Appendix 6 for Math 632: 2025 Winter Notes

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1 Review on smoothness and regular varieties

1.1 Zariski Tangent space

In this part of the review, we discuss the fundamental notion of when a variety is smooth. This is a refined version of the notion of regularity, which is easy to define but subtle in practice. Our first goal is to define the **Zariski Tangent Space**, this is a general notion that makes sense on any scheme but in addition behaves well as you expect at smooth points where we will define later:

Definition 1.1. The **Zariski cotangent space** of a local ring (A, m) is defined to be m/m^2, which is a vector space over the residue field A/m, the dual vector space is called the **Zariski tangent space**.

If X is a scheme, the Zariski cotangent space $T^\vee_{X,p}$ at a point p is defined to be the Zariski cotangent space of the local ring $\mathscr{O}_{X,p}$ and similarly for the **Zariski tangent space** $T_{X,p}$, which is defined to be the dual of the Zariski cotangent space at p.

Elements of the Zariski cotangent space are called **cotangent vectors** or **differentials** and elements of the tangent space are called tangent vectors.

Before we proceed to try to justify this seemingly weird definition, let's settle an important algebra result:

Proposition 1.2. For any ring A and a maximal ideal m of A, we have a canonical isomorphism between m/m^2 and mA_m/m^2A_m as an isomorphism of $A/m \cong A_m/mA_m = k$ spaces.

Proof. Consider the following diagram:

$$\begin{tikzcd} A \arrow[r] & A_m \\ m \arrow[u, hook] \arrow[r] & mA_m \arrow[u, hook] \end{tikzcd}$$

where we get a natural A-module morphism from m to mA_m via restriction of the localization map $A \to A_m$. Notice that you can then consider:

$$m \to mA_m \to mA_m/m^2A_m$$

then it's easy to notice that m^2 is contained in the kernel of the map, thus we have a well defined morphism $m/m^2 \to mA_m/m^2A_m$, which is a A/m-vector space morphism. We want to show that this is an isomorphism. Notice that the dimension of mA_m/m^2A_m is always less than or equal to m/m^2 since you can see that if $f_1, \cdots, f_r$ generates m, then their image in A_m generate mA_m, then suppose that m/m^2 is of dimension n, then take a basis of m/m^2 they lift to A which then generates m, then their image generates mA_m, then by Nakayama's lemma, the dimension of mA_m/m^2A_m is less than or equal to n(since we are looking for minimal set of generators). Here is also a small technical point that you need to notice, since mA_m/m^2A_m here is in general considered as a A_m/mA_m-module, but recall that we have a canonical isomorphism $A/m \cong A_m/mA_m$ via $\overline{a} \mapsto \overline{\frac{a}{1}}$. Then you can show that the A/m vector space structure that we have on mA_m/m^2A_m is compatible with the A_m/mA_m vector space structure.

Then we only have to prove this is an injective morphism to conclude that this is an isomorphism. Now suppose that a is mapped to $\frac{a}{1}$ is 0, then we know that this is equivalent to $\frac{a}{1}$ is in m^2A_m, then it's left to you to conclude that this is equivalent to $ax \in m^2$ for some $x \notin m$. Notice that m^2 is a m-primary ideal, then we know that in A/m^2, any zerodivisor is nilpotent, since $x \notin m$, it's not zero in A/m^2 and we know that any power of x is not in m, thus the image of x in A/m^2 is not nilpotent, thus it's not a zerodivisor, thus we know that since the image of ax is 0, then the image of a in A/m^2 must be 0, thus $a \in m^2$, we are done. $\square$

Here are two plausibility arguments you can make to justify the definition of the cotangent and tangent space, the first one tends to mimic differential geometry:

First, recall that in differential geometry, the tangent space at a point is defined to be the space of derivations at that point, which is an $\mathbb{R}$-linear operator that satisfies the **Leibniz rule**:

$$(fg)' = f'g + g'f$$

we will later review derivations in a more general setting, but for now, let's just use the above notion which is enough. The argument below if a preview of what you will see later.

You probably have proved that this is the same information as a $\mathbb{R}$-linear map from m_p to $\mathbb{R}$ that satisfies the Leibniz rule where m_p is the maximal ideal of $\mathscr{O}_p$. This is because given a derivation at p, you can easily obtain a $\mathbb{R}$-linear map that satisfies the Leibniz rule just by restriction. Conversely, given a $\mathbb{R}$-linear map that satisfies the Leibniz rule from m_p to $\mathbb{R}$ we can define a map from $\mathscr{O}_p$ to $\mathbb{R}$ by $f \mapsto f - f(p)$ then apply the map from m. Now consider $fg \mapsto fg - f(p)g(p)$, notice $fg - f(p)g(p)$ is the same as $(f - f(p))(g - g(p)) - 2f(p)g(p) + f(p)g + g(p)f$ since the map we defined from $\mathscr{O}_p$ to $\mathbb{R}$ is linear, we know that this is the same as we consider separately for each term and then add them up. Then you again use the definition of the map from $\mathscr{O}_p$ to $\mathbb{R}$, you obtain that this is indeed satisfies the Leibniz rule. Sorry for the possibly confusing notation, if you are doing this more carefully, you should use different symbol to differentiate derivation from m_p and the map from $\mathscr{O}$ we defined using the derivation from m_p to $\mathbb{R}$, possibly using superscript $\circ$ and $'$.

You can check that the above two operations are inverses of each other. Therefore, you know that they have the same amount of information. One key point to notice is that if we use the point of view of think of derivations as $\mathbb{R}$-linear maps from m_p to $\mathbb{R}$ that satisfies the Leibniz rule, then we can check that elements in m^2 is now mapped to 0 by Leibniz Rule, thus a derivation gives you a map from m/m^2 to $\mathbb{R}$, i.e a derivation gives a element in $(m/m^2)^\vee$, our definition of the tangent space.

Surprisingly, this process is reversible in the sense that given an $\mathbb{R}$-linear map from $m/m^2 \to \mathbb{R}$, we naturally have a $\mathbb{R}$-linear map from m to $\mathbb{R}$ such that it satisfies the Leibniz rule. To see how this works, we define $f \in m$ is mapped to the image of $\overline{f}$, or in other words, you consider:

$$m \to m/m^2 \to \mathbb{R}$$

This map naturally satisfies Leibniz rule since fg is mapped to 0 and it's equal to $f(p)g' + g(p)f'$ which is also 0. Also, this construction is the inverse of constructing a $\mathbb{R}$-linear map from m/m^2 to $\mathbb{R}$ via a derivation. Therefore, derivation at the point p indeed corresponds to $(m_p/m_p^2)^\vee$!

Also, we have the second simpler argument (but vaguer) to convince you if the first one is too complicated. We want to figure out what is the cotangent space of the origin of $\mathbb{A}^n$. In general, you should think of the cotangent space as the linear functionals of the tangent space, thus what is the tangent space. Our intuition in $\mathbb{R}^n$ is that it's naturally identified as $\mathbb{R}^n$, spanned by the standard basis, thus we can think of the linear functionals as linear combination of x_i where $x_i(e_j) = \delta_{ij}$, thus here for example we consider $\mathbb{A}^2$, then the linear functionals should be spanned by x, y, z, the linear polynomials in x, y, z, this is exactly $(x, y, z)/(x, y, z)^2$! If you are not totally convinced, don't worry, this is just some ways you can go to think about them, and as we develop further results, you will gradually think this is a reasonable definition.

Let's take a look at the below old-fashioned example to get a feeling of the Zariski cotangent and tangent space:

Example 1.3. Say we work in $\mathbb{A}^3$, and you have two curves cut out by two equations $x + y + z^2 + xyz = 0$ and $x - 2y + z + x^2y^2z^3 = 0$. You can actually check that the dimension of the closed subscheme is indeed 1 by using Krull's Principal ideal theorem, but it's not important here. Let's assume that it's smooth near the origin, then what is the tangent line there. In very primitive geometric view, the first surface looks like $x + y = 0$ near the origin and the second surface looks like $x - 2y + z = 0$ near the origin, thus you should expect the tangent line is cut out by these two linear equations, and indeed it is, which is a point we will make clear in the following result.

Proposition 1.4. Suppose that A is a ring and m is a maximal ideal of A. If $f \in m$, then the Zariski tangent space of A/f at m is cut out in the Zariski tangent space of A at m by f mod m^2.

Proof. Notice that by definition, we are viewing $SpecA$ and $SpecA/f$ as schemes, the Zariski tangent space of $SpecA$ at m is the dual of mA_m/m^2A_m which is canonically isomorphic to m/m^2 as we proved earlier. Then what about $SpecA/(f)$, the tangent space at m/f is the dual of $(m/f)(A/(f))_m/[(m/f)(A/(f))_m]^2$, which is isomorphic to $(m/f)/(m^2/f)$ as a $(A/f)/(m/f)$-vector space. But notice that there is a canonical isomorphism of rings $(A/f)/(m/f)$, thus we know we can view the dual of $(m/f)/(m^2/f)$ as a A/m-vector space with obvious action of A/m.

Now, it's fairly easy to identify the tangent space of $SpecA/f$ at $m/(f)$ as a subspace of the tangent space of $SpecA$ at m, this is done by considering the following surjection induced by the universal property of quotients:

$$\begin{tikzcd} m \arrow[r, "quotient"] \arrow[d, "quotient"'] & m/f \arrow[d, "quotient"] \\ m/m^2 \arrow[r, dashed, two heads] & (m/f)/(m/f)^2 \end{tikzcd}$$

the result surjection is a morphism of A/m-vector spaces, thus induces an injection of the dual space, which identifies the tangent space of the quotient as a subspace of the tangent space of the original one and you can easily check that everything in the image should vanish at f mod m^2, but we would like to say explicitly that everything in the tangent space that vanish at f mod m^2 is in the image. This is the argument below.

One isomorphism is crucial in the proof, let's consider the submodule of m/f generated by products of elements in m/f, if we denote it by m^2/f, then we claim that the quotient $m/f/m^2/f$ is isomorphic to the quotient of m/m^2 quotient the subspace generated by $\overline{f}$, to see this consider the SES:

$$0 \longrightarrow m^2 \longrightarrow m \longrightarrow m/m^2 \longrightarrow 0$$

which can all be viewed as A-modules, then we tensor this SES with A/f over A, since all tensor product is right exact, we get another exact sequence:

$$m^2 \otimes_A A/f \longrightarrow m \otimes_A A/f \longrightarrow (m/m^2) \otimes_A A/f \longrightarrow 0$$

notice that if you think about what is this sequence, the first term is just the quotient of m^2 by the submodule fm^2, and similarly, the second is m/f and the third is just m/m^2 quotient the submodule generated by $\overline{f}$. Notice that the image of the first map in $m \otimes_A A/f$ is just m^2/f then you have $m/f/m^2/f$ is isomorphic to the last term, then we can use that universal property of quotient, say $\varphi : m/m^2 \to A/m$ that vanishes on the f, then you can check that if you think of this as a morphism of A-modules, then φ vanishes on the submodule of m/m^2 generated by $\overline{f}$, thus by the universal property of quotient, it uniquely descends to a morphism from $(m/m^2) \otimes_A A/f$ to A/m, which is still A-linear. But as we mentioned above, $(m/f)/(m^2/f)$ is isomorphic as an A-module to $(m/m^2) \otimes_A A/f$, thus we uniquely get an A-module morphism from $(m/f)/(m^2/f)$ to A/m denoted by Φ which is defined to be $\Phi(\overline{x}) = \varphi(x)$, then you can check that this is A/m-linear. In conclusion, you have:

$$\begin{tikzcd} m/m^2 \arrow[r, "\varphi"] \arrow[d] & A/m \\ m/(f)/m^2/(f) \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

you can also prove that this is an A/m-linear morphism from the explicit formula we gave above then by the universal property, this is a bijection, thus we are done.

More over, if you have a vector space V over k and you consider it's dual $V^\vee$, then the hypersurface defined by those which evaluate to 0 at f, if f is not 0, is of dimension 1 less (assume that $V^\vee$ is finite dimensional), since you always have the following morphism:

$$0 \longrightarrow Ker \longrightarrow V^\vee \xrightarrow{f} k \longrightarrow 0$$

where the second map is defined to be evaluate at f, then if f is not zero, the morphism is surjective, then you have the desired assertion. $\square$

Remark 1.5. The above technique can be applied to more than just one f, suppose that you have $(f_1, \cdot, f_r)$ where all f_i's are in m, then the above argument still holds and will let you end up with the conclusion that any element in the dual of m/m^2 such that they vanish at $(f_1, \cdots, f_r)$ mod m^2 corresponds to a unique element in the tangent space of $A/(f_1, \cdots, f_r)$ at $m/(f_1, \cdots, f_r)$.

Thus this justifies the discussion we had in the example above, where you see that the tangent space of the line cut out by $x - 2y + z + x^2y^2z^3 = 0$ and $x + y + z^2 + xyz = 0$ at the origin is defined to be $x + y$ and $x - 2y + z$ vanish in the tangent space of $\mathbb{A}^3$ which can be though of as k^3. Here $x + y$ and $x - 2y + z$ are the two equations $x - 2y + z + x^2y^2z^3 = 0$ and $x + y + z^2 + xyz = 0$ mod m^2 (deleting all the non linear terms), which aligns with our result in the above proposition.

This also agrees with our geometric intuition since we naturally think of the tangent space of a smooth curve to be one dimensional, and you can see that since the two linear functionals $x + y$ and $x - 2y + z$ are linearly independent, we will have the subspace defined by they vanish is actually of dimension 1. However, this is because that the curve is smooth at the origin, if it is not, then you will have something rather surprising, as the below example shows.

Example 1.6. Consider the curve cut out by $x + y + z^2 = 0$ and $x + y + x^2 + y^4 + z^5 = 0$ which passes through the origin. As usual you can check using Krull's Theorem that this is indeed a curve, but again it's not so important here. So, if we consider the above technique, you will find out that the tangent space at the origin is not cut out by the vanishing of $x + y$ and $x + y$, which is a two dimensional space.

You will be surprised to see that the tangent space of a curve is not a one dimensional space, but this is precisely because the curve is not smooth at the origin. Indeed, we will later use this as the definition of not having regularity or in other words, it's singular at the origin.

One last point you need to pay attention to is that notice that the dimension of the tangent space which is the dimension of the cotangent space jumps up, rather than going down(in our example above this jumps to 2, not down to 0). This is not a coincidence since if you recall that for any Noetherian local ring (A, m, k) we already proved that $dimA$ is less than or equal to $dim_k(m/m^2)$. In our case, we are looking at the local ring of $k[x, y, z]/(x + y + z^2, x + y + x^2 + y^4 + z^5)$ localized at the maximal ideal corresponding to (x, y, z). Since we already know that the ring $k[x, y, z]/(x + y + z^2, x + y + x^2 + y^4 + z^5)$ is of dimension 1 then it's easy to see that the dimension of the local ring at (x, y, z) is also 1 since you can see that $(x + y, z)$ are also a prime ideal contained in (x, y, z) that contains the ideal we are quotient out.

Then we know that the cotangent space is at least of dimension 1.

Now we explore in general how the tangent space of closed subschemes interactw with the tangent space of the ambient scheme:

Proposition 1.7. Suppose that Y, Z are closed subschemes of X both containing the point p, then the tangent space $T_{Z,p}$ is naturally a sub-$k(p)$-vector space of $T_{X,p}$.

Proof. Notice that by definition, $T_{Z,p}$ is defined to be the dual of $m_{Z,p}/m_{Z,p}^2$, notice that by choosing an affine open in X containing p, we naturally identify $m_{Z,p} = m_{X,p}/I_p$, then you have a natural surjection:

$$\begin{tikzcd} m_{X,p} \arrow[r] \arrow[d] & m_{X,p}/I_p \arrow[d] \\ m_{X,p}/m_{X,p}^2 \arrow[r, dashed, two heads] & (m_{X,p}/I_p)/(m_{X,p}/I_p)^2 \end{tikzcd}$$

this can be viewed as a surjection of $k(p)$-vector spaces, which induces an injection of the dual as a $k(p)$-vector space, thus we get the desired injective $k(p)$-vector space morphism:

$$T_{Z,p} \hookrightarrow T_{X,p}$$

Let's be even more specific about what dose this mean since it is helpful for our intuition and the use for later results.

Say that you have a k-linear map $V \to W$ of k-vector spaces, what is the dual of it, $f^* : W^* \to V^*$, it's defined to be $\varphi \mapsto \varphi \circ f$. Then translate this to our setting here, we see that an element in the dual of $(m_{X,p}/I_p)/(m_{X,p}/I_p)^2$ is precomposed by the induced surjection, and the map you get will vanish on $I_p/m_{X,p}^2$.

Again, we would like to ask the question: does the image consists all the linear functional on $m_{X,p}/m_{X,p}^2$ that vanishes on I_p mod $m_{X,p}^2$. But notice that the techniques we used in the previous proposition is not special about the ideal generated by one element. Rather, it works for all I, thus you run exactly the same argument with (f) replaced by I_p, then we are done. □

Proposition 1.8. Suppose we have the same setup as the above proposition, then $T_{Y\cap Z,p} = T_{Y,p} \cap T_{Z,p}$, where the first intersection is the scheme-theoretic intersection and the second intersection is the intersection as subspaces of $T_{X,p}$.

Proof. Notice that if $\mathscr{I}$ and $\mathscr{J}$ are the ideal sheaves defining $Y \cap Z$, then as usually, notice that the tangent space at p of $Y \cap Z$ is the dual of $(m_{X,p}/(I + J)_p)/(m_{X,p}/(I + J)_p)^2$. Notice by the previous result, we know that this induces a injection which identifies the tangent space of $Y \cap Z$ as the subspace of $T_{X,p}$ that vanishes at $(I + J)_p/m_{X,p}^2$.

Let's see a general result first, suppose that you have a vector space V and it's dual $V*$. Then if you also have two subspaces of V^* defined by linear functionals vanishing on subspaces of V, which are V_1 and V_2. If we denote them by V_1^* and V_2^* we claim that $V_1^* \cap V_2^* = (V_1 + V_2)^*$.

Apply this to our setting, if we can show that the subspace of $m_{X,p}/m_{X,p}^2$ spanned by the images $(I+J)_p$ is the sum of the subspaces spanned by I_p and J_p but this is just to show that in A_p, $I_p + J_p = (I + J)_p$, but this is easy to show. □

Proposition 1.9. Again, with the same setup, $T_{Y\cup Z,p}$ contains the span of $T_{Z,p}$ and $T_{Y,p}$ in $T_{X,p}$, however, it can strictly contain the span of $T_{Y,p}$ in $T_{X,p}$.

Proof. Similar to the proof above, we are considering $(I \cap J)_p$ this time, notice that localization commutes with finite intersection, thus we have $I_p \cap J_p$ in A_p. Then if something vanishes on I_p or J_p then it vanish on the intersection, therefore the span of $T_{Z,p}$ and $T_{Y,p}$ is contained in $T_{Z\cap Y,p}$.

To give an example of strict containment, we first see this is definitely possible since there is no reason for something that vanishes on the intersection to also vanish on the individual space. To see this, consider the tangent space of the origin in the closed subscheme:

$$Speck[x, y]/(y(y - x^2)) = Speck[x, y]/(y^2 - yx^2)$$

Then we will view the tangent space of the origin to be those that vanish on $y^2 - yx^2$ mod $(x, y)^2$, thus is the whole space since everything vanish on 0. Then we can also consider the individual tangent spaces, which are both defined to be those vanishes on y, then you can see that this is indeed a strict containment. □

Using the notion of Zariski's tangent space, we can prove that:

Proposition 1.10. Consider the ring $A = k[x, y, z]/(xy - z^2)$, then the ideal (x, z) is not principal.

