheafification, the shefified version $\mathscr{G}$ is naturally a $\mathscr{R}$-module. The definition of the action on the sheafified version should be obvious if you view it as locally constant functions on stalks.

Then here if we want to show that $\pi^{-1}\mathscr{G}$ is a $\pi^{-1}\mathscr{G}$-module. We only have to consider the stuff before we do the sheafification, which are two filtered colimits, indexed by the same set, thus we only have to prove that if we have a filtered set, which indexes families of rings $\{R_i\}$ and families of R_i-modules $\{M_i\}$, then $colim_{i \in I} R_i$ is a $colim_{i \in I} R_i$-module, which is easy to prove since we have an explicity description of the colimits. □

Notice that, in general, if we have two sheaves of rings on X, denoted by $\mathscr{R}$ and $\mathscr{S}$, with a morphism of sheaves of rings: $\mathscr{R} \to \mathscr{S}$, suppose that $\mathscr{G}$ is a sheaf of $\mathscr{R}$-module, then we can consider the tensor product $\mathscr{G} \otimes_{\mathscr{R}} \mathscr{S}$ where we view $\mathscr{S}$ also as a module over $\mathscr{R}$. Then we can prove that $\mathscr{G} \otimes_{\mathscr{R}} \mathscr{S}$ is also an $\mathscr{S}$-module. To prove this, we still only have to prove this for presheaves, if the presheaf has a structure of $\mathscr{S}$-action such that it's compatible with restrictions, then the sheafified version also has one. But the presehaf is just the extension of scalars, which can be checked easily to be compatible with the restriction. This is the following definition:

Definition 2.5. We define $\pi^*\mathscr{G} =:$

$$\pi^{-1}\mathscr{G} \otimes_{\pi^{-1}\mathscr{G}} \mathscr{O}_X$$

where the morphism of sheaves of rings:

$$\pi^{-1}\mathscr{G} \to \mathscr{O}_X$$

is the morphism given by adjointness and $\pi : X \to Y$.

Proposition 2.6. This π^* is a covariant functor from $\mathscr{O}_Y$-modules to $\mathscr{O}_X$-modules.

Proof. We will actually prove that π^{-1} is a covariant functor from $\mathscr{O}_Y$-modules to $\pi^{-1}\mathscr{O}_Y$-modules and $\otimes_{\pi^{-1}\mathscr{O}_Y} \mathscr{O}_X$ is a covaraint functor form $\pi^{-1}\mathscr{O}_Y$-modules to $\mathscr{O}_X$-modules, then the composition give π^* is a covariant functor.

Suppose that $f : \mathscr{F} \to \mathscr{G}$ is a morphism of $\mathscr{O}_Y$-modules, we need to figure out what is $\pi^* f$. First, we know that π^{-1} is a covariant functor from $\mathscr{O}_Y$-modules to $\pi^{-1}\mathscr{O}_Y$, we have $\pi^{-1}f : \pi^{-1}\mathscr{F} \to \pi^{-1}\mathscr{G}$, which is a morphism of $\pi^{-1}\mathscr{O}_Y$-modules, defined by the sheafification of the morphism of the presheaf defined on colimits (you should make sure you know what is the map on colimits look like!).

Then, why it's a $\pi^{-1}\mathscr{O}_Y$-module? This is still the interplay of presheaves and it's sheafified version. It's easy to show that if the morphism before you sheafify is a morphism that is compatible with the actions and restriction, then after you sheafify, it should also be, as long as you remember you sheafification is locally constant functions on stalks!

Then we consider the tensor product, the morphism is still defined to be the sheafification of the morphism of presheaves, the difference from the above π^{-1} is we don't have to sheafify the sheaf of rings. $\square$

Now we check how will this pullback behave under compositions:

Proposition 2.7. Suppose that we have $\rho : (W, \mathscr{O}_W) \to (X, \mathscr{O}_X)$, another morphism of ringed spaces, then we have $\rho^*\pi^* = (\pi \circ \rho)^*$. (Here we actually mean that these two functors are naturally isomorphism, not exactly the same.)

Proof. The key to this question is that we can prove inverse image functor commute with tensor product in a natural sense. As you can guess, this property holds since sheafification always plays well with colimits.

We want to show that $\pi^{-1}(\mathscr{F} \otimes \mathscr{G}) \cong \pi^{-1}\mathscr{F} \otimes \pi^{-1}\mathscr{G}$ by a natural isomorphism. Notice that they are both sheafification of certain presheaves, we would like to show everything for presheaves instead of their sheafification which is a lot easier. Following this intention, we prove a few preliminary results, which is also of value intellectually.

First, we want to prove $\pi^{-1}(\mathscr{G}^{\#}) \cong (colim(\mathscr{G}))^{\#}$ where $\mathscr{G}$ is a presheaf. This follows that we can easily define a morphism from the right hand side to the left hand side, and show that this is an isomorphism. What is the morphism, you need to specify a morphism form the colimit presheaf to the colimit of $\mathscr{G}^{\#}$, which is compatible with restriction, which is obvious. Then we only have to show that this is an isomorphism on each stalks, this is left to you, which should not be hard as long as you are familiar with the stalk identification. This isomorphsim is compatible with the $\pi^{-1}\mathscr{O}_X$ action, but we omit it since it's just to tedious.

Similarly, we also have $(\mathscr{G} \otimes \mathscr{F})^{\#} \cong \mathscr{G}^{\#} \otimes \mathscr{F}^{\#}$, which can also be proved by using universal properties to give morphisms and checking that this is isomorphism on stalks.

Then back to our question, we want to show that:

$$\pi^{-1}(\mathscr{F} \otimes \mathscr{G}) \cong \pi^{-1}\mathscr{F} \otimes \pi^{-1}\mathscr{G}$$

First, we know that the left is isomorphic to the associated sheaf of the comit of the tensor presheaf by the first assertion. Then the RHS is the associated sheaf the tensor of the colimit over the colimit. Therefore, we only have to show that there is a natural isomorphism of:

$$\varinjlim M_i \otimes_{\varinjlim A_i} \varinjlim N_i = \varinjlim (M_i \otimes_{A_i} N_i)$$

over a filtered system such that it's compatible with restrictions, it's easy to define maps between both sides, form the left side to the right side, by universal property of colimits. To define the map from left to right, we need to keep in mind we have maps $A_i \to \varinjlim A_i$, thus $\varinjlim A_i$-modules are also A_i-modules for any A_i, then you can check that they are inverses of each other.

Then the property we want to prove in the first place is easy, but you need to make sure you understand why it's obvious! □

Proposition 2.8. Like the adjointness between (π^{-1}, π_*), we have an adjoint pair $(\pi^*, \pi*)$. You should make sure you understand between whatcategories these functors are going back and forth, and you should describe a natural isomorphism between:

$$Hom_{(X,\mathscr{O}_X)}(\pi^*\mathscr{G}, \mathscr{F}) \to Hom_{(Y,\mathscr{O}_Y)}(\mathscr{G}, \pi_*\mathscr{F})$$

such that it's functorial in both $\mathscr{G}$ and $\mathscr{F}$.

Proof. **** later* □

Now, we discuss a notion that is new to us, called the push-pull map, we do it first for inverse image functor and then for the pullback functor:

Suppose that we have the following commutative diagram, not necessarily Cartesian:

$$\begin{tikzcd} W \arrow[r, "\beta'"] \arrow[d, "\alpha'"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & Z \end{tikzcd}$$

Suppose that we have a sheaf $\mathscr{F}$ on X, then we can define the push-pull map:

$$\beta^{-1}\alpha_*\mathscr{F} \to \alpha'_*(\beta')^{-1}\mathscr{F}$$

as follows: We start with the identity map $(\beta')^{-1}\mathscr{F} \to (\beta')^{-1}\mathscr{F}$ on W, then by the adjointness, we know this gives $\mathscr{F} \to \beta'_*(\beta')^{-1}\mathscr{F}$ on X, then we apply α_* to both side to get $\alpha_*\mathscr{F} \to \alpha_*\beta'_*(\beta')^{-1}\mathscr{F}$. By the commutativity of the diagram, we have $\alpha_*\beta'_* = \beta_*\alpha'_*$, thus we get a morphism: $\alpha_*\mathscr{F} \to \beta_*\alpha'_*(\beta')^{-1}\mathscr{F}$, use adjointness again we get a morphism: $(\beta)^{-1}\alpha_*\mathscr{F} \to \alpha'_*(\beta')^{-1}\mathscr{F}$. You can see that it's called a push-pull map since we start from a sheaf on X, we pushforward or pullback to get morphism on sheaves on Y. Notice that this construction is functorial in $\mathscr{F}$, in the sense that if we have a morphism of sheaves on X, $\mathscr{F} \to \mathscr{G}$, we apply $(\beta')^{-1}$ to get a commutative diagram, then the adjointness is also functorial, we keep going, geting a commutative diagram of sheaves on Y in the end.

Here we make a small detour about support of a sheaf since I don't know where to put it eleswhere.

Definition 2.9. Suppose that $\mathscr{F}$ is a sheaf of abelian groups on X and s is a global section of $\mathscr{F}$, we define **the support of the section** s, to be $Supp(s)$, which is the set of points of X where the germ of s is not zero, the complement where s is the 0 section.

An easy result of support of a section is that it's closed since you can see that the complement is open very easily.

Definition 2.10. In the above setup, we define the **support of a sheaf** of abelian groups to be: the locus where the stalk is nontrivial, i.e. not the zero group.

Clearly, this is a stalk local notion and thus you can pass to open subsets and get the restriction of support. It's also the union of supports of sections over all open subsets.

The most important exercise why we want to introduce this notion is because it plays well with inverse image functors and closed subsets:

Proposition 2.11. Suppose that $Z \subset Y$ is a closed subset and $i : Z \hookrightarrow Y$ is the inclusion. If $\mathscr{F}$ is a sheaf of abelian groups on Z, then the stalk $(i_*\mathscr{F})_q$ is the trivial group if $q \notin Z$ and $\mathscr{F}_q$ if $q \in Z$.

Proof. Notice that when q is not in Z, since Z is closed, then the complement of Z is open, which has empty intersection with Z, this explains why the stalk is the trivial group.

Now if $q \in Z$, then we see that the stalk of $i_*\mathscr{F}$ at q is the colimit over all opens containing q, then since the intersection with Z is also open in Z, this colimit is the same as taken over all the opens in Z that contains q. □

Proposition 2.12. Suppose that $Supp\mathscr{G} \subset Z$ where Z is closed, then we have a natural isomorphism $\mathscr{G} \to i_*i^{-1}\mathscr{G}$. This tells that for those sheaves whose support is contained in a closed subset (and of course in particular if it's support is closed itself), it can be viewed as $i^{-1}\mathscr{G}$.

Proof. First, what is the morphism $\mathscr{G} \to i_*i^{-1}\mathscr{G}$, it's the one coming from the identity of $i^{-1}\mathscr{G} \to i^{-1}\mathscr{G}$ and adjointness.

To show it's an isomorphism, we only have to show that it's an isomorphism on each stalk. First, for q outside Z, the induced map is zero since both stalks are zero. Then, for points in Z, but not in support of $\mathscr{G}$, the map on stalks is still an isomorphism since we know that $i_*i^{-1}\mathscr{G}$ has stalk at q the stalk of $i^{-1}\mathscr{G}$ at p, which is the stalk of $\mathscr{G}$ at p, which is 0.

Therefore, only those points in the support of $\mathscr{G}$ matters. But notice that first since the support is in Z, we are still thinking about points in Z, thus we can use the stalk identification. We are looking at what kind of maps is:

$$\mathscr{G}(U) \to i^{-1}\mathscr{G}(U \cap Z)$$

and what is the induced map on stalks. Recall our compatible diagram characterization of the adjointness, we first figure what is the diagram that characterize the identity $i^{-1}\mathscr{G} \to i^{-1}\mathscr{G}$, it's a diagram from $\mathscr{G} \to i^{-1}\mathscr{G}$, and you should also be familiar with how to get a morphism $i^{-1}\mathscr{G} \to i^{-1}\mathscr{G}$ from the diagram. Finally, notice that the identity map $i^{-1}\mathscr{G} \to i^{-1}\mathscr{G}$ is induced by the identity map in the colimits, then the rest of the details are left to you.

In conclusion, you see that the map:

$$\mathscr{G}(U) \to i^{-1}\mathscr{G}(U \cap Z)$$

is induced by $\mathscr{G}(U) \to \varinjlim_{V \supset U \cap Z} \mathscr{G}(V)$ which is again induced by the identity $\mathscr{G}(U) \to \mathscr{G}(U)$. Therefore, when $q \in Supp\mathscr{G}$, you should also be familiar with the map we used to identify the stalk, which is a double colimit to a single colimit, then you will see that this is just the map that induces the identity.

This is another example that we always only care about the presheaf before we do sheafification, especially when we only care about the stalk. □

Now back to our question of push-pull map of the pullback functor instead of the inverse image functor: the setup is very similar as the push-pull map of the inverse image functor, we still have a not necessarily Cartesian commutative diagram:

$$\begin{tikzcd} W \arrow[r, "\beta'"] \arrow[d, "\alpha'"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & Z \end{tikzcd}$$

since the adjointness of the inverse image functor and the pushforward has a exact same version for the pullback and pushforward functor, we can use the same method to define the push-full map.

Now we begin to state and prove a few essential results for pullback of locally free sheaves of rank n:

Proposition 2.13. Suppose that $\pi : X \to Y$ is a morphism of locally ringed spaces, then $\pi^*(\mathscr{O}_Y^{\oplus n}) \cong \mathscr{O}_X^{\oplus n}$ canonically.

Proof. This result boils down to a few easier canonical isomorphisms, the first is the canonical isomorphism showing inverse image functor commutes with finite direct sums, which is the result that sheafifications commutes with direct sums, which is easily proved since they satisfy the same universal property. Now, the only stuff left is to prove that colimits commute with direct sums, but notice that in general colimit commutes with colimit. Finally, we have to prove that tensor commute with finite sums, which are also easy since we can still do it on presheaves. □

Proposition 2.14. Suppose that $\mathscr{G}$ is a locally free sheaf of rank n, then so is $\pi^*\mathscr{G}$.

Proof. Choose the trivializing neighborhood of $\mathscr{G}$, suppose that this is an cover U_i, we claim that the trivializing neighborhoods of $\pi^*\mathscr{G}$ are $\pi^{-1}U_i$. Again, this result is a result of several smaller results. First, we prove that tensor commutes with restriction, you should figure out what this means! This comes from the fact that sheafification commutes with restriction, then we only have to prove that the presheaves are isomorphic.

Another result we need is the interplay between restrictions and inverse image functors. We need to phrase this carefully, suppose that $\pi : X \to Y$ and $\mathscr{G}$ is a map on Y, then suppose that $U \subset Y$ open, then we claim that $(\pi^{-1}\mathscr{G})|_{\pi^{-1}U} \cong \pi^{-1}(\mathscr{G}|_{U_i})$ where we view π as a map from $\pi^{-1}U \to U$. We again use the fact that sheafification commutes with restriction, thus we only have to show the presheaves are isomorphic, which should be easy.

Finally, it's left to you how to use the above two results to get the desired result. □

Remark 2.15. The above proposition only deals with trivializing neighborhoods, but we can even be precise on the transition functions after we pullback. Suppose that T_{ij} are the original transition function, we want to find new functions denoted by π^*T_{ij}, which are defined on $\pi^{-1}U_i \cap \pi^{-1}U_j$ as functions in $GL_n(\mathscr{O}_X(\pi^{-1}U_i \cap \pi^{-1}U_j))$ such that they are the new transition functions.

Well, we are using π^*T_{ij} to indicate that these are the functions after we apply π^* to T_{ij}, and indeed we can view T_{ij}'s as invertible morphisms $\mathscr{O}^{\oplus n}_{U_i\cap U_j}$ to itself, i.e. automorphisms of $\mathscr{O}^{\oplus n}_{U_i\cap U_j}$, then notice that as a functor π^* takes invertible morphisms to invertible morphisms, thus we get invertible morphisms from $\pi^*(\mathscr{O}^{\oplus n}_{U_i\cap U_j})$ to itself, which is an automorphism of $\mathscr{O}^{\oplus n}_{\pi^{-1}U_i\cap\pi^{-1}U_j}$.

We also should show that these π^*T_{ij}'s satisfy the cocycle condition so that they are indeed transition functions. This is first because that you can see that π^* is a functor that respects restrictions which should be easy to show. Then we know that it will satisfy the cocycle condition before we do the identification: $\pi^*(\mathscr{O}^{\oplus n}_{U_i\cap U_j}) \cong \mathscr{O}^{\oplus n}_{\pi^{-1}U_i\cap\pi^{-1}U_j}$, thus we know that automorphisms on $\pi^*(\mathscr{O}^{\oplus n}_{U_i\cap U_j\cap U_k})$, they satisfy the cocycle condition. Then, it's left to you why this satisfies the cocycle condition even after we do the identification.

Next, we consider a few useful constructions on locally free sheaves:

Proposition 2.16. Suppose that $\mathscr{F}$ and $\mathscr{G}$ are locally free sheaves on X of rank m, n, then $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ are locally free of rank mn.

Proof. First, we can assume that there is an open cover of X on which both $\mathscr{F}$ and $\mathscr{G}$ are free. $\mathscr{H}om$ commutes with restrictions since a morphism of $\mathscr{O}_X$-modules restricted to an open subset U is a morphism of $\mathscr{O}_U$-modules. Then, we know that this is just a local notion, and we can assume that both $\mathscr{F}$ and $\mathscr{G}$ are free. Thus, this becomes to prove $\mathscr{H}om$ commutes with finite direct sums, in both slots. But this is just the universal property of direct sums. □

Definition 2.17. Suppose that $\mathscr{E}$ is a locally free sheaf of rank n on X, then the above result shows that $\mathscr{E}^\vee$ is locally free of rank n, this is called the **dual** of $\mathscr{E}$.

Now, given transition functions of $\mathscr{E}$, we want to understand the transition functions of $\mathscr{E}^\vee$. Let's first do it algebraically, suppose that we have an invertible function $GL_n(R)$ for a ring R, to be clear about the target and the source, we will call this T_{12}, then we also get it's inverse T_{21}. Then notice that this induces an invertible morphism:

$$Hom_1(R^n, R) \to Hom_2(R^n, R)$$

by mapping $f : R^n \to R$ to $f \circ T_{21}$. If we identify $Hom(R^n, R)$ with R^n using the isomorphism that specifies where e_i goes in R^n, we see that this will correspond to a matrix multiplication, this is actually a linear algebra problem, this matrix will correspond to the transpose of T_{21}, we will use T_{21}^T.

Then, we do it globally, for $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{O}_X)$, we know that locally over the trivializing neighborhood of $\mathscr{E}$, this is just exactly the same set up as above, thus the transistion functions will be $K_{ij} = T_{ji}^T = (T_{ij}^{-1})^T$, notice that since T_{ij} satisfies the cocycle condition, then so is the inverse, then so will the transpose. In conclusion, we have the following proposition:

Proposition 2.18. $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{O}_X)$ is also locally free of rank n, with transition functions $K_{ij} = (T_{ij}^{-1})^T$ if the transition function of $\mathscr{E}$ is T_{ij}.

Proposition 2.19. There is a canonical isomorphism $\mathscr{E} \cong (\mathscr{E}^\vee)^\vee$.

Proof. You have two ways to do this, the first is by the characterization of trivializing neighborhoods and transition functions, if these two data agree, then the two locally free sheaves can be viewed as glued together via the above data, thus there is a unique isomorphism between them. Or, you can construct a canonical morphism and show that this is an isomorphism.

To use the first method, notice that the transition functions of $(\mathscr{E}^\vee)^\vee$ is $(((T_{ij}^{-1})^T)^{-1})^T$, notice that the inverse and the transpose commute with each other, thus we get the transition function of $(\mathscr{E}^\vee)^\vee$ is the same as the transition function of $\mathscr{E}$, they also have the same trivializing neighborhoods.