Proof. Let's first set the stage for the discussion afterwards.

We first claim that $dimA = 2$, this is because that $xy - z^2$ is a irreducible polynomial and in particular it's not a zero divisor, then by Krull theorem it codimension in $SpecA$ is 1, then since $SpecA$ is pure dimensional locally of finite type over k, then for irreducible closed subsets inside, we can apply dimension of the larger one is the sum of the codimension and dimension of the smaller one.

Then notice that $A/(x, z) \cong k[y]$ which is of dimension 1. Therefore since (x, y) is again prime, we use the result of dimension and codimension again to conclude that (x, y) is of codimension 1 in A.

Now we are ready to prove our assertion. But before that, some geometric picture might be useful. Geometrically speaking, this $SpecA$ is a cone in $\mathbb{A}^3$ and then you cut it with the plane $x = 0, y = 0$, you get a line on the cone passing through the origin, as what you will see later, the problem occurs at the origin, which is easily seen as a special point.

Now consider the the tangent space of $SpecA/(x, z)$ at (x, y, z), which is the tangent space of $Speck[y]$ at (y), which is one dimensional. Then we claim that, by all the previous arguments, the tangent space of

SpecA at the origin is of dimension 3. But notice that if (x, z) is generated by one element say f, then notice that the tangent space of A/f at the origin is the subspace of the tangent space of A at the origin, defined by f modulo m^2 vanishes, thus is at least dimension $3 - 1 = 2$, which is a contradiction, we are done. □

Remark 1.11. Notice that we proved something more in the above argument: (x, z) is of codimension 1 in A but it's not principal, thus by an earlier result we showed, A is not a UFD.

Also, here is another way to solve the above problem without using the notion of Zariski tangent space.

Notice that the fact that A is still graded since you are quotienting out a homogeneous ideal and you can define x, y, z to have degree 1, even more (x, z) is still homogeneous in the quotient ring. We won't write out the exact way of doing it using the grading technique, but we do want to point out that the advantage of using the Zariski tangent space is you can handle more general cases where the ideal you are quotienting out is not homogeneous.

However, this graded method is still important since it involves completion and associated graded rings we will discuss later.

As a practice of the tangent space technique, we can also prove:

Proposition 1.12. (x, z) is a codimension 1 but not principal ideal in $A = k[x, y, z, w]/(wz - xy)$.

Proof. First, still notice that the ideal $(wz - xy)$ is prime since $wz - xy$ is irreducible, then it's codimension is 1 by Krull's Theorem, and again we know that it's dimension is 3 by codimension and dimension relation.

Then we know that if we again quotient out (x, z), we get $k[y, w]$, which means that the tangent space at the origin is 2, but again notice that the tangent space of $k[x, y, z, w]/(wz - xy)$ at 0 is of dimension 4, then if it's principal, then the tangent space at the origin after you quotient out x, z should at least be 3, which is a contradiction.

Again, we proved that there is a codimension 1 prime ideal that is not principal, thus' it's not a UFD. In fact that we can prove that *SpecA* is not factorial, which means that there is a local ring of *SpecA* which is not a UFD. A is not a UFD, but this is not enough, we still want to prove that if we localize it at (x, y, z, w), it's again not a UFD.

We already saw that (x, y) is of codimension 1 and since it's contained in (w, y, z, w), after you localize at (x, y, z, w), (x, y) remains codimension 1, we claim that it's not principal. To see this, notice that again, after the localization, the tangent space at the origin is still dimension 4, but again, we quotient out $(x, z)_{x,y,z,w}$, you get $k[y, w]_{(y,w)}$ and the quotient at the origin is 2, then we got the contradiction similarly.

Moreover, if $char k \neq 2$, this ring is integrally closed, thus it's normal, but not factorial! This is our first nontrivial example of a normal but nonfactorial ring. □

Below is a fun exercise using the higher order data of the maximal ideal, we first begin with an observation:

Lemma 1.13. Suppose that you work in $\mathbb{A}^3_{\mathbb{C}}$, then set theoretically the three axis is cut out by an ideal generated by 3 elements.

Proof. We claim that $I(S)$ where S is the union of the three axis is generated by xy, xz, yz since xy, xz, yz are the generic point of the three axis. □

What is interesting is that we claim that this ideal can't be generated by less than 3 element:

Proposition 1.14. The ideal (xy, xz, yz) is not generated by less than 3 element in $k[x, y, z]$.

Proof. Let's assume that for contradiction, (xy, xz, yz) is generated by 2 elements, say f_1 and f_2, then first we notice that for purely dimensional reasons, one of the two generators must contain some linear term, otherwise since xy, yz, xz are linearly dependent, we can not get a there dimension k-vector space by two elements. However, if any of them contains a linear term, the tangent space at the origin is never three dimensional, a contradiction to a quick computation that the dimension of the tangent space of $k[x, y, z]/(xy, yz, xz)$ at the origin is 3. □

Corollary 1.15. As a result of the above lemma, we can show that this closed subvariety, the union of three axis, is not regularly embedded in $\mathbb{A}^3$.

Proof. This is not exactly a corollary of the previous proposition, but the idea is similar, we can show that the ideal $(xy, xz, yz)_{(x,y,z)}$ is never generated by 2 elements in $k[x, y, z]_{(x,y,z)}$, thus if it's regularly embedded, it will be generated by a regular sequence of at least 3 elements, but the Krull's height theorem implies that any minimal prime over $(xy, xz, yz)_{(x,y,z)}$ is of codimension 3, which is the maximal ideal of $k[x, y, z]_{(x,y,z)}$ since it's dimension is 3. However, this is a contradiction since we can easily find $(x, y)_{(x,y,z)}$ over this ideal which is not maximal.

The way we show that the ideal $(xy, xz, yz)_{(x,y,z)}$ is never generated by 2 elements is by Nakayama's lemma which allows us to conclude that since $(xy, xz, yz)_{(x,y,z)}/m(xy, xz, yz)_{(x,y,z)}$ is three dimensional, the ideal at least have 3 generators. □

Now we begin to explore how morphisms of schemes play with tangent spaces:

Suppose that $\pi : X \to Y$ is a morphism of schemes with $\pi(p) = q$, then if we are in the category of manifolds and π is a smooth map between X, Y smooth manifolds, then we expect a tangent map, which can also be examined algebraically. Indeed, notice that π induces a morphism on the stalks $\mathscr{O}_{Y,q} \to \mathscr{O}_{X,p}$ by precomposition, and you can verify that it maps the maximal ideal of $\mathscr{O}_{Y,q}$ to the maximal ideal of $\mathscr{O}_{X,p}$, thus $m_{Y,q}^2$ it mapped to $m_{X,p}^2$, then we can define a well defined morphism of $\mathscr{O}_{Y,q}/m_{Y,q}$-vector spaces:

$$m_{Y,q}/m_{Y,q}^2 \to m_{X,p}/m_{X,p}^2$$

But notice that both $\mathscr{O}_{Y,q}/m_{Y,q}$ and $\mathscr{O}_{X,p}/m_{X,p}$ are isomorphic to $\mathbb{R}$, thus we can think of the above morphism as an $\mathbb{R}$-vector space morphism, which induces an $\mathbb{R}$-vector space isomorphism of the dual in the opposite direction, which is the tangent space morphism.

You will find the same discussion above naturally carries over to the case of morphism of schemes if the point p and q naturally have the same residue field. The point is that the map of the cotangent spaces always exists, but in order to apply the dual, we need both of the cotangent space to be over the the same residue field. To have some practice, we can do the following exercise:

Proposition 1.16. Suppose that X is a k-scheme, then there is a natural bijection from $Mor_k(Speck[\epsilon]/\epsilon^2, X)$ to the data of of a point $p \in X$ with a residue field k and a tangent vector at p.

Proof. First, notice that if we have a diagram like this:

$$\begin{tikzcd} Speck[\epsilon]/\epsilon^2 \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & Speck & \end{tikzcd}$$

then the image of $Speck[\epsilon]/\epsilon^2$ is necessarily a closed point as you can check $Speck[\epsilon]/\epsilon^2 \to X$ is finite from the fact that $Speck[\epsilon]/\epsilon^2 \to Speck$ is finite and use the fact that finite morphism is necessarily closed.

Then we also see that the residue field of the image point is necessarily k since the residue point of $Speck$ and $Speck[\epsilon]/(\epsilon^2)$ are all k and you will have the following diagram of field extensions:

$$\begin{tikzcd} k & & k(p) \arrow[ll, hook'] \\ & k \arrow[ul, "id"] \arrow[ur, hook] & \end{tikzcd}$$

Therefore, given $f \in Mor_k(Speck[\epsilon]/\epsilon^2 \to X)$, we naturally get a morphism of tangent spaces from the tangent space of $Speck[\epsilon]/\epsilon^2$ at the origin to the tangent space of X at p by what we discussed previously.

We can naturally identify the tangent space of $Speck[\epsilon]/\epsilon^2$ at the origin with the field k, then as a k-vector space morphism, we know that the image is a one dimensional subspace of the tangent space at p. We will define a map that maps f to the image of 1. We claim that this is a bijection, to see so, we need to construct an inverse.

To obtain it, consider the induced morphism induced by f:

$$m_p/m_p^2 \to k$$

it gives you $k \to (m_p/m_p^2)^\vee$, suppose that we want that in this map the map that maps 1 to 1 is mapped to v^*, which correspond to the image of v in m_p/m_p^2 under the dual map. Notice that if we denote that maps 1 to 1 by φ, then it's mapped to $\varphi \circ f$, we want it to be v^*, which means that we need to find a suitable f, if f is the map that maps v to 1, then we are done, thus we are trying to find which morphism will give you this f.

The key here is to notice that you you have a k-algebra morphism form a local k-algebra (A, m, k) to $k[\epsilon]/(\epsilon^2)$, then it completely determine by a k-linear map:

$$m/m^2 \to (\epsilon)/(\epsilon^2) \cong k$$

To see how this works, notice that any element in A, say x, is uniquely written as $a + b$ where $a \in k$ and $b \in m$. This is again because you have the following sequence:

$$k \hookrightarrow A \longrightarrow A/m \cong k$$

notice that in this case $k \to k$ is just the identity. Then suppose that $x \in A$, then first consider it's image in A/m, which is the image of a unique $a \in k$ to A, thus we can claim that $x = a + b$ for a unique $b \in m$ and $a \in k$. Therefore proving our claim that any element in A is a unique summation of something in k and something in m.

Then given $f' : m/m^2 \to (\epsilon)/(\epsilon^2)$, we can define $f : A \to k[\epsilon]/\epsilon^2$ by defining $f(x) = f(a + b) = a + f'(b)$, we only have to verify this is indeed a ring homomorphism. It's obviously additive, and for $f(xy) = f(a_1a_2 + a_1b_2 + a_2b_1 + b_1b_2)$ since $b_1b_2 \in m^2$, their image under f' is 0, thus you get $a_1a_2 + a_1f'(b_2) + a_2f'(b_1) = (a_1 + f'(b_1)) \cdot (a_2 + f'(b_2))$. Notice that $f'(b_1)f'(b_2) \in (\epsilon^2)$, thus is zero in $k[\epsilon]/(\epsilon^2)$, we are done.

Back to our question, a morphism from $Speck[\epsilon]/(\epsilon^2) \to X$ is the same as morphism(if you don't understand why, see the proposition after this):

$$Speck[\epsilon]/(\epsilon^2) \to Spec\mathscr{O}_{X,p} \to X$$

and in particular, $\mathscr{O}_{X,p}$ is a local k-algebra with residue field k, thus it's really just a morphism $Speck[\epsilon]/(\epsilon^2) \to Spec\mathscr{O}_{X,p}$ since the morphism $Spec\mathscr{O}_{X,p} \to X$ is canonical, which by the above assumption correspond to a k-linear morphism from $m_p/m_p^2 \to k$. Finally, we just need to choose a any k-linear map that maps v to 1 and other basis to 0, we are done. □

In the above proof, we are implicitly using the fact that there is a bijection between $Mor(Speck[\epsilon]/(\epsilon^2) \to X$ and local ring homomorphisms $\mathscr{O}_{X,p} \to k[\epsilon]/(\epsilon)^2$, the following few propositions makes this clear:

Proposition 1.17. Suppose that $p \in X$ is a point in a scheme X, then there is a canonical or in other words, choice-free morphism $Spec\mathscr{O}_{X,p} \to X$.

Proof. Notice that if X is an affine scheme $SpecA$, then we can identify the morphism $Spec\mathscr{O}_{X,p} \to X$ as a ring homomorphism $A \to A_p$, then the canonical choice is just the localization map, then when X is not affine, we choose any affine open around p, say $SpecA$, then notice that you can define a canonical morphsim first from $Spec\mathscr{O}_{X,p}$ to $SpecA$ then use the inclusion to include $SpecA \hookrightarrow X$. The only issue here is to check that this is independent of choice of $SpecA$, the way you do it is to the affine communication lemma, suppose that $SpecA$ and $SpecB$ are two affine opens, then there is an affine open U that is distinguished in both of them. Then if we can prove that both $Spec\mathscr{O}_{X,p} \to SpecA \hookrightarrow X$ and $Spec\mathscr{O}_{X,p} \to SpecB \hookrightarrow X$ all agree with $Spec\mathscr{O}_{X,p} \to U \hookrightarrow X$. But notice that you can first include U into $SpecA$ and $SpecB$, suppose that $U = SpecA_f$ and $SpecB_g$ viewed in $SpecA$ and $SpecB$ respectively. Then we only have to show that we have the following diagram commute:

$$\begin{tikzcd}
 & & SpecA \\
Spec\mathscr{O}_{X,p} \arrow[urr] \arrow[r] \arrow[drr] & U \arrow[ur, hook] \arrow[dr, hook] & \\
 & & SpecB
\end{tikzcd}$$

which boils down to the commutativity of the smaller two triangles. Let's first see the upper triangle, it's commutativity can be translated to show the following diagram is commutative(make sure you understand why!):

$$\begin{tikzcd} SpecA_p \arrow[r] \arrow[d, leftrightarrow, "iso"'] & SpecA \\ Spec(A_f)_{p_f} \arrow[r] & SpecA_f \arrow[u] \end{tikzcd}$$

which boils down to show a standard diagram of of localizations of rings is commutative. ☐

Corollary 1.18. Based on the above result, we have a canonical morphism $Speck(p) \to X$.

Proof. Consider the canonical morphism of affine schemes induced by $\mathscr{O}_{X,p} \to k(p)$. ☐

Also, based on the canonical morphism from $Spec\mathscr{O}_{X,p} \to X$, we can prove the result we were implicitly using:

Proposition 1.19. Suppose that X is any scheme and (A, m) is a local ring. Suppose we have a scheme morphism $\pi : SpecA \to X$ that sends $[m]$ to p, then any open subset of X that contains p contains the set-theoretical image of π.

In particular, there is a bijection between $Mor(SpecA, X)$ and:

$$\{p \in X : \text{some local homomorphism } \mathscr{O}_{X,p} \to A\}$$

Proof. Notice that to prove the first assertion we only have to prove it for affine opens. Then say that $SpecB$ is an affine open in X that contains the image of $[m]$, the maximal ideal. We want to show that all the image of $SpecA$ is in $SpecB$, or equivalently we want to show that $\pi^{-1}SpecB = SpecA$. First notice that the inverse image of $SpecB$ is an affine open in $SpecA$ which contains the maximal ideal, we claim that any open that contains the unique maximal ideal contains everything. To see this, notice that for any p prime in A since $p \subset m$, we know that $m \in V(p)$, thus we know that the open intersects with $V(p)$ nontrivially, thus since p is the generic point, p is also in the open intersecting with $V(p)$.

Then to prove the second assertion, suppose that we are given a morphism $SpecA \to X$, then recall how we defined $Spec\mathscr{O}_{X,p} \to X$: we choose any affine open $SpecB$ and identify this morphism as $SpecB_p \to SpecB$, then you can consider the following diagram:

$$\begin{tikzcd} SpecA \arrow[r] & SpecB \arrow[r, hook] & X \\ & SpecB_p \arrow[u] \arrow[ur] & \end{tikzcd}$$

notice that the above composition is just $SpecA \to X$ since we proved that any affine open contains the image of $SpecA$, then notice that we have a diagram of rings:

$$\begin{tikzcd} A & B \arrow[l] \arrow[d] \\ & B_p \arrow[ul, dashed] \end{tikzcd}$$

the dotted induced morphism comes from the universal property of localization since we assumed that the inverse image of m is p, thus everything outside p is mapped to elements outside m, which are invertible, and you can easily check this is indeed a local ring homomorphism.

Conversely, given any local ring morphism $\mathscr{O}_{X,p} \to A$, we naturally get a morphism $SpecA \to X$ that maps $[m]$ to p by choosing an affine open $SpecB$ then the ring morphism $B \to B_p \to B_p \to A$ induces the morphism of schemes we want.

You can easily check that these are inverses of each other, thus a bijection. ☐

This is enough detour we want to have right now, we now turn to a very important topic of tangent spaces of a scheme: what will happen if we only work with locally finite type k-schemes.

But to begin with, let's do a little bit review on field-valued, ring valued or even scheme valued points:

Definition 1.20. If Z is a scheme, then Z**-valued points** of a scheme X, denoted by $X(Z)$, are defined to be morphisms $Z \to X$, if A is a ring, then the A**-valued points** of a scheme X, denoted by $X(A)$, are defined to be $SpecA$-valued points of X. Mostly commonly, A is a field, and we are considering field valued points.

Remark 1.21. If you are working with some base scheme B, for example, complex algebraic geometers will only consider scheme X over $B = Spec\mathbb{C}$, then in the above definition of Z-valued points of X is implicitly assumed to be morphisms of B-schemes, i.e. it comes with a structure morphism $Z \to B$. For example, if we are considering a scheme X over $\mathbb{C}$, then $\mathbb{C}(t)$-valued points are $\mathbb{C}$-morphisms from $Spec\mathbb{C}(t)$ to X, or in other words, morphims $Spec\mathbb{C}(t) \to X$ such that the following diagram commutes:

$$\begin{tikzcd} Spec\mathbb{C}(t) \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & Spec\mathbb{C} & \end{tikzcd}$$

Here are some easy properties of scheme valued points:

Proposition 1.22. A morphism of schemes $X \to Y$ induces a map of Z-valued points of X and Y.

Proof. Suppose that you have a morphism $Z \to X$, then compose it with $X \to Y$. □

Proposition 1.23. Notice that we have proved that morphism of schemes $X \to Y$ are not determined by it's underlying map of sets(a classic example is $Spec\mathbb{C} \to Spec\mathbb{R}$), but they are determined by their induced map of Z-valued points as Z varies over all possible schemes.

Proof. A remark before we begin the proof: you probably have noticed that our notation $X(Z)$ is indicating that Z is the argument of a (contravariant) functor associated with X, which we still denote by X. This is a functor from the category of schemes to the category of sets. In this interpretation, you will notice that a morphism $X \to Y$ induces a natural transformation of the two functors X, Y, and we are claiming that if two morphisms $\pi_1, \pi_2 : X \to Y$ induces the same natural transformation, then the two morphisms of schemes must be the same. Make sure you understand this interpretation, since this is an example of the very abstract Yoneda lemma.

Now we begin to prove this, and surely without the direct quote of Yoneda's lemma, since we want to show how the abstract method we used to prove Yoneda lemma looks like here, you will be surprised to see how simple this is. To begin with, let's see what happens if $Z = X$, then π_1, π_2 induces morphism that maps morphism from X to X to morphisms from X to Y by post-composing π_1 and π_2, then since we assumed that π_1 and π_2 induces the same natural transformation, then we know that they map $id : X \to X$ to the same morphism, thus $\pi_1 = \pi_2$. When you go to the section where we review Yoneda's lemma, you should think of this example. □

Here is another exercise you can have that is related to scheme valued points:

Proposition 1.24. Suppose that B is a ring, and X is a B-scheme. If $f_0, \cdots, f_n$ are $n+1$ functions on X with no common zeros, then they give a morphism of B-schemes $X \to \mathbb{P}^n_B$.

Proof. We will use $D(f_i)$ to indicate the locus where f_i doesn't vanish. Notice that the restriction of f_i to $D(f_i)$ is an invertible function of $\Gamma(D(f_i), \mathscr{O}_X)$. Also, it's easy to check that these $D(f_i)$ will cover X.

Then notice that we can define local morphisms:

$$D_(f_i) \to D_+(x_i) \cong SpecB[\frac{x_j}{x_i}|j \neq i]$$

by the ring homomorphism $k[\frac{x_j}{x_i}|j \neq i] \to \Gamma(X, D(f_i))$ via the B-algebra homomorphism that maps $\frac{x_j}{x_i}$ to $\frac{f_j}{f_i}$. We only have to show that they agree on the intersections.

Consider the restriction of $D(f_i) \to D_+(x_i)$ and $D(f_j) \to D_+(x_j)$ to $D(f_i) \cap D(f_j) \to D_+(x_i) \cap D_+(x_j)$, notice that this is determined by ring homomorphisms:

$$\begin{tikzcd}
k[\frac{x_k}{x_j}|k \neq j] \arrow[r] \arrow[d, "\frac{x_k}{x_j} \mapsto \frac{x_k}{x_j}"'] & \Gamma(D_+(f_j), \mathscr{O}_X) \arrow[d, "res"] \\
k[\frac{x_k}{x_j}, \frac{x_j}{x_i}] \arrow[r] & \Gamma(D_+(f_i) \cap D(f_j), \mathscr{O}_X) \\
k[\frac{x_k}{x_i}|k \neq i] \arrow[r] \arrow[u, "\frac{x_k}{x_i} \mapsto \frac{x_k}{x_i} \cdot \frac{x_i}{x_j}"] & \Gamma(D_+(f_i), \mathscr{O}_X) \arrow[u, "res"']
\end{tikzcd}$$

Then it's left to you to check this agrees. ☐

Remark 1.25. The above proposition gives you a convient way to construct maps from a scheme to the projective space, but this method doesn't give you all morphisms from a scheme to projective space. An example is the identity morphism $\mathbb{P}^1_k \to \mathbb{P}^1_k$. Notice that on $\mathbb{P}^1_k$ there are no nonconstant global functions, thus you can easily check given any two global functions, since each one of them doesn't vanish anywhere we can each one of them gives the same morphism from $\mathbb{P}^1_k$ to $D_+(x_0)$ and $D_+(x_1)$, thus the image of the morphism is the intersection, not the whole $\mathbb{P}^1$, thus not the identity.

The reason why this fails to give all the morphisms will later be one of our motivations to understand line bundles on a scheme.

Remark 1.26. Another remark we want to give is that if you have a nonwhere vanishing function on X, and f_i's as in the above proposition, then gf_i and f_i will give the same morphism, as you can easily check.

Proposition 1.27. Suppose that X is a finite type k-scheme, then locally it's of the form $Speck[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$, then suppose that p is a k-valued point of X (closed point with residue field k), it's cotangent space at p is given by the cokernel of the k-linear map: $k^r \to k^n$ given by the Jacobian matrix:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_r}{\partial x_1}(p) \\ \cdots & & \cdots \\ \frac{\partial f_1}{\partial x_n}(p) & \cdots & \frac{\partial f_r}{\partial x_n}(p) \end{bmatrix}$$

Proof. Cotangent space of a point is determined by any open subset containing that point, then we can just assume that X is $Speck[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$. Notice that since a field k is in particular local, then k-valued points of a scheme X can be interpreted as local morphisms $\mathscr{O}_{X,p} \to k$, where you can see that the maximal ideal is mapped to 0 and since 1 is mapped to 1, the kernel of the map is exactly the maximal ideal of $\mathscr{O}_{X,p}$, thus the residue field of p is k. We can also do this more explicitly, since we know that a k-mophism $Speck \to Speck[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$ correspond to a k-algebra homomorphism $k[x_1, \cdots, x_n]/(f_1, \cdots, f_r) \to k$, from which you can see that the k-valued points are exactly of the form $(k_1, \cdots, f_n)$.

Then let's first consider a special case where m is the maximal ideal corresponding the origin, or in other words, it's the ideal $(x_1, \cdots, x_n)$. In this case, what is the cotangent space, by definition and our previous result, it can be identified by m/m^2, in our case, we see that m/m^2 can be spanned over k by the image of $x_1, \cdots, x_n$ in m/m^2, however, they might not be linearly independent, thus we need to quotient out the relations, this is what we will explain later about the map defined by the matrix J.

Notice that J defines a map from k^r to k^n, and the image can be identified to the n-dimensional space spanned by $x_1, \cdots, x_n$ where we pretend they are linearly independent and we will use the image to impose the relations. Notice that since we are only allowed to do k-linear combination, then everything in $f_1, \cdots, f_r$ that is not linear doesn't impose any condition on the cotangent space. Moreover, if there are constant terms in these $f_1, \cdots, f_r$, they also doesn't do anything to the cotangent space, thus we are looking only at the linear terms, and more precisely, only the coefficients of the linear terms. Also, since we are working in the origin, if we take the partial derivative and evaluate at $(0, \cdots, 0)$, we kill all the constant and nonlinear terms. And more importantly, when we take the partial derivative of x_i, you only pick out the coefficient of x_i since when you evaluate at 0 all the other terms that don't involve x_i vanish. Then in general, what this matrix J is capturing is that it's i,j-th term is pick out the coefficient of x_j in f_i, then how can you interpret the image of this map.

To answer this question, you need to figure out that is the relation we want to impose on $x_1, \cdots, x_n$. By our discussion above, we are just quotient out all the linear combinations of the linear terms in $f_1, \cdots, f_r$. Then notice that if we are considering the linear terms in $k_1 f_1 + \cdots + k_r f_r$, what is the coefficients of x_1, it's exactly:

$$\sum_i \frac{\partial f_i}{\partial x_1}(p) k_i$$

and it's similar for all the other x_j's, which is exactly the image of J, we are done.

To finally settle down the general case where we want to see what is the cotangent space at $m = (x_1 - k_1, \cdots, x_n - k_n)$. We claim that we can reduce the problem to the case where all $k_i = 0$, which is the previous case we settled. The reason we can do this reduction is because that the ideal here is basically the same with the previous case, since we are still trying to first pretend $x_i - k_i$ are independent and then quotient out all the combinations of $x_i - k_i$ in $f_1, \cdots, f_r$, thus we can write $x_i = (x_i - k_i + k_i)$ in $f_1, \cdots, f_n$ and then use new variables to denote $x_i - k_i = y_i$, then take partial derivatives of y_i and evaluate at $y_i = 0$ for all y_i. But you can use the chain rule to show that take partial derivatives of y_i and evaluate at 0, is the same as take partial derivatives of x_i and evaluate at $(k_1, \cdots, k_n)$, we are done. □

Remark 1.28. This result will be generalized later to closed points of X whose residue field is perfect over k, thus in particular if $char(k) = 0$, we can generalize this result to all closed points of k. But this will wait for later discussion of the cotangent sheaf and related notions.

Now, after we interpret the tangent space as the cokernel of the map J, we can discuss an important numerical invariant, the corank: we begin by try to extend the notion of the matrix J we have before to points that are not necessarily k-valued:

Definition 1.29. Suppose that $X = Speck[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$, and $p \in X$ is any point, not necessarily closed or k-valued or anything. How can we get a similar Jacobian matrix J as what we had before. Notice that the partial derivatives of $f_1, \cdots, f_r$ still makes sense as they are just formally defined, the issue here is the when p is not a k-valued point, we can't just identify p as a n-tuple of elements in k, thus we can't just say that we can evaluate the partial derivatives at the n-tuple.

However, if you really think about the process, you will notice that the evaluation map $k[x_1, \cdots, x_n]/(f_1, \cdots, f_r) \to k$ is just viewing k as $k[x_1, \cdots, x_n]/(f_1, \cdots, f_r)/p$ where $p = (x_1 - k_1, \cdots, x_n - k_n)$. And notice that this map can be identified to be the map from $k[x_1, \cdots, x_n]$ to the residue field at p! Therefore, when p is just an arbitrary point in $Speck[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$, we can still view it as a prime in $k[x_1, \cdots, x_n]$ and we can evaluate the partial derivatives of $f_1, \cdots, f_r$ which are in $k[x_1, \cdots, x_n]$ at the residue field at p, which is $k[x_1, \cdots, x_n]_p/pk[x_1, \cdots, x_n]_p$ which is the same as the residue field of $Speck[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$ at p. We will define the matrix defined with entries in $k(p)$, the Jacobian matrix of $Speck[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$ at p, which is a $k(p)$-linear map from $k(p)^r$ to $k(p)^n$, thus it makes sense to discuss it's corank and everything that makes sense for a vector space morphism.

Proposition 1.30. Let A be a finitely generated k-algebra, generated by $x_1, \cdots, x_n$ with relation ideal I generated by $f_1, \cdots, f_r$, let p be a point in $SpecA$. Then suppose that g is in I, then appending the column of partial derivative of g evaluated at p doesn't change the corank of the Jacobian matrix we defined above(after adding the column, we now have a $k(p)$-linear morphism from k^{r+1} to k^n). Hence the corank of the Jacobian matrix doesn't depend on the choice of generators of I.

Proof. Let's first see if $g \in I$, then we can write $g = \sum h_i f_i$, then you see that by the Leibniz rule of taking partial derivatives, we have $\frac{\partial g}{\partial x_j} = \sum_i \frac{\partial h_i}{\partial x_j} f_i + \frac{\partial f_i}{\partial x_j} h_i$, then we are considering it's image under the map $k[x_1, \cdots, x_n] \to k[x_1, \cdots, x_n]_p/pk[x_1, \cdots, x_n]_p$ where p is in $V(f_1, \cdots, f_r)$, which means that $p \supset (f_1, \cdots, f_n)$. Then notice that under this aasumption, f_i is mapped to 0 in $k[x_1, \cdots, x_n]_p/pk[x_1, \cdots, x_n]_p$, thus the terms $\frac{\partial h_i}{\partial x_j} f_i$ in $\frac{\partial g}{\partial x_j}$ doesn't contribute anything. But all the other terms image in $k(p)$ will be linear combinations of partial derivatives of f_i, which is in the image of J before we appended the partial derivatives of g.

Based on this result, we can prove that the corank of J doesn't depend on the generators of I. Say $f_1, \cdots, f_r$ and $g_1, \cdots, g_s$ are two sets of generators of I, then we can first consider the corank of J defined by f_i, then adding g_i once at a time, which won't change the corank, as we proved before. And after you get the J defined by f_i and g_i's, you delete f_i's one at a time, which still won't change the corank. $\square$

Proposition 1.31. Suppose that $q \in k[x_1, \cdots, x_n]$ any polynomial in the variables $x_1, \cdots, x_n$, let h be the polynomial $y - q$ in $k[x_1, \cdots, x_n, y]$, then the Jacobian matrix of $(f_1, \cdots, f_r, h)$ with respect to the variables $(x_1, \cdots, x_n, y)$ has the same corank at p as the Jacobian matrix of $f_1, \cdots, f_r$ with respect to $(x_1, \cdots, x_n)$. Hence the corank of the Jacobian matrix of p at p is independent of the generators of A.

Proof. Let's first make it clear what is this p here. Notice that we have a natural closed embedding $k[x_1, \cdots, x_n, y] \to k[x_1, \cdots, x_n]$ by quotient out y, thus if we start by a prime p in $k[x_1, \cdots, x_n]$, it's naturally a prime in $k[x_1, \cdots, x_n, y]$.

Now, this implies that the residue field $k(p)$ at p computed from $k[x_1, \cdots, x_n, y]$ or from $k[x_1, \cdots, x_n]$ are naturally isomorphic. Therefore, we can consider two matrix J_y and J, where J is the $n \times r$ matrix defined by $f_1, \cdots, f_r$, as a map from k^r to k^n and J_y is the $(n+1) \times (r+1)$ matrix defined by $f_1, \cdots, f_r, y$ viewed as map from $k^{r+1} \to k^{n+1}$.

Now we claim that the two maps have the same corank. To see so, suppose that corank of J is a. Then notice that for the matrix J_y, the left upper sub $r \times n$ matrix is just J fop the last row, since $f_1, \cdots, f_r$ doesn't involve y, thus you get 0 on those entries. For the last one, the y partial derivative of h is 1, thus it's image in $k(p)$ is also 1, there fore we see that J_y is of the form:

$$J_y = \begin{bmatrix} J & \cdots \\ 0 & 1 \end{bmatrix}$$

then you can easily see from linear algebra that the rank of J_y is the rank of J plus 1, but the dimension of the target of J_y is also 1 larger, thus the corank of J_y is the same as that of J.

Finally, to show the assertion that the corank is independent of the generators. Suppose that A is generated by $x_1, \cdots, x_n$ with relation polynomials $f_1, \cdots, f_r$ and also generated by $y_1, \cdots, y_m$ with relation polynomials $g_1, \cdots, g_s$. Let's see if we add on a new variable y_1 to $k[x_1, \cdots, x_n]/(f_1, \cdots, f_n)$, we have $k[x_1, \cdots, x_n, y_1]/(f_1, \cdots, f_n)$, but we need to quotient out y_1 to make it isomorphic to the ring we started with. Then you keep going until you reach $k[x_1, \cdots, x_n, y_1, \cdots, y_m]/(f_1, \cdots, f_r, y_1, \cdots, y_m)$. Similarly, you also get $k[x_1, \cdots, x_n, y_1, \cdots, y_m]/(g_1, \cdots, g_s, x_1, \cdots, x_n)$. However, if you think of how these two rings are isomorphic to A, you will get the following commutative diagram:

$$\begin{tikzcd}
 & k[x_1, \cdots, x_n, y_1, \cdots, y_m]/(g_1, \cdots, g_s, x_1, \cdots, x_n) \arrow[d, dashed, "iso"] \\
k[x_1, \cdots, x_n, y_1, \cdots, y_m] \arrow[ur, "quotient"] \arrow[r, "x_i \mapsto x_i \quad y_j \mapsto y_j"] \arrow[dr, "quotient"'] & A \\
 & k[x_1, \cdots, x_n, y_1, \cdots, y_m]/(f_1, \cdots, f_r, y_1, \cdots, y_m) \arrow[u, dashed, "iso"']
\end{tikzcd}$$

Therefore, the two ideals we are quotienting out are both kernel of the map $k[x_1, \cdots, x_n, y_1, \cdots, y_m] \to A$, thus by the previous result about the corank doesn't depend on the generators of the ideal, and also by the result we just discussed in this proposition, adding a variable and then quotient out it doesn't affect the corank, thus we are done. $\square$

Based on these above two result, suppose that X is a finite type k-scheme, and $p \in X$ is any point, we know that there will be an affine open $SpecA$ containing p where A is a finitely generated k-algebra. Then we can use it to define the corank of the Jacobian matrix at p, which have a corank. What we proved above is that this is independent of the choice of A as long as you choose something that contains p. In conlcusion, every finite type k-scheme comes with an intrinsic **Jacobian corank function** that comes from the underlying set of X to $\mathbb{Z}_{\geqslant 0}$.

Moreover, we also prove that when p is a k-valued point, then the Jacobian corank is equal to the dimension of the cotangent space at p. We will temporaily call this function $JC : X \to \mathbb{Z}_{\geqslant 0}$. Later after we discussed the cotangent sheaf, you will find out the this is the rank of the cotangent sheaf.

Proposition 1.32. *JC* defined above is an uppersemicontinuous function.

Proof. As we discussed in the review section of dimensions, we only have to show that the set $JC^{-1}(-\infty, n)$ is open for every $n \in \mathbb{Z}_{\geqslant 0}$. To prove this, we only have to prove that if p is a point such that the Jacobian corank is less than n, which is equivalent to the rank of the Jacobian matrix is at least some number, then there is an open neighborhood of p such that each point in the neighborhood has corank less than n, which is equivalent to there is an open neighborhood around p such that the rank of the Jacobian matrix is at least that number.

To prove this, we can just assume that $X = Speck[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$, suppose that at p, the rank of the Jacobian matrix is at least some number, which is equivalent to some minors not vanishing at p, which is an open condition for each of the minor. Since we only have finitely many such minors to consider and each of the open associated to the minors contains p, we get a nonempty open in $Speck[x_1, \cdots, x_n]$ that contains p, take the intersection with $V(f_1, \cdots, f_r)$, we are done. □

Now, let's prove that this Jacobian corank is preserve ba any field extension:

Proposition 1.33. If X is a finite type k-scheme and l/k is any field extension of k, then under the base change map $X \times_k l \to X$, if $p \mapsto q$, we have $JC(p) = JC(q)$.

Proof. First consider the following diagram, which consist of two small Cartesian diagrams:

$$\begin{tikzcd} Specl[x_1, \cdots, x_n]/(f_1, \cdots, f_r) \arrow[r] \arrow[d] & X \times_k l \arrow[r] \arrow[d] & l \arrow[d] \\ Speck[x_1, \cdots, x_n]/(f_1, \cdots, f_r) \arrow[r, hook] & X \arrow[r] & k \end{tikzcd}$$

where the map from $Specl[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$ to $Speck[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$ is defined by the morphism induced by the inclusion of $k[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$ to $l[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$. Then notice that suppose that the Jacobian corank at point $p \in Specl[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$ is n, where q is the inverse image of p. Notice that if the corank of $(f_1, \cdots, f_r)$ at p is r, then we can assume that the rank of the Jacobian matrix is $k = n - r$. We want to show that the Jacobian matrix of $f_1, \cdots, f_r$ at q is also k. First consider the following diagram:

$$\begin{tikzcd} k[x_1, \cdots, x_n] \arrow[r, hook] \arrow[d] & l[x_1, \cdots, x_n] \arrow[d] \\ k[x_1, \cdots, x_n]_q \arrow[r, dashed] \arrow[d] & l[x_1, \cdots, x_n]_p \arrow[d] \\ k[x_1, \cdots, x_n]_q/qk[x_1, \cdots, x_n]_q \arrow[r, hook, dashed] & l[x_1, \cdots, x_n]_p/pl[x_1, \cdots, x_n]_p \end{tikzcd}$$

where all the dotted maps are induced by some universal properties. Then we see that determinant of any minor of the Jacobian matrix of $f_1, \cdots, f_r$ vanishes or doesn't vanish in $k(q)$ iff it doesn't vanish in $k(p)$, thus the rank is preserved, we are done. □

Proposition 1.34. Again, suppose that X is finite type over k, then for any $p \in X$ (not necessarily closed), we can prove that $JC(p) \geqslant dim_X p$.

Proof. Let's first consider the case when k is algebraically closed, then we will use the field extension $k \to \overline{k}$ and prove that JC and $dimX_p$ are preserved under base change of field extensions to deal with the case where k is not algebraically closed.