You can also use an explicit construction, which means that we need to describe a canonical morphism:

$$\mathscr{E} \to \mathcal{H}om(\mathcal{H}om(\mathscr{E}, \mathscr{O}_X), \mathscr{O}_X)$$

Explicitly, on each open U, you need to define a canonical map:

$$\Phi : \mathscr{E}(U) \to \mathcal{H}om(\mathcal{H}om(\mathscr{E}|_U, \mathscr{O}_U), \mathscr{O}_U)$$

such that it's compatible with restrictions. The way we do it is by the evaluation operation, where given $x \in \mathscr{E}(U)$, we define the moprhism $\Phi(x)$ to be the a morphism:

$$\Phi(x) : \mathcal{H}om(\mathscr{E}|_U, \mathscr{O}_U) \to \mathscr{O}|_U$$

this means that suppose that $f \in \mathcal{H}om(\mathscr{E}|_U, \mathscr{O}_U)$, we need to specify on each $V \subset U$, where does $f|_V$ go, which should be an element in $\mathscr{O}_U(V)$, but as you can guess, this should be the evaluation of $x|_V$ by f.

Now, why this is an isomorphism, you can check this locally, when $\mathscr{E}$ is free of rank n, this is an isomorphism which is left to you, there is still something to check, make sure you know what's going on, the key is understanding why:

$$\mathcal{H}om(\mathcal{H}om(\mathscr{O}_X, \mathscr{O}_x), \mathscr{O}_X) \cong \mathscr{O}_X$$

□

Remark 2.20. Notice that even though we have $\mathscr{E}$ is isomorphic to it's double dual, we don't have $\mathscr{E} \cong \mathscr{E}^\vee$, even though this is true if $\mathscr{E}$ is not just locally free, but rather free. You will later see and example, which is the Serre's twisted sheaf on projective spaces.

Proposition 2.21. Tensor product of locally free sheaf is locally free.

Proof. Again, since tensor product commutes with restriction, we can just assume that we are tensoring free sheaves. Then we use the fact that tensor product commute with direct sums. □

Proposition 2.22. If $\mathscr{F}$ is an invertible sheaf, then $\mathscr{F} \otimes \mathscr{F}^\vee \cong \mathscr{O}_X$

Proof. Notice that there is a natural and canonical morphism from $\mathscr{F} \otimes \mathscr{F}^\vee$ to $\mathscr{O}_X$, defined by the universal property of tensor product and then evaluation, figure out what is it and we claim that when $\mathscr{F}$ is invertible, this is an isomorphism. Again, this is a local result, we only have to show that $\mathscr{O}_X \otimes (\mathscr{O}_X)^\vee \cong \mathscr{O}_X$, but notice that $\mathscr{O}_X^\vee \cong \mathscr{O}_X$, we are done. □

Remark 2.23. Notice that if $\mathscr{E}$ is any locally free sheaf of finite rank, there is a natural morphism:

$$\mathscr{E} \otimes \mathscr{E}^\vee \to \mathscr{O}_X$$

defined by universal property of tensor product and then evaluation, in general this is not an isomorphism, but at least we have a canonical morphism, which will turn out to be useful later.

Now, we prove a extremely useful result we will use repeatedly later on. Recall that tensor product is in general only right exact, but we claim that tensoring by locally free sheaves are exact:

Proposition 2.24. Suppose that $\mathscr{F}$ is a locally free sheaf and $\mathscr{G}' \to \mathscr{G} \to \mathscr{G}''$ is exact, then $\mathscr{G}' \otimes \mathscr{F} \to \mathscr{G} \otimes \mathscr{F} \to \mathscr{G}'' \otimes \mathscr{F}$ is still exact.

Proof. It suffices to check on stalks, it's left to you to check that applying $\otimes \mathscr{F}$ induces $\otimes \mathscr{F}_x$ on stalks. Now, the map on stalks is:

$$\mathscr{G}'_x \otimes \mathscr{F}_x \to \mathscr{G}_x \otimes \mathscr{F}_x \to \mathscr{G}''_x \otimes \mathscr{F}_x$$

We only have to show it's exact. Now, this is a local notion, thus we can assume $\mathscr{F}$ is free, (make sure you understand what does this means and justify why we can assume it's free). Then, it boils down to the fact that tensoring with free stuff is exact. $\square$

Proposition 2.25. Suppose that $\mathscr{E}$ is locally free of finite rank and $\mathscr{F}$ and $\mathscr{G}$ are just $\mathscr{O}_X$-modules, then we have the following natural isomorphisms:

$$\mathcal{H}om(\mathscr{F}, \mathscr{G} \otimes \mathscr{E}) \cong \mathcal{H}om(\mathscr{F} \otimes \mathscr{E}^\vee, \mathscr{G}) \cong \mathcal{H}om(\mathscr{F}, \mathscr{G}) \otimes \mathscr{E}$$

Proof. Let's first prove $\mathcal{H}om(\mathcal{F}, \mathcal{G} \otimes \mathcal{E}) \simeq \mathcal{H}om(\mathcal{F} \otimes \mathcal{E}^\vee, \mathcal{G})$. To do this, we first need a canonically defined moprhism. To do this, notice that for any $f : \mathcal{F} \to \mathcal{G} \otimes \mathcal{G}$, we can apply $- \otimes \mathcal{E}^\vee$ and get a morphism $f \otimes \mathcal{E}^\vee : \mathcal{F} \otimes \mathcal{E}^\vee \to \mathcal{G} \otimes \mathcal{E} \otimes \mathcal{E}^\vee$. Now, recall that $\mathcal{E} \otimes \mathcal{E}^\vee = \mathcal{E} \otimes \mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$, thus we can use the universal property of tensor to get a canonical morphism $\mathcal{E} \otimes \mathcal{E}^\vee \to \mathcal{O}_X$, which give you a canonical morphism $G : \mathcal{G} \otimes \mathcal{E} \otimes \mathcal{E}^\vee \to \mathcal{G}$. We post compose this with $f \otimes \mathcal{E}^\vee$ to get the morphism $G \circ (f \otimes \mathcal{E}^\vee) : \mathcal{F} \otimes \mathcal{E}^\vee \to \mathcal{G}$. We can define this on each open subset of X and to show this is indeed a morphism, we need to check that this is compatible with the restriction. First, let's check tensor commutes with restrictions, which is equivalent to show that the following diagram commutes:

$$\begin{tikzcd}
Hom(\mathcal{F}, \mathcal{G}) \arrow[r, "\otimes \mathcal{E}"] \arrow[d, "res"] & Hom(\mathcal{F} \otimes \mathcal{E}, \mathcal{G} \otimes \mathcal{E}) \arrow[d, "res"] \\
Hom(\mathcal{F}|_U, \mathcal{G}|_U) \arrow[r, "\otimes \mathcal{E}|_U"] & Hom(\mathcal{F}|_U \otimes \mathcal{E}|_U, \mathcal{G}|_U \otimes \mathcal{E}|_U)
\end{tikzcd}$$

where we identify $(\mathcal{F}, \mathcal{G})|_U$ with $\mathcal{F}|_U, \mathcal{G}|_U$ and similarly. The key to see this diagram is commutative is to understand what is the restriction after we did the identification. I will let you do the details. You also need to prove $\mathcal{E} \otimes \mathcal{E}^\bullet \to \mathcal{O}_X$ is also compatible with restrictions in a similar way, then we are done since we know that composition always respect restriction.

Now we claim that this is an isomorphism. To do this, it suffice to check locally on an open subset U, thus we can just assume that $\mathcal{E}$ is $\mathcal{O}_X^n$. Now to prove this is an isomorphism, it suffice to prove that this is an isomorphism for each open U, i.e. it's surjective and injective.

The way we do this is use the previous result that the two morphisms of sheaves are identical iff they induce the same map on each stalk. Let's first consider if it's injective. To see this, we only have to prove that if the image of f, g induce the same morphism on each stalk, then f, g induce the same morphism on the stalks as well. Then we can pass to the stalk (recall that we don't in general have $\mathcal{H}om$ commute with the stalk, but here we avoided argue like that), and the question is just a algebra problem, and we will let you do it.

To see it's surjective, we notice that when $\mathcal{E}$ is free, it's dual is also free and we can identify morphism from $\mathcal{F} \otimes (\mathcal{O}_X^\vee)^n \to \mathcal{G}$ with $\mathcal{F}^n \to \mathcal{G}$ via:

$$\begin{tikzcd}
\mathcal{F} \otimes (\mathcal{O}_X^\vee)^n \arrow[r] \arrow[d, "\simeq"'] & \mathcal{G} \arrow[d] \\
\mathcal{F} \otimes (\mathcal{O}_X^n) \arrow[r] \arrow[d, "\simeq"'] & \mathcal{G} \arrow[d] \\
\mathcal{F}^n \arrow[r] & G
\end{tikzcd}$$

Where the map $x \otimes (r_1, \cdots, r_n) \mapsto r_1\phi_1(x) + \cdots + r_n\phi_n(x)$ is identified with the morphism $(x_1, \cdots, x_n) \mapsto \phi_1(x_1) + \cdots + \phi_n(x_n)$. Then I will let you figure out why the morphism $\mathcal{F} \to \mathcal{G} \otimes \mathcal{O}_X^n$ defined by $\sum_i \phi_i \otimes e_i$ will be mapped to the desired morphism.

We still have to prove $\mathcal{H}om(\mathcal{F}, \mathcal{G}) \otimes \mathcal{E} \simeq \mathcal{H}om(\mathcal{F}, \mathcal{G} \otimes \mathcal{E})$, the srategy is similar, you first figure out how to define a canonical moprhism, then show that it's compatible with restrictions, then assume $\mathcal{E}$ is free. $\square$

Remark 2.26. Notice that the above result in particular shows that if $\mathscr{D}$ is a locally free sheaf of finite rank, then we have $\mathcal{H}om(\mathscr{D}, \mathscr{G}) \cong \mathscr{D}^\vee \otimes \mathscr{G}$ for any $\mathscr{G}$.

But this can also be proved directly more easily, notice that there is a canonical morphism from the RHS to the LHS defined globally(what is it), then we only have to show that it's an isomorphism locally, then we can assume $\mathscr{D}$ is free. Then, we need to show that the above defined morphism is an isomorphism:

$$\mathcal{H}om(\mathscr{O}_X^n, \mathscr{O}_X) \otimes \mathscr{G} \to \mathcal{H}om(\mathscr{O}_X^n, \mathscr{G})$$

one way to do this is to check that the map on stalks are isomorphisms, but notice that we don't really understand the stalks of homs, therefore you need to relate both of this to $\mathscr{G}^n$, and use it to show that the maps on stalks are indeed isomorphisms.

Now we can again introduce the important notion, Picard group of a ringed space:

Definition 2.27. The above results shows that the invertible sheaves on a ringed space, under the identification of isomorphic classes, is an abelian group on X under tensor products, called the **Picard group** on X.

It's also a quick observation that Pic, taking Picard groups, is a **contravariant functor** form the category of ringed spaces to the category of abelian groups by pullback, as we already showed that pullback preserve local freeness and rank.

2.1 Locally free sheaves on schemes in particular

We now discuss aspects of locally free sheaves that is specific to schemes. There are a lot interesting properties when you focus on schemes, for example, you may expect from you experience of smooth manifolds that every point has a small open such that all line bundles or finite rank vector bundles on that open is trivial. However, this is not true in the world of schemes, for example, on elliptic curves, which you will see later.

To start our discussion, notice that finite rank vector bundles are quasi-coherent. But it can happen that even if $\mathcal{O}_X$ is not coherent if we don't assume some finiteness condition. Remember that coherent is finitely generated plus morphism from something of finite rank to it has finitely generated kernel. If $\mathcal{O}_X$ is not coherent, then coherence is a pretty useless condition on X, but if it is coherent, we know that any finite rank vector bundle will be coherent, since we can prove that finite direct sum of coherent modules are still coherent.

As our first example of vector bundles on a scheme, consider the following proposition:

Proposition 2.28. Every finite rank n vector bundle on $\mathbb{A}_k^1$ is trivial of rank n.

Proof. Recall that on a locally Noetherian scheme, finitely generated, coherent, finitely presented is the same thing for quasi-coherent sheaves. Then since we already know that the structure sheaf is coherent over itself, then we know by the above discussion that any finite rank vector bundle is finitely generated.

Then suppose that the vector bundle is $\mathcal{E}$ of rank n, and $\Gamma(X, \mathcal{E}) = E$, we only have to show that E is free of finite rank n. To do so, notice that any finitely generated module on a PID is a direct sum of cyclic modules, say $E = E_1 \oplus \cdots E_n$. We claim that each of the cyclic module is free, otherwise, say E_i is generated by e_i and $re_i = 0$ for a nonzero $r \in R$. Then suppose that $\mathcal{E}$ is trivial on $D(f)$ where r doesn't vanish on $D(f)$, then in particular, the global section of $\mathcal{E}$ is torsion free there, however, we know that e there is $re = 0$, since r doesn't vanish there, this implies that E is a torsion element, which is a contradiction. $\square$

The generalization to $\mathbb{A}_k^n$ for $n > 1$ is going to be very hard, as you can expect, since we don't have the PID condition any more.

Definition 2.29. We will define the regular sections or rational sections on a locally free sheaf on a scheme X just as what we did for regular and rational functions on a scheme.

Remark 2.30. Here is a good point to review a little bit on the definition of rational functions.

First, for the simplest case, suppose that X is an integral scheme with generic point η. Then the elements in the function field $K(X)$ are called rational functions on X.

This definition can be generalized with the help of associated points.

Suppose that now we have a locally Noetherian scheme X, we can define the notion of associated points of X, then we define the rational points to be the colimit of $\Gamma(U, \mathcal{O}_X)$ where U is an open subset containing all the associated points of X (we can prove that in this sense the restriction is injective). This indeed generalize the definition for integral schemes since for an integral locally Noetherian scheme, the only associated point is the generic point, thus we know that the colimit is taken over all nonempty opens, thus the function field.

Proposition 2.31. Suppose that s is a section of a locally free sheaf $\mathcal{F}$ on X, there is a closed subscheme of X cut out by $s = 0$.

Proof. We will figure how to find such a scheme in this proposition.

Recall that closed subschemes correspond to quasi-coherent ideal sheaves of $\mathcal{O}_X$, thus we just need to figure out how to appropriately define this ideal sheaf. Let's first define this abstractly and see how locally this corresponds to a pretty concrete description.

Recall that the dual of $\mathcal{E}$, where $\mathcal{E}$ is a locally free sheaf on X is defined to be $\mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$, then we know that for any global section $s \in \mathcal{E}$, we have a natural morphism:

$$s : \mathcal{E}^\vee \to \mathcal{O}_X$$

which is evaluation of s. Notice that when $\mathcal{E}$ is locally free, the dual is quasi-coherent, then since the category of quasi-coherent is abelian, we know that the image is a quasi-coherent ideal sheaf of $\mathcal{O}_X$, we will define the closed subscheme cut out by s to be the closed subscheme defined by the image of $s : \mathcal{E}^\vee \to \mathcal{O}_X$.

Now, in order to understand this definition. Let's see on each of the trivializing affine opens, what does this closed subscheme look like locally.

Suppose that $U = SpecA$ is a trivializing neighborhood of $\mathcal{E}$, then we see that on $U = SpecA$, s is juts the morphism:

$$s : \mathcal{O}_U^n \to \mathcal{O}_U \qquad (r_1, \cdots, r_n) \mapsto \sum_i r_i f_i$$

where $s|_U$ is identified with $(f_1, \cdots, f_n)$ and $\mathcal{E}^\vee|_U$ is identified with $\mathcal{O}_U^n$. Thus you can think of this ideal sheaf on U to be the sheafification of the one generated by f_i on each $V \subset U$. ☐

Remark 2.32. There is another way to think of this closed subscheme.

Recall that if X is a scheme, then we have seen that if $f_1, \cdots, f_n$ are regular functions on X, then they cut out a closed subscheme correponding to the quasicoherent ideal sheaf defined to be on each affine open it's generated by $f_1, \cdots, f_n$ restricted to that affine open. This is indeed a quasicoherent sheaf since you can easily see that the restriction can be identified with localization.

Moreover, suppose that M is an invertible matrix whose entries are invertible regular functions, then $M(f_1, \cdots, f_n)$ will define the same closed subscheme since you can easily see that they give you the same ideal sheaf. This construction is also compatible with restriction, you can first define this ideal sheaf then restrict to an open subscheme or you can first restrict the global functions to that open subscheme and define the ideal sheaf, you end up with the same ideal sheaf of $\mathcal{O}_U$.

Then using this, suppose that $\mathcal{E}$ is locally free of finite rank and s is a global section, say U_i trivializes $\mathcal{E}$, then on each U_i, we define an ideal sheaf on U_i using the above discussion. Since on the intersection, two localization differs by a transition function, then you can see that on the intersection, they are isomorphic via the identity transition function, then they give you a unique global ideal sheaf. To show this definition is independent of the choice of trivializations. But notice that two sets of trivializing opens can be unioned and give a larger trivializing set, then you only have to prove the two ideal sheaves defined using the smaller sets agree with the ideal sheaf defined using the larger set, and I will let you check it.

But notice that no matter which way we choose to think of this ideal sheaf, we only have some control on it over a trivializing open of $\mathcal{E}$, if that happens to be an affine, then we know that it's exactly $\widetilde{(f_1, \cdots, f_r)}$, then this gives you some handle on what the closed subscheme looks like locally on affine open.

Proposition 2.33. Locally free sheaves on locally Noetherian normal schemes satisfy Algebraic Hartog's lemma: Sections defined away a set of codimension 2 extend over that set.

Before we prove this, let's spend some time tell the story of poles of regular of functions since we didn't manage to make it clear before. As you will see, this definition of poles and zeros only make sense if the point p is a regular codimension 1 point and algebraic Hartog's lemma only works for normal locally Noetherian schemes, thus we will assume throughout that X is a normal locally Noetherian scheme.

Recall the definition of associated points of X, which it's defined to be the associated points of $\mathcal{O}_X$, which can be defined affine locally. If we work with normal locally Noetherian schemes, we can cover X with affine opens $SpecA$ where A is a Noetherian integral domain. Now as we know the associated point of an integral domain is only the 0 ideal. Thus we can make sure that each affine open contains exactly 1 associated point of X.

Now recall the definition of regular functions f, which are elements of the colimit $\Gamma(U)$ where U is an open that contains all the associated points. Suppose that p is a regular codimension 1 point in X. We claim $\mathcal{O}_{X,p}$ has a natural morphism to $\prod_{q \in AssX} \mathcal{O}_{X,q}$. Suppose that $f \in \mathcal{O}_{X,p}$, which is the restriction of f in $U = SpecA$ and A is a domain, then we know that U only contains 1 associated point, say q, then we can consider the inclusion:

$$\mathcal{O}_{X,p} \hookrightarrow K(A) = \mathcal{O}_{X,q} \hookrightarrow \prod_{q \in AssX} \mathcal{O}_{X,q}$$

This is well defined, and let's you view $\mathcal{O}_{X,p}$ as a subring of a unique associated point, which only works in normal locally Noetherian schemes. Then you probably will ask whether a rational function f is in $\mathcal{O}_{X,p}$ or if it's in the fraction field of $\mathcal{O}_{X,p}$. But this doesn't make sense, since you find that the fraction field of $\mathcal{O}_{X,p}$ is just $\mathcal{O}_{X,q}$! If you want the image in $\mathcal{O}_{X,q}$, it should vanish on every associated point except for q, which is not true for an arbitrary rational functions. Instead, from our intuition from complex geometry, whether a function has a pole or not at a point can be determined by it's restriction around that point. Therefore, we can consider f restricted to $U \cap SpecA$, which is not empty since at least q, that associated point is in that open. And you also have an injective morphism:

$$\Gamma(SpecA \cap U) \hookrightarrow K(A)$$

Only this does the definition makes sense, which allows you ask whether f is in $\mathcal{O}_{X,p}$ or $K(A)$. We will use this as the definition for f has a pole or zero at p.

Suppose that f is the rational function defined on U with no poles, then for any affine open X, recall that we can decompose it into a disjoint union of integral affine opens. Then algebraic Hartog's lemma tells you that the regular function f restricted on $SpecA \cap U$ can be extended to all of $SpecA$ and we have the following diagram:

$$\begin{tikzcd}
\Gamma(SpecA) \arrow[d] \arrow[rr] & & \Gamma(SpecB) \arrow[dl] & \\
\Gamma(SpecA \cap U) \arrow[dr] & \Gamma(SpecA \cap SpecB) \arrow[d, hook] & \Gamma(U \cap SpecB) \arrow[dl] \\
& \Gamma(SpecA \cap SpecB \cap U) &
\end{tikzcd}$$

where the injection follows from we are working locally with integral schemes.