Now assume that k is algebraically closed, we claim that we can assume that $X = Speck[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$. To see so, we notice that the corank is determined locally, thus no problem. Moreover, we have proved that even for a general scheme X, the dimension of the irreducible component of X may not be equal to the dimension of the intersection of an open subset with the irreducible component, this doesn't happen for locally finite type over k-schemes(this is basically because we have closed points are dense and the relation between dimension and codimension).

Then we claim that we only have to prove this for maximal ideals. To see why, suppose that $p \subset m$ where m is a maximal ideal and p is a prime ideal of $k[x_1, \cdots, x_n]$, then we notice that we have a canonical inclusion $k(m) \to k(p)$ as shown in the following diagram:

$$\begin{tikzcd} k[x_1, \cdots, x_n]_m \arrow[r] \arrow[d] & k[x_1, \cdots, x_n]_p \arrow[d] \\ k[x_1, \cdots, x_n]_m/mk[x_1, \cdots, x_n]_m \arrow[r] & k[x_1, \cdots, x_n]_p/pk[x_1, \cdots, x_n]_p \end{tikzcd}$$

where morphisms of rings are all induced by universal properties.

Then, you will notice that if a polynomial $f \in k[x_1, \cdots, x_n]$ is mapped to 0 in $k(m) = k[x_1, \cdots, x_n]_m/mk[x_1, \cdots, x_n]_m$, iff it also mapped to 0 in $k(p)$, which proves our assertion since this implies that the rank of the Jacobian matrix at p is equal to that at m since the rank is determined by vanishing determinant of certain minors.

Now, to prove this for closed points, since we are working with an algebraically closed field, the closed points are exactly the k-valued points, thus we already know that corank is the dimension of the cotangent space at the maximal ideal m. Therefore we only have to show that the dimension at m is less than or equal to $dim_k(m/m^2)$. But notice that the dimension of that is just the codimension of m, or the dimension of the ring localized at m, but notice that if we write m' the corresponding maximal ideal of m after we localize at m, then the dimension $dim_k(m'/(m')^2) = dim_k m/m^2$, then notice that this is a Noetherian local ring, then the dimension of the ring is less than or equal to the dimension of $dim_k(m/m^2)$, we are done. $\square$

The last hands on example of computing the corank before we head to the discussion of regularity and smoothness is given to projective space:

Proposition 1.35. Suppose that $X \subset \mathbb{P}^n_k$ is a closed subscheme cut out by a finite number of homogeneous polynomials $f_1, \cdots, f_r$. Then we claim that the corank of the Jacobian matrix of X at any point is one less than the Jacobian corank of the so called **projective Jacobian matrix**:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_r}{\partial x_1}(p) \\ \cdots & & \cdots \\ \frac{\partial f_1}{\partial x_n}(p) & \cdots & \frac{\partial f_r}{\partial x_n}(p) \end{bmatrix}$$

Proof. We have to first make sense of the right hand side, since these f_i's are not even global functions on $\mathbb{P}^n_k$. The following discussion is helpful in both finding the corank of the actually Jacobian matrix of X at a point p and how to interpret the projective Jacobian matrix.

Notice that by definition, to compute the Jacobian corank of X at p, we only have to know that what X looks like locally around o, then suppose that $p \in D_+(x_k)$, we know that locally X in $D_+(x_k)$ can be written as $Speck[\frac{x_i}{x_k}]/(f_{1,k}, \cdots, f_{r,k})$ where the $f_{i,k}$ correspond to the polynomial f_i homogenized at x_k, i.e. a polynomial with variables $\frac{x_i}{x_k}$ where $i \neq k$ you get after you divide f_i by x_k^d where d is the degree of f_i.

Now, we will interpret the elements in the matrix J we defined above as just the ordinary Jacobian matrix you have when you work with $k[x_0, \cdots, x_n]$ since p is still a homogeneous prime ideal in $k[x_0, \cdots, x_n]$, which is in particular still a prime ideal of $k[x_1, \cdots, x_n]$. Then we have to show that the corank of J is one larger than the actually Jacobian matrix corank of X at p. To see it, let's first consider the following diagram:

$$\begin{tikzcd} k[\frac{x_0}{x_k}, \cdots, \frac{x_n}{x_k}] \arrow[r, hook] \arrow[d] & k[x_1, \cdots, x_n]_{x_k} \arrow[d] \\ k[\frac{x_0}{x_k}, \cdots, \frac{x_n}{x_k}]_{p_k} \arrow[r] & (k[x_1, \cdots, x_n]_{x_k})_{p_{x_k}} \cong k[x_1, \cdots, x_n]_p \end{tikzcd}$$

where p_k is the homogenized version of p. We can do the identification at the bottom right because we assumed in the beginning that x_k doesn't lie in p.

Then notice that via this, we can naturally identify the residue field at p_k as a subfield of the residue field at p. Moreover, if you analyze what happens when you compute the actual Jacobian matrix of X, you find that you basically are only treating all the x_k's in $f_1, \cdots, f_r$ as 1 and compute the partial derivatives of all the other variables the same as what you will do if you are computing the Jacobian matrix of $f_1, \cdots, f_r$ in

$k[x_0, \cdots, x_n]$. Then, I will let you to figure out what before you evaluate the actual Jacobian matrix at the residue field of p_k, each row of the actual Jacobian matrix times a power of x_k is that row of the projective matrix J (if that row is not the k-th row since it doesn't exist in the actual Jacobian matrix). Thus when you compute the determinant minors of the two Jacobian matrix, one of the minor is the one you obtained from the other after you do a certain elementary operation, thus the determinant doesn't change, thus we obtain that the rank of the projective matrix is the same as the one of the actual matrix. But notice that the target of the actual Jacobian matrix is one less than that of the projective matrix, thus we know that the corank is also one less, we are done. (Of course here you have to consider the case where the rank of the projective Jacobian matrix is the same as the number of the number of varaibles since in this case the usual vanishing minor condition is not enough to justify the argument, but this case implies that the corank of the projective Jacobian matrix is $0 = n - n$, then you can show that the rank of the actual Jacobian matrix is $n - 1$ together with image of the actual Jacobian matrix is of dimension one less, which is $n - 1$, in this case we still have the corank equal to $0 = n - 1 - (n - 1)$.) □

Now we finally begin to discuss regularity and smoothness of varieties, we start by the notion of regularity, and you will soon discover that the single absolute notion of regularity is not entirely desirable which forces us to use the more flexible version of smoothness:

Definition 1.36. Suppose that (A, m, k) is a Noetherian local ring, if the dimension of A is equal to the k dimension of m/m^2, we will say that A is a **regular local ring**, thus in particular, regular local rings are necessarily Noetherian.

If now you only have a Noetherian ring A, it's called regular if A_p is a regular local ring at all prime ideals p. We will show later that you only have to check this for maximal ideals, since you p is contained in m, and A_m is regular, then A_p is automatically regular. Thus, regular local rings are in particular regular rings.

A locally Noetherian scheme X is called **regular at a point** p if $\mathscr{O}_{X,p}$ is a regular local ring and it's singular at p if otherwise and we will say that this point is a singularity. X is called **regular** if it's regular at all points $p \in X$ and it's called singular if otherwise.

Proposition 1.37. If (A, m) is a Noetherian local ring of dimension 0 then it's regular iff it's a field.

Proof. Suppose that it's a field, then easy to see that it's regular.

Conversely, suppose that A is regular, then $dim_k m/m^2 = 0$, thus 0 will generate m, thus $m = 0$, thus A is a field. □

As we develop the theory further, you will gradually be convinced that this regularity is the right notion of smoothness of schemes. Krull first introduced it as a purely algebraic notion until Zariski later find this is a fundamental notion in geometry.

Proposition 1.38. Suppose that (A, m) is a regular local ring of dimension $n > 0$, and $f \in A$, then we claim that A/f is still a regular local ring iff $f \in m - m^2$.

Proof. Notice that if $f \in m - m^2$, we have already saw that the cotangent space of $A/(f)$ at m is of dimension 1 less, also we proved that $dim A/(f) \geqslant dim A - 1$, we also always have the dimension of the cotangent space at $m/(f)$ is always at least $dim A/(f)$, we thus have:

$$dim_k(m/(f)/(m/(f))^2) \leqslant dim A/(f) \leqslant dim_k(m/(f)/(m/(f))^2)$$

we are done.

Now suppose that $dim A/(f)$ is regular local of dimension $n - 1$, we first show that $f \in m$. If not, notice that if $f \notin m$, then $A/(f) = 0$ a contradiction. Then in order for A/f to still be regular local, the dimension of the cotangent space has to drop by 1, thus by what we showed before $f \notin m^2$. □

The above result also has an geometric version:

Proposition 1.39. Suppose that X is a Noetherian scheme and D is an effective Cartier divisor on X, then if $p \in D$ is a regular point of D, then p is also a regular point of X.

Proof. Recall that, in an earlier proposition, we proved that if (A, m) is a Noetherian local ring and $f \in m$ is an element, then $dimA/f \geqslant dimA - 1$. Here we refine this result: we claim that if f is a non-zerodivisor, then $dimA/f = dimA - 1$.

This is basically a result of Krull's Principal ideal theorem: Notice that if $f \in m$ and a non-zerodivisor then notice that the irreducible components of $V(f)$ are of codimension 1, then suppose that the following chain of prime ideals gives the dimension of A/f:

$$(f) \subsetneq p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_n$$

since p_0 is of codimension 1, there is a prime strictly below p_0, say p, thus we know that the dimension of A is at least 1 larger than $dimA/f$, then we have:

$$dimA - 1 \leqslant dimA/f \leqslant dimA - 1$$

where the first inequality is what we had before and the second one is the above discussion.

You might wonder why only applying Krull's principal ideal theorem doesn't suffice to show $dimA = dimA/f + 1$. This is because that as what we showed above, we only can conclude that $dimA \geqslant dimA/f + 1$ but to conclude that this is an equality, we also need to ensure that the chain: $p \subsetneq p_0 \subsetneq \cdots \subsetneq p_n$ is the chain that gives the dimension of A, but this may not hold.

Then in this problem, notice that a point is regular or nor is determined locally, thus we can assume that $X = SpecA$ and $f \in m$ is not a zerodivisor $D = SpecA/f$. Then we can assume that $f \in p$ for $p \in D$, then suppose that D is regular at p, then we know that the local ring A_p/fA_p is regular. We want to show that A_p is also regular, notice that being a nonzerodivisor is preserved under localization, thus $dimA_p/fA_p$ is $dimA_p - 1$ by what we just discussed. Now, if we can show that the dimension of the cotangent space at pA_p in A_p is one larger than that in A_p/fA_p, we are done. Notice that we have an inequality that always holds:

$$dimA_p \leqslant dim_k(pA_p/(pA_p)^2)$$

but notice that $dimA_p = dimA_p/fA_p + 1$ while the k-dimension of the cotangent space at pA_p/f is $dimA_p/fA_p$, which is at most one smaller than the dimension of pA_p, thus we have the following inequality:

$$dim_k(pA_p/(pA_p)^2) - 1 + 1 \leqslant dimA_p = dimA_p/fA_p + 1 \leqslant dim_k(pA_p/(pA_p)^2)$$

We are done. □

Now, for a finite type k-scheme, we have a very hands on way to determine if a point is regular or not if the point is a k-valued point and k is algebraically closed:

Proposition 1.40. Suppose that $X = Speck[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$ is of pure dimension d for an arbitrary field k, then if p is a k-valued point then it's a regular point iff the Jacobian corank function at p is d.

Proof. Suppose that m is a k-valued point, then first suppose that it's regular at m, this implies that the local ring at m is regular. We claim that the dimension of the ring is d.

This is because we assume that X has pure dimension d and we also know that every k-valued point in X should be a closed point, which correspond to a maximal ideal, thus we know that for any minimal prime p over $(f_1, \cdots, f_r)$, $Speck[x_1, \cdots, x_n]/p$ is of dimension d. Then suppose that m is a maximal ideal which contains some minimal prime over $(f_1, \cdots, f_r)$, then we know that the $V(m)$ is 0, which implies that it's codimension in $V(p)$ is the dimension of $V(p)$, which is d, thus localize at m give you a local ring of dimension d. Therefore, being a regular local ring, we know that the cotangent space at m should have dimension d, but we already proved that for k-valued points, the dimension of the cotangent space is the corank of the Jacobian matrix. Hence, it's a regular point iff the corank of the Jacobian matrix is d. □

One pretty easy consequence of the above result is:

Corollary 1.41. Suppose that $k = \overline{k}$, then the singular closed points of the hypersurface $f(x_1, \cdots, x_n)$ in $\mathbb{A}^n_k$ are given by the equations:

$$f = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$$

Proof. Let's first show that the hyperfurface is of pure dimension $n-1$, this holds since we have Krull's principal ideal theorem and f is not a zero divisor.

Since $k = \overline{k}$, then the Hilbert's Nullstllensatz gives that all the closed points are k-valued points, thus you apply the above proposition, noting that the Jacobian matrix in this case is just a row vector, thus the corank is not $n-1$ (notice that it can only get larger) iff the rank of the Jacobian matrix is 0, thus everything is 0. $\square$

1.2 Regularity and smoothness over a field

Now we begin to formally introduce what is smoothness over a field k:

We begin with a useful remark to help you understand why we want the notion of smoothness over a field and what difference it has in contrast to the notion of regularity: As you have seen via the above results, to check a finite type k-scheme X is regular, we can only check for closed points, while for non closed points, we don't have a clue how to proceed at this point. The problem is solved by a later result showing that if you get a regular local ring after you localize at a maximal ideal, then for any prime ideal contained in the maximal ideal, localizing at it also gives you a regular local ring. But even we can prove this result, we are still under the assmption that $k = \overline{k}$, even though we would like to have a way of checking things for arbitrary k. All of these drawbacks of the notion of regularity seems to suggest that the old-fashioned Jacobian matrix criterion of smoothness is not suitable for the modern notion of regularity and we need to come up with new techniques of checking regularity for non closed points.

However, this is the wrong though, it's not the notion of Jacobian matrix and the geometric intuition we had from differential geometry is to blame, the fault is due to the notion of regularity. This is not the perfect nothing we can use to describe smoothness. As what you will see below, we will define even correctly though not perfectly what does it mean to say that a scheme X is smooth over a filed, which generalizes to the notion of smooth morphisms later. This is another example which testifies Grothendieck's ideal, the relative notion of properties of morphisms is better than the absolute notion of certain properties of the scheme itself.

Definition 1.42. A k-scheme X is **k-smooth of dimension d** or **smooth of dimension d over k** if it's pure dimensional of dimension d and there exists an affine open cover by $Speck[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$ where the Jacobian matrix has corank d at all points not just closed points. In particular, we always have smooth k-schemes are locally of finite type.

We will also say that X is smooth over k, this means that it's smooth over k of some dimension, we will also possible omit the k if it's clear from the context. By the theory of Jacobian corank we developed, this can be checked using any affine open cover.

Let's now formally state and prove a result we implicitly used in the process of developing the theory of Jacobian coranks:

Proposition 1.43. Suppose that $X = Speck[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$ then if the Jacobian matrix of $(f_1, \cdots, f_r)$ has corank d at a closed point m, then for all prime ideals $p \subset m$, the Jacobian matrix also have corank d at p.

Proof. As what we have implicitly proved before, we have a canonical inclusion of residue field at m to the residue field at p, which is compatible with the evaluation map from $k[x_1, \cdots, x_n]$ to $k(p)$ and $k(m)$:

$$\begin{tikzcd}
k[x_1, \cdots, x_n]_m \arrow[r] \arrow[d] & k[x_1, \cdots, x_n]_p \arrow[d] \\
k[x_1, \cdots, x_n]_m/mk[x_1, \cdots, x_n]_m \arrow[r] & k[x_1, \cdots, x_n]_p/pk[x_1, \cdots, x_n]_p
\end{tikzcd}$$

thus using the vanishing determinant of minors condition, we are done. $\square$

Now let's introduce the first examples of smooth k-schemes we have:

Proposition 1.44. $\mathbb{A}_k^n$ is smooth for any n and k.

Proof. First notice that $\mathbb{A}_k^n$ is irreducible of dimension n, in this case the Jacobian matrix is just 0, thus in any case, the rank at any prime ideal p is just 0, the the corank is always n, we are done.

Notice that we don't have to check regularity or anything, and we don't have to care if the point p is closed or not. □

Proposition 1.45. We claim that for a field k with characteristic not 2 or 3, the curve $y^2z = x^3 - xz^2$ in $\mathbb{P}_k^2$ is smooth.

Proof. Let's first notice that $y^2z - x^3 - xz^2$ is an irreducible polynomial, you can show this just by supposing this is not irreducible, then it's the product of a linear term and a quadratic term, thus we already proved that this defines an integral scheme of dimension 1.

We already proved that at p, a homogeneous prime ideal, the corank of the actual Jacobian matrix is one less than the projective Jacobian matrix. Then notice that we want this to be corank 2 since it's dimension is 1, which should be equal to the corank minus 1.

To compute the corank, you write out the projective Jacobian matrix:

$$(3x^2 - z^2, 2yz, -2xz)$$

It's of corank 2 at each point p iff not every entry is 0 when we evaluate it at any homogeneous prime ideals.

For the sake of contradiction, we assume that, there is a homogeneous prime ideal p, not containing (x, y, z), contains $3x^2 - z^2, 2yz, -2xz$, when the characteristic of the field is not 2 or 3, you can prove that this implies that p contains x, y, z, which is a contradicton, thus we are done. □

Proposition 1.46. Suppose that $f \in k[x_1, \cdots, x_n]$ is a polynomial such that the system of equations:

$$f = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$$

has no solution in $\overline{k}$, then the hypersurface $f = 0$ in $\mathbb{A}_k^n$ is smooth.

Proof. Recall that the hypersurface defines a dimension $n - 1$ closed subscheme. Then it's smooth iff the entries $\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}$ do not evaluate to 0 at a prime p that contains f.

For the sake of contradiction, there is a prime p such that $f \in p$ and $\frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$ when you evaluate at p, then if you consider the canonical inclusion:

$$k[x_1, \cdots, x_n] \hookrightarrow \overline{k}[x_1, \cdots, x_n]$$

there is a maximal $\overline{m} = (x_i - k_i) \subset \overline{k}[x_1, \cdots, x_n]$ such that $f \in \overline{m}$ and $\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n} \in \overline{m}$, but this is a contradiction, since the above condition means that in $\overline{k}$, if we set $x_i = k_i \in \overline{k}$, we have:

$$f = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$$

□

The above proposition is suggesting that being smooth is independent of a field extension, which is indeed true since we have the following proposition

Proposition 1.47. Suppose that X is a locally finite type k-scheme and $k \subset l$ is a field extension, then if X is smooth over k of dimension d if and only if $X \times_k Spec l$ is smooth over l of dimension d.

Proof. We have proved in the section where we studied dimension that $X \times_k Spec l$ is still of dimension d. To prove that it's still smooth, recall that by the construction of fibered product, we can just assume that $Spec l[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$ covers $X \times_k Spec l$ where $Spec k[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$ covers X.

For the sake of contradiction, if you assume that $X \times_k Spec l$ is not smooth, say at the prime ideal $q \subset l[x_1, \cdots, x_n]$, then if you consider the natural inclusion of:

$$k[x_1, \cdots, x_n]_p / pk[x_1, \cdots, x_n]_p \hookrightarrow l[x_1, \cdots, x_n]_q / ql[x_1, \cdots, x_n]_q$$

where p is the inverse image of q under the canonical inclusion $k[x_1, \cdots, x_n] \to l[x_1, \cdots, x_n]$, we know that the corank of the Jacobian matrix at p is equal to the corank at q, a contradiction. □

Even though we have said that regularity is not a good notion for smoothness, it's still useful in it's own right for several reasons. Now we want to some comparison between the notion of smoothness and regularity. A lot of the statements will not be proved at this point, since we will revisit them later with much more powerful tools. The only reason we want to state them here is that it makes a few statements cleaner.

Proposition 1.48. Suppose that X is a locally finite type scheme of pure dimension d over an algebraically closed field $k = \overline{k}$. Then X is regular at all of it's closed points iff it's smooth.

Proof. Let's first prove that regular at closed points implies it's smooth. This still uses the characterization of closed points in a scheme which is locally of finite type over a scheme k. It's a closed point iff the residue field is a finite extension of k, thus you can prove that when you work with an algebraically closed field $\overline{k} = k$, the closed points are just those that locally looks like tuples of elements in k. Therefore, let's just assume we are working with $Speck[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$ that is of pure dimension d.

Then, if X is regular at all closed points (notice that since k is algebraically closed, then k-valued points are just closed points), we use the Jacobian criterion for regularity to conclude that the Jacobian corank here is d, thus it's smooth at all closed point. As what we have explained, smooth at all closed points is enough to show that it's smooth.

The converse is just an easy application of Jacobian criterion for regularity. $\square$

The following theorem is only stated without proof. However, once we have enough machinery, it's just an easy consequence of that:

Theorem 1.49.

1. If k is perfect then every regular finite type k-scheme is smooth over k.
2. Every smooth k-scheme is regular, without any assumption on k.

Remark 1.50. The theorem above basically tells you the full picture of how regularity and smoothness are related: you always have regular if you have smoothness, but smoothness is not guaranteed by regularity.

It coincides with the proposition preceding it since we know that an algebraically closed field is always perfect. And we will prove later that regular at closed points implies that its regular everywhere.

The reason why we can't use the technique which proved the preceding proposition to prove the theorem is that we only know the Jacobian criterion at k-valued points, but when k is not algebraically closed, closed points are not necessarily k-valued points, which is a huge obstacle we will try to overcome later.

Example 1.51. Here is an example that tells you that regularity doesn't imply smoothness when the base field is not perfect.

Let $k = \mathbb{F}_p(u)$, which is not perfect. Consider the hypersurface $X = Speck[x]/(x^p - u)$. Notice that $x^p - u$ is an irreducible polynomial, thus $k[x]/x^p - u$ is a field, thus regular, but as you can check, this is not a smooth variety since at the point $(x^p - u)$, both $x^p - u$ and $d(x^p - u) = 0$ is in this prime ideal, thus Jacobian criterion fails, it's not smooth over k.

One observation is that, if you only consider a field k and $Speck$, then any $Speck$ is smooth over itself, thus we know that X defined above is smooth over $k[x]/x^p - u$, but not smooth over k. This reminds you to remember that smoothness is a relative notion, it depends what base field we choose, and indeed, later we will define what it means to be smooth over an arbitrary scheme.

Therefore, you should never use regular and smooth interchangeably.

Below are some geometric properties enjoyed by regular local rings:

First, we already know, even didn't prove, any scheme smooth over k is going to be regular, the following property of regular local rings tells you that if you have a smooth scheme, then it's always locally irreducible:

Theorem 1.52. Suppose that (A, m, k) is a regular local ring of dimension n, then A is an integral domain.

Before we prove it, we can give some consequences of this theorem:

Proposition 1.53. Suppose that p is a regular point of a locally Noetherian scheme X, then only one irreducible component of X passes through p.

Proof. Let's assume that for the sake of contradiction, you can find two components X_1, X_2 passing through p, recall that if you pick any affine open $SpecA$ near p, where A is a Noetherian ring, $SpecA \cap X_1$ and $SpecA \cap X_2$ are two distinct irreducible components of $SpecA$. We will call the prime corresponding to the two irreducible component p_1, p_2, which coorepond to minimal primes under p and they are not equal. Then since we know that X is regular at p, A_p is a regular local ring, say of dimension n, then it's an integral domain, which means that $(p_1)_p = (p_2)_p = 0$, thus you take the contraction of them to conclude that $p_1 = p_2$, contradiction since $p_1 \neq p_2$. □

Proposition 1.54. A nonempty regular Noetherian scheme is irreducible iff it's connected.

Proof. We already know that irreducible implies connected.

For the converse, assume that it's connected, but not irreducible, then since it's a Noetherian scheme, we can assume that we have finitely many irreducible components $X_1, \cdots, X_n$. Also, since it's connected, we claim that there is at least a point that lies in the intersection of two components. This is because we can find a component that is completely disjoint with the union of the other, which is a contradiction to connectedness since we are only doing finite union. This is a contradiction to X being regular according to the above proposition. □

Proposition 1.55. Suppose that (A, m, k) is a regular local ring of dimension n, and $I \subset A$ is an ideal of A cutting out a regular local ring of dimension d, let $r = n - d$, then $SpecA/I \hookrightarrow SpecA$ is a regular embedding into $SpecA$.

Proof. Let's first do some algebra. Form the assumption of regular local rings, we know that m/m^2 is of dimension n over k. Also, $(m/I)/(m^2/I) \simeq m/m^2 + I$ is of dimension d over k. Also, you have a canonical surjection $m/m^2 \hookrightarrow m/m^2 + I$ induced by the universal property of quotients, then the kernel is easily seen to be $m^2 + I/m^2$, thus if you view everything as k-vector spaces, you know the dimension is additive, thus the dimension of $m^2 + I/m^2 \simeq I/(m^2 \cap I)$ is r, thus, there are $f_1, \cdots, f_r$ in I such that the image in $I/(m^2 \cap I)$ form a basis. Now we claim that:

$$A/(f_1, \cdots, f_r) \qquad A/I$$

are both regular local. To prove this, first notice that $f_1, f_2, \cdots, f_r$ forms a regular sequence, this is because we choose then such that they are a basis in $I/I + m^2$. This implies that the dimension of $A/(f_1, \cdots, f_r)$ is just d, the same as that of A/I.

Then notice that by our choice of $f_1, \cdots, f_r$, the dimension of $m/(f_1, \cdots, f_r) + m^2$ is at most d, this is because that this is isomorphic to the quotient $(m/m^2)/(f_1 \cdots, f_r)/m^2$, thus $A/(f_1, \cdots, f_r)$ is still regular local and thus a domain.

Then suppose that $p \supset I$, then we have a surjection:

$$A_p/(f_1, \cdots, f_r)_p \to A_p/I_p$$

which is a surjection between domains of the same dimension, then is an isomorphism. Thus we conclude that $I_p = (f_1, \cdots, f_r)_p$, then since $f_1, \cdots, f_r$ is still a regular sequence in A_p, we are done.

In particular, we proved that if you consider the inclusion:

$$(f_1, \cdots, f_r) \hookrightarrow I$$

it an isomorphism for every p prime in A (we prove for $p \supset I$ above and if not, localize at p just gives you 0), thus this is isomorphism, which means that $(f_1, \cdots, f_r) = I$, and we know $f_1, \cdots, f_r$ is a regular sequence, thus I is also generated by a regular sequence. □

Corollary 1.56. Suppose that $\pi : X \hookrightarrow Y$ is a embedding of regular schemes, then π is a regular embedding.

Proof. Suppose that $p \in X$, we want to show that in the local ring $\mathcal{O}_{Y,p}$, the ideal of X is generated by a regular sequence. By our assumption, we know that we have a surjection:

$$A \to A/I$$

of regular local rings where I is an ideal of A, we need to show that I is generated by a regular sequence, but this is just the above proposition. □

The above proposition and corollary has the following striking geometric consequence:

Proposition 1.57. Suppose that W is a smooth variety of pure dimension d over an algebraically closed field k, and X, Y are pure dimensional subvarieties possibly singular of W with codimension m, n respectively, then every component of $X \cap Y$ has codimension at most $m + n$ in W.

Proof. First, notice that we are assuming W is smooth, not just regular. We are also assuming k is algebraically closed. The reason is that we can prove $W \times_k W$ is still smooth of dimension $2d$. The key here is that you can locally represent $W \times_k W$ by:

$$Speck[x_1, \cdots, x_n, y_1, \cdots, y_m]/(f_1, \cdots, f_r, g_1, \cdots, g_s)$$

where the Jacobian matrix is a block matrix and when the field is algebraically closed, we know exactly what are the closed points, also, to check smoothness, you only have to care about the closed points.

Therefore, since smoothness implies regularity, we know the embedding:

$$W \simeq \Delta \hookrightarrow W \times_k W$$

is a regular embedding of codimension d.

Notice that we are also assuming varieties here in this proposition. The reason for this is we can quote the result in dimension theory that if you have X, Y pure dimensional k-varieties of dimension m, n, then $X \times_k Y$ is of dimension $m + n$. Here you can use this to conclude the dimension of $W \times_k W$ is $2d$. Similarly, $X \times_k Y$ is also of pure dimension.

Now we again use the identification of $X \cap Y \subset W$ as $X \times_k Y \cap \Delta \subset W \times_k W$. Notice that $\Delta \hookrightarrow W \times_k W$ being a regular embedding codimension d, we claim that $X \cap Y$ is locally cut out in $X \times Y$ by d equations. To show this, consider the following fibered product:

$$\begin{tikzcd} X \cap Y \arrow[r, hook] \arrow[d, hook] & X \times_k Y \arrow[d, hook] \\ \Delta \arrow[r, hook] & W \times_k W \end{tikzcd}$$

The claim is that there is an open cover U_i of $X \times_k Y$ such that the inverse image under $X \cap Y \hookrightarrow X \times_k Y$ has codimension at most d in U_i, which will prove that the codimension of $X \cap Y$ in $X \times_k Y$ is at most d according to krull's Theorem, thus recall that everything here are pure dimensional k-varieties, then the dimension codimension equality still holds, thus you have the dimension of $X \cap Y$ is at least $d - m + d - n - d = d - m - n$, thus the codimension of $X \cap Y$ in W is at most $d - d + m + n = m + n$, which settles the proposition.

Now we are only left with the proof of the assertion. To show this, notice that you have the usual open cover of $W \times_k W$, which gives you a cover of $X \times_k Y$, the idea is that this cover suffices. Recall that regular embedding is an open condition, thus we can always find the open neighborhood small enough to ensure that $\Delta \hookrightarrow W \times_k W$ is cut out by an ideal generated by a sequence of length d. Then I will let you figure out that in this case $X \cap Y$ locally looks like the spec of:

$$k[x_1, \cdots, x_n, y_1, \cdots, y_m]/(I, J, U, V) \otimes_{k[x_1, \cdots, x_n, y_1, \cdots, y_m][I, J]} k[x_1, \cdots, x_n, y_1, \cdots, y_m][I, J, K]$$

where K is an ideal generated by d polynomials, thus it's just the Spec of: $k[x_1, \cdots, x_n, y_1, \cdots, y_m][I, J, K, U, V]$ in the Spec of $k[x_1, \cdots, x_n, y_1, \cdots, y_m][I, J, U, V]$, thus cut out by d polynomials, which proves our assertion. $\square$

Remark 1.58. In the above proposition, we assumed k is algebraically closed, now we use this remark to drop that condition.

First, recall that being smooth is preserved by a field extension, then $W_{\overline{k}}$ is smooth over $\overline{k}$, and below we will prove that any smooth variety is reduced, thus it's still a $\overline{k}$-variety.

Moreover, being pure dimensional of some dimension is also preserved under pass to the algebraic closure if we are working with locally finite type k-schemes. Then, we can prove that $X_{\overline{k}} \times_{\overline{k}} Y_{\overline{k}}$ is at most of codimension $m + n$ in $W_{\overline{k}}$, but this means that the dimension of $X \times_k Y$ is at least $d - m - n$. Then the key point is that extension of the base field commutes with intersections in the sense that:

$$(X \cap Y)_l = X_l \cap Y_l$$

where you view X_l, Y_l as subschemes of W_l. The proof is in essence not enlightening, but it does uses some properties of Cartesian diagrams, which I will let you do it if you are particularly interested.

But if you know this, the dimension of $(X \cap Y)_l$ is at most $d-m-n$ implies that the dimension of $X \cap Y$ is at most $d-m-n$, which proves the assertion.

One good property about smooth k-schemes is that they are always reduced:

Proposition 1.59. Smooth k-schemes are reduced.

Proof. Let X be a smooth k-variety, then $X_{\overline{k}}$ is also smooth, thus we know that in the case of working with an algebraically closed field, being smooth is the same as being regular, thus in particular, all closed points are regular, then the local ring is a regular local ring, thus an integral domain, thus reduced. Since any smooth k-variety is locally finite type over k, we know that the reduced points are open which means the nonreduced points are closed, thus suppose that there is a point that is not reduced, then everything in the closure is nonreduced, which is a contradiction since we know there is always a closed point in the closure. This proves $X_{\overline{k}}$ is reduced, which implies that X is reduced, since we know that there is an affine open cover of X, which are of the form $Speck[x_1, \cdots, x_n]/I$, then you know that $Spec\overline{k}[x_1, \cdots, x_n]/(I)$ is an affine open cover of $X_{\overline{k}}$, which implies $\overline{k}[x_1, \cdots, x_n]/(I)$ are non reduced, thus if there is a polynomial f in $k[x_1, \cdots, x_n]$ such that some power of it is 0, then you know that the morphism:

$$k[x_1, \cdots, x_n]/I \to \overline{k}[x_1, \cdots, x_n]/(I)$$

is actually $k[x_1, \cdots, x_n]/I \to k[x_1, \cdots, x_n]/I \otimes_k \overline{k}$, and f is mapped to $f \otimes 1$, we know that for any $x \in k[x_1, \cdots, x_n]/I$, if $x \neq 0$, $x \otimes 1$ is not zero since we know that we have a basis of $k[x_1, \cdots, x_n]/I \otimes_k \overline{k}$, but notice that $(f \otimes 1)^n = f^n \otimes 1$, which is 0 if $f^n = 0$, a contradiction. ☐

Proposition 1.60. Suppose that p is a closed point of a smooth k-scheme X of dimension n, then $\mathcal{O}_{X,p}$ is an integral domain of dimension n.

Proof. The reason why $\mathcal{O}_{X,p}$ is of dimension n is because the codimension of p is n.

To show that it's a domain, of course we could use the fact that smooth implies regular, then we use the result that a regular local ring of dimension is a domain. But since we didn't prove smoothness implies regularity at this point, we will refrain from doing that.

Instead, we consider the algebraic closure of k, $k \hookrightarrow \overline{k}$ and the extension $X_{\overline{k}}$. Consider the following diagram of fibers:

$$\begin{tikzcd}
Speck(p) \otimes_k \overline{k} \arrow[r] \arrow[d] & X_{\overline{k}} \arrow[r] \arrow[d] & \overline{k} \arrow[d] \\
Speck(p) \arrow[r] & X \arrow[r] & k
\end{tikzcd}$$

Recall that when p is a closed point, the fiber is also a closed subset, thus since $X_{\overline{k}}$ is locally finite type over $\overline{k}$, then we know that any closed subset contains a closed point, then choose that closed point, call it q. Then we know that the local ring is just $\mathcal{O}_{X,p} \otimes_k \overline{k}$(figure out why!), which is a regular local ring, thus a domain, which implies that $\mathcal{O}_{X,p}$ is a domain. ☐

Let's now prove Thm 1.52

Proof. We will prove this result by induction on n.

The case when $n = 0$ is clear since we know that regular local rings of dimension 0 are necessarily a field.

So assume $n > 0$, and the results holds for smaller dimensions. To show that (A, m, k) is a domain, it's equivalent to show that the minimal ideal is 0 or in other words, if you have a prime p such that $dimA/p = n$, then $p = 0$.

The first step is to show that A/p is still a regular local ring of dimension n. To show this, we only have to show that the dimension $m/m^2 + p$ has dimension n, but notice that we have a surjection of k-vector spaces from $m/m^2 \to m/m^2 + p$, thus we know that the dimension won't grow, thus we are done.

This implies that p is in m^2, otherwise, there is definitely something nonzero being killed. Then pick some $f \in m$ such that the image in m/m^2 is not 0, thus f is not in p.

Now if you consider the two rings: $A/(f)$ and $A/(p, f)$, we know that the dimension are both greater than $n-1$, but the dimension of the cotangent space of $A/(p, f)$ and A/f is $n-1$, thus they are both regular local ring of dimension $n-1$, thus are domains. However, you have a surjection:

$$A/f \to A/(p, f)$$

which is between integral domains of the same dimension, thus is an isomorpihsm, thus $p \subset (f)$, since $f \notin p$, we know that $p \subset fp$, thus $p = fp$, thus $p = fp \subset mp \subset p$, thus $p = mp$, thus $p = 0$ by Nakayama's lemma, we are done. □

1.3 Examples

In this section, we explore the notion of regularity and smoothness via some examples:

Proposition 1.61. Let k be any field, then $\mathbb{A}^1_k$ and $\mathbb{A}^2_{\bar{k}}$ are regular, then $\mathbb{P}^1_k$ and $\mathbb{P}^2_{\bar{k}}$ are regular.

Proof. Let's first consider $\mathbb{A}^1_k$, notice that the primes in $A = k[x]$ are generated by an irreducible polynomial f, then we have to show that:

$$dim_{A_p/pA_p}(pA_p/p^2A_p) = 1$$

but notice that the quotient is just $(f)/(f^2)$, we claim that over A_p/pA_p, f generates this quotient, this is because for any $y \in (f)/(f^2)$, not 0, it's of the form fg where $g \notin (f)$, thus g is a nonzero element in A_p/pA_p, we are done.

Then if k is algebraically closed, we claim that $\mathbb{A}^2_k$ is also regular, this comes from the fact that when k is algebraically closed, we know exactly what are the primes in $k[x, y]$, then you just need to follow the same argument as above.

Obviously this implies that $\mathbb{P}^n_k$ for $n = 1, 2$ are smooth since you can just check this locally and pretend they are $\mathbb{A}^n_k$. □

However, to generalize the above to higher dimensions, without tools, it can be really hard, however, you can check easily by what we have right now that $\mathbb{A}^n_k$ are smooth for each n since the Jacobian matrix is just 0, thus the corank is of course n for any point. You don't even have to calculate anything!

Proposition 1.62. The non-smooth points of the hypersurface $f = 0$ in $\mathbb{P}^n_k$ correspond to the locus where:

$$f = \frac{\partial f}{\partial x_0} = \cdots = \frac{\partial f}{\partial x_n} = 0$$

In particular, if the degree of f is not divisible by $char(k)$, it suffice to check $frac\partial f \partial x_0 = \cdots = \frac{\partial f}{\partial x_n} = 0$.

Moreover, if k is algebraically closed, then the singular closed points are given by the same locus.

Proof. Recall that the dimension of the hypersurface is just $n-1$ from Krull's Theorem, then what's left is just the application of the discussion we had about projective Jacobian matrix. The second assertion comes from the fact that:

$$(deg f)f = \sum_i x_i \frac{\partial f}{\partial x_i}$$

As for the last assertion, when k is algebraically closed, we know the maximal ideals are precises $(x_i - a_i)$, then, recall that these are the k-valued points, then we have proved that the Zariski-cotangent space is the cokernel of the Jacobian matrix, thus the we can still use the corank to determine if it's regular. □

Proposition 1.63. Suppose that $char(k) \neq 2$, then $y^2z = x^3 - xz^2$ in $\mathbb{P}^2_k$ defines an irreducible smooth curve.