This diagram tells you that f_{SpecA}'s agree on intersection, that there is a unique regular function f_X restricting to all of them. We claim that $f_X|_U$ is f. This still comes from the above diagram, where we prove that the restriction to $SpecA \cap U$'s agree on the intersection, but f is also such an element, then the sheaf axiom tells you that f is $f_X|_U$.

Now suppose that f is a regular function defined on U where U is defined away from something of codimension at least 2, then we know that U contains all the codimension 1 point. Thus this implies that f has no poles in particular, then you can apply the discussion above.

Proof. Now let's consider the case of a finite rank n locally free sheaf $\mathcal{E}$ on X, a normal locally Noetherian scheme.

Now we can still cover X with specs of integral Noetherian rings A such that $\mathcal{E}$ is trivial in that $SpecA$. Then suppose that s is a rational section of $\mathcal{E}$ which is defined to be an element $s \in \Gamma(U, \mathcal{E})$ where U is an open containing all the associated points of X. Then we know that s restricted to $U \cap SpecA$ just correspond to n function on $\Gamma(U \cap SpecA, \mathcal{O}_X)$, then the above discussion allows you to extend each of them to $SpecA$, thus you get an element in $\Gamma(SpecA, \mathcal{E})$, for each $SpecA$ and they agree on intersection $SpecA \cap SpecB$, thus extend to all of X. ☐

Similarly, we have the following result:

Proposition 2.34. Suppose that s is a nonzero rational section of an invertible sheaf $\mathcal{L}$ on a locally Noetherian normal scheme, then if s has no poles, s is regular.

Proof. You again apply the above argument where you first find trivializing neighborhoods of $\mathcal{L}$ and show that the section extend, and they agree on intersections. ☐

Now, let's see what happens for SES of locally free sheaves:

Proposition 2.35. Suppose that:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is a short exact sequence of quasi-coherent sheaves on X. Suppose that $U = SpecA$ is an affine open subset where $\mathcal{F}$ and $\mathcal{F}''$ are free. Say $\mathcal{F}'|_{SpecA} = \widetilde{A}^{\oplus a}$ and $\mathcal{F}''|_{SpecA} = \widetilde{A}^{\oplus b}$.

Then $\mathcal{F}$ is also free on $SpecA$, and this exact sequence can be thought of as coming from the tautological exact sequence:

$$0 \to A^a \to A^{a+b} \to A^b \to 0$$

As a consequence, if $\mathcal{F}'$ and $\mathcal{F}''$ are free, then $\mathcal{F}$ is free.

Proof. Since they are quasi-coherent, then we know that on $SpecA$, the SES is just:

$$0 \to \widetilde{A}^{\oplus a} \to \widetilde{M} \to \widetilde{A}^{\oplus b} \to 0$$

which comes from a SES of A-modules:

$$0 \to A^a \to M \to A^b \to 0$$

The reason this sequence is exact is because we can check this on the stalks and we know that stalkification is an exact functor.

Then you can consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & A^a \arrow[r] \arrow[d, dashed, "="] & M \arrow[r] & A^b \arrow[r] \arrow[d, dashed, "="] \arrow[l, dashed, bend left] & 0 \\
0 \arrow[r] & A^a \arrow[r] & A^{a+b} \arrow[r] \arrow[u] & A^b \arrow[r] & 0
\end{tikzcd}$$

where the dotted morphism is because A^n is projective, then you can check it helps to define $A^{a+b} \to M$, then five lemma shows that it's an isomorphism. Therefore, we are done. ☐

Remark 2.36. Here is a good point to review some stuff on quasi-coherent sheaves we didn't talk about before since we used it in the proof above.

Let $X = SpecA$, an affine scheme, suppose that $\mathcal{M} = \widetilde{M}$ is a quasi-coherent sheaf on X while $\mathcal{N}$ is just any sheaf of $\mathcal{O}_X$-modules. Then we claim that we have an isomorphism induced by the global section functor:

$$Hom_{SpecA}(\mathcal{M}, \mathcal{N}) \to Hom_A(M, \Gamma(SpecA, \mathcal{N}))$$

To see why, we need to recall the notion of distinguished affine base, which can be viewed as the first example of a Grothendieck topology, recall that we have an equivalence between sheaf on a distinguished affine base and sheaf on the Zariski-topology. Then we have the following diagram:

$$\begin{tikzcd}
Hom_{SpecA}(\mathcal{M}, \mathcal{N}) \arrow[r, dashed] & Hom_A(M, \Gamma(SpecA, \mathcal{N})) \\
Hom_{SpecA^b}(\mathcal{M}^b, \mathcal{N}^b) \arrow[u, "\simeq"] \arrow[r, "\simeq"'] & Hom_A(M, \Gamma(SpecA, \mathcal{N})) \arrow[u, "="']
\end{tikzcd}$$

recall that since we are working with an affine scheme, then if you think about how we defined the left vertical morphism, you find that for any ϕ, a morphism of the distinguished base, the image in $Hom_{SpecA}(\mathcal{M},\mathcal{N})$ induce a morphism on the global section which is precisely the image of f in $Hom_A(M,\Gamma(SpecA,\mathcal{N}))$, thus the induced horizontal morphism above is just the global section functor.

In particular, if $\mathcal{N}$ is also quasi-coherent, then this isomorphism becomes:

$$Hom_{SpecA}(\widetilde{M},\widetilde{N}) \to Hom_A(M,N)$$

and you can easily see that this is compatible with taking stalks and localization at a prime.

Moreover, it's compatible with post- and pre- compositions, which can be checked readily.

Proposition 2.37. Assume we work with the same setup as before. Then choose an trivializing affine open cover of both $\mathcal{F}'$ and $\mathcal{F}''$, then the transition function of $\mathcal{F}$ can be characterized by the block matrix whose upper $a \times a$ block is the transition function of $\mathcal{F}'$ and the lower $b \times b$ block is the transition function of $\mathcal{F}'$.

Proof. This is basically the content of the diagram below:

$$\begin{tikzcd}
0 \arrow[r] & A^a \arrow[r] \arrow[d, "T_{ij}"] & A^{a+b} \arrow[r] \arrow[d] & A^b \arrow[r] \arrow[d, dashed, "S_{ij}"] & 0 \\
0 \arrow[r] & A^a \arrow[r] & A^{a+b} \arrow[r] & A^b \arrow[r] & 0
\end{tikzcd}$$

You find the block matrix in the statement is an isomorphism and it satisfies the condition of the transition function, i.e. the composition is as expected. This is because the composition of T_{ij} and S_{ij} is as expected and the composition of the block matrix doesn't have mixed terms of T_{ij} and S_{ij}. ☐

We continue discussion of the properties of the same sort as above:

Proposition 2.38. We again assume that we have a SES of quasi-coherent sheaves on X:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

Then if $\mathcal{F}''$ and $\mathcal{F}$ are locally free of finite rank, then $\mathcal{F}'$ is too.

Proof. This is a local question, thus we can choose affine opens $SpecA$ which trivializes $\mathcal{F}$ and $\mathcal{F}''$. Then we get a diagram:

$$0 \to \widetilde{M} \to \widetilde{A}^{\oplus m} \to \widetilde{A}^{\oplus n} \to 0$$

We want to prove M is also free of rank $m-n$. To see this, we apply the global section functor, which gives you an exact sequence of A-modules:

$$0 \to M \to A^{\oplus m} \to A^{\oplus n} \to 0$$

Now for every $p \in SpecA$, we can localize at p and still get the exact sequence. Thus M_p as the kernel of $A_p^{\oplus m} \to A_p^{\oplus n}$, suppose that the map ${}^{\oplus m} \to A^{\oplus n}$ is given by a $n \times m$ matrix T with values in A. Then we know that for each $p \in SpecA$, there is n-columns of T such that the determinant of the $n \times n$ submatrix on the top is invertible in A_p. This is becasue that you can tensor $\otimes_{A_p} A_p/pA_p$ and know that the rank of the matrix $T \otimes k(p)$ is n, thus we know that the determinant of the top $n \times n$ submatrix is not 0 in $k(p)$, thus invertible at A_p. But notice that being invertible is an open property, thus there is a distinguished open near p such that the determinant is invertible.

Then by restricting to that open and apply a change of coordinates on $A^{\oplus m}$, we can assume that the first n columns and rows of T gives you nonvanishing determinant. Then apply another change of coordinates, we can assume that T is the matrix where the top left block is the $n \times n$ matrix and other entries are 0. (You can do linear algebra in the residue field of p and the lift back to A, which give you a matrix with the top left block a diagonal matrix with invertible entries, then shrinking the distinguished open again if needed and do another change of coordinates).

Then after all of these you can identify M_f with $A_f^{\oplus(m-n)}$ since M_f is the kernel. This works for each $p \in SpecA$, then we are done. ☐

However, we see in the following example that even if $\mathcal{F}'$ and $\mathcal{F}$ are locally free of finite rank, $\mathcal{F}''$ may not be:

Example 2.39. Consider the affine scheme $\mathbb{A}^1_k$. If you consider the following exact sequence of quasi-coherent sheaves defined by the following exact sequence of $k[t]$-modules: $0 \to tk[t] \hookrightarrow k[t] \to k \to 0$.

Notice that the sheaf $\widetilde{tk[t]}$ is free since as an $k[t]$ module, we can literally identify it with $k[t]$ via the $k[t]$-module morphism:

$$tk[t] \to k[t]$$

where t is mapped to 1. However, we know that $\widetilde{k}$ is not locally free since it's only supported at a single point (t) (can you see why this implies that it's not locally free?)

Proposition 2.40. Suppose that:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is a SES of finite rank locally free sheaves, then:

$$0 \to \mathcal{F}''^{\vee} \to \mathcal{F}^{\vee} \to \mathcal{F}'^{\vee} \to 0$$

is also exact.

Proof. Without to much surprise, we still need to do this locally. Notice that the dual sequence is defined by pre-composition. You can easily check that this is compatible with restriction. Also notice that checking a sequence is exact, we can do it locally since we know that to check a sequence is exact we can check the induced sequence on stalks is exact.

Then, we can assume that they are all free and the sequence:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$$

is the tautological exact sequence:

$$0 \to \mathcal{O}_X^{\oplus m} \to \mathcal{O}_X^{\oplus m+n} \to \mathcal{O}_X^{\oplus n} \to 0$$

Apply the dual, I will let you check that the result is identified with the tautological sequence:

$$0 \to \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus m+n} \to \mathcal{O}_X^{\oplus m} \to 0$$

□

Now recall our example of sequences of vector bundles before, it is not as desired that you have a SES where the first two objects in the sequence are vector bundles, but the last object, i.e. the cokernel is not a vector bundle. This is an example that the if we consider morphisms between vector bundles are those coming from morphisms of $\mathcal{O}_X$-modules, then the result category is not abelian. To resolve this problem, we consider the following class of more restrictive morphisms between vector bundles. They have no accepted name, so we just choose one of them:

Definition 2.41. Let X be a scheme and let $\mathcal{E}$ and $\mathcal{F}$ be locally free sheaves on X with finite rank $a + b$ and $a + c$ respectively.

We say that $\phi : \mathcal{E} \to \mathcal{F}$ is a **rank a map of vector bundles** if for every point $p \in X$, there is an open neighborhood U of p with a commutative diagram:

$$\begin{tikzcd}
\mathcal{E}|_U \arrow[rr, "\phi|_U"'] \arrow[d, "\simeq"] & & \mathcal{F}|_U \arrow[d, "\simeq"'] \\
\mathcal{O}_U^{\oplus a+b} \arrow[r, two heads] & \mathcal{O}_U^{\oplus a} \arrow[r, hook] & \mathcal{O}_U^{\oplus a+c}
\end{tikzcd}$$

where the morphisms at the bottom is just the projection and injection.

We say that ϕ is a map of vector bundles if near every point p, ϕ is a rank a map of vector bundles for some a. If $b = 0$ we will say it's an injection of vector bundles.

Here are some important observations of maps of vector bundles:

Proposition 2.42. Suppose that $f : X \to Y$ is a morphism of schemes, then if $\phi : \mathcal{E} \to \mathcal{F}$ is a map of vector bundles of rank a, then the pullback $f^*\phi$ is also a map of vector bundles of rank a.

Proof. First we need to show that the pullback functor is compatible with restriction in the sense that:

$$f^*\phi|_{f^{-1}U} = f^*(\phi|_U)$$

To prove this, it's easy to prove that f^{-1} satisfies this property, and we already proved in a previous exercise that tensor functor is compatible with restrictions.

Moreover, the pullback functor clearly commutes with direct sums, thus you only have to apply the pullback functor to the above diagram to get the conclusion. □

Proposition 2.43. If ϕ is a map of vector bundles, then the kernel, cokernel and image are all vector bundles.

Proof. Using shefifications commute with restrictions, you can see that kernel, image cokernel all commute with restrictions, then the diagram tells you everything. □

Proposition 2.44. If $\phi : \mathcal{E} \to \mathcal{F}$ is a rank a map of vector bundles, then at a point $p \in X$, the rank of $\phi|_p : \mathcal{E}|_p \to \mathcal{F}|_p$ is a.

Proof. Omitted. □

We give several criteria to show a morphism of finite rank locally free sheaves is a map of vector bundles:

Proposition 2.45. Let $\phi : \mathcal{E} \to \mathcal{F}$ be a morphism of finite rank locally free sheaves where $\mathcal{F}$ has rank $a + c$, then ϕ is a rank a map of vector bundles if and only if $coker\phi$ is locally free of rank c.

In particular, every surjection of finite rank locally free sheaves is a map of vector bundles.

Proof. This is still a local statement, then we can assume that we have a diagram like this:

$$\begin{tikzcd}
\mathcal{E}|_U \arrow[r, "\phi|_U"'] \arrow[d, "\simeq"] & \mathcal{F}|_U \arrow[r] \arrow[d, "\simeq"'] & Coker|_U \arrow[d, dashed, "\simeq"] \\
\mathcal{O}_U^{\oplus a+b} \arrow[r] & \mathcal{O}_U^{\oplus a+c} \arrow[r] & \mathcal{O}_U^c
\end{tikzcd}$$

Now, suppose that the morphism $\mathcal{O}_U^{\oplus a+b} \to \mathcal{O}_U^{a+c}$ is given by a matrix M, then for any p in U, by passing to $k(p)$ and do the linear algebra, we can use the technique of shrinking U and doing a change of coordinate to assume the matrix is given by a upper block matrix where the first $a \times a$ block is the identity and all the other entries are 0. Then you can easily verify that the morphism can be decomposed into the desired form.

Conversely, if this is a map of vector bundles of rank a, the cokernel is easy to compute. □

Remark 2.46. What we proved above is actually something stronger, i.e. any cokernel $\mathcal{C}$ of morphisms of locally free sheaves of finite rank is also locally free if it's rank at p, defined to be the dimension as a $k(p)$-vector space of $\mathcal{C}_p/m_{X,p}\mathcal{C}_p$, is constant for every $p \in X$.

Another useful characterization is:

Proposition 2.47. Suppose that $\mathcal{E} \to \mathcal{F}$ is a morphism of finite rank locally free sheaves and let $p \in X$ be a point.

Then the natural map $(ker\phi)_p \to ker(\phi|_p)$ is surjective iff ϕ is a map of vector bundles in some neighborhoods of p.

Proof. Recall that the natural map $Ker(\phi)|_p \to Ker(\phi|_p)$ comes from the following diagram:

$$\begin{tikzcd} Ker(\phi|_p) \arrow[r] & \mathcal{E}|_p \arrow[r, "\phi|_p"] & \mathcal{F}|_p \\ (Ker\phi)|_p \arrow[u, dashed] \arrow[ur] & & \end{tikzcd}$$

where the dotted line comes from the universal property. If ϕ is a map of vector bundles around a neighborhood of p, then it's easy to see that the natural map is an isomorphism, thus surjective.

Please pay attention that we are working here with vector spaces, not just the stalk, we are tensoring with the residue field. Otherwise, the induced morphism is always an isomorphism.

Conversely, if we know the natural morphism is surjective. Suppose that the dimension of $Ker(\phi|_p)$ is a, then we can find a elements in $(Ker\phi)|_p$ which maps to the basis of $Ker(\phi|_p)$.

Suppose that the rank of $\mathcal{E}|_p$ is m and the rank of $\mathcal{F}|_p$ is n. If the dimension of $Ker(\phi|_p)$ is not m, then we can use the previous technique to locally convert ϕ to the standard form $\mathcal{O}_U^m \to \mathcal{O}_X^n$, this tells you it's locally a map of vector bundles. The problem occurs when the dimension of $Ker(\phi|_p)$ is m, which means that the map $\phi|_p$ is 0 at p. If this happens, we are not able to conclude that ϕ is 0 around a neighborhood of p without further condition, this is where the surjectivity condition comes in.

The first thought could be to show that $\phi|_q$ is 0 for all q in a neighborhood of p and then prove ϕ is locally 0. But this doesn't work since we will need reducedness condition, as you will see later. However, we can consider the following argument: think about the diagram:

$$\begin{tikzcd} (Ker\phi)_p \arrow[r, hook] \arrow[d] & \mathcal{E}_p \arrow[d] \\ (Ker\phi)|_p \arrow[r] \arrow[dr, two heads] & \mathcal{E}|_p \\ & Ker(\phi|_p) \arrow[u, two heads] \end{tikzcd}$$

we know that the morphism $(Ker\phi)|_p \to \mathcal{E}|_p$ is surjective, then we know that there are m elements $f_1, \cdots, f_n$ in $(Ker\phi)_p$ such that their image in $\mathcal{E}|_p$ generates $\mathcal{E}|_p$, but notice that by Nakayama's lemma, $\mathcal{E}_p$ is just $\mathcal{O}_{X,p}^m$, finitely generated, then the images in $\mathcal{E}_p$ also generates $\mathcal{E}_p$, by multiply these elements with suitable coeefienients, we can assume that $f_1, \cdots, f_m$ is in $SpecA$ where we assume that locally $\mathcal{E}$ is $\widetilde{A}^n$. Then, again since everything is finitely generated here, the support of $A/(f_1, \cdots, f_m)$ is closed, then there is a distinguished open near p such that for each q in that open, $A_q/(f_1, \cdots, f_m)_q$ is 0, thus we know that $Ker\phi \to \mathcal{E}$ is an isomorphism on that distinguished open, thus on that open, $Ker\phi \simeq \mathcal{E}$, thus this is the 0 morphism, of course a vector bundle morphism. $\square$

Remark 2.48. From the above proof, you will see that why we want vector bundles of finite rank. We need finite generation to apply nakayama and use the fact that the support is closed.

You should also have an interpretation of the surjective condition on kernels. Intuitively speaking, this surjection condition makes sure that the rank of $\phi|_p$ won't jump up when you don't want it to, i.e. it won't happen that the rank at a point is 0, but there is an open around that point such that the rank of the map is 0 only at that point. So, in the end, you see that we are only trying to impose a constant rank condition on the morphism.

Now we would like to discuss on the notion of various tensor powers of an $\mathcal{O}_X$-module $\mathcal{F}$, let's start with some definitions:

Definition 2.49. Suppose that $\mathcal{F}$ is an $\mathcal{O}_X$-module. Then we define the tensor algebra, symmetric algebra and exterior algebra of $\mathcal{F}$, written as $T^\bullet(\mathcal{F}), Sym^\bullet(\mathcal{F})$ and $\bigwedge^\bullet(\mathcal{F})$, by the sheafification of the presheaf:

$$U \mapsto T^\bullet(\Gamma(U, \mathcal{F})), \quad Sym^\bullet(\Gamma(U, \mathcal{F})), \quad \bigwedge^\bullet(\Gamma(U, \mathcal{F}))$$

We also define the n-th tensor, symmetric and exterior power of $\mathcal{F}$, written as $T^n(\mathcal{F}), Sym^n(\mathcal{F})$ and $\bigwedge^n(\mathcal{F})$ to be the sheafification of the presheaf:

$$U \mapsto T^n(\Gamma(U,\mathcal{F})), \quad Sym^n(\Gamma(U,\mathcal{F})), \quad \bigwedge^n(\Gamma(U,\mathcal{F}))$$

Remark 2.50. There are several stuff you need to check on the definition above.