Proof. We claim that this polynomial is irreducible, then we already proved by using Krull's Theorem that this defines an integral scheme of dimension 1, thus a curve.

To show it's smooth, consider the projective Jacobian matrix:

$$(3x^2 - z^2, 2yz, y^2 + 2xz)$$

Recall that we just prove that when the characteristic doesn't divide the degree, we only have to show that there is not locus such that these three polynomials are 0 at the same time.

Then if we assume that the Jacobian matrix is 0, then there are two possibilities, coming from $yz = 0$. First, if $y = 0$, then you get $xz = 0$, and $-2x^3 = 0$, thus $x = 0$ and $z = 0$, which is a contradiction. Also, if you assume $z = 0$, then $y = 0$, then $x = 0$, but this is again a contradiction. □

Proposition 1.64. Suppose that $char(k) \neq 2$, show that a quadratic hypersurface in $\mathbb{P}^n_k$ is smooth if and only if it's of maximal rank.

Proof. Recall that any quadratic form in $k[x_1, \cdots, x_n]$ can be written as squares of linearly independent linear forms and the number of these squares are called the rank of the quadratic form, moreover, if the number of the squares is equal to the number of the variables, we will say that the quadratic form if of full rank.

Then, since $char(k) \neq 2$, and 2 is the degree of the polynomial, then we only have to care about the corank of the Jacobian matrix to determine smoothness.

Suppose that the quadratic form is of full rank, for the sake of contradiction, we assume there is a point such that all the partial derivatives vanish, then there is a prime that contains all the partial derivatives, but I claim that this implies that the prime contains each variable, which is a contradiction.

The reason is that if the squares in the quadratic form are $u_1, \cdots, u_n$, linearly independent, we claim that the prime contains all the partials implies that the prime contains all u_i, thus all the x_i, since $u_1, \cdots, u_n$ spans all the linear forms.

Conversely, if the quadratic form is not full rank, then we claim that the ideal generated by all the partial derivatives is a homogeneous prime. It's obviously homogeneous, to show it's prime, we first notice that it can not contain every x_i, otherwise, the dimension doesn't match. Thus, we can suppose that x_i is not in the prime use the bijection of homogeneous prime ideal in $k[x_0, \cdots, x_n]$ such that they don't contain x_i with prime ideals in $\mathbb{A}^{n-1}$ to see that it's indeed prime. Then notice that this implies that this prime ideal is the locus where every partial vanishes, thus it's not smooth there, a contradiction. □

Proposition 1.65. Suppose that k is algebraically closed with $char(k) = 0$, then there is a regular projective plane curve of degree d for any $d \geqslant 0$.

Proof. Actually, if we are willing to use the fact that smoothness implies regularity. We can drop the assumption that k is algebraically closed.

Now, in $\mathbb{P}^2_k$, we can consider the Fermat curve: $x^d + y^d + z^d = 0$ is of dimension 1 since you know that the codimension is exactly 1 thus the dimension is just 1.

Now to show that it's smooth, you know that the Jacobian matrix is just: $(dx^{d-1}, dy^{d-1}, dz^{d-1})$ Then it's obvious that there is no homogeneous prime that contains them without containing x, y, z. □

Now, we see a few classical examples of singular (or nonsmooth if you wish) plane curves:

Example 1.66. Throughout this example, we will assume that k is algebraically closed with $char(k) = 0$.

1. Consider the curve $y^2 = x^2 + x^3$. We will try to study it's singular closed points. Notice that when k is algebraically closed, we can just use Jacobian matrix to determine singular closed points. And if you write out the matrix it's $(2x + 3x^2, 2y)$, then we know that on $\mathbb{A}^2$, the only closed point where it's not smooth is $(0, 0)$. You probably will think that $(\frac{2}{3})$ also works, but notice that this is not a point on the curve.

   This is usually known as a **node**.

2. Consider again the plane curve $y^2 = x^3$, again you compute the Jacobian matrix and get the answer that it's not smooth iff at (x, y), i.e. the prime corresponding to the origin.

   This is usually known as a **cusp**.

3. Consider the plane curve $y^2 = x^4$, whose Jacobian matrix is degenerate again at the origin.

   This is usually known as a **tacnode**.

Remark 1.67. We could put different definitions to nodes, cusps and tachnodes, one of which is going to be discussed at a later point.

Here for example, we give an alternative definition of a node here, when k is algebraically closed. Suppose that C is a k-variety of pure dimension 1, then a closed point p on C, with the local ring $(A = \mathcal{O}_{C,p}, m)$ is called a node if $dim_k m/m^2 = 2$ and the kernel of the natural map:

$$Sym^2(m/m^2) \to m^2/m^3$$

has dimension 1 and the kernel is not spanned by the square of an element in m/m^2.

We won't use the definition any further later, thus you don't have to memorize it, but you find that this definition makes sense.

To see this, notice that the first condition is saying that the Jacobian cotangent space has dimension 2, which correspond to the fact that at the note, there are two tangent directions.

Then for the condition on the kernel, first recall the universal property of symmetric powers, the k-linear morphism from $Sym^2(m/m^2)$ is defined by the symmetric bilinear morphism:

$$m/m^2 \times m/m^2 \to m^2/m^3$$

$(\overline{x}, \overline{y}) \mapsto \overline{xy}$ Then you can think of the morphism $Sym^2 \to m^2/m^3$ as mapping formal quadratic forms to their product in m^2/m^3, thus the kernel is just those quadratic forms such that they evaluate to 0. In other words, we are doing here is that we are assembling all the possible quadratic terms and see which of these terms vanishes.

The result of the inspection above tells you that indeed one of the possible terms vanishes, but this vanishing term is not a square, which implies that the geometry near the point we are studying is likely to have two branches.

It's possibly good to really compute an example of $Sym^2(m/m^2) \to m^2/m^3$ using the example we provided before. It's also possible to define the cusp and tacnode similarly, but we won't do it here.

Proposition 1.68. The twisted cubic curve $Projk[x, y, z, w]/(wz - xy, wy - x^2, xz - y^2)$ is smooth.

Proof. We already know that the twisted cubic is isomorphic to $\mathbb{P}^1$, thus this should be a smooth curve, but we will again use the Jacobian matrix just for the purpose of practicing.

The projective Jacobian matrix is:

$$\begin{bmatrix} -y & -x & w & z \\ -2x & w & 0 & y \\ z & -2y & x & 0 \end{bmatrix}$$

We expect the corank of this matrix to be 2, which means that the rank is 2. Pay special attention that the expected rank is not 1 since the dimension of the source is 4. But it's too complicated to compute all the 3×3 minors in this case.

Recall that we know this twisted cubic is covered by $w \neq 0$ and $z \neq 0$, then you can restrict to the affine case and consider the normal matrix, the good thing in the affine case is that the three polynomials are now dependent, thus only two of the polynomials generated the ideal, simplifying the computation, we will leave it for you to do the computation. □

Definition 1.69. Suppose that a scheme $X \subset \mathbb{A}^n_k$ is cut out by equations $f_1, \cdots, f_r$ and X is smooth of pure dimension d at the k-valuded point $(a_1, \cdots, a_n)$, then the tangent d-plane to X at a sometimes denoted by T_aX is defined by r equations in $\mathbb{A}^n$:

$$(\frac{\partial f_i}{\partial x_1}(a))(x_1 - a_1) + \cdots + (\frac{\partial f_i}{\partial x_n}(a))(x_n - a_1) = 0$$

where $1 \leqslant i \leqslant r$.

The first thing we notice is that the definition of the tangent plane is independent of the choice of the defining equations $f_1, \cdots, f_r$:

Proposition 1.70. The definition of the tangent plane is independent of the choice of the defining equations $f_1, \cdots, f_r$.

Proof. Recall that we proved before that if both $f_1, \cdots, f_r$ and $g_1, \cdots, g_s$ cut out X, then the corank of the Jacobian matrix is the same. This implies that the rank of the Jacobian matrix is also the same, in particular, the following two matrices have the same rank:

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \cdots & \cdots & \cdots \\ \frac{\partial f_r}{\partial x_1}(a) & \cdots & \frac{\partial f_r}{\partial x_n}(a) \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial g_1}{\partial x_1}(a) & \cdots & \frac{\partial g_1}{\partial x_n}(a) \\ \cdots & \cdots & \cdots \\ \frac{\partial g_s}{\partial x_1}(a) & \cdots & \frac{\partial g_s}{\partial x_n}(a) \end{bmatrix}$$

Moreover, we actually proved that when we fix the variables and only change the defining equations, the image of the two Jacobian matrix is the same, therefore, we have proved that linear combinations of $(\frac{\partial f_i}{\partial x_1}(a))(x_1 - a_1) + \cdots + (\frac{\partial f_i}{\partial x_n}(a))(x_n - a_1)$ can give you $(\frac{\partial g_i}{\partial x_1}(a))(x_1 - a_1) + \cdots + (\frac{\partial g_i}{\partial x_n}(a))(x_n - a_1)$, thus we are done. $\square$

Remark 1.71. Recall that for the k-valued points, we can use the Jacobian criterion to determine regularity, thus these r equations indeed defines a d-plane, which is defined to be closed subschemes of dimension d in $\mathbb{A}^n$ defined by linear equations. This is because we know in the r-equations there are $n - d$ linearly independent ones, thus you can use Krull's theorem to conclude.

Remark 1.72. If now you have a closed subscheme $X \subset \mathbb{P}^n$ cut out by homogeneous equations $f_1, \cdots, f_r$, $p \in X$ is a k-valued point that is smooth of pure dimension d, we can also define the projective tangent d-plane in a similar manner.

Suppose that this k-valued point correspond to the projective coordinate $a = [a_0 : \cdots : a_n]$, then we define the projective tangent d-plane to X at a is given by the equations (they cut out a projective plane in $\mathbb{P}^n$):

$$(\frac{\partial f_i}{\partial x_1}(a))(x_1 - a_1) + \cdots + (\frac{\partial f_i}{\partial x_n}(a))(x_n - a_1) = 0$$

you should first notice that this is independent of the projective coordinate we choose since partials of homogeneous polynomials are still homogeneous.

Also, this definition is again, independent of the choice of defining functions. This uses a result we will talk about much later which allows us to conclude that two subschemes of $\mathbb{P}^n$ are the same iff they are cut out but the same ideal, thus, we can just quote the result we had when we discuss the tangent plane for a subcsheme of the affine space.

Remark 1.73. Here is a side remark to help you think about this T_pX in a coordinate free way, which is cleaner.

Now we wind back a little bit and show that $\mathbb{A}^n_k$ is regular for every n and every field k, which comes to the following proposition:

Proposition 1.74. Suppose that (B, n, k) is a regular local ring of dimension d, let $\phi : B \to B[x]$ be the natural map, then suppose that p is a prime ideal of $A := B[x]$ such that $nB[x] \subset p$, then A_p is a regular local ring.

Proof. Geometrically speaking, we have a morphism $\pi : SpecB[x] \to SpecB$ such that $\pi([p]) = [n]$, then if you consider the fiber of π at $[n]$, you know:

$$[p] \in \pi^{-1}([n])$$

Then since $\pi^{-1}([n])$ is a closed subset, either the closure of $[p]$ is all of the fiber, which is $\mathbb{A}^1_k$, or $[p]$ is a closed point in the fiber.

To see why this holds, notice that there are only two kinds of primes in the PID $k[x]$, it's either the generic point, or a maximal ideal.

Recall that being a regular local ring of dimension d, there is a system of generators $f_1, \cdots, f_d$ in $n \subset B$ and there is a chain of prime ideal:

$$0 = q_0 \subset q_1 \subset \cdots \subset q_d = n$$

Then let's consider case 1 where the closure of $[p]$ is all of $\mathbb{A}^1$. In this case, we claim that $p = nB[x]$, this is because we know that the inverse image of p in $k[x]$ is just $p/nB[x] = 0$.

Now, notice that p is generated by n elements, thus the codimension of p is at most d, however $0 = q_0 \subset q_1 \subset \cdots \subset q_d = n$ implies that the codimension of p is at least d, thus the codimension is just d. Thus the dimension of A_p is just d, also since pA_p is generated by d elements, the dimension of the cotangent space is at most d, then the ring is regular local.

The second case is the closure of p is a closed point of A^1_k. This way, we know that $p/nB[x]$ correspond to a monic irreducible polynomial in $k[x]$, say g, lift g back to B, with an arbitrary choice of coeeficients, we know that p is generated by $(f_1, \cdots, f_r, g)$, thus the codimension of p is at most $d+1$. On the other hand, we have:

$$0 = q_0B[x] \subset q_1B[x] \subset \cdots \subset q_dB[x] = nB[x] \subset p$$

This proves the codimension of p is $d+1$ and is generated by $d+1$ elements, thus A_p is regular. ☐

One easy consequence is that:

Corollary 1.75. Let X be a locally Noetherian scheme over k, then if X is regular, so is $X \otimes_k \mathbb{A}^1_k$.

Proof. Notice that a point being regular is only a local condition, thus we can just assume that $X = SpecB$ for a Noetherian ring B, then we are require to prove that if B is regular, then $B[x]$ is regular.

To show this, suppose that you have a prime p in $B[x]$, whose inverse image in B is q, then you have the following diagram:

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & B[x] \arrow[d] \arrow[dr] & \\
B_q \arrow[r] & B_q[x] \arrow[r] & B[x]_p
\end{tikzcd}$$

our goal is to prove that $B[x]_p$ is a regular local ring. To show this, recall that the $B_q[x] \to B[x]_p$ is the same as localize at $pB_q[x]$, which is a prime ideal in B_q such that the inverse image in B_q contains qB_q, thus the localization at this prime, according to the above proposition, gives you a regular local ring, we are done. ☐

1.4 Bertini's Theorem

We now discuss Bertini's Theorem a fundamental classical result. Many of the arguments in this section need to be justified later using the language of line bundles and sections of line bundles, thus if you feel confused with some of the statements and arguments, don't worry, you can come back later after with much more confidence:

Definition 1.76. We start by the definition of the so called dual projective space, denoted usually by $\mathbb{P}^{n\vee}_k$.

Formally speaking, this is still an n-projective space, but with coordinate $a_0, \cdots, a_n$, or in other words, it's just $Proj(k[a_0, \cdots, a_n])$. But it comes with an extra piece of data, called the incidence variety or incidence correspondence, which is a closed subscheme $I \subset P^n_k \times_k P^{n\vee}_k$ that is defined by the equation $a_0x_0 + a_1x_1 + \cdots + a_nx_n = 0$.

This could be the first point where you feel confused, since we haven't talked about how to think about a_i, x_i here, obviously, they are not interpreted as coordinates in $P^n_k \times_k P^{n\vee}_k$ since we didn't discuss how to view the fibered product as the projective space whose coordinates contain x_i, a_i. But you shouldn't worry about this here in this proposition both beacause we will discuss this later in great detail and also because this makes sense intuitively if you think of $P^{n,\vee}_k$ as the projective space that parametrizes the hyperplanes in $\mathbb{P}^n_k$.

Indeed, if you think about the k-valued points in $\mathbb{P}^{n,\vee}_k$, say $[a_0 : a_1 : \cdots : a_n]$ where $a_i \in k$, it corresponds to the hyperplane $a_0x_0 + \cdots + a_nx_n = 0$ in $\mathbb{P}^n_k$.

With the above definition, we can state the following theorem:

Theorem 1.77. Suppose that X is a smooth subvariety of $\mathbb{P}^n_k$ of pure dimension d, then there is a nonempty (=dense) open U in $\mathbb{P}^{n\vee}_k$ such that for each point $p = [H] \in U$, H doesn't contain any component of X and the scheme $H \cap X$ is smooth over $k(p)$ of pure dimension $d-1$.

Before we prove this theorem, we spend some time making some useful remarks and state the strategy we will use to prove this theorem.

First, to understand this theorem, it's probably better to think of the case where $k(p) = k$, i.e. those k-valued points. This is the origin of this theorem since people are concerned with the case of closed points of a variety over an algebraically closed field. However, even if the point is not k-valued, this still makes sense. To see how this works, we have to look at the following diagram:

$$\begin{tikzcd}
 & \mathbb{P}^n & \\
H \arrow[r, hook] \arrow[d] & \mathbb{P}^n \times_k k(p) = \mathbb{P}^n_{k(p)} \arrow[u, "p"] \arrow[r] \arrow[d] & Speck(p) \arrow[d] \\
I \arrow[r, hook] & \mathbb{P}^n \times_k \mathbb{P}^{n\vee} \arrow[r] & \mathbb{P}^{n\vee}
\end{tikzcd}$$

when $k(p) = k$, it's simple, however, when $k(p)$ is not k, but an extension of k, we have to use the projection of $\mathbb{P}^n_{k(p)}$ to $\mathbb{P}^n_k$, the scheme theoretic image is H, which may not be a hyperplane of $\mathbb{P}^n$, but at least still a subscheme of $\mathbb{P}^n_k$, thus makes sense to intersect with X.

Remark 1.78. Here is a useful application of Bertini's Theorem:

A general degree $d > 0$ hypersurface in $\mathbb{P}^n_k$ intersects a smooth subvareity X in a smooth subvariety of codimension 1. This can be deduced from Bertini's Theorem and d-th Veronese embedding. Consider the following composition:

$$X \hookrightarrow \mathbb{P}^n \overset{v_d}{\hookrightarrow} \mathbb{P}^N$$

this identifies X as a smooth subvareity of $\mathbb{P}^N$ and a degree d-hypersurface correspond to a degree 1-hypersurface in $\mathbb{P}^N$ and vice versa. By Bertini's Theorem, there is a dense open in $\mathbb{P}^{N\vee}$ such that for any $p = [H]$ there, $H \cap X$ is smooth of pure dimension $d-1$, then using the fact that we are working with k-varieties, dimension and codimension is additive, thus the codimension of $X \cap H$ in X is 1, which is summarized in the following diagram:

$$\begin{tikzcd}
X \cap H' = X \cap H \arrow[r, hook] \arrow[d, hook] & H' \arrow[r, hook] \arrow[d, hook] & H \arrow[d, hook] \\
X \arrow[r, hook] & \mathbb{P}^n_k \arrow[r, hook, "v_d"] & \mathbb{P}^N
\end{tikzcd}$$

Here this general holds the usual meaning in algebraic geometry, which just means that there is a dense open in $\mathbb{P}^{N\vee}$. Here you know that $\mathbb{P}^{B\vee}$ parametrizes degree d hypersurfaces in $\mathbb{P}^n$ if we choose N appropriately.

A useful consequence of this is to take $X = \mathbb{P}^n$ itself, then there is a degree d hypersurface with pure dimension $n-1$ such that it's smooth over k. This need some explanation that any nonempty open subset of $\mathbb{P}^n$ have a k-valued point, this is easy to see when we work with algebraically closed fields since in this case this is equivalent to find a closed point, which is dense. If k is not algebraically closed, this is equivalent to show that any nonempty open in $\mathbb{A}^n$ contains a k-valued point, which suffices to prove that any ideal I in $k[x_1, \cdots, x_n]$ which is not the whole ring doesn't contain all the classical points, which is easy to prove since is so, then I contains $x, x+1$, thus contains 1, thus the whole ring, we are done.

Now let's prove this theorem:

Proof. Let's first spend a little bit of time explaining the strategy of the proof.

First, we can suppose that X is irreducible, since if we proved this for irreducible smooth varieties, we can take the irreducible component of X, which is also going to be smooth proved by a much later tool called unramified morphism which we will introduce after we know what are the cotangent sheaf. But this allows us to find a dense open for each component of X, and then by taking the intersection, we are done. Therefore, in what follows we will just assume that X is irreducible.