First, you need to check that the assignment we described above indeed gives you a presheaf, in particular, the restriction should make sense. For example, the way we define restriction on the tensor algebra is via the universal property:

$$T^\bullet(\Gamma(U,\mathcal{F})) \rightharpoonup T^\bullet(\Gamma(V,\mathcal{F}))$$

is define by the $\Gamma(V,\mathcal{O}_X)$-module morphism:

$$\Gamma(U,\mathcal{F}) \to \Gamma(V,\mathcal{F}) \hookrightarrow T^\bullet(\Gamma(V,\mathcal{F}))$$

this morphism via the universal property of tensor algebras gives you the desired $\Gamma(V,\mathcal{O}_X)$-algebra morphism $T^\bullet(\Gamma(U,\mathcal{F})) \rightharpoonup T^\bullet(\Gamma(V,\mathcal{F}))$ such that restriction on $\Gamma(U,\mathcal{O}_X)$ is $\Gamma(U,\mathcal{F}) \to \Gamma(V,\mathcal{F}) \hookrightarrow T^\bullet(\Gamma(V,\mathcal{F}))$. I will let you figure out why this universal property tells you the restriction gives you a presheaf.

You can also do this similarly for symmetric and exterior algebra and tensor, symmetric and exterior powers.

Remark 2.51. We also want to study what does the stalk of the stuff we described above look like. For example for the tensor algebra, we claim that we have a natural and canonical isomorphism:

$$T^\bullet(\mathcal{F})_x \simeq T^\bullet(\mathcal{F}_x)$$

Again, since we defined this to be the sheafification of a presheaf, to study stalks we only have to care about the presheaf. Then the claim boils down to a natural isomorphism:

$$colim_{p\in U}(T^\bullet(\mathcal{F}(U))) \simeq T^\bullet(colim_{p\in U}(\mathcal{F}(U)))$$

since we know that direct sums commute with colimts, we only have to prove that tensor powers commute with colimts, which is left to you since it's just a universal property manipulation.

The more concrete stuff we should care is that for $x_1 \otimes x_2 \otimes \cdots \otimes x_n$ in $T^\bullet(\mathcal{F}(U))$, it's image in the stalk is just $x_1 \otimes x_2 \otimes \cdots \otimes x_n$ where we know think of x_i's as their image in $\mathcal{F}_x$ and the tensor is computed over $\mathcal{O}_{X,x}$.

You can do similar stuff for symmetric and exterior algebras and their components.

Remark 2.52. Another stuff you need to notice is that since sheafifications of pre-$\mathcal{O}_X$ algebras will be $\mathcal{O}_X$-algebras, all the algebras we define above are indeed algebras.

Moreover, they carry a graded structure. For example for the tensor algebra $T^\bullet(\mathcal{F})$ we have an injective morphism of sheaves of $\mathcal{O}_X$-modules:

$$T^n(\mathcal{F}) \hookrightarrow T^\bullet(\mathcal{F})$$

defined by the corresponding morphism on the presheaves and then take sheafification. This inclusion preserves the degree in the sense that $T^n(\mathcal{F}) \cdot T^m(\mathcal{F}) \subset T^{m+n}(\mathcal{F})$ where we view the multiplication inside $T^\bullet(\mathcal{F})$, you can check this easily via the locally constant stalk point of view of the sheafification. Finally, you have an induced isomorphism of $\mathcal{O}_X$-modules:

$$\oplus_n T^n(\mathcal{F}) \rightharpoonup T^\bullet(\mathcal{F})$$

this is induced by the universal property of direct sums and you check it's an isomorphism still by check the induced morphism on stalks are isomorphisms thanks to the previous remark we had about stalks.

Remark 2.53. Just like we obtained the universal property of tensor product of $\mathcal{O}_X$-modules from the universal property tensors, we can obtain the universal property of all the stuff defined above from the universal property, which I will let you do.

Now, let's prove something concrete and useful about these algebra construction:

Proposition 2.54. If $\mathcal{F}$ is locally free of rank m, then $T^n\mathcal{F}$, $Sym^n\mathcal{F}$ and $\bigwedge^n \mathcal{F}$ are all locally free of finite rank. The rank of $T^n\mathcal{F}$ is m^n and the rank of $Sym^n\mathcal{F}$ is $\binom{m+n+1}{n}$ and the rank of $\bigwedge^n \mathcal{F}$ is $\binom{m}{n}$, where is $n > m$, we just have 0.

Proof. Notice that sheafification commutes with restriction, then we can just assume $\mathcal{F}$ is $\mathcal{O}_X^m$.

We claim that $T^n\mathcal{O}_X^m \simeq \mathcal{O}_X^{m^n}$. We can use the universal property of T^n to construct a natural morphism to $\mathcal{O}_X^n$, which comes from the presheaf of T^n, and you can easily see that the presheaf morphism is an isomorphism since this is literally just algebra without anything involved in sheaves, then the induced morphism of sheaves is also an isomorphism.

Similarly you reduce the other cases to algebra question, and you find all you need to do is to extablish isomorphisms:

$$T^nR^m \simeq R^{m^n} \qquad Sym^nR^m \simeq R^{\binom{m+n+1}{n}} \qquad \bigwedge^n R^m \simeq R^{\binom{m}{n}}$$

You only have to use universal property to give morphisms back and forth and see that they are really inverses of each other. $\square$

Definition 2.55. If $\mathcal{F}$ is a locally free sheaf of finite rank $rank\mathcal{F}$, we define $\bigwedge^{rank\mathcal{F}} \mathcal{F}$, denoted by $det\mathcal{F}$, to be the determinant line bundle. Some people also call it the determinant locally free sheaf.

Proposition 2.56. Suppose that $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a SES of locally free sheaves of finite rank, then for any r we have a filtration of $Sym^r\mathcal{F}$:

$$Sym^r\mathcal{F} = \mathcal{G}^0 \supset \mathcal{G}^1 \supset \cdots \supset \mathcal{G}^r \supset \mathcal{G}^{r+1} \supset 0$$

with subquotients $\mathcal{G}^p/\mathcal{G}^{p+1} \simeq (Sym^p\mathcal{F}') \otimes (Sym^{r-p}\mathcal{F}'')$

Proof. To have some motivation of this proposition, let's first consider an algebra fact. Let A be a ring, then we claim that we have the following canonical isomorphism:

$$Sym^r A^{p+q} = \oplus_{i=0}^r (Sym^i A^p \otimes Sym^{r-i} A^q)$$

The intuition behind this is that if we consider the basis of A^{p+q} to be $e_1, \cdots, e_p$ and $f_1, \cdots, f_q$. Then Sym^r can be think of as the set of homogeneous degree r polynomials in variables $e_1, \cdots, e_p$ and $f_1, \cdots, f_q$. You can classify the polynomials by the number of variable from $e_1, \cdots, e_p$ appear, if there are i variables coming from $e_1, \cdots, e_p$, then the rest of them should come from $f_1, \cdots, f_q$. To establish this isomorphism, we can first define a morphism going from the left to the right hand side, which is the same as a r-linear symmetric map:

$$A^{p+q} \times \cdots \times A^{p+q} \to \oplus_{i=0}^r (Sym^i A^p \otimes Sym^{r-i} A^q)$$

It's hard to write this map down since the notation is just too complicated, but the idea behind this is to still classify the polynomial by the number of variables coming from $e_1, \cdots, e_p$. Similarly, you can define a morphism from the right hand side to the left hand side, which by the universal property of direct sums, is same as define morphisms from $Sym^i A^p \otimes Sym^{r-i} A^q$ for each i.

Then if you define $G_p = \oplus_{i=p}^r (Sym^i A^p \otimes Sym^{r-i} A^q)$ you naturally gets a filtration of $Sym^r A^{p+q}$ such that the subquotients looks like the one we want. This exercise aims to generalize this local algebra result to the global case.

We won't write out every detail since it's too complicated, you only need to know the idea behind this so that you won't feel scared with the statement and don't know where it comes from next time you see it. The idea here is to choose an affine open cover of X where on each affine open cover $\mathcal{F}', \mathcal{F}$ and $\mathcal{F}''$ are trivial. We already know that if we denote the transition matrix of $\mathcal{F}'$ and $\mathcal{F}''$ to be g' and g'', then the transition matrix of $\mathcal{F}$ is the upper triangular matrix where g', g'' lie on the diagonal. Then in this way, it's not hard to convince yourself that the transition matrix of $Sym^r\mathcal{F}$ is also upper triangular with $r+1$ blocks if we use the trivialization of $Sym^r\mathcal{F} \simeq Sym^r\mathcal{O}_X^{p+q} \simeq \oplus_{i=0}^r (Sym^i\mathcal{O}_X^p \otimes Sym^{r-i}\mathcal{O}_X^q)$. The reason behind this is also simple, each of the direct sum correspond to a block. And intuitively, you should think of this as mapping the monimial $e_{r_1} \cdots e_{r_i} f_{r,1} \cdots f_{r,(r-i)}$ is mapped to the polynomial of $g(e_{i_1}) \cdots g(e_{i_p}) g''(f_{i,1}) \cdots g''(f_{i,q})$ which is still a polynomial whose term only consists of those with i variables coming from $e_1, \cdots, e_p$ and $r-i$ variables coming from $f_1, \cdots, f_q$.

Now being a transition function, we know that it should satisfy the cocycle condition, but notice that the composition of block diagonal matrix is simple in the sense that we are just composing each block. This implies that we can pick any number of the blocks in the matrix, put them into a new block matrix and get a well defined transition matrix!

Now we define $\mathcal{G}^p$ locally on each of the affine opens as:

$$\mathcal{G}^p = \oplus_{i=p}^{r}(Sym^i\mathcal{F}' \otimes Sym^{r-i}\mathcal{F}'')$$

with the trivialiaztion induced by the trivialization of $\mathcal{F}'$ and $\mathcal{F}''$, we know that the transition matrix is just the transition function of $Sym^r\mathcal{F}$, but only with some blocks of that, thus this transition function is well defined, thus $\mathcal{G}^p$ gives you a well defined global locally free sheaf.

Notice that $\mathcal{G}^0$ is isomorphic to $Sym^r\mathcal{F}$ since they share the same transition function. And we have injections $\mathcal{G}^{p+1} \hookrightarrow \mathcal{G}^p$, which is defined locally and checked to glue. Notice that locally this injection is obvious to define, we only need to check that they agree on intersections. But this is trivial to check, notice that the way we define the local morphism is by first identifying $\mathcal{G}^p$ locally with $\oplus_{i=p}^{r}(Sym^i\mathcal{F}' \otimes Sym^{r-i}\mathcal{F}'')$ and then do the inclusion, but notice that on intersections the two identifications are connected by the gluing isomorphism, thus they agree.

The assertion about the quotient is also checked readily since what we proved above really is $\mathcal{G}^p = \oplus_{i=p}^{r}(Sym^i\mathcal{F}' \otimes Sym^{r-i}\mathcal{F}'')$ globally and the morphisms between them are just inclusion morphisms. ☐

Following similar way of thinking we can prove another useful result about exterior powers:

Proposition 2.57. Suppose $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a SES of locally free sheaves of finite rank, then for every r, there is a filtration of $\bigwedge^r \mathcal{F}$:

$$\bigwedge^r \mathcal{F} = \mathcal{G}^0 \supset \mathcal{G}^1 \supset \cdots \supset \mathcal{G}^r \supset \mathcal{G}^{r+1} = 0$$

where each subquotient is:

$$\mathcal{G}^p/\mathcal{G}^{p+1} \simeq \bigwedge^p \mathcal{F}' \otimes \bigwedge^{r-p} \mathcal{F}''$$

In particular, if we take $r = rank\mathcal{F}$, we know that $det(\mathcal{F}) \simeq det(\mathcal{F}'') \otimes det(\mathcal{F}')$

Proof. The proof of the filtration and quotient is similar to that of the above proposition. For the last assertion, suppose that the rank of $\mathcal{F}'$ and $\mathcal{F}''$ are m, n respectively, then we know the rank of $\mathcal{F}$ is $m+n$, thus the subquotient is nonzero only if $p = m$, this tells you the result. ☐

Proposition 2.58. Suppose that $\mathcal{F}$ is a locally free sheaf of rank n, then there is a map $\bigwedge^r \mathcal{F} \times \bigwedge^{n-r} \mathcal{F} \to \bigwedge^n \mathcal{F}$ that induces an isomorphism $\bigwedge^r \mathscr{F} \to (\bigwedge^{n-r})^\vee \otimes \bigwedge^n \mathcal{F}$.

Proof. Let's see first how this is related to the perfect pairing of a vector space V. Consider the following pairing:

$$\bigwedge^r V \times \bigwedge^{n-r} V \to \bigwedge^n V$$

which is the same as defining a k-linear map from $\bigwedge^r V$ to $Hom_k(\bigwedge^{n-r} V, \bigwedge^n V)$. By the universal property of wedge products, this is the same as giving a skew-symmetric map:

$$V \times V \times \cdots \times V \to Hom_k(\bigwedge^{n-r} V, \bigwedge^n V)$$

where $a_1 \wedge a_2 \cdots \wedge a_r$ is mapped to the map that maps $b_1 \wedge \cdots \wedge b_{n-r}$ to $a_1 \wedge \cdots \wedge a_r \wedge b_1 \wedge \cdots \wedge b_{n-r}$.

You check this is well defined, and notice that we have an isomorphism:

$$Hom_k(\bigwedge^{n-r} V, \bigwedge^n V) \simeq Hom_k(\bigwedge^{n-r} V, k) \otimes \bigwedge^n V$$

this gives you a morphism $\bigwedge^r V \to (\bigwedge^{n-r} V)^\vee \otimes \bigwedge^n V$, and I claim that this is an isomorphism, which I will let you check. There is nothing deep there, the only thing you need to pay attention to is that $\bigwedge^n V$ is just k where the identification is via a basis of V.

This argument works more generally when V is a finite rank free R-module.

Then let's see how this generalizes to the sheaf statement we are working here. First, a morphism $\bigwedge^r \mathcal{F} \times \bigwedge^{n-r} \mathcal{F} \to \bigwedge^n \mathcal{F}$ is still the same as a morphism $\bigwedge^r \mathcal{F} \to \mathcal{H}om(\bigwedge^{n-r} \mathcal{F}, \bigwedge^n \mathcal{F})$ which can be proved generally. Then recall that $\mathcal{H}om(\bigwedge^{n-r} \mathcal{F}, \bigwedge^n \mathcal{F}) \simeq \mathcal{H}om(\bigwedge^{n-r} \mathcal{F}, \mathcal{O}_X) \otimes \bigwedge^n \mathcal{F}$ since $\bigwedge^n \mathcal{F}$ is locally free. This gives the induced morpism $\bigwedge^r \mathcal{F} \to (\bigwedge^{n-r} \mathcal{F})^\vee \otimes \bigwedge^n \mathcal{F}$. We claim that this is still an isomorphism, which reduces to a local statement, which reduces to the discussion we had above. $\square$

Proposition 2.59. Suppose that $0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_n \to 0$ is an exact sequence of finite rank locally free sheaves on X, then the alternating product of determinant bundles is trivial, i.e:

$$det(\mathcal{F}_1) \otimes det(\mathcal{F}_2)^\vee \otimes det(\mathcal{F}_3) \otimes det(\mathcal{F}_4)^\vee \otimes \cdots \otimes det(\mathcal{F}_n)^{(-1)^{n+1}} \simeq \mathcal{O}_X$$

Proof. First notice that we have a SES:

$$0 \to Ker_{n-1} \to \mathcal{F}_{n-1} \to \mathcal{F}_n \to 0$$

then we know that since $\mathcal{F}_{n-1}$ and $\mathcal{F}_n$ are locally free, the kernel Ker_{n-1} is also locally free. But this gives you another SES:

$$0 \to Ker_{n-2} \to \mathcal{F}_{n-2} \to Ker_{n-1} \to 0$$

Keep going, we can break this LES into several SESs, then we use that determinant behaves well wrt SES to conclude the result. $\square$

Before we finish these tensor product constructions, we would like also to remind you that even though we have focused on the study of locally free sheaves, if $\mathcal{F}$ is quasi-coherent, we also have a clear structure of these tensor products, i.e. they are also quasi-coherent and we understand what they look like on each affine open:

Proposition 2.60. Suppose that $\mathcal{F}$ is a quasi-coherent sheaf, then $T^n\mathcal{F}$, $Sym^n\mathcal{F}$ and $\bigwedge^n \mathcal{F}$ are all quasi-coherent.

Proof. We will redefine all of the sheaves, prove they are quasi-coherent and show this new definition agrees with the definition we had before.

For example, we will define $T^n\mathcal{F}$ using the sheaf on a distinguished affine base where on an affine open $SpecA$, we just define $T^n\mathcal{F}(SpecA) = T^n(\mathcal{F}(SpecA))$ and the restrictions are induced by the restrictions on $\mathcal{F}$. Recall that $\mathcal{F}$ is quasi-coherent, thus the restriction to a distinguished open is isomorphic to localization, which tells you that the following diagram is commutative:

$$\begin{tikzcd}
T^n(\mathcal{F}(SpecA)) \arrow[r, "res"] \arrow[d, "loc"'] & T^n(\mathcal{F}(SpecA_f)) \\
T^n(\mathcal{F}(SpecA))_f \arrow[r, "\simeq"'] & T^n(\mathcal{F}(SpecA)_f) \arrow[u, "\simeq"']
\end{tikzcd}$$

which is equivalent of the associated sheaf is quasi-coherent.

Moreover, this is isomorphic to the definition we had before using sheafification. The way you prove it is to define a morphism from the distinguished affine base we just defined to $T^n\mathcal{F}$ restricted the the distinguished affine base, $T^n\mathcal{F}^b$. Since $T^n\mathcal{F}^b$ on any affine base is just locally constant stalks of the presheaf, you can see this the natural morphism induces isomorphism on each stalk, thus a global isomorphism, we are done.

The other powers are constructed and proved similarly. $\square$

2.2 More Pleasant Properties of Finite Type and Coherent Sheaves

Now we describe a few good properties of quasicoherent sheaves when we have some finiteness property. First, we will see that the $\mathcal{H}om$ functor behaves pretty well if the source is finitely presented, and thus if we work with locally Noetherian schemes, it behaves well if the source is finite type or coherent:

Proposition 2.61. Suppose that $\mathcal{F}$ is a finitely presented quasicoherent sheaf on X and $\mathcal{G}$ is quasicoherent sheaf on X, then $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is quasicoherent.

Proof. Recall that the functor $\mathcal{H}om(-, \mathcal{G})$ is compatible with localizations then $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is just $\mathcal{H}om(\mathcal{F}|_U, \mathcal{G}|_U)$, then since being quasicoherent is a local statement, we can assume we work with an affine scheme $SpecA$ and $\mathcal{F}$ is just $\widetilde{M}$ with a finite presentation of M given by:

$$A^m \to A^n \to M \to 0$$

this finite presentation of M gives you a finite presentation of $\mathcal{F}$:

$$\mathcal{O}_X^m \to \mathcal{O}_X^n \to \mathcal{F} \to 0$$

by taking the associated sheaf. Then apply the left exact functor $\mathcal{H}om(-, \mathcal{G})$, which gives you a exact sequence:

$$0 \to \mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om(\mathcal{O}_X^n, \mathcal{G}) \to \mathcal{H}om(\mathcal{O}_X^m, \mathcal{G})$$

this identifies $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ as the kernel of $\mathcal{G}^n \to \mathcal{G}^m$, thus $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is quasi-coherent since quasicoherent sheaves form an abelian category.

In particular, since we know that $\mathcal{F}$ and $\mathcal{G}$ are quasicoherent, they are $\widetilde{F}$ and $\widetilde{G}$, thus the global section of $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is just $Hom(\widetilde{F}, \widetilde{G})$, by the isomorphism we mentioned before, this is identified with $Hom(F, G)$, thus locally on an affine open $SpecA$ $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is just the associated sheaf of $Hom(\mathcal{F}(SpecA), \mathcal{G}(SpecA))$. □

Remark 2.62. If the source is not finitely presented, there are examples where even if you work an affine scheme, $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is not quasi-coherent. So the finite condition is necessary.

Proposition 2.63. Assume the same set up as the above proposition, if we further more assume that $\mathcal{G}$ is coherent, then $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is also coherent.