The central idea is quite simple, we will describe those bad hypersurfaces in the sense that either the intersection with X is not smooth of pure dimension or it contains X. Then we will show that they form a closed subset of $\mathbb{P}^{n\vee}$ of dimension at most $n-1$, which means that it's a proper closed subset, thus we are done since the complement, which are those good hyperplanes now form a dense open subset.

Informally speaking, we will describe a subvariety $Z \subset X \times \mathbb{P}^{n\vee}$ that looks like the stuff below:

$$Z = \{(p \in X, H \subset \mathbb{P}^n_k) : p \in H, \text{ either } p \text{ is a singular point of } X \cap H \text{ ,or } X \subset H\}$$

After we spend some time carefully defining Z, we will see that the projection $\pi : Z \to X$ has fibers at closed points that are projective spaces of dimension $n-1-dimX$, then we apply the dimension theory and get the inequality:

$$codim_Z p \leqslant dimX + n - 1 - dimX = n - 1$$

However, notice that if we assume that p maps to a closed point, then p is also a closed point, thus each irreducible component of Z has dimension less than or equal to $n-1$, thus $dimZ \leqslant n-1$. Now consider the projection $p : X \to \mathbb{P}^{n\vee}$, we first claim that it's closed, to see so, we know that $\mathbb{P}^n \to Speck$ is proper, thus it's universally closed, thus we can do a base change to get the projection $\mathbb{P}^{n\vee} \times \mathbb{P}^n_k \to \mathbb{P}^{n\vee}_k$ is a closed morphism, then since we are thinking about the composition $Z \hookrightarrow X \times \mathbb{P}^{n\vee} \hookrightarrow \mathbb{P}^n \times \mathbb{P}^{n\vee} \to \mathbb{P}^{n\vee}$, where each of these morphisms is closed, thus the composition is closed. This shows that the image of Z in $\mathbb{P}^{n\vee}$ is a closed subset. I claim that this dimension is at most $n-1$, this is because the dimension of Z is at most $n-1$. To see why, we need to discuss some lemma that is not so related to our goal here which is not discussed before, we will put them to the appendix of this section, which you can check later.

Nevertheless, we will put the argument here for you to have a feeling of what's going on. First, you can directly prove that the image of an irreducible subset is irreducible, thus, if you consider any irreducible component of Z, say Z_i, we know that if we endow it with the reduced subscheme structure, we can consider $Z_i \hookrightarrow Z \to \mathbb{P}^{n\vee}$, whose image is irreducible closed, and it will be an irreducible component of the image of $Z \to \mathbb{P}^n$ (which can also be proved using elementary arguments), now if you consider the scheme theoretic image of $Z_i \to \mathbb{P}^{n\vee}$, and the following diagram:

$$\begin{tikzcd}
 & Im(p)_{red} \arrow[r, hook] & Im(p) \arrow[d, hook] \\
Z_i \arrow[ur, dashed] \arrow[urr, dashed] \arrow[rr, "p"] & & \mathbb{P}^{n\vee}
\end{tikzcd}$$

where the dotted arrows represent $Z_i \to \mathbb{P}^{n\vee}$ factors through the scheme-theoretic image and the reduced version of that, which is what we will discuss in the appendix. The factorization through $Im(p)_{red}$ comes from the fact that Z_i is reduced.

Via this, you get a morphism between integral k-varieties, with nonempty fiber for any point in $Im(p)_{red}$, thus you can apply the fiber dimension equality in a dense open of $Im(p)_{red}$ to conclude that $dim(Im(p)_{red}) \leqslant dimZ_i$. Since the underlying topological space of $Im(p)$ and $Im(p)_{red}$ is the same one, thus the dimension is the same, thus the dimension of the image of $Z \to \mathbb{P}^{n\vee}$ is at most $dimZ \leqslant n-1$. This conclude the proof.

Therefore, our goal is just to carefully describe Z, show the fiber dimension, and show that the complement of the image $Z \to \mathbb{P}^{n\vee}$ satisfies the desired property.

Firstly, let's define Z, in terms of equations on $\mathbb{P}^n \times \mathbb{P}^{n\vee}$, where the coordinate of $\mathbb{P}^n$ and $\mathbb{P}^{n\vee}$ are $x_0, \cdots, x_n$ and $a_0, \cdots, a_n$. Suppose that $X \hookrightarrow \mathbb{P}^n$ is cut out by homogeneous equations $f_1, \cdots, f_r$, we then first record these equations. We also record the equation $a_0x_0 + \cdots + a_nx_n = 0$, but you will find that Z will be described by more equations.

Recall that in $X \hookrightarrow \mathbb{P}^n$ is smooth, thus it's projective Jacobian matrix:

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_0}(p) & \cdots & \frac{\partial f_1}{\partial x_n}(p) \\ \cdots & \cdots & \cdots \\ \frac{\partial f_r}{\partial x_0}(p) & \cdots & \frac{\partial f_r}{\partial x_n}(p) \end{bmatrix}$$

if of corank $d+1$ at all points of X, now we will append a new column to this matrix and get:

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_0} & \cdots & \frac{\partial f_1}{\partial x_n} & a_0 \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_r}{\partial x_0} & \cdots & \frac{\partial f_r}{\partial x_n} & a_n \end{bmatrix}$$

In the above matrix, we forget we are evaluating all the partials at some point $p \in X$, we are just treating them as polynomials, and we record the equations that makes the matrix has corank at least $d+1$, but notice that adding a column can only decrease the corank, thus we are indeed requiring that the matrix to still have corank $d+1$, which can be translated to the rank condition and then translated to the vanish determinants of minors condition. Finally, we define Z to be the closed subscheme of $\mathbb{P}^n \times \mathbb{P}^{n\vee}$ that is cut out by all these equations we recorded.

You probably will find that this is confusing again since we know that in general, the coordinate on $\mathbb{P}^n \times \mathbb{P}^{n\vee}$ are not in terms of x_i, a_i, but again, we are using the knowledge of line bundles over $\mathbb{P}^n \times \mathbb{P}^{n\vee}$. Don't worry about this, just try to intuitively understand them and we will come back at a later point.

Now let's think about why the complement of the image $Z \to \mathbb{P}^{n\vee}$ satisfies our conditions. First, for clarity, we need to emphasize that even if we made a comment before about what happens if we consider the non $-$-valued points in $\mathbb{P}^{n\vee}$, these points do determine a closed subscheme of $\mathbb{P}^n$, but they are not in general hyperplanes and it's hard to describe them. In practice, we only consider the intersection of X with a hyperplane H, which means that we only consider those k-valued points of $\mathbb{P}^{n\vee}$.

Then, suppose that you have a k-valued point $[a_0 : \cdots : a_n]$ where $a_i \in k$, this will correspond to the hyperplane $H = a_0x_0 + \cdots + a_0x_n = 0$ in $\mathbb{P}^n$, then the intersection of $H \cap X$ will be cut out by equations $f_1, \cdots, f_r, a_0x_0 + \cdots + a_0x_n$. We are trying to find those hyperplanes such that the intersection is not smooth or $X \subset H$. First, if $H \cap X$ is not smooth, then it could be that $H \cap X$ is not pure dimensional any more. I claim that this can't happen: recall that X is an irreducible closed subset, thus it's cut out by a homogeneous prime ideal, and since we assumed that X is not in H, then we know the defining equation of H is not in the ideal, thus you can use Krull's theorem and the dimension codimension additivity to prove that each component of $X \cap H$ have dimension is $d-1$.

Therefore, the only possible cases are either X is in H or there are points on $X \cap H$ such that the corank of the Jacobian matrix is $d+1$. Obviously if there is nonsmooth points, the corank conditions takes care of that. If $X \subset H$, then f is in the ideal cutting out X, therefore we know by a previous proved proposition that adding the partials of a function in the ideal doesn't change the corank of the matrix, thus the corank is again $d+1$, thus the corank condition also takes care of this case.

These discussion above shows that if a k-valued point $[a_0 : \cdots : a_n]$ is a bad point, then it necessarily lies in the image of $Z \to \mathbb{P}^{n\vee}$. Again, all these are just intuitive understanding since at this point we don't have the right language to define Z properly and thus don't know what is the image of a specific point under $\mathbb{P}^n \times \mathbb{P}^{n\vee} \to \mathbb{P}^{n\vee}$.

Finally, let's prove that the dimension of Z is at most $n-1$. Now as what we explained in the beginning of the proof, consider a k-valued point in X, say p, we want to understand what is the fiber of this point of the morphism $Z \to X$.

After some arguments, you will find that the fiber is isomorphic to a closed subscheme of $\mathbb{P}^{n\vee}$ cut out by equations of the condition that $p \in H$ $H \cap X$ is singular at p or $X \subset H$. (Notice that since we work with k-valued points, the condition of being regular is the same as the condition of smoothness.) Then notice that this point is singular iff the dimension of the cotangent space of the intersection at p didn't drop by 1, which is the same as the dimension of the tangent space of the intersection at p didn't drop by 1. Recall that since the intersection is a closed subscheme of X, the tangent space of the intersection at p is a subspace of the tangent space of X, which is identified by the kernel of the linear functional $a_0x_0 + \cdots + a_nx_n$.

Also recall that the dimension of the tangent space of X at p is d, then the dimension of the tangent space of the intersection at p is also d, which is the kernel of $a_0x_0 + \cdots + a_nx_n$, thus the condition for this to hold is you have d linearly independent elements in T_pX such that they evaluate at 0 by $a_0x_0 + \cdots + a_nx_n$, which gives you d-linear equations of a_i.

If $X \subset H$, the reasoning is similar and again gives you d-equations. The additional one more condition is given by $p \in H$, thus you have $d+1$ linear conditions of a_i in total and they are all linearly independent (to see why you can use a chart to pretend everything is in the affine space, then via a automorphism of the affine space to pretend everything is at the origin, then you find that the condition that $p \in H$ is to set every $\frac{x_i}{x_j} = 0$, but the subspace condition is to set $\frac{x_i}{x_j}$ as some nonzero elements, thus the conditions on a_i can't be linearly dependent), therefore they cut out a dimension $n-d-1$ closed subscheme in $\mathbb{P}^{n\vee}$, thus we are done. □

Remark 1.79. We have indeed used a lot of stuff we promise to prove in the future, so keep this proof in mind and come back when you have the right tool and then make sure to do adjustment to this proof until you are satisfied.

But once we established this result, a lot of consequences follow readily:

Proposition 1.80. If $p \in \mathbb{P}^n$ is a point, then in $\mathbb{P}^{n\vee}$, there is a dense open such that for each $p = [H]$ in that open, the hyperplane avoids that point.

Proof. □

Proposition 1.81. The above theorem holds true even if we assume X is a pure dimensional subvariety of $\mathbb{P}^n$ that has finitely many non-smooth points.

Proof. If you look back at the proof of Bertini's Theorem, even if we start with a non-smooth pure dimensional k-variety, the proof still goes through except when we try to prove the closed subscheme Z we defined is of dimension at most $n-1$, which requires us to show that the fiber W_p is of codimension $d+1$ for a k-valued point p(probably a closed point also works but I need to figure out why this works). To do so, we need X to be smooth at p, □

Proposition 1.82. Bertini's Theorem still works if $X \hookrightarrow \mathbb{P}^n$ is only a locally closed smooth pure dimensional subvariety.

Proof. **** later* □

Proposition 1.83. If $X \hookrightarrow \mathbb{P}^n$ is a projective variety of dimension n in $\mathbb{P}^m$, then the intersection of X with n general hyperplanes consists of a finite number of reduced points.

Proof. We can't apply Bertini's result repeatedly and take the intersection of the dense open in $\mathbb{P}^{n\vee}$.
**** later* □

Proposition 1.84. Suppose that X is a smooth subvariety of $\mathbb{P}^n_k$ of pure dimension $d \geqslant r$, suppose that $e_1, \cdots, e_r$ are positive integers, so:

$$\mathbb{P}^{\binom{e_1+n}{n}-1} \times \cdots \times \mathbb{P}^{\binom{e_r+n}{n}-1}$$

parametrizes r-tuple of hypersurfaces $(H_1, \cdots, H_r)$ of degree $e_1, \cdots, e_r$, then there is a dense open of $\mathbb{P}^{\binom{e_1+n}{n}-1} \times \cdots \times \mathbb{P}^{\binom{e_r+n}{n}-1}$ such that for each point in that open, the corresponding scheme theoretic intersection $H_1 \cap \cdots \cap H_r$ doesn't contain any irreducible component of X, and $H_1 \cap \cdots \cap H_r \cap X$ is smooth over $k(p)$ of pure dimension $d-r$.

Proof. **** later* □

1.5 Discrete Valuation Rings and Algebraic Hartogs's Lemma

In this section, we are mostly concerned with regular local rings of dimension 1. Which can be though as the germ of a smooth curve, which makes sense since it's of dimension and local and regular. One example to keep in mind is $k[x]_{(x)}$, which is a germ of the smooth curve $Speck[x]$.

In the following discussions, we will try to give several equivalent definitions of such rings and you will see that the fact that we are working in dimension 1 is a huge benefit which makes a lot of good properties of a Noetherian local ring equivalent, for example, it's a PID, it's a UFD, it's normal and it's regular. Moreover, you will also find several other stuff equivalent in dimension 1 for local rings, for example, m is principal or all ideals are powers of m or the ring has a discrete valuation, which measures the size of elements in the ring in some sense.

Theorem 1.85. Suppose that (A, m) is a regular local ring of dimension 1, then the following notions are equivalent:

1. (A, m) is regular.

2. m is principal.

Proof. The proof is simple: 1 implies 2 because the dimension of m/m^2 is 1, thus a single element will generate m by Nakayama's lemma.

2 implies 1 because if m is generated by 1 element, the dimension of m/m^2 is at most one by Nakayama's lemma, but we always know $dimm/m^2 \geqslant dimA = 1$, thus we are done. □

The following useful fact will be used later. Even though it looks like an algebra result, it's actually geometrically based:

Proposition 1.86. If (A, m) is Noetherian local, then $\cap_i m^i = 0$.

Before proving it, the geometric meaning of this proposition is that if some function vanishes at all orders at a point, then it actually vanishes at a small neighborhood of that point. Or in other words, if a function is analytically zero at a point p, then it's locally zero around that point. You can think of this as a special case of the Artin-Rees lemma, which will be introduced later.

Using this proposition, we can get another equivalent definition:

Theorem 1.87. Suppose that (A, m) is a Noetherian local ring of dimension 1, then 1 and 2 in the previous theorem is equivalent to:

$3 :$ A is an integral domain and all the ideals are of the form m^n for $n \geqslant 0$ or 0.

Proof. First suppose that (A, m) is regular local, then suppose that t is the generator of m, then I claim that $m^n \neq m^{n+1}$ for all n, otherwise by Nakayama's lemma $m^n = 0$ thus $t^n = 0$, but since regular local rings are domains $t = 0$, but this implies that A is a field, a contradiction.

Now, I claim that m^n/m^{n+1} is of dimension one over $k = A/m$. To see this, notice that t^n generates this ideal m^n, thus it generates m^n/m^{n+1}, thus the dimension of this is at most 1 by Nakayama's lemma, but it's not 0, thus of dimension 1.

I claim that this suffice to prove that there are no other ideals of the form m^n or (0). The idea is simple, you find that you have a decreasing sequence:

$$A \supset m \supset m^2 \supset \cdots$$

such that each quotient is of dimension 1 over k, thus there is no other room for other ideals. To be more precise, suppose that I is an ideal in A not (0), since we know $\cap_i m^i = 0$, there is a n such that $I \subset m^n$ but I is not contained in m^{n+1}, thus choose some $u \in I$ not in m^{n+1}, we know that the image of u in m^n/m^{n+1} is not zero thus generating this vector space since it's dimension 1, but this implies that u also generates m^n by Nakayama's lemma. Thus $(u) = m^n \subset I \subset m^n$, thus $I = m^n$, we are done.

Conversely, suppose that (A, m) is not regular, then we know that the dimension of m/m^2 is at least 2, suppose that u is in $m - m^2$, then we know that $m^2 \subset (u, m^2) \subset m$ with each inclusion being strict. The reason for the strict $(u, m^2) \subset m$ is because if (u, m^2) is m, then we know that m/m^2 is generated by u, thus of dimension 1, a contradiction. Therefore, we find an ideal that is not m^n and (0), a contradiction to our assumption. □

With the above equivalences, we can show:

Proposition 1.88. Suppose that (A, m) is a Noetherian local ring of dimension 1, then $1, 2$ and 3 above is equivalent to:

$4 :$ A is a PID.

Proof. Suppose that A is a PID then m is principal, thus this implies 2.

Conversely, if (A, m) is regular local, then all the ideals are m^n or (0) where $m = (t)$, thus all the ideals are (t^n) or (0), thus A is a PID. □

To proceed further, we need the notion of a discrete valuation, which is a special case of a valuation. Though only the discrete version is useful for us.

Definition 1.89. Suppose that K is a field, then a discrete valuation on K is a surjective group homomorphism:

$$v : K^{\times} \to \mathbb{Z}$$

satisfying:

$$V(x+y) \geqslant min(v(x), v(y))$$

except when $x + y = 0$ since in this case the left hand side is not defined.

Key examples to keep in mind are:

Example 1.90.
1. Let $K = \mathbb{Q}$ and v is just the power of 5 appearing in $r \in K$. For example, $v(35/2) = 1$ and $v(2/125) = -3$. You can easily see that $v(x + y) \geqslant min(v(x), v(y))$.
2. Let $K = k(x)$, then v is the power of x appearing in f. For example $v(x) = 1$ and $v(1/x^3) = -3$.
3. Again $K = k(x)$, v is the negative of the degree, this is a special case of the second example with x replaced with $\frac{1}{x}$.

A key observation here is that $0 \cup \{x \in K : v(x) \geqslant 0\}$ is a subring, which will be denoted by $\mathcal{O}_v$, called the valuation ring of v. Notice that for example the valuation ring of $K = k(x)$ and v is the power of x appearing in f is just $k[x]_{(x)}$, which is a regular local ring. This is not a coincidence, as you will see in what follows.

Proposition 1.91. Every valuation ring of a discrete valuation is a local ring.

Proof. We will prove that the elements $0 \cup \{x \in K : v(x) > 0\}$ is the unique maximal ideal by showing that everything not in this set is invertible.

For example $v(x) = 0$, then notice that $v(1) = 0 = v(xx^{-1}) = v(x) + v(x^{-1}) = v(x^{-1})$, thus x^{-1} is also in the valuation ring. ☐

Definition 1.92. An integral domain is called a discrete valuation domain or DVR if there is a discrete valuation v on $FracA$ such that A is $\mathcal{O}_v$.

One key observation of regular local rings is that every regular local ring is a discrete valuation domain

1.6 Smooth and Etale Morphisms: First Definition

Now we have learned enough to give the right definition of smooth morphisms (and as a special case, etale morphisms) in general. Our definition will be the right definition, however imperfect in a few aspects. It will look a little surprising and as another sign, it will be not be obvious from this definition that this definition generalizes the definition of smooth over a field. A third flaw is that this definition of of the form "there exists an open cover that satisfies some property", it's in general hard to show that a morphism is not smooth using this definition.

The advantage of this definition will be that it's easy to define without more complicated notions and a lot of properties of smooth morphisms will be straight forward to show.