Proof. We first know that it's quasicoherent, then notice that coherent sheaves also form an abelian category, then if $\mathcal{G}$ is coherent, we already know that any direct sum of coherent sheaf will be coherent, thus the kernel of $\mathcal{G}^n \to \mathcal{G}^m$ is also coherent. □

Remark 2.64. The above to propositions show that if $\mathcal{F}$ is finitely presented, then the functor $\mathcal{H}om(\mathcal{F}, -)$ is a covariant left exact functor $Qcoh(X) \to Qcoh(X)$ and $Coh(X) \to Coh(X)$. If $\mathcal{G}$ is quasicoherent, then the functor $\mathcal{H}om(-, \mathcal{G})$ is a contravariant functor from finitely presented sheaves to $Qcoh(X)$.

Definition 2.65. Assume that $\mathcal{O}_X$ is coherent (since otherwise we are in a particularly bad situation). Suppose that $\mathcal{F}$ is finitely presented, then notice that the above propositions tell us the **dual** of $\mathcal{F}$, again denoted by $\mathcal{F}^\vee$ is coherent $\mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$. You again have the natural morphism $\mathcal{F} \to \mathcal{F}^{\vee\vee}$, but not similar to the case of locally free sheaves, this morphism is not always isomorphisms. If it happens to be an isomorphism, we will call $\mathcal{F}$ a **reflexive sheaf**, though we wont use this notion later.

You also have the natural morphism $\mathcal{F} \otimes \mathcal{F}^\vee \to \mathcal{O}_X$, called the **trace map**.

Proposition 2.66. Suppose that:

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is a short exact sequence of quasicoherent sheaves, with $\mathcal{H}$ locally free. If $\mathcal{E}$ is quasicoherent, then we claim that:

$$o \to \mathcal{H}om(\mathcal{H}, \mathcal{E}) \to \mathcal{H}om(\mathcal{G}, \mathcal{E}) \to \mathcal{H}om(\mathcal{F}, \mathcal{E}) \to 0$$

is also exact.

Proof. Again since the statement is a local statement and the functor $\mathcal{H}om(-, \mathcal{E})$ is compatible with restrictions to opens, we can just assume we are working with an affine open and the sequence of sheaves correspond to an exact module sequence:

$$0 \to M \to N \to A^{\oplus I} \to 0$$

here we don't have to assume $\mathcal{H}$ is finite rank since the argument works more generally.

Then since free modules are projective, we know this sequence splits and $N \simeq M \oplus A^{\oplus I}$ and locally this sequence is identified with:

$$0 \to \widetilde{M} \to \widetilde{M} \oplus \widetilde{A}^{\oplus I} \to \widetilde{A}^{\oplus I} \to 0$$

since taking the associated sheaf commutes with direct sums, then since $\mathcal{H}om(, \mathcal{E})$ also commutes with directs, after you apply it, you get:

$$0 \to \mathcal{H}om(\widetilde{A^{\oplus I}}, \mathcal{E}) \to \mathcal{H}om(\widetilde{A^{\oplus I}}, \mathcal{E}) \oplus \mathcal{H}om(\widetilde{M}, \mathcal{E}) \to \mathcal{H}om(\widetilde{M}, \mathcal{E}) \to 0$$

since this functor is always left exact, we only have to consider if the sequence is surjective on the right, which is obvious. ☐

Remark 2.67. In particular, if we assume further more that $\mathcal{F}, \mathcal{G}$ and $\mathcal{H}$ are coherent, we have an exact sequence of coherent sheaves:

$$0 \to \mathcal{H}^\vee \to \mathcal{G}^\vee \to \mathcal{F}^\vee \to 0$$

induced by applying $\mathcal{H}om(, \mathcal{O}_X)$

Before we proceed further, it's right now a good point to review some notions on the support of a sheaf on a scheme: We have previously defined the support of a section of a sheaf to be the locus where it's not 0 in the stalk. We also defined the support of a sheaf to be the locus where the stalk is not trivial. Or equivalently, the union of all sections over all opens. Notice that even though the support of a single global section is closed, we don't know in general if the support of a sheaf is closed or not, however, if the sheaf is a finite type quasicoherent sheaf on X, the support is indeed closed:

Proposition 2.68. The support of a finite type quasicoherent sheaf on X is a closed subset.

Proof. Recall that a subset is closed iff we have an open cover of the space and the set is closed in every open.

Cover this scheme with affine opens and we know that on each open, the support of the sheaf on the open is the support of restriction of the sheaf. However, since the sheaf is quasicoherent, we know that $\mathcal{F}|_U \simeq \widetilde{M}$, then we know that the support of the restriction is just $SuppM$, however, since the sheaf is finite type, M is finitely generated, thus we know that it's support is closed, we are done. ☐

Remark 2.69. The support of a quasicoherent sheaf without any finiteness condition is not necessarily closed. For example, let $A = \mathbb{C}[t]$, then $\mathbb{C}[t]/(t-a)$ is an A-module supported in a single point $(t-a)$ for $a \in \mathbb{C}$. Consider the A-module defined by $\oplus_{a \in \mathbb{C}} \mathbb{C}[t]/(t-a)$, then you can verify that except each nonzero prime ideal is in the support since localization commutes with arbitrary direct sum. But we know exactly that closed subsets in $\mathbb{A}^1_{\mathbb{C}}$ are finite union of closed points, thus we know that the support is not closed.

We now proceed to see the geometric interpretation of Nakayama's lemma which explain why Nakayama's lemma should be considered as a geometry fact which is proved by algebra. We have already implicitly used it, here let's state it clearly:

Proposition 2.70. Let X be a scheme and let $\mathcal{F}$ be a finite type quasicoherent sheaf on X then if $U \subset X$ is an open neighborhood of $p \in X$ and we have sections $a_1, \cdots, a_n \in \mathcal{F}(U)$ so that their image $a_1|_p, \cdots, a_n|_p$ generate the fiber $\mathcal{F}|_p$ (i.e. $\mathcal{F}_p \otimes k(p)$, which will be explictly defined later), then there is an affine open $p \in SpecA \subset U$ such that $a_1|_{SpecA}, \cdots, a_n|_{SpecA}$ generates $\mathcal{F}|_{SpecA}$ in the following sense:

1. $a_1|_{SpecA}, \cdots, a_n|_{SpecA}$ generates $\mathcal{F}(SpecA)$ as an A-module.
2. For any $q \in SpecA$, the image of $a_1, \cdots, a_n$ generates $\mathcal{F}_q$ as an $\mathcal{O}_{X,q}$-module and in particular their in the fiber generates the fiber as well.

In particular, if $\mathcal{F}_p$ is 0, then there is an open neighborhood V of p such that $\mathcal{F}|_V = 0$.

Proof. Recall that since $\mathcal{F}$ is a finite type quasicoherent sheaf, then for any $p \in SpecA$, we can assume that $\mathcal{F}|_{SpecA} \simeq \widetilde{M}$, where M is a finitely generated A-module.

Then if $a_1, \cdots, a_n$'s image generates $\mathcal{F}|_p$, we know that since $\mathcal{F}_p$ is finitely generated over $\mathcal{O}_{X,p}$, then Nakayama's lemma implies that $a_1, \cdots, a_n$'s image in $\mathcal{F}_p$ also generates p. Translate to the local statement where we work with $\widetilde{M}$ and $SpecA$, we know that there are $a_1, \cdots, a_n \in M$ such that their image in M_p generates M_p as an A_p-module.

Then consider the finitely generated A module $M/(a_1, \cdots, a_n)$, we know that p is not in the support, which is a closed subset of $SpecA$, then we know that there is a distinguished open $D(f)$ around p such that for each q in the open we know that $a_1, \cdots, a_n$ generates M_q, but this implies that $a_1, \cdots, a_n$ generates M_f, this distinguished open will be the one in the statement of the proposition.

The last assertion follows similarly except now we don't have to consider any quotient in the argument. □

Remark 2.71. Pay attention to how we utilized the fact that the sheaf is finite type, which gives you closed support and allows you to use Nakayama's lemma.

Now you will see that for finitely presented sheaves, locally free is equivalent to have a free stalk:

Proposition 2.72. Suppose that $\mathcal{F}$ is a finitely presented sheaf on X then $\mathcal{F}_p$ is a free $\mathcal{O}_{X,p}$-module iff $\mathcal{F}$ is locally free in some neighborhoods of p.

Therefore, $\mathcal{F}$ is locally free iff $\mathcal{F}_p$ is a free $\mathcal{O}_{X,p}$-module for every point $p \in X$.

Proof. One direction of the assertion is easy: locally free certainly implies that the stalk is free.

Conversely, suppose that $\mathcal{F}_p$ is free over $\mathcal{O}_{X,p}$ it's also of finite rank since finitely presented implies finite type, then suppose that the basis are $e_1, \cdots, e_n$ which are images of $e_1, \cdots, e_n$ in $\mathcal{F}(U)$, then the geometric Nakayama above tells you that there is an affine open in U, say $Y = SpecA$ such that the restrictions of $e_1, \cdots, e_n$ to $SpecA$ generates $\mathcal{F}(SpecA)$ and their image generates $\mathcal{F}_q$ for each $q \in SpecA$, this gives you a surjection:

$$\mathcal{O}_Y^n \to \mathcal{F}|_Y$$

(recall that surjection is defined by the surjection of stalks). This correspond to a surjective A-module morphism:

$$A^n \to M$$

then since M is finitely presented, we know that it's always finite presented which implies that the kernel is finitely generated, then we know that the support of the kernel is closed in Y, say the support is Z, then we see that the surjective morphism $\mathcal{O}_Y^n \to \mathcal{F}|_Y$ will be an isomorphism on $Y - Z$ since you can check this on the stalk at each $q \in Y - Z$, you know that taking stalks preserves being surjective, but the kernel vanishes after we take the stalk, thus we are done. □

Remark 2.73. This is an enlightening exercise in a few ways: First you know that being locally free for a finitely presented sheaf is a stalk local property.

Furthermore, if you are willing to add on some extra assumptions on the scheme X, for example, if X is integral, then we claim that every finitely presented sheaf on X is automatically free near the generic point. This is because the stalk $\mathcal{F}_p$ is automatically a vector space over $K(X)$, thus always free, thus $\mathcal{F}$ is free over a dense open in X.

Also, if we work with an irreducible scheme, then if $\mathcal{F}_p$ is 0 then $\mathcal{F}$ is automatically 0 on a dense open. These statements are pretty mysterious in classical algebraic geometry but with the right language we see that this is not that complicated.

Now we see how torsion and torsion free can be put into the picture. Let's review the basic definitions: Recall that that an A-module M is said to be torsion free if for any $a \in A$ and any $m \in M$ $am = 0$ implies that a is a zero-divisor in A or $m = 0$ in M. As you will see below, the notion of torsion and torsion free will only be used in integral schemes and integral domains, thus suppose from now on that A is an integral domain. Then this definition of torsion free can be restated as $am = 0$ implies $a = 0$ or $m = 0$. Motivated by this, we define a submodule of M, called the torsion submodule of M and denoted by M_{tor}, which consists of elements of M that is killed by a nonzero element of A. Pay special attention that if A is not an integral domain, this even may not be a submodule, which is part of the reason we insist A is an integral domain.

Clearly in this case M is torsion free iff $M_{tor} = 0$ and we will say M is torsion if $M = M_{tor}$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module on a scheme X, then it's called torsion free if each $\mathcal{F}_p$ is a torsion free $\mathcal{O}_{X,p}$-module. Notice that M is torsion iff $M \otimes_A K(A) \simeq 0$.

We also have the definition of a torsion sheaf, which is a little bit surprising at first sight:

Definition 2.74. If $\mathcal{F}$ is a quasicoherent sheaf on a reduced scheme X, then $\mathcal{F}$ is torsion if it's stalk at the generic point of each irreducible components is 0.

Remark 2.75. The reason we define it like this can be thought of as the generalization of the torsion sheaf on an integral scheme, notice that if X is integral, then the stalk at the generic point is $M \otimes_A K(A)$ for a affine open of the generic point and M is torsion iff $M \otimes_A K(A)$ is 0.

We will only use this notion when dealing with regular curves, which is a particularly easy situation.

Now let's see how this notion are applied in the context of regular curves, which we will use later:

Proposition 2.76. Torsion free coherent sheaves on a regular (thus locally Noetherian is assumed) curve are locally free.

Proof. Recall that on a locally Noetherian scheme, the three finiteness condition coincide. Then to prove it's locally free, we only have to prove that the stalk at each point is free.

Recall that for a regular curve, the codimension of a point is either 1 or 0, then notice that if p is of codimension 0, then $\mathcal{O}_{X,p}$ is a regular local ring of dimension 0, thus a field, thus the field is necessarily free. On the other hand, if p is of codimension 1, we know that $\mathcal{O}_{X,p}$ is a regular local ring of dimension 1, thus a PID, then we know that any finitely generated module over a PID is a direct sum of cyclic modules, then since we know $\mathcal{F}_p$ will be a finitely generated torsion free module over $\mathcal{O}_{X,p}$, it's finite rank free. $\square$

Proposition 2.77. A torsion coherent sheaf on a quasicompact regular integral curve are supported at a finite number of closed points.

Proof. Again, the three finiteness condition coincide and you find that a torsion coherent sheaf should have 0 stalk at the generic point, therefore, this sheaf is zero on an dense open.

Notice that the support of $\mathcal{F}$ is a closed subset since the generic point is not in the support, since the curve is integral, we know the dimension of the support is 0, otherwise it will be all of the curve, which is a contradiction. Recall that this space is quasicompact and locally Noetherian, thus Noetherian, then we know that in a Noetherian space, each closed subset decomposes into finite union of irreducible components. Here, in particular, each irreducible components of the support is also zero dimensional, which is a point, since every Noetherian space has at least a closed point, which is proved by repeatedly taking the closure of a point and use the descending chain condition. $\square$

Proposition 2.78. Let $\mathcal{F}$ be a coherent sheaf on a quasi-compact regular curve, then there is a canonical SES:

$$0 \to \mathcal{F}_{tor} \to \mathcal{F} \to \mathcal{F}_{lf} \to 0$$

such that $\mathcal{F}_{tor}$ is a torsion sheaf and $\mathcal{F}_{lf}$ is locally free.

Proof. First notice that for the generic points of each irreducible components, the local ring there is regular local of dimension 0, thus a field. Thus $\mathcal{F}_p$ is always free. Then we know that $\mathcal{F}$ is locally free in a neighborhood of each of the generic point. The complement is a closed subset of dimension at most 0 since it doesn't contain any generic point of the irreducible components. Then again, we are dealing with Noetherian spaces, thus these are just a finite union of closed points. Then we define $\mathcal{F}_{tor}$ to be the finite direct sum of skyscraper sheaf supported at these points with the torsion submodule of $\mathcal{F}_p$, then we have a canonical inclusion of $\mathcal{F}_{tor}$ into $\mathcal{F}$. It's indeed torsion since the stalk at each of the generic point is 0.

Then we take $\mathcal{F}_{lf}$ to be the cokernel, which you can see to have torsion free stalks, thus locally free by the prebvious proposition, we are done. $\square$

Recall the definition we had about the fiber of an $\mathcal{O}_X$-module, it's denoted by $\mathcal{F}|_p = \mathcal{F}_p \otimes_{\mathcal{O}_{X,p}} k(p) = \mathcal{F}_p/m_p\mathcal{F}_p$ where m_p is the maximal ideal of $\mathcal{O}_{X,p}$. The rank of $\mathcal{F}$ at p, denoted by $rank_p\mathcal{F}$ is just the dimension of $\mathcal{F}|_p$ as a $k(p)$ vector space.

If we are dealing with a quasicoherent sheaf $\mathcal{F}$, then we can compute the fiber and rank over any affine open, say $SpecA$, if $\mathcal{F}(SpecA) = M$, then we know that the fiber at $p \in SpecA$ is just M_p/pA_pM_p and the rank of $\mathcal{F}$ at p is just $dim_{K(A/p)}M_p/pA_pM_p$, here we use the canonical identification of $A_p/pA_p \simeq K(A/p)$.

Example 2.79. Suppose that $X = \mathbb{A}^1_k$ and $\mathcal{F}$ is the quasicoherent sheaf corresponding to the module $k[t]/(t)$. We will compute it's rank at each prime.

First notice that the support of this module is just (t), thus we know that for any prime not (t), the stalk is 0, thus the vector space is 0, thus the fiber is 0, the rank is also 0, this is the same even at the generic point. By the geometric Nakayama we proved before, this implies that this sheaf is 0 on a dense open. At (t), notice that $(k[t]/(t))_{(t)}$ is just k, thus the dimension after you quotient the maximal ideal (0) is still 1.

Now we see that rank at a point is additive with respect direct sums and multiplicative with respect to tensor products:

Proposition 2.80. Let $\mathcal{F}$, $\mathcal{G}$ be two $\mathcal{O}_X$-modules on X, not necessarily quasi-coherent, then for any point $p \in X$, we have $rank_p(\mathcal{F} \oplus \mathcal{G}) = rank_p(\mathcal{F}) + rank_p(\mathcal{G})$ and $rank_p(\mathcal{F} \otimes \mathcal{G}) = rank_p(\mathcal{F}) \cdot rank_p(\mathcal{G})$.

Proof. Recall that taking stalk and tensor commutes with direct sums, thus $(\mathcal{F} \oplus \mathcal{G}) \simeq \mathcal{F}_q \otimes \mathcal{G}_q$, then if we tensor with $- \otimes_{\mathcal{O}_{X,p}} k(p)$, it's the direct sum of two fibers. We are done since dimension is additive.

For the tensor product, recall that we have a canonical isomorphism $(\mathcal{F} \otimes \mathcal{G})_p \simeq \mathcal{F}_p \otimes \mathcal{G}_p$, then tensor with $\otimes_{\mathcal{O}_{X,p}} k(p)$. Recall the general result that if M, N are R-modules and I is an ideal of R, then we have:

$$(M \otimes_R N) \otimes_R (R/I) \simeq (M \otimes_R R/I) \otimes_{R/I} (N \otimes_R R/I)$$

First, you view both sides as R-modules then universal property of R-modules gives you a morphism from left to right as an R-module morphism, but you find that it's also a R/I-module morphism. Then you can see that universal property of the right hand side can be phrased as giving morphism:

$$M \times R/I \times N \times N$$

such that is R-multilinear and R/I bilinear at two R/I-slots and you can see that they are inverses of each other. $\square$

Definition 2.81. If X is irreeucible and $\mathcal{F}$ is a quasicoherent sheaf on X, then we will say that the rank of $\mathcal{F}$, not just the rank at a point p, to be the rank of $\mathcal{F}$ at the generic point.

Proposition 2.82. The rank of coherent sheaves on an integral scheme is additive in short exact sequences on integral schemes.

Proof. Caution, this only holds for the rank at the generic point, certainly not any point, for the reason that will be clear in the proof.

Recall that since we work with an integral scheme, then at the generic point η we have $\mathcal{O}_{X,\eta}$ is already a field, thus $\mathcal{F}_p$ is $\mathcal{F}|_p$, then if you consider the short exact sequence of stalks at the generic point, we conclude the assertion from the fact that dimension of vector spaces is additive in a short exact sequence.

This doesn't work more generally in any closed point since even if the stalification functor is exact, we still have to tensor with the residue field, which is not exact in general. $\square$

Now, we show that rank function for a finite type quasicoherent sheaf is upper semicontinuous:

Proposition 2.83. Let $\mathcal{F}$ be a finite type quasicoherent sheaf, then for any $n \in \mathbb{Z}$, the preimage of (∞, n) via the rank function is open.

Proof. Suppose that $p \in X$ is a point such that the rank of p is $k \leqslant n$. Then since $\mathcal{F}$ is finite type, we can apply the geometric Nakayama's lemma to conclude that there is an open neighborhood around p such that for each q in that open, $\mathcal{F}_q$ is generated by k elements, then we can again apply Nakayama's lemma to conclude that the dimension of $\mathcal{F}|_q$ is at most k, we are done. $\square$

This yields an important result: For finite rank quasicoherent sheaves on reduced schemes, constant rank implies it's a vector bundle.

Proposition 2.84. Suppose that X is a reduced scheme and $\mathcal{F}$ is a finite type quasicoherent sheaf on X, and the rank of $\mathcal{F}$ is constant, then $\mathcal{F}$ is locally free.

Proof. Let's first make it clear why we insist on the scheme to be reduced. For any reduced ring, since it has no nilpotents, we know that the intersection of all primes is 0, thus for any nonzero element x, there is a prime p such that $x \notin p$, therefore we know that after we localize at p, x becomes invertible.

Suppose that the rank of this sheaf is n at every point, then for any $p \in X$ by geometric Nakayama, there is an affine open $SpecA$ of that point such that $\mathcal{F}$ is $\widetilde{M}$ where M is a finitely generated A-module and $m_1, \cdots, m_n$ generates M and each M_q for $q \in SpecA$. Now consider the surjection:

$$A^n \to M$$

defined by mapping each standard basis to m_i. We claim that the kernel is 0, otherwise, suppose that $(a_1, \cdots, a_n)$ is in the kernel with a_i not 0, then since we work with a reduced scheme A is reduced, then we know that there is a $q \in SpecA$ such that $a_i \notin q$, then localize at q, we get a surjection:

$$A_q^n \to M_q$$

such that one of the factors in A_q is mapped to 0 completely, thus the generators of M_q are one less, thus the rank of $\mathcal{F}$ at q is only less, contradicting to the canstant rank condition.

This proves that locally $\mathcal{F}$ looks like $\mathcal{O}_X^n$ where n is the rank of $\mathcal{F}$. ☐

Proposition 2.85. Finite type quasicoherent sheaf on a reduced scheme is locally free on a dense open.

Proof. Recall that the upper semicontinuous is saying that $(-\infty, n)$ for each $n \in \mathbb{Z}$ is open, then since we are obly dealing with the integers, we know that $(-\infty, n])$ is also open. Suppose that p is a point in X such that the rank at p is minimum, say n_p, then we know that the subset of X consisting of those where the rank is n_p is open. Then it consists the generic points of the irreducible component p is in. Then we consider X to be the space minus that component, which is still open. Then find a point with the minimal rank, keep repeating this, we are done. ☐

Remark 2.86. The assertions above all uses the important assumption that the scheme in question is reduced. To see what happens without the assumption, consider $Speck[x]/(x^2)$, which is just a one point scheme, notice that the qusicoherent sheaf associated to k is of rank 1 at the only point, but it's not locally free.

These two propositions play well with the notion of map of vector bundles and you will see that for a reduced scheme, a morphism between vector bundles is a map of vector bundles iff it's of constant rank:

Proposition 2.87. Suppose that X is a reduced scheme and $\phi : \mathcal{E} \to \mathcal{F}$ is a morphism between vector bundles, then ϕ is a rank a map of vector bundles iff the vetor space map $\phi|_p : \mathcal{E}|_p \to \mathcal{F}|_p$ is of rank a for each p.

Proof. Now, consider the cokernel of this morphism, since we know tensor preserves cokernel, we know that the cokernel of $\phi|_p$ is just $(Coker\phi)|_p = Coker(\phi|_p)$, thus the assumption tells you that the cokernel is of constant rank, and since it's on a reduced scheme, we know that it's locally free, then a previous criterion tells you that this implies that ϕ is a rank a map of vector spaces. ☐

2.3 Pushforwards of Quasicoherent Sheaves

In this section and the next, we will discuss pushforwards and pullback of a quasicoherent sheaf, including their properties and some applications. To give some motivation, consider a ring map $B \to A$, then $(- \otimes_B A, -_B)$ is an adjoint pair between the category of A-modules and B-modules and we have the bifunctorial bijection:

$$Hom_B(M \otimes_B A, N) \simeq Hom_A(M, N_B)$$

You will see later that this construction plays well with localization and hence is compatible with quasi-coherent sheaves. In this point of view, the easier construction $M \mapsto M_B$ will correspond to pushforwards and the tensor product will correspond to the pull back. We will start in this section with the easier part, pushforward.

The main goal of this section is to tell you that in reasonable situations the pushforward of a quasi-coherent sheaf is quasi-coherent, and can be understood in terms of the construction discussed above. Let's first take a look at what happens in the affine case:

Proposition 2.88. Let $\pi : SpecA \to SpecB$ be a morphism of schemes induced by $B \to A$, suppose that M is an A-module, then we claim that there is an isomorphism $\pi_* \widetilde{M} \simeq \widetilde{M_B}$.

Proof. Recall how we defined the associated sheaf, they are the associated sheaf of sheaf on distinguished basis, thus if we can produce an isomorphism between the sheaf on associated basis, we are done by the equivalence between sheaves on a basis and sheaves on the space.

Let $f : B \to A$ be the ring map, recall that $\pi^{-1}(D(x)) = D(f(x))$ for $x \in B$. Then we know that on each distinguished basis $\pi\widetilde{M}(D(x))$, by definition is just $M_{f(x)}$. For $D(y) \subset D(x)$, we know that $D(f(y)) \subset D(f(x))$ and the restriction:

$$res : \pi\widetilde{M}(D(x)) \to \pi\widetilde{M}(D(y))$$

is just the restriction:

$$res : M_{f(x)} \to M_{f(y)}$$

Then what about for $\widetilde{M_B}$, on $D(x)$, it's just $(M_B)_x$ and the restriction is just the natural map:

$$res : (M_B)_x \to (M_B)_y$$

But notice that here we are viewing M as a B-module via restriction. Notice that via the natural ring morphism $B_x \to A_{f(x)}$, we can view $M_{f(x)}$ as a B_x-module, and the universal property of $(M_B)_x$ gives you a natural morphism of B_x-modules:

$$(M_B)_x \to M_{f(x)}$$

You can check this is an isomorphism since surjective is easy to check and injectivity is also easy to check if you keep in mind how we view $M_{f(x)}$ as an B_x-module. It's also easy to check this commutes with the restrictions, thus an isomorphism on the distinguished basis, we are done. □

In particular, $\pi_* \widetilde{M}$ is also quasi-coherent. Therefore, it's not a surprise to expect:

Proposition 2.89. Suppose that $f : X \to Y$ is an affine morphism of schemes, and $\mathcal{F}$ is quasicoherent on X, then $f_* \mathcal{F}$ is quasicoherent on Y.

Proof. Suppose that $SpecB$ is an affine open on Y, then since π is affine, we know that locally on the inverse image of $SpecB$, we just have $\pi : SpecA \to SpecB$. There are some compatibility condition you need to check, for example, suppose that we have a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "g"] & Y \\ X \arrow[u, "\simeq"] \arrow[r, "f"] & Y \arrow[u, "\simeq"] \end{tikzcd}$$

then the pushforward of $f_* \mathcal{F}$ and $g_* \mathcal{F}$ are isomorphic. And also you need to check that pushforward is compatible with restrictions, but these are all strightforward.

At the end of the day, you will reduce the problem to the affine case, which is the one we discussed already. □

The following exercise generalize the above situation and tells you that qcqs morphisms always push forward quasicoherent to quasicoherent sheaves:

Proposition 2.90. Suppose that $\pi : X \to Y$ is a qcqs morphism and let $\mathcal{F}$ be a quasicoherent sheaf on X, then $\pi_* \mathcal{F}$ is also quasicoherent.

Proof. This is a good reminder of the qcqs lemma we prove before. It states that if X is a qcqs scheme and if f is a global section of $\mathcal{O}_X$, then the restriction from $\Gamma(X,\mathcal{F})$ to $\Gamma(X_f,\mathcal{F})$ is precisely the localization of f. This is a direct consequence of localization commutes with direct sums, but not arbitrary direct products.

To see how this helps our discussion here, again suppose that $SpecB$ is an affine open of Y, then the assumption that π is qcqs implies that the preimage is a qcqs space. Suppose that this is induced by the ring morphism $f: B \to \Gamma(\pi^{-1}SpecB, \mathcal{O}_X)$ Then we know that by definition, $\pi_*\mathcal{F}(SpecB) = \mathcal{F}(\pi^{-1}SpecB)$ and the restriction to $SpecB_x$ is induced by:

$$\begin{tikzcd}
 & \mathcal{F}(\pi^{-1}SpecB) \arrow[r, "="] \arrow[dl, "f(x)"'] \arrow[d, "res"'] & \pi_*(\mathcal{F})(SpecB) \arrow[d, "res"] \\
\mathcal{F}(\pi^{-1}SpecB)_{f(x)} \arrow[r, "\simeq"'] & \mathcal{F}((\pi^{-1}SpecB)_{f(x)}) \arrow[r, "="'] & \pi_*(\mathcal{F})(SpecB_x)
\end{tikzcd}$$

where the localization on the left comes from the qcqs lemma.

Thus by this diagram you can easily identify the vertical restriction on the right to be localization at x when viewed as a B-module instead of a $\Gamma(\pi^{-1}SpecB,\mathcal{O}_X)$-module. Then recall the criterion we had before saying that it's quasi-coherent sheaf iff the restriction is localization for each affine open, then we are doe. $\square$

Remark 2.91. If we don't have qcqs property, there are counterexamples where a quasicoherent sheaf is not pushed forward to a quasicoherent sheaf.

On the other hand, even if we have qcqs condition and work with quite nice schemes, coherent sheaf is not usually pushed forward to a coherent sheaf. For example consider the structure sheaf on $Speck[t]$ and consider the affine morphism $Speck[t] \to Speck$, notice that the pushforward is just $\widetilde{k[t]}$ viewed as a k-vector space, this is clearly not a finitely generated k-vector space.

However, when the morphism is finite between locally Noetherian schemes, then coherent sheaves indeed gets pushed forward to coherent sheaves:

Proposition 2.92. Let $\pi: X \to Y$ be a finite morphism of locally Noetherian scheme and $\mathcal{F}$ is a coherent sheaf on X, then $\pi_*\mathcal{F}$ is coherent on Y.

Proof. We already showed $f_*\mathcal{F}$ is quasicoherent since finite morphism is a special class of affine morphisms. Then we only have to show that when the ring map is finite $B \to A$ a finitely generated A-module M, is also a finitely generated B module when we think of M as M_B, which is easy to show. $\square$

Remark 2.93. What we proved above without the Noetherain hypothesis is just finite type quasicoherent sheaves are pushed forward to finite type quasicoherent sheaves by finite type morphisms.

In a much later point, you will see that pushforwards of coherent sheaves by projective and proper morphisms also tend to be coherent if the unerlying space is locally Noetherian.

Before we end the discussion of pushforward, we give a definition that's useful later:

Definition 2.94. Suppose that $\pi: X \to Y$ is a finite morphism then we already saw that $\pi_*\mathcal{O}_X$ is a finite type quasicoherent sheaf on Y. Let $p \in Y$ be a point, then the rank of $(\pi\mathcal{O}_X)_p$ at the point p is called the degree of the finite morphism at p.

Recall that this is a upper semicontinuous function on Y.

Proposition 2.95. Let $\pi: X \to Y$ be a finite morphism, then the degree of π at p is the dimension of the spaces of functions on the scheme theoretic preimage of p, considered as a vector space over $k(p)$. In particular if $\pi^{-1}(p)$ is empty, then the rank is 0 at p.

Proof. This problem is only a local question since both the fiber of p and the rank are only local properties. Then suppose that $p \in SpecB$ and the inverse image is $SpecA$ where $B \to A$ is finite. Then consider the fiber diagram:

$$\begin{tikzcd}
SpecA \otimes_B k(p) \arrow[r] \arrow[d] & Speck(p) \arrow[d] \\
SpecA \arrow[r] & SpecB
\end{tikzcd}$$

notice that $\widetilde{A}$ is pushed forward to $\widetilde{A_B}$, then we know that the fiber at p is just $(A \otimes_B B_p) \otimes_{B_p} B_p/pB_p$, then it's isomorphic to $A \otimes_B (B_p \otimes k(p)) \simeq A \otimes_B k(p)$, which proves the assertion. $\square$

2.4 Pullbacks of Quasicoherent Sheaves: Three Different Perspectives

Now, let's investigate the pullback of quasicoherent sheaf. Recall that we have already seen the idea of the pullback of an $\mathcal{O}_Y$-module defined by $\pi^*\mathcal{G} = \pi^{-1}\mathcal{G} \otimes_{\pi^{-1}\mathcal{O}_Y} \mathcal{O}_X$. And we have already seen how we can pullback vector bundles and still get a vector bundle. Also, you can easily check that if $U \hookrightarrow X$ is an open subscheme, then the pullback is just restriction to the open subscheme. Also, for example, if $p \in Y$ is a point in a scheme, then pulling back of any $\mathcal{O}_Y$-module $\mathcal{F}$ along $p \hookrightarrow Y$, is just computing the fiber of $\mathcal{F}$ at p, which you can also easily verify. The reason we are using similar notations for $\mathcal{F}|_U$ and $\mathcal{F}|_p$ is precisely because that they both come from pullbacks of the sheaf along some kind of inclusion.

More generally speaking, if $\pi : X \to Y$ is some sort of inclusion morphism, for example, a locally closed embedding or inclusion of some point, we often call $\pi^*\mathcal{G}$ the restriction of $\mathcal{G}$ to X and often written as $\mathcal{G}|_X$.

Before we study what happens when pulling back quasicoherent sheaves, let's make our strategy clear: We will introduce three different approaches to understand pullbacks of quasicoherent sheaves. Each of them has advantage and disanvantage, that is why we should know all of them, which together fit into a good picture.

1. Since we usually understand quasicoherent sheaves in terms of affine opens and the module over that ring. The first interpretation is along this vein, which will be useful in proving and understanding facts. However, since this is a local interpretation, the gluing argument follows that makes the definition confusing. Therefore, we will use other approaches and the come back to show that it agrees with our local expectation.

2. The second approach is similar to how we understand fibered products. The gluing arguments there is also confuing and annoying, which is simplified a lot using universal property argument. Therefore, we will try to imitate this experience and give the universal property definition.

   This will be elegant but still needs a construction, and because it works in $\mathcal{O}_X$-modules, it's not clear from the universal property that the object is qusicoherent. However, our construction in the first approach tells you that if the target is affine, then it satisfies the universal property. Furthermore, you will see the universal property is local on the target, in the sense that if $\pi : X \to Y$ is a morphism and $U \subset Y$ is an open subscheme, for any quasicoherent sheaf $\mathcal{G}$ on Y, if $\pi^*\mathcal{G}$ exists, then it's restriction to $\pi^{-1}U$ is canonically identified with $(\pi|_U)^*(\mathcal{G}|_U)$. Therefore if the pullback exists in general you can show that locally it looks like the stuff we did in the first approach.

3. To see that the pullback in general exists, we give the third perspective, which is a short abstract construction that can be proved to satisfies the property easily, then thanks to the discussion in the first and second approach, the understanding now is complete.

Approach 1: Description when the target is affine To begin our first approach, let's suppose that $\pi : X \to Y$ is a morphism of schemes and $\mathcal{G}$ is a quasicoherent sheaf on Y. Suppose that $SpecB$ is an affine open and $\mathcal{G}|_{SpecB} = \widetilde{N}$, we would like to have the fact that if $SpecA$ is in $\pi^{-1}SpecB$, then $\Gamma(SpecA, \pi^*\mathcal{G}) = N \otimes_B A$, which is motivated that pullback has some kind of tensors built in and this is the most natural way we can think of right now. To do this, let's assume that $\phi : \pi^{-1}SpecB \to SpecB$ is the restriction of π. We will use the distinguished affine base to define a quasicherent sheaf on $\pi^{-1}SpecB$.

Suppose that $SpecA$ is in the inverse image. We define:

$$\Gamma(SpecA, \phi^*\mathcal{G}) = N \otimes_B A$$

for each such affine open. Then you can define the restriction to the distinguished open $SpecA_f$ to be the natural map:

$$N \otimes_B A \to N \otimes_B (A_f)$$

Then you find out that you have a diagram:

$$\begin{tikzcd} & N\otimes_B A \arrow[dl, "loc"'] \arrow[dr, "res"] & \\ (N\otimes_B A)_f \arrow[rr, "\simeq"] & & N\otimes_B A_f \end{tikzcd}$$

then the cirteria of quasicoherent sheaves we had before tells you that this indeed defines a quasicoherent sheaf on $\pi^{-1}SpecB$ by taking the associated sheaf. You can try to glue this construction together, but we don't encourage you to do this, since it's not so enlightening and we have a better way of doing this.

Remark 2.96. You should notice that the above construction, when Y is affine clearly defined a functor from $Qcoh(Y) \to Qcoh(X)$ and we will temporarily call this functor P which try to suggest this will be the pullback functor once we know that pullback exists in general.

Approach 2: Universal property description: Again, let $\pi : X \to Y$ be a morphism of ringed spaces, we define the pullback functor π^* from the category of $\mathcal{O}_Y$-modules to the category of $\mathcal{O}_X$-modules to be the left adjoint of π_*, which means that there is a bifunctorial isomorphism:

$$Hom_{\mathcal{O}_X}(\pi^*\mathcal{G}, \mathcal{F}) \simeq Hom_{\mathcal{O}_Y}(\mathcal{G}, \pi_*\mathcal{F})$$

Recall that if the right adjoint π^* functor exists, it's unique up to isomorphism, we just need to show that there is such a functor that satisfies the property.

Proposition 2.97. If Y is affine and $\mathcal{G}$ is quasicoherent, then the construction in approach 1 satisfies the bifunctorial isomorphism property.

Proof. Remember that only $\mathcal{G}$ here is assumed to be quasicoherent, while $\mathcal{F}$ is only assumed to be an $\mathcal{O}_X$-module.

Recall that we have a bifunctorial isomorphism:

$$Hom_{\mathcal{O}_Y}(\mathcal{G}, \pi_*\mathcal{F}) \simeq Hom_B(N, \Gamma(X, \mathcal{F}))$$

defined by taking the global section functor (you should check that this is really functorial in both $\mathcal{F}$ and $\mathcal{G}$ since we didn't mention it when we talked about the isomorphism).

Now we claim that given any morphism in $Hom_B(N, \Gamma(X, \mathcal{O}_X))$, we can construct a morphism in $Hom_{\mathcal{O}_X}(\pi^*\mathcal{G}, \mathcal{F})$ in a bijective and bifunctorial way. To see this recall that we have a bifunctorail bijective homomoprhism between $Hom_{\mathcal{O}_X}(\pi^*\mathcal{G}, \mathcal{F})$ and the morphism between their restriction on the distinguished basis, which form the following sequence:

$$Hom_{\mathcal{O}_X}(\pi^*\mathcal{G}, \mathcal{F}) \simeq Hom_{\mathcal{O}_X^b}(\pi^*\mathcal{G}^b, \mathcal{F}^b) \simeq Hom_{\mathcal{O}_X}(\pi^*\mathcal{G}^{ba}, \mathcal{F}^{ba}) \simeq Hom_{\mathcal{O}_X}(\pi^*\mathcal{G}, \mathcal{F})$$

where a stands for taking the associated sheaf and b stands for taking the restriction to the distinguished basis. This comes from that $-^b$ and ^{a} from an equivalence from the category of $\mathcal{O}_X$-modules to the category of $\mathcal{O}_X$-modules on a distinguished basis. The last morphism comes from the fact that $-^{ba} \simeq id$.

There if we prove that we have a bifunctorial isomorphism between $Hom_B(N, \mathcal{F}(X))$ and $Hom_{\mathcal{O}_X^b}(\pi^*\mathcal{G}^b, \mathcal{F}^b)$, we are done. First, recall that from the ring morphism $B \to \Gamma(X, \mathcal{O}_X)$, we have an adjoint pair:

$$Hom_B(N, \mathcal{F}(X)) \simeq Hom_{\Gamma(X,\mathcal{O}_X)}(N \otimes_B \Gamma(X, \mathcal{O}_X), \Gamma(X, \mathcal{F})_B)$$

Then given a morphism in $Hom_{\Gamma(X,\mathcal{O}_X)}(N \otimes_B \Gamma(X, \mathcal{O}_X), \Gamma(X, \mathcal{F})_B)$, we can form a morphism on the distinguished basis as following diagram suggests:

$$\begin{tikzcd} N\otimes_B \Gamma(X,\mathcal{O}_X) \arrow[r, "\psi"] \arrow[d] & \Gamma(X,\mathcal{F}) \arrow[d, dashed, "res"] \\ N\otimes_B \Gamma(SpecA,\mathcal{O}_X) \arrow[r] \arrow[d, "res"] & \Gamma(SpecA,\mathcal{F}) \arrow[d, dashed, "res"] \\ N\otimes_B \Gamma(SpecA_f,\mathcal{O}_X) \arrow[r] & \Gamma(SpecA_f,\mathcal{F}) \end{tikzcd}$$

where the two vertical morphisms down below are defined by $x \otimes a \mapsto a \cdot \psi(x \otimes 1)|_{res}$. Then you can check this is indeed a morphism of the distinguished basis.

We still have to show this is a bijective assignment, which is accomplished by constructing an inverse. To do this ,suppose that we have a morphism on the distinguished basis, we define a morphism $N \otimes \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{F})$ still according to the diagram above, the sheaf axioms tells you that it's well defined.

Then you have to check that the composition give you identity, which is easy to do. For example, since the top horizontal morphism is $\Gamma(X, \mathcal{O}_X)$-linear, you only have to show that $n \otimes 1$ is mapped to the same element by the morphism you start with and the one you have later. But you will find this is nothing but the identity axiom of sheaves. I will leave the other direction for you to check.

You also will have to check the bifunctoriality (you can still interpret N and $\Gamma(X, \mathcal{F})_b$ as some objects you get by applying functors to $\mathcal{G}$ and $\mathcal{F}$ thus the bifunctoriality here still refers to $\mathcal{F}$ and $\mathcal{G}$), which I will also left for you. □

Therefore, at least in the case of affine targets, we indeed find something that satisfies the universal property and is indeed quasicoherent and behaves as well as what we expected in affine opens of the source.

Remark 2.98. Notice that this doesn't guarantee the existence of π^* in general since we only considered quasicoherent sheaves $\mathcal{G}$ while this pullback functor should be defined for all $\mathcal{O}_Y$-modules. What the above proposition tells you is that if π^* exists in general, then since the isomorphism is functorial in $\mathcal{G}$ then what you get is for a quasicoherent sheaf $\mathcal{G}$, we have a bifunctorial isomrophism:

$$Hom_{\mathcal{O}_X}(\pi^*\mathcal{G}, \mathcal{F}) \simeq Hom_{\mathcal{O}_X}(P\mathcal{G}, \mathcal{F})$$

Then the general Yoneda lemma argument gives you a isomorphism between $P\mathcal{G}$ and $\pi^*\mathcal{G}$.

Another important case where we can find something quasicoherent without much effort is the case of an open embedding:

Proposition 2.99. Suppose that $i : U \hookrightarrow X$ is an open embedding of ringed spaces and $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $\mathcal{F}|_U$ satisfies the universal property $i^*\mathcal{F}$:

$$Hom_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{E}) \simeq Hom_{\mathcal{O}_X}(\mathcal{F}, i_*\mathcal{E})$$

thus $\mathcal{F}|_U$ is just $i^*\mathcal{F}$ by the uniqueness of adjoint.

Proof. This is just an easy check since if you think about what is a morphism $\varphi : \mathcal{F} \to i_*\mathcal{E}$, you will find that for each $V \subset X$, it gives you a map:

$$\mathcal{F}(V) \to \mathcal{E}(U \cap V)$$

in a way that is compatible with restrictions, thus this readily gives you a morphism $\mathcal{F}|_U \to \mathcal{E}$ by just take those $V \subset U$.

Conversely, given any morphism $\mathcal{F}|_U \to \mathcal{E}$, we obtain a morphism $\mathcal{F} \to i_*\mathcal{E}$ where we assign the following morphism for each V:

$$\mathcal{F}(V) \to \mathcal{F}(V \cap U) \to \mathcal{E}(V \cap U)$$

You only have to check that these two constructions are inverses of each other, which I left for you. □

Next let's show that this universal property is local on the target, in the sense that if π^* exists with the morphism $\pi : X \to Y$, then if $V \hookrightarrow Y$ is any open subset let $U = \pi^{-1}V$ and we consider the following Cartesian diagram of open embeddings:

$$\begin{tikzcd} U \arrow[r, hook, "i"] \arrow[d, "\pi|_U"] & X \arrow[d, "\pi"] \\ V \arrow[r, hook, "j"] & Y \end{tikzcd}$$

We claim that $\pi|_U^*$ also exists. To describe what it looks like, recall that we hvae the extension of 0 functor $j_!$ and takes each $\mathcal{O}_V$-module $\mathcal{G}'$ to an $\mathcal{O}_Y$-module, and we know that $(j_!\mathcal{G})|_V = \mathcal{G}$, then you can consider the following sequence for each $\mathcal{O}_U$-module $\mathcal{F}$ and each $\mathcal{O}_V$-module $\mathcal{G}$:

$$\begin{aligned} Hom_{\mathcal{O}_U}((\pi^*(j_!\mathcal{G}))|_U, \mathcal{F}) &\simeq Hom_{\mathcal{O}_X}(\pi^*(j_!\mathcal{G}), i_*\mathcal{F}) \\ &\simeq Hom_{\mathcal{O}_Y}(j_!\mathcal{G}, \pi_* i_* \mathcal{F}) \\ &\simeq Hom_{\mathcal{O}_Y}(j_!\mathcal{G}, j_*(\pi|_U)_*\mathcal{F}) \\ &\simeq Hom_{\mathcal{O}_V}((j_!\mathcal{G})|_V, (\pi|_U)_*\mathcal{F}) \\ &= Hom_{\mathcal{O}_V}(\mathcal{G}, (\pi|_U)_*\mathcal{F}) \end{aligned}$$

Each isomorphism is bifunctorial, thus the composition is bifunctorial therefore we know that the functor $(\pi^*(j_!-))|_U$ is the adjoint of $(\pi|_U)^*$ by the uniqueness of adjoint.

Then thanks to this, we know that if Y is an open subscheme of an affine open, then for any $\pi : X \to Y$, we know that π^* exists since we can consider $i \circ \pi : X \to Y \hookrightarrow SpecB$, which has an adjoint.

Approach 3: Pullback exists via explicit construction without gluing

Imitating the definition for pulling back vector bundles, we will define:

Definition 2.100. Suppose that $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces and $\mathcal{G}$ is an $\mathcal{O}_Y$-module, we define the pullback of $\mathcal{G}$, denoted by $\pi^*\mathcal{G}$ to be:

$$\pi^*\mathcal{G} = \pi^{-1} \otimes_{\pi^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

Recall that we have showed when we discussed that the above definition indeed gives you a left adjoint of π_*, at least as a functor from $\mathcal{O}_Y$-modules to $\mathcal{O}_X$-modules. Now we begin to show that this indeed gives a functor from $Qcoh_Y \to Qcoh_X$.

Proposition 2.101. Suppose that $\mathcal{G}$ is a quasicoherent shaef on Y and $\pi : X \to Y$ is a morphism of schemes, then we claim that $\pi^*\mathcal{G}$ is quasicoherent on X.

Proof. Suppose $V \subset Y$ is such an affine open, we denote it's inverse image by U, then we know from the previous proposition that the pullback $\pi|_U : U \to V$ exists and we know that $(\pi^*\mathcal{G})|_U \simeq (\pi|_U)^*(\mathcal{G}|_V)$, since $\mathcal{G}|_V$ is quasicoherent on V, we proved that the pullback of a quasicoherent sheaf on an affine scheme is quasicoherent, thus we know that $(\pi^*\mathcal{G})|_U$ is quasicoherent, since when V varies all the affine opens we know that their images U cover X, then $\pi^*\mathcal{G}$ is quasicoherent since $(\pi^*\mathcal{G})|_U$ is quasicoherent. □

Remark 2.102. Let $\mathcal{G}$ be a quasicoherent sheaf on Y, then we can interpret a section s of $\mathcal{G}$ as a morphism of $\mathcal{O}_X$-modules:

$$s : \mathcal{O}_Y \to \mathcal{G}$$

that maps the global section 1 to s.

Now if you apply π^* to this morphism, you get a morphism:

$$\pi^*s : \mathcal{O}_X \to \pi^*\mathcal{G}$$

and we will call this section the pullback of the section s.

Recall that when $\pi : X \to Y$ is qcqs, we proved that it pushes quasicoherent sheaves to qusicoherent sheaves, then the following proposition is immediate:

Proposition 2.103. Let $\pi : X \to Y$ be a qcqs morphism, then (π^*, π_*) is an adjoint pair between $Qcoh_X$ and $Qcoh_Y$.

Proof. Clear. □

Remark 2.104. We already proved that the inverse image functor is exact and the tensor functor is right exact, this tells that the pullback functor is right exact.

However, later we will see that for an important class of morphisms called flat morphisms, the pullback is exact.

We will now see that pullback preserves finite type and finitely presented quasicoherent sheaves:

Proposition 2.105. Suppose that $\pi : X \to Y$ is a morphism of schemes and $\mathcal{G}$ is a finite type or finitely presented quasicoherent sheaf on Y, then $\pi^*\mathcal{G}$ is finite type or finitely presented as well.

Proof. let's first deal with the finite type case: Recall that if $\mathcal{G}$ is quasicoherent on X, then $\mathcal{G}$ is quasicoherent if we can find an open cover U of X such that on each U we have a surjective morphism:

$$\mathcal{O}_U^n \to \mathcal{G}|_U$$

In our question, since $\mathcal{G}$ is finite type on Y, then for any affine open V of Y, we have a surjective morphism:

$$\mathcal{O}_V^n \to \mathcal{G}|_V$$

since the pullback functor is right exact, we can apply it and still gets a surjective morphism, we can apply $(\pi|_U)^*$ to the above surjection and get:

$$\mathcal{O}_U^n \to \pi^*(\mathcal{G})|_U$$

since $(\pi|_U(-))^* \simeq (\pi^*(-))|_U$.

Similarly, we know that a quasicoherent sheaf is finitely presented on X if there is an open cover on X such that on each U in the cover we have an exact sequence:

$$\mathcal{O}_U^m \rightharpoonup \mathcal{O}_U^n \to \mathcal{G}|_U \to 0$$

again we use the right exactness of π^* and $\mathcal{G}$ is finitely presented on Y to get the desired result. ☐

Proposition 2.106. Suppose that $\xi : W \to X$ and $\pi : X \to Y$ are two morphism of schemes, then there is a canonical functor isomorphism:

$$\xi^* \circ \pi^* \simeq (\pi \circ \xi)^*$$

Proof. This is again an easy application of the Yoneda argument where we can show that $\xi^* \circ \pi^*$ and $(\pi \circ \xi)^*$ are all left adjoint to $(\pi \circ \xi)_* \simeq (\pi_* \circ \xi_*)$. I will let you do it. ☐

We also want to know what the stalk and fiber of the pullback looks like:

Proposition 2.107. Suppose that $\pi : X \to Y$ is a morphism of schemes with $\pi(p) = q$ and $\mathcal{G}$ is a quasicoherent sheaf on Y, then we have natural isomorphisms:

$$(\pi^*\mathcal{G})_p \simeq \mathcal{G}_q \otimes_{\mathcal{O}_{Y,q}} \mathcal{O}_{X,p}$$

and

$$(\pi^*\mathcal{G})|_p \simeq \mathcal{G}|_q \otimes_{k(q)} k(p)$$

Proof. Actually, you don't even have to use the quasicoherent hypothesis.

The first identification comes from the identification of stalks of tensors and stalks of inverse images. To see the second identification, we need a small algebra fact: Suppose that $B \to A$ such that the maximal ideal $m \subset A$ has inverse image $n \subset B$, then we have a canonical morphism $B/n \to A/m$. Suppose we have a B-module N, then we have the following two tensor product:

$$(N \otimes_B A) \otimes_A (A/m) \qquad (N \otimes_B B/n) \otimes_{B/n} (A/m)$$

We claim that they are isomorphic A/m-modules since they both isomorphic to: $N \otimes_B (A/m)$, you just need to use the universal property to define morphisms and check they are inverses, which I will let you do.

Then for the second identification, first notice we have a local ring morphism $\mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$, then you have $(\pi^*\mathcal{G})|_p = (\pi^*\mathcal{G})_p \otimes_{\mathcal{O}_{X,p}} k(p)$, which is isomorphic to $(\pi^*\mathcal{G}_p \otimes_{\mathcal{O}_{Y,q}} k(q)) \otimes_{k(q)} k(p)$ by the isomorphism we just had, then we are done.

And if you are willing to use the quasicoherent condition, you can do the computation on an affine open $SpecA \to SpecB$, which still will lead to to similar computations we did above. ☐

Pullback also plays well with tensor products in the sense that:

Proposition 2.108. Suppose that $\pi : X \to Y$ is a morphism of schemes and $\mathcal{G}$ and $\mathcal{F}$ are quasicoherent sheaves on Y, we have:

$$\pi^*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}) \simeq \pi^*\mathcal{F} \otimes_{\mathcal{O}_X} \pi^*\mathcal{G}$$

Proof. We will use general universal property Yoneda argument to prove this. To see how this works, we first claim that for any $\mathcal{O}_Y$-module $\mathcal{G}$ and $\mathcal{O}_X$-module $\mathcal{F}$, we have the following equation:

$$\pi_*\mathcal{H}om_{\mathcal{O}_X}(\pi^*\mathcal{G}.\mathcal{F}) = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{G}, \pi_*\mathcal{F})$$

To see how this works, suppose that U is any open of Y and let V be the inverse image under π, then $\pi_*\mathcal{H}om_{\mathcal{O}_X}(\pi^*\mathcal{G}.\mathcal{F})(U) = Hom_{\mathcal{O}_V}(\pi^*\mathcal{G}|_V, \mathcal{F}|_V) = Hom_{\mathcal{O}_U}(\mathcal{G}|_U, (\pi|_U)_*\mathcal{F}|_V) \simeq Hom_{\mathcal{O}_U}(\mathcal{G}|_U, (\pi_*\mathcal{F})|_U) = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{G}, (\pi_*\mathcal{F}))(U)$ and the restriction is as expected, therefore we have the canonical identification which is bifunctorial in both $\mathcal{F}$ and $\mathcal{G}$.

Then consider the sheaf version of tensor hom adjunction:

$$\begin{aligned} Hom_{\mathcal{O}_X}(\pi^*\mathcal{G} \otimes_{\mathcal{O}_X} \pi^*\mathcal{G}', \mathcal{F}) &\simeq Hom_{\mathcal{O}_X}(\pi^*\mathcal{G}, \mathcal{H}om_{\mathcal{O}_X}(\pi^*\mathcal{G}', \mathcal{F})) \\ &\simeq Hom_{\mathcal{O}_Y}(\mathcal{G}, \pi_*\mathcal{H}om_{\mathcal{O}_X}(\pi^*\mathcal{G}', \mathcal{F})) \\ &\simeq Hom_{\mathcal{O}_Y}(\mathcal{G}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}', \pi_*\mathcal{F})) \\ &\simeq Hom_{\mathcal{O}_Y}(\mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{G}', \pi_*\mathcal{F}) \end{aligned}$$

Notice that this sequence of isomorphisms are all functorial in $\mathcal{F}$, even though it's not so easy to make a statement about the functoriality of $\mathcal{G}$ and $\mathcal{G}'$. But this is enough for us to conclude that $\pi^*\mathcal{G} \otimes_{\mathcal{O}_X} \pi^*\mathcal{G}'$ is canonically isomorphic to $\pi^*(\mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{G}')$.

Here being quasicoherent only helps us prove that there is an affine cover of X such that locally on each affine cover, the both sides are isomorphic. But this doesn't help the global isomorphism since there may be affine opens whose images are not contained in any affine opens of Y, thus we are not able to use the distinguished affine base to conclude they are isomorphic. □

As usual we have the push-pull diagram:

Definition 2.109. Suppose that we have a commutative diagram of schemes:

$$\begin{tikzcd} X' \arrow[r, "\phi"] \arrow[d, "\tau"'] & X \arrow[d, "\pi"] \\ Y' \arrow[r, "\psi"'] & Y \end{tikzcd}$$

where π, τ are both, then we have a natural functor morphism between:

$$\psi^* \circ \pi_* \to \tau_* \circ \phi^*$$

where both functors are from $Qcoh_X$ to $Qcoh_{Y'}$.

Remark 2.110. The morphism is again from the following composition:

$$Hom_{\mathcal{O}_{X'}}(\phi^*\mathcal{F}, \phi^*\mathcal{F}) \to Hom_{\mathcal{O}_X}(\mathcal{F}, \phi_*\phi^*\mathcal{F}) \xrightarrow{\pi_*} Hom_{\mathcal{O}_Y}(\pi_*\mathcal{F}, \pi_*\phi_*\phi^*\mathcal{F}) = Hom_{\mathcal{O}_Y}(\pi_*\mathcal{F}, \psi_*\tau_*\phi^*\mathcal{F}) \simeq Hom_{\mathcal{O}_{Y'}}(\psi^*\pi_*\mathcal{F}, \tau_*\phi$$

where we look at the image of $id \in Hom_{\mathcal{O}_{X'}}(\phi^*\mathcal{F}, \phi^*\mathcal{F})$, notice that each morphism is bifunctorail, which can be used to show the functoraility of the of the morphism:

$$\psi^*\pi_*\mathcal{F} \to \tau_*\pi^*(\mathcal{F})$$

I will let you do it.

Here we are requiring π, τ to be qcqs since we want the pushforward to still be quasicoherent.

Example 2.111. Suppose that Y' is the canonical inclusion of a point q in Y, and the diagranm is the fiber diagram, then the above push-pull map becomes:

$$(\pi_* \mathcal{F})_q \otimes_{\mathcal{O}_{Y,q}} k(q) \to \Gamma(X', \phi^* \mathcal{F}) = \Gamma(\pi^{-1}q, \mathcal{F}|_{\pi^{-1}q})$$

You might expect that this is an isomorphism, but this is not generally true except you assume just the right condition. We will explore this later.

The following proposition is called **the projection formula** for pullback and pushforward, which can be generalized to higher direct images and higher inverse images after we know cohomology:

Proposition 2.112. Let $\pi : X \to Y$ be qcqs and $\mathcal{F}, \mathcal{G}$ are quasicoherent on X, Y respectively, then there is a natural morphism:

$$(\pi_* \mathcal{F}) \otimes \mathcal{G} \to \pi_*(\mathcal{F} \otimes \pi^* \mathcal{G})$$

moreover, if $\mathcal{G}$ is locally free or if π is affine this is an isomorphism.

Proof. We will only sketch the proof since the full detail is just annoying.

Let's first see what is the natural morphism. Recall that by the adjointness property, a morphism $(\pi_* \mathcal{F}) \otimes \mathcal{G} \to \pi_*(\mathcal{F} \otimes \pi^* \mathcal{G})$ is the same as a morphism:

$$\pi^* \pi_* \mathcal{F} \otimes \pi^* \mathcal{G} \to \mathcal{F} \otimes \pi^* \mathcal{G}$$

then this morphism is given by the identity $\pi_* \mathcal{F} \to \pi_* \mathcal{F}$ and the identity $\pi^* \mathcal{G} \to \pi^* \mathcal{G}$.

Now we claim that when $\mathcal{G}$ is free, then the natural morphism is an isomorphism. To see this, first we claim that the adjointness property "commutes" with direct sums, in the sense that if you look at the adjunction isomorphism:

$$Hom(\pi^* \mathcal{F}_1 \oplus \pi^* \mathcal{F}_2, \mathcal{G}_1 \oplus \mathcal{G}_2) \simeq Hom(\mathcal{F}_1 \oplus \mathcal{F}_2, \pi_* \mathcal{G}_1 \oplus \pi_* \mathcal{G}_2)$$

and you start with some diagonal morphism, in the sense that $\pi \mathcal{F}_i \to \mathcal{G}_j = 0$ when $i \neq j$, then the image on the right hand side is also a diagonal morphism. To prove this, it reduces to prove that the inverse image functor satisfies the property, since the tensor functor satisfies the property.

Thus we can assume that $\mathcal{G}$ is just the structure sheaf. And we know that this adjunction property respects restrictions that we can check that when $\mathcal{G}$ is locally free, we can reduce the problem to an open where $\mathcal{G}$ is free, we are done.

To prove this prove for an affine morphism, we still use the natural morphism, but we can see how it looks like in the affine case $SpecA \to SpecB$ say $\mathcal{F} = \widetilde{M}$ and $\mathcal{G} = \widetilde{N}$, then the morphism correspond to a morphism of modules:

$$M_B \otimes_B N \to (M \otimes_A (N \otimes_B A))_B$$

which comes from the adjunction map:

$$M_B \otimes_B A \otimes_A (N \otimes_B A) \to M \otimes_A (N \otimes_B A)$$

Then you can check that it's just correspond to the identity, thus we are done. □

2.5 The Quasicoherent Sheaf Corresponding to a Graded Module

The last section of this review will be focused on describe quasicoherent sheaves on a projective A-scheme. Recall that a projective A-scheme is produced from a finitely generated graded ring over A which is graded by $\mathbb{Z}_{\geqslant 0}$.

Now we consider a graded $S_\bullet$-module $M_\bullet$, graded by $\mathbb{Z}$. Here we are using $\mathbb{Z}$ as the grading not $\mathbb{Z}_{\geqslant 0}$, but you will see that this doesn't affect our construction below, just makes some statement cleaner.

We will describe a sheaf $\widetilde{M_\bullet}$ on $ProjS_\bullet$ as follows (notice that this sheaf is not graded in any sense, it's just an associated sheaf of a graded module): Recall that $ProjS_\bullet$ is covered by $D(f)$ for all f homogeneous of positive degree and we know that $D(f) \simeq Spec((S_\bullet)_f)_0$. Similarly, recall that for any such f, we can consider the graded $(S_\bullet)_f$-module $(M_\bullet)_f$, where the ring and the module are all $\mathbb{Z}$-graded. Now consider the zero degree piece $((M_\bullet)_f)_0$, which is a $((S_\bullet)_f)_0$-module. Then we define a sheaf on $D(f)$ by $\widetilde{((M_\bullet)_f)_0}$. We now only have to show that they glue together on $ProjS_\bullet$ by showing the cocycle condition.

Suppose that we have $D(f)$ and $D(g)$, then recall that on the intersection $D(fg)$ is, the sheaves $\widetilde{((M_\bullet)_f)_0}$ and $\widetilde{((M_\bullet)_g)_0}$ restrict to $\widetilde{((M_\bullet)_{fg})_0}$ just like what we did when we describe $ProjS_\bullet$, thus there is an obvious isomorphism. You should appropriately interpret the statement we made before, which can be really confusing when you first thing about it. We are of course think the restriction of $\widetilde{((M_\bullet)_f)_0}$ and $\widetilde{((M_\bullet)_g)_0}$ as the restricting usual quasicoherent sheaves on a affine space to a distinguished open, but what is special here is that after we restrict, there is another affine open and a quasicoherent sheaf on it, such that both restrictions are isomorphic to the sheaf on the new affine scheme. It's probably easier to sit down and draw a diagram and see what happens there.

But once you understand what is going on in the previous paragraph, the cocycle condition is clear since what you do to check the cocycle condition will again be to find a common object and show that on the triple intersection, the isomorphisms are related by the third object, it's just what happens here is more complicated, which has three layers of information you need unpack, but since every construction here is canonical, to check that, you only have to be very careful and spend enough time, so we don't do it and just tell you to believe that it's true.

Notice that by construction, this is a quasicoherent sheaf on $ProjS_\bullet$.

Let's first take a look at what the stalk looks like:

Proposition 2.113. Suppose that $p \subset S_\bullet$ is a homogeneous prime and $\widetilde{M_\bullet}$ is the quasicoherent sheaf associated to the graded module $M_\bullet$, then the stalk of $\widetilde{M_\bullet}$ is isomorphic to $((M_\bullet)_p)_0$ which is the zero degree piece of the homogeneous localization at p.

Proof. Using the definition of the stalk, since we know that $D(f)$ where f is a positive degree homogeneous element, forms a basis, then we can compute the stalk at p by the following colimit:

$$colim_{f \notin p}((M_\bullet)_f)_0$$

Then notice that for each $((M_\bullet)_f)_0$, we have a canonical morphism to $((M_\bullet)_p)_0$ induced by the map $((M_\bullet)_f) \to ((M_\bullet)_p)$, it induces a canonical morphism from the colimit to $((M_\bullet)_p)_0$, which is clearly surjective. To prove it's injective, notice that if $\frac{x}{f^n}$ is mapped to 0 in $((M_\bullet)_p)_0$, then we know that $xg = 0$ for some $g \notin p$, then restrict $\frac{x}{f^n}$ to $D(fg)$, we are done. $\square$

Remark 2.114. The above exercise can also be used in the beginning to give an alternative definition of $\widetilde{M_\bullet}$ in terms of compatible germs.

Remark 2.115. Given a map of graded modules $\phi : M_\bullet \to N_\bullet$ we get a induced morphism of $\widetilde{M_\bullet} \to \widetilde{N_\bullet}$. To be precise, recall that to give a morphism of sheaves, we only have to know what are the actions on a basis, then consider the following diagram:

$$\begin{tikzcd}
M_\bullet \arrow[r, "\phi"] \arrow[d] & N_\bullet \arrow[d] \\
((M_\bullet)_f)_0 \arrow[r, "\phi_{f_0}"] \arrow[d] & ((N_\bullet)_f)_0 \arrow[d] \\
((M_\bullet)_g)_0 \arrow[r, "\phi_{g_0}"] & ((N_\bullet)_g)_0
\end{tikzcd}$$

if you can prove this commutes for all $D(g) \subset D(f)$ where ϕ_{f_0} means that take the 0 degree piece of the induced map ϕ_f and similarly for ϕ_{g_0}, then it gives the desired morphism, which is easy to check since everything here is canonically defined.

This makes $\widetilde{-}$ a functor from the category of graded $S_\bullet$-modules to quasicoherent sheaves on $ProjS_\bullet$.

Proposition 2.116. The functor $\widetilde{-}$ defined above is an exact functor.

Proof. From the way we define the functor, you know that if you have a $\phi : M_\bullet \to M_\bullet$, then the induced morphism from $\widetilde{M_\bullet}$ to $\widetilde{N_\bullet}$ will further induce a moprhism on the stalks $(M_p)_0 \to (N_p)_0$ which is just first do the homogeneous localization to ϕ and then take the 0's piece.

Recall that localization is exact, and for a graded morphism, you can also easily prove that it takes SES to SES, thus also exact, moreover composition of exact functor is exact, thus the functor is exact. $\square$

Proposition 2.117. If $\phi : M_\bullet \to N_\bullet$ is a morphism of graded modules such that it restrict to isomorphisms of abelian groups when the degree is high enough, then $\widetilde{M_\bullet} \simeq \widetilde{N_\bullet}$.

Proof. Consider the morphism $\widetilde{M_\bullet} \to \widetilde{N_\bullet}$ induced by ϕ, we claim that it's an isomorphism. To do this, we only have to prove the induced morphisms on stalks are isomorphisms for each point p.

To show this, we claim that for a homogeneous element f of high enough degree the induced morphism:

$$\phi_{f_0} : ((M_\bullet)_f)_0 \to ((N_\bullet)_f)_0$$

is an isomorphism. You can check this by hand: suppose that $\frac{m}{f^n}$ is mapped to $0 = \frac{\phi(m)}{f^n}$, this implies that $\phi(mf^t) = 0$ for some t, we can assume that $t \neq 0$ since we can always multiply both sides by f, but when f has high enough degree ϕ is an isomorphism there, thus $mf^t = 0$, thus $\frac{f}{f^n} = 0$, thus injective. To show it's surjective, suppose that $\frac{n}{f^n}$ is in $((N_\bullet)_f)_0$, then it's equal to $\frac{nf^k}{f^{n+k}}$, but when f has high enough degree nf^k has high enough degree, thus there is some m that is mapped to nf^k by ϕ, this show that it's surjective.

However, notice that for p a homogeneous prime not containing S_+, there is some positive degree homogeneous element not in p, thus it's powers are not in p, then say f^n is of high enough degree, the above discussion tells you that ϕ induces an isomorphism on $D(f^n)$, thus we are done. $\square$

Remark 2.118. The above proposition tells you that the assignment where $M \mapsto \widetilde{M}$ is probably not an equivalence of categories. The reason is that the exactly argument tells you that if ϕ, ψ are two morphisms of graded module that agree when the degree is high enough, then they induce isomorphisms on the associated sheaf, this implies that two distinct morphism may be mapped to the same morphism, but we know an equivalence of category must be fully fiathful, contradiction.

On order to know how ti fix such problems, we need the notion of a **saturation**, which will be developed later, but here we can do something in that direction, which is the following proposition.

Proposition 2.119. There is a natural map of S_0-modules:

$$M_0 \to \Gamma(ProjS_\bullet, \widetilde{M_\bullet})$$

Proof. First, by the sheaf axioms the right hand side can put in an exact sequence:

$$0 \to \Gamma(ProjS_\bullet, \widetilde{M_\bullet}) \to \prod_i \Gamma(U_i, \widetilde{M_\bullet}) \to \prod_i \Gamma(U_i \cap U_j, \widetilde{M_\bullet})$$

for any open cover U_i of $ProjS_\bullet$, and if we choose them to be $D(f)$ where f are all the positive degree homogeneous elements, we know that each of the stuff in the product is a S_0 module via $S_0 \to (S_f)_0$, then the global section is naturally a S_0-module.

To give such a map, we can use the universal property of kernel and give a morphism from M_0 to the product such that they agree on intersections. Consider the natural morphism:

$$M_0 \to (M_f)_0$$

Then it's easy to see this works. $\square$

Recall that given a homogeneous ideal $I_\bullet$ of $S_\bullet$, we have a induced closed embedding $i : Proj(S_\bullet/I_\bullet) \hookrightarrow ProjS_\bullet$:

Proposition 2.120. Apply the functor $\widetilde{-}$ to the SES:

$$0 \to I_\bullet \to S_\bullet \to S_\bullet/I_\bullet \to 0$$

we get the SES associated to the embedding $Proj(S_\bullet/I_\bullet) \hookrightarrow ProjS_\bullet$

Proof. First, recall that $\widetilde{-}$ is exact, thus the sequence:

$$0 \to \widetilde{I_\bullet} \to \widetilde{S_\bullet} \to \widetilde{S_\bullet/I_\bullet} \to 0$$

is still exact, then if you can prove $\widetilde{S_\bullet} \to \widetilde{S_\bullet/I_\bullet}$ is just the morphism $\mathcal{O}_{ProjS_\bullet} \to i_*\mathcal{O}_{Proj(S_\bullet/I_\bullet)}$, we can identify the two sequences since $\widetilde{S_\bullet}$ is the structure sheaf on $ProjS_\bullet$ and similarly for $S_\bullet/I_\bullet$. Then you only have to check this locally, which is left for you. □

Remark 2.121. We will prove later that all closed embeddings of $ProjS_\bullet$ arise in this way, corresponding to a homogeneous ideal.

3 Yoneda Lemma and Representable functors

In this section we first briefly review Yoneda's lemma, which is also useful in another our notes series. We will basically work with sets, but if you think if through, you can easily find the same line of thinking and reasoning also work for sets with additional structures.

For the setup, let's consider a category $\mathscr{C}$ and an object in $A \in \mathscr{C}$, now for any object $C \in \mathscr{C}$, we have a set of morphisms from C to A, $Mor(C, A)$ (this can also be a set with additional structures depending on the stuff you are working with, but this is not so important since the argument below also works in that case), now if you have a specific morphism $f : B \to C$, you get a map from $Mor(C, A)$ to $Mor(B, A)$ by precompsing:

$$Mor(C, A) \to Mor(B, A)$$

This rule gives a functor from $\mathscr{C}$ to the category of sets, since this rule is based on this object $A \in \mathscr{C}$, we will use h_A to denote this functor. To put Yoneda's lemma in simple words, this functor h_A determines this object A up to unique isomorphism. We will expand on this below. We start with an important exercise that you should at least do once in you life until it becomes your second instinct:

Proposition 3.1. Suppose that you have two objects A, A' in $\mathscr{C}$ then you can consider functors h_A and $h_{A'}$, then we also suppose that for every object $C \in \mathscr{C}$, you have a map:

$$i_C : Mor(C, A) \to Mor(C, A')$$

such that the following diagram commutes:

$$\begin{tikzcd} Mor(C,A) \arrow[r] \arrow[d, dashed, "i_C"] & Mor(B,A) \arrow[d, dashed, "i_B"] \\ Mor(C,A') \arrow[r] & Mor(B,A') \end{tikzcd}$$

where the top and bottom horizontal map are induced by the same morphism $f : B \to C$. Or in other words, you have a natural transformation between functor h_A and $h_{A'}$. The claim is that these i_C's as C range over all objects in $\mathscr{C}$ are all induced by a unique morphism $g : A \to A'$ via post composition. Or in simple words, we just found all possible natural transformations between h_A and $h_{A'}$.

Proof. How can you find the unique morphism $g : A \to A'$ in the claim, well you don't have too much choice, since the information you hold at your hands right now is just these map i_C, but you find that if you consider i_A, which is a map:

$$i_A : Mor(A, A) \to Mor(A, A')$$

and you also need the following diagram to commute:

$$\begin{tikzcd} Mor(A,A) \arrow[r] \arrow[d, dashed, "i_A"] & Mor(B,A) \arrow[d, dashed, "i_B"] \\ Mor(A,A') \arrow[r] & Mor(B,A') \end{tikzcd}$$

and you expect i_A and i_B are induced by postcomposing a morphism $g : A \to A'$, then if this is true, the identity in $Mor(A, A)$ is mapped to g, thus the image $i_A(id_A)$ is the only candidate for g, and what we need to do is just to check this is indeed the case.

How to check it, let's consider the following diagram:

$$\begin{tikzcd} Mor(A,A) \arrow[r] \arrow[d, dashed, "i_A"] & Mor(C,A) \arrow[d, dashed, "i_C"] \\ Mor(A,A') \arrow[r] & Mor(C,A') \end{tikzcd}$$

for any $h \in Mor(C, A)$, we want to show that $i_C(h) = g \circ h$, to show this we need the top and bottom horizontal map to be induced by h, then we know that id_A is mapped to $h \in Mor(C, A)$ and g in $Mor(A, A')$, then g is mapped to $g \circ h$ in $Mor(C, A')$, thus $h \in Mor(C, A)$ must be mapped to $h \circ g$, we are done. ☐

Proposition 3.2. We still use the set up in the above proposition, then if for all $C \in \mathscr{C}$, these i_C's are all bijections, then g is an isomorphism. Or in other words, if the natural transformation between h_A and $h_{A'}$ is a natural isomorphism, then A and A' are isomorphic.

Proof. This is also easy if you know what to look for. We know that i_C's are all bijections, which we will denote by j_C, then we know $j_C : Mor(C, A') \to Mor(C, A)$ now assemble to a natural transformation from $h_{A'}$ to h_A instead of from h_A to $h_{A'}$, therefore by the previous proposition, this correspond to a unique morphism $g' : A' \to A$. A natural thought is that g' and g are inverses of each other, which turns out to be the correct thought, and we only have to verify this.

This is done via the following commutative diagram:

$$\begin{tikzcd} Mor(A,A) \arrow[r] \arrow[d, shift left, "i_A"] & Mor(A'A) \arrow[d, shift left, "i_{A'}"] \\ Mor(A,A') \arrow[r] \arrow[u, shift left, "j_A"] & Mor(A',A') \arrow[u, shift left, "j_{A'}"] \end{tikzcd}$$

follow where the identity $id_A \in Mor(A, A)$ goes will verify g and g' are inverses. ☐

Now, let's try to further conclude what is the upshot of the above proposition. We already saw that if h_A and $h_{A'}$ are isomorphic, then A and A' are isomorphic. But what's more important is that A and A' is isomorphic via **a unique isomorphism** determined by the natural isomorphism between h_A and $h_{A'}$. Therefore, the conclusion is that the natural transformations between h_A and $h_{A'}$ are in bijection with morphisms $A \to A'$ and natural isomorphisms of functors correspond to isomorphisms between A and A'. And to put everything in fancy terms, what we did above is the following: given a category $\mathscr{C}$, we can create new category called it's functor category whose objects are contravariant functors from $\mathscr{C}$ to the category of sets (or sets with additional structures) and whose morphisms are natural transformations between functors, then we are saying that the functor that maps A to h_A is a fully faithful functor, which called the **Yoneda embedding**.

Also, the above discussion is for contravariant functor h_A, we can also define h^A where C is mapped to $Mor(A, C)$, which is going to be a covariant functor and all the above discussion have similar versions in this case. In the most general case, we can prove(for simplicity we will show the covariant version):

Proposition 3.3. Suppose that F is a covariant functor from $\mathscr{C}$ to sets, then there is a bijection between natural transformations $h^A \to F$ and $F(A)$. If we use $F = h^A$, this is the content we proved above.

Proof. Suppose that the natural transformation is shown in the following diagram:

$$\begin{tikzcd} Mor(A,B) \arrow[r] \arrow[d, "i_B"] & Mor(A,C) \arrow[d, "i_c"'] \\ F(B) \arrow[r] & F(C) \end{tikzcd}$$

now, what happens if $B = A$, then we get the following diagram:

$$\begin{tikzcd} Mor(A,A) \arrow[r] \arrow[d, "i_A"] & Mor(A,C) \arrow[d, "i_c"'] \\ F(A) \arrow[r] & F(C) \end{tikzcd}$$

let's look at the image of the identity id_A in $F(A)$, we claim that for different natural transformations you will get different image in $F(A)$, and I will leave that to you to see why. This is the easy part of the proposition.

Conversely, given any object in $x \in F(A)$, how can we define a natural transformation, i.e. we need to define i_C for each $C \in \mathscr{C}$ and see if this is really a natural transformation. Well, suppose that $h \in Mor(A,C)$, we have $F(h) : F(A) \to F(C)$, then if we define $i_C(h) = F(h)(x)$, we get a map from $Mor(A,C)$ to $F(C)$, now we only have to show this is a natural transformation. This is the content of the following diagram:

$$\begin{tikzcd} & Mor(A,A) \arrow[dl] \arrow[dr] \arrow[ddd] & \\ Mor(A,B) \arrow[rr] \arrow[d, "i_B"] & & Mor(A,C) \arrow[d, "i_C"'] \\ F(B) \arrow[rr] & & F(C) \\ & F(A) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Finally, we need to show these two constructions are inverses of each other, but this is easy to check. □

To compliment other parts of the AG notes series, we introduce a notion that can be useful in other parts of the note series.

Definition 3.4. A contravariant functor F from C to sets is called **representable** if there is an $A \in \mathscr{C}$ such that $F \cong h_A$. Then by what we discussed above, the representing object are unique up to a unique isomorphism in the sense that if $h_{A'} \cong F \cong h_A$, then the isomorphism $h_{A'} \cong h_A$ induces a unique isomorphism between A and A'.

We also want to give a few scenario where we apply Yoneda's lemma a lot and you will find this discussion useful in the notes about Derived categories:

To begin with, recall that we already proved that if $Hom(-,A) \simeq Hom(-,B)$, then there is a unique isomorphism between A and B inducing the functor isomorphism, depending how we choose A, B, we can further explore this result. One of the most often cases you will see is that you have two functors F, G and $Hom(-,FA) \simeq Hom(-,GA)$ for every A. The most obvious conclusion is that $GA \simeq FA$ for each A, but you will see in other notes that we always want these isomorphism for each A to assemble to a functor isomorphism between F and G, which requires this isomorphism $FA \simeq GA$ to be functorial in the sense that the following diagram commutes for $f : A_1 \to A_2$:

$$\begin{tikzcd} FA_1 \arrow[r] \arrow[d, "Ff"'] & GA_1 \arrow[d, "Gf"] \\ FA_2 \arrow[r] & GA_2 \end{tikzcd}$$

can you get this? The answer is fortunately yes, if you know that $Hom(-,FA) \simeq Hom(-,GA)$ is functorial in A, in the sense that the following diagram commutes:

$$\begin{tikzcd} Hom(-,FA_1) \arrow[r] \arrow[d, "Ff"'] & Hom(-,GA_1) \arrow[d, "Gf"] \\ Hom(-,FA_2) \arrow[r] & Hom(-,GA_2) \end{tikzcd}$$

once you have this, plug FA_1 in the first slot, and see where the identity $id : FA_1 \to FA_1$ is mapped to.

The converse of this is also true, i.e. if $F \simeq G$, then $Hom(-, FA) \simeq Hom(-, GA)$ for every A, via an isomorphism functorial in A. Thus, what we saw above is that the following diagram commutes:

$$\begin{array}{ccc} FA_1 & \longrightarrow & GA_1 \\ {\scriptstyle Ff}\big\downarrow & & \big\downarrow{\scriptstyle Gf} \\ FA_2 & \longrightarrow & GA_2 \end{array}$$

is equivalent to the following diagram commutes:

$$\begin{array}{ccc} Hom(-, FA_1) & \longrightarrow & Hom(-, GA_1) \\ {\scriptstyle Ff}\big\downarrow & & \big\downarrow{\scriptstyle Gf} \\ Hom(-, FA_2) & \longrightarrow & Hom(-, GA_2) \end{array}$$

this is not the content of Yoneda's lemma, but it's also extremely useful when you apply this result along with Yoneda's lemma.

References