Before give the real definition, let's consider some motivation coming from differential geometry. As a spoiler, our definition of smoothness will look like the notion of a smooth submersion in differential geometry, and we know that a morphism $f : X \to Y$ of smooth manifolds, is called smooth of relative dimension n, or a submersion if it locally looks like the projection $Y' \times \mathbb{R}^n \to Y'$. Or in other words, this morphism is locally on a source of smooth fibration. In particular, the notion of smooth of relative dimension 0 is the important notion of a locally isomorphism, which can be translated to being an isomorphism locally on the source.

However, if you try to carry this definition to algebraic geometry, you get the "wrong" definition, in the sense that it doesn't behave the way we expected. There will be an example later to explain this, but right now, let's see how we give a better definition:

Definition 1.93. A morphism of schemes $f : X \to Y$ is called smooth of relative dimension n if there is an affine open cover $\{V_i\}$ of Y and an open cover $\{U_i\}$ of X such that $f(U_i) \subset V_i$ and we have a commutative diagram for each i:

$$\begin{tikzcd}
U_i \arrow[r, leftrightarrow, "iso"] \arrow[d, "\pi|_{U_i}"'] & W \arrow[r, hook, "open"] \arrow[d, "\rho|_W"] & SpecB[x_1, \cdots, x_{n+r}]/(f_1, \cdots, f_r) \arrow[dl, "\rho"] \\
V_i \arrow[r, leftrightarrow, "iso"'] & SpecB &
\end{tikzcd}$$

and the morphism ρ is just induced by the natural ring morphism $B \to B[x_1, \cdots, x_{n+r}]/(f_1, \cdots, f_r)$ and W is an open subscheme of $SpecB[x_1, \cdots, x_{n+r}]/(f_1, \cdots, f_r)$ such that the determinant of the Jacobian matrix of the f_i's with respect to the first r x_i's

$$det(\frac{\partial f_j}{\partial x_i})_{i,j \leqslant r}$$

is an invertible function on W (no where vanishing).

In particular, etale means smooth of relative dimension 0.

We will say that $f : X \to Y$ is smooth of relative dimension n at a point $p \in X$ if there is an open U of p and a open V of $f(p)$, such that the commutative diagram of the same sort as above exists.

Let's make some quick observations of this definition right now:

Remark 1.94. From the definition, this morphism is locally of finite presentation, this is because you can cover W by distinguished opens of $SpecB[x_1, \cdots, x_{n+r}]/(f_1, \cdots, f_r)$ which will be finitely presented over B. Recall that locally of finite presentation always implies locally of finite type.

Also, by the definition, for any ring B, we know that:

$$\mathbb{A}^n_k \to SpecB$$

is smooth of relative dimension n since we have the following diagram:

$$\begin{tikzcd}
X \arrow[r, "="] \arrow[d] & \mathbb{A}^n_B \arrow[r, hook, "open"] \arrow[d] & \mathbb{A}^n_B = SpecB[x_1, \cdots, x_n] \arrow[dl, "\rho"] \\
Y \arrow[r, "="] & SpecB &
\end{tikzcd}$$

Also, from the definition, we know tha the locus where f is smooth of relative dimension n is open, and in particular, the locus where f is etale is open.

Remark 1.95. According to the definition, we see that in the commutative diagram, we can always let U_i to be small enough such that W is an distinguished open of $SpecB[x_1, \cdots, x_{n+r}]/(f_1, \cdots, f_r)$

As two easy example, we know that if $X \hookrightarrow Y$ is an open embedding, then it's etale. Also, suppose that Y is a scheme over k, then the projection:

$$\mathbb{A}^n \times_k Y \to Y$$

is always smooth of relative dimension n since we can choose any affine open cover of Y which gives you an affine open cover $\mathbb{A}^n_B$ of $\mathbb{A}^n \times Y$ and the morphism is locally just induced by the ring map: $B \hookrightarrow B[x_1, \cdots, x_n]$, we are done. But you probably have already noticed that if the morphism is not given with the right form of variables and relations, it can be very hard to tell that if it's smooth.

However, as what we have promised, using this definition, it's pretty straight-forward to prove quite a lot of the properties we want:

Proposition 1.96. Let $f : X \to Y$ be a morphism of schemes, then f being smooth is both local on the source and target, thus in particular, being an etale morphism is local on the source and target.

Proof. First, suppose that there is an open cover U_i of X such that $f|_{U_i} : U_i \to Y$ is smooth of relative dimension n, then we claim that $f : X \to Y$ is indeed smooth. This is pretty clear just from definition thus omitted here.

Now suppose that there is an open cover V_i of Y, such that $f|_{f^{-1}V_i} : f^{-1}V_i \to V_i$ is smooth for each i, we also claim that $f : X \to Y$ is smooth of relative dimension n, this is also clear from the definition as long as you sit down and think a little bit.

We also have to show another direction. Suppose that $f : X \to Y$ is smooth, we want to show that for any open $V \subset Y$, the restricted morphism $f : f^{-1}V \to V$ is smooth, this comes down to the following diagram of rings:

$$\begin{tikzcd} B[x_1, \cdots, x_{n+r}]/(f_1, \cdots, f_r) \arrow[r] & B_f[x_1, \cdots, x_{n+r}]/(f_1, \cdots, f_r) \\ B \arrow[u] \arrow[r] & B_f \arrow[u] \end{tikzcd}$$

I will let you figure out why this suffices.

Finally, we need to show that if $f : X \to Y$ is smooth, then for any open $U \subset X$, $f : X \to Y$ is smooth. I claim that this is still true, the main reason is just composition of open embeddings are still open embeddings. $\square$

If we only refer to the phrase smooth morphism, it typically means that some morphism is smooth of some relative dimension.

Proposition 1.97. The notion of a smooth morphism of relative dimension n is preserved by base change.

Proof. Consider a fibered product where $f : X \to Y$ is smooth of relative dimension n:

$$\begin{tikzcd} X \times_Z Y \arrow[r, "F"'] \arrow[d] & Z \arrow[d, "g"'] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Since we assumed f is smooth of relative dimension n, then we have the affine open cover V_i and U_i such that the restriction $U_i \to V_i$ looks like the composition:

$$U_i \overset{open}{\hookrightarrow} SpecB[x_1, \cdots, x_{n+r}]/(f_1, \cdots, f_r) \to SpecB$$

such that the determinant of the special Jacobian matrix is nonzero at each point of U_i.

Now consider $g^{-1}V_i$, we can find an affine open cover W_{ij} of these $g^{-1}V_i$'s. Now, notice that $U_i \times_{V_i} W_{ij}$, when i, j ranges over all possible values, forms an open cover of $X \times_Z Y$. Now suppose that $SpecB = V_i$ and $SpecC = W_{ij}$ with g restricted on $W_{ij} \to V_i$ being $SpecC \to SpecB$, in this case the morphism:

$$F|_{U_i \times_{V_i} W_{ij}} : U_i \times_{V_i} W_{ij} \to W_{ij}$$

is just determined by the local Fibered product (You can also choose U_i's to be affine if you want to):

$$\begin{tikzcd} U_i \times_B C \arrow[r] \arrow[d] & SpecC \arrow[d] \\ U_i \arrow[r] & SpecB \end{tikzcd}$$

However, recall that $U_i \to SpecB$ looks like the composition, thus we can form:

$$\begin{tikzcd} U_i \times_{V_i} W_{ij} \arrow[r, hook, "open"'] \arrow[d] & SpecC[x_1, \cdots, x_{n+r}]/(f_1, \cdots, f_r) \arrow[r] \arrow[d] & SpecC \arrow[d] \\ U_i \arrow[r, hook, "open"] & SpecB[x_1, \cdots, x_{n+r}]/(f_1, \cdots, f_r) \arrow[r] & SpecB \end{tikzcd}$$

thus if you think compute the determinant again, you obtain the same equation, then we only have to prove that it's invertible on $U_i \times_{V_i} W_{ij}$. Notice that we have a local ring morphism for each p in $SpecC[x_1, \cdots, x_{n+r}]/(f_1, \cdots, f_r)$ thus if the polynomial vanishes at p, it will vanish at q which is the image of p under $U \times_{V_i} W_{ij}$, thus we are done. $\square$

Recall that we proved before that if A is a finitely presented B-algebra, say $B[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$, then A_g for any $g \in A$ is also finitely presented and it will be isomorphic to $B[x_1, \cdots, x_n, x_{n+1}]/(f_1, \cdots, f_r, 1-gx_{n+1})$

Proposition 1.98. Suppose that $\pi : X \to Y$ is smooth of relative dimension m and $\rho : Y \to Z$ is smooth of relative dimension n, then $\rho \circ \pi$ is smooth of relative dimension $m + n$.

In particular, the composition of etale morphisms is etale.

Proof. Recall that we can always take U_i's to be a distinguished open of $B[x_1, \cdots, x_{n+r}]/(f_1, \cdots, f_r)$, then by the above discussion, if $f : X \to Y$ is smooth, we can always assume that we have affine open cover V_i of Y and affine open cover U_i of X such that we have the following diagram:

$$\begin{tikzcd}
U_i \arrow[r, "="] \arrow[d] & SpecB[x_1, \cdots, x_r, y, x_{r+1}, \cdots, x_{n+r}]/(f_1, \cdots, f_r, 1-yg) \arrow[d] \\
V_i \arrow[r, "="] & SpecB
\end{tikzcd}$$

where g is a polynomial in all variables except y. Recall that the determinant with repsect to the first $r + 1$ variables is just:

$$det(\frac{\partial f_j}{\partial x_i})_{i,j \leqslant r} \cdot g$$

recall that $det(\frac{\partial f_j}{\partial x_i})_{i,j \leqslant r}$ doesn't vanish on an open subset larger than U_i, and g doesn't vanish on U_i, so is their product, thus we indeed know that we can use U_i and V_i to demonstrate the smoothness of f.

Now, suppose that you have $f : X \to Y$ and $g : Y \to Z$ smooth, we pick affine open cover U_i, V_i of Y and Z such that the above condition holds. Then recall that being smooth is open in the target, thus $f^{-1}U_i \to U_i$ is again smooth, then I claim we can choose open cover U_{ig} of $f^{-1}U_i$ and open cover W_i of U_i such that we have the following diagram:

$$\begin{tikzcd}
V_i \arrow[rr, "="] & & SpecC \\
U_i \arrow[u] \arrow[rr, "="] & & SpecC[x_1, \cdots, x_{n+r}]/(f_1, \cdots, f_r) \arrow[u] \\
U_{ig} \arrow[u] & = SpecC[x_1, \cdots, x_{n+r}, y]/(f_1, \cdots, f_r, 1-gy) & = SpecB \arrow[u] \\
W_i \arrow[u] \arrow[rr, "="] & & SpecB[z_1, \cdots, z_{m+s}]/(h_1, \cdots, s_s) \arrow[u]
\end{tikzcd}$$

where $B[z_1, \cdots, z_{m+s}]/(h_1, \cdots, s_s)$ can be viewed as a quotient of polynomial rings of C, and the determinant of the matrix of partials is just a product of nonvanishing functions, thus the result is nonvanishing, we are done. ☐

These several properties tell you that being smooth is a nice class of morphisms in the sense we discussed before.

Here is a quick observation:

Remark 1.99. Suppose that $f : X \to Y$ is smooth of relative dimension n, then locally on X, π can be described as an etale cover of $\mathbb{A}^n_Y$ in the sense that for every point $p \in X$, there is an open neighborhood U of p, such that $f|_U$ can be factored into:

$$U \to \mathbb{A}^n_Y \to Y$$

where the first morphism is etale and the second morphism is the projection.

This is easy to see by the following commutative diagram:

$$\begin{tikzcd}
U \arrow[r, hook] & SpecB[x_1, \cdots, x_{n+r}]/(f_1, \cdots, f_r) \arrow[r] \arrow[d] & SpecB[x_1, \cdots, x_n] \arrow[dl] \\
& SpecB &
\end{tikzcd}$$

Proposition 1.100. The natural morphism $Speck \to Speck[\epsilon]/(\epsilon^2)$ is not etale.

Proof. As what we discussed before, you should in some sense, view etale morphisms as local isomorphisms, which means more than a bijective morphism, rather, it should preserve the scheme structure. Here, even if both schemes are just a single point space, the first one is a reduced point and the second has infinitesimal thickenings, thus you will expect that this morphism is not etale.

This is indeed the case, if this map were etale, then by definition, you have a diagram:

$$\begin{tikzcd} Speck \arrow[r, hook, "open"] \arrow[dr] & Speck[\epsilon, x_1, \cdots, x_r]/(f_1, \cdots, f_r.\epsilon^2) \arrow[d] \\ & Speck[\epsilon]/(\epsilon^2) \end{tikzcd}$$

then as an open embedding, $Speck \hookrightarrow Speck[\epsilon, x_1, \cdots, x_r]/(f_1, \cdots, f_r.\epsilon^2)$ will induce an isomorphism on the cotangent space, and in particular, the dimension should match. However, the dimension of the cotangent space of $Speck$ at (0) is 0. The image of (0) in $Speck[\epsilon, x_1, \cdots, x_r]/(f_1, \cdots, f_r.\epsilon^2)$ is a prime that contain ϵ, thus, you can prove that the dimension of the cotangent space is at least 1 since ϵ is always in the tangent space and is not 0, a contradiction. $\square$

Even though we said that in some sense, the etale morphism comes from the notion of local isomorphisms in differential geometry, but in algebraic geometry, it's not the case, below is an example

Proposition 1.101. Let k be a field of characteristic not 2, let Y be $Speck[t]$ and $X = Speck[u, \frac{1}{u}]$, then the morphism $f : X \to Y$ defined by $t \mapsto u^2$ is etale.

Moreover, there is no nonempty open U of X such that f is an isomorphism, thus the naive extension of the differential geometry definition is wrong.

Proof. It's easy to see why this is etale:

$$\begin{tikzcd} Speck[u, \frac{1}{u}] \arrow[r, "t \mapsto u"] \arrow[dr] & Speck[t] \arrow[d, "t \mapsto t^2"] \\ & Speck[t] \end{tikzcd}$$

Now to prove that there is no open on $Speck[u, \frac{1}{u}]$ such that f restricts to an isomorphism, we will suppose that there is such for the sake of contradiction.

Now if there is such an open, we can assume that, by restricting further, that is a distinguished open, $D(f)$, now we know that the morphism will be induced by the ring homomorphism $k[t] \to k[u]_f$ where $t \mapsto u^2$. Since we are assuming this is an isomorphism, we can assume that there is a distinguished open $D(g(t))$ in the image such that f induced an isomorphism between $D(g(t))$ and it's inverse image, thus the ring homomorphism $k[t]_{g(t)} \to k[u]_{g(u^2)}$. But we can easily see this can't be an isomorphism since $\frac{1}{u}$ is not in the image.

If $\frac{h}{g^n}$ is mapped to u where h is coprime to g^n, we will have $h(u^2)g^m(u^2) = g^n(u^2)$, but this implies that $n \geqslant m$ and $h = g^{n-m}$, a contradiction. $\square$

We now show that our definition before is a special case of this new definition:

Theorem 1.102. Let X be a k-scheme, then the following are equivalent:

1. X is smooth of relative dimension n over $Speck$.

2. X is smooth over k.

This theorem allows us to use smooth over k and smooth over $Speck$ interchangeably. We will say that an A-scheme X is smooth over a ring A if the structure morphism $X \to SpecA$ is smooth of some relative dimension.

Proof. Let's first show that $X \to Speck$ is smooth of relative dimension n implies that X is smooth over k. Recall that by the definition of smooth over k, we need to check that X is pure dimensional and locally of the form $Speck[x_1, \cdots, x_m]/(f_1, \cdots, f_r)$ such that the corank of the Jacobian matrix is n.

The central part of the proof lies in showing X is pure dimensional. If we showed this, we are done since we know that X is locally of the form $Speck[x_1, \cdots, x_{n+r}]/(f_1, \cdots, f_r)$ such that the Jacobian matrix is of rank r, thus corank n.

To show that X is pure dimensional, recall that if we can prove that there is an open cover of X such that each open subscheme in the cover is of pure dimension n, then X is pure dimension of dimension n. If not, choose a irreducible component of X not of dimension n, consider the generic point of that component, then there is an open subset in the cover containing that point, then the intersection is one irreducible component of that open, but that is of dimension not n, contradiction to the assumption. Thus, we can just work locally and assume X is affine.

Recall that when X is an affine k-scheme, X is pure dimensional iff $X_{\overline{k}}$ is pure dimensional of the same dimension. Thus, let's assume that k is algebraically closed and X is just $Speck[x_1, \cdots, x_{n+r}]/(f_1, \cdots, f_r)$ such that the Jacobian matrix is of rank r. Let p be a closed point of X, since X is finite type over k, we know that the dimension of the component containing p is the codimension of p. However, notice that the codimension of components X in $\mathbb{A}_k^n$ is at most n, again, use the dimension codimension additivity, we know that the dimension of that component is at least n, thus the codimension of p is at least n.

Now we still consider the point p, since k is algebraically closed, the cotangent space at this point is the cokernel of the Jacobian matrix, thus we know the dimension of the cokernel is n. However, notice that the dimension of the cotangent space bounds the dimension of the local ring at p, which is the codimension of p in the irreducible component containing it, thus the codimension of p is at most n, thus the codimension of p is just n, which implies the dimension of the component containing p is n. Now each component contains at least a closed point, we are done.

Conversely, suppose that X is smooth over k, we claim that any union of open neighborhoods of all closed points is the space X. To see this, suppose that the union is not X, then the complement is closed, give it the reduced scheme structure, we know it's still locally finite type over k, thus we know that it has closed points, which is going to be closed in X, a contradiction.

This allows us to focus on a neighborhood of a closed point. Therefore, we can assume that $X = Speck[x_1, \cdots, x_{n+r}]/(f_1, \cdots, f_N)$ and $p \in X$ is a closed point, and we can also assume X is of pure dimension n since the irreducible components of an open subset are just intersections of irreducible components of the total space with that open, and we use the fact that closed points are dense. Of course, since X is smooth, we can assume that the corank of the Jacobian matrix is n, thus of rank r.

Then by permuting the f_i's and x_j's, we may assume that the Jacobian matrix of the first r f_i's with respect to the first r variables is of rank r at each point of X, thus if you consider the scheme $Y = Spec[x_1, \cdots, x_{n+r}]/(f_1, \cdots, f_r)$, Y is smooth over $Speck$ of relative dimension n, thus pure dimensional of dimension n. We also have a closed embedding:

$$X \hookrightarrow Y$$

induced by the surjective ring homomorphism. Then notice that if we consider $p \in Y$ now, $\mathcal{O}_{Y,p}$ is a regular local ring of dimension n since p is also a closed point in Y and Y is smooth over $Speck$, thus smooth over k, thus we can quote the result before that tells you that for closed points, the local ring there is a regular local ring of dimension n, in particular a domain.

Now consider the surjective morphism:

$$\mathcal{O}_{Y,p} \to \mathcal{O}_{Y,p}/(f_{r+1}, \cdots, f_{r+n}) = \mathcal{O}_{X,p}$$

since $\mathcal{O}_{X,p}$ is also a domain of dimension n, we know that a surjective morphism between domains of the same dimension is an isomorphism, thus $f_{r+1}, \cdots, f_{r+n}$ is 0 in the local ring, thus they are zero around a small open near p, thus we know that there is a small distinguished open around p, such that on that open we have the following diagram:

$$\begin{tikzcd}
U = Speck[x_1,\cdots,x_{n+r}]_g/(f_1,\cdots,f_r)_g \arrow[r, hook, "open"] & Speck[x_1,\cdots,x_{n+r}]/(f_1,\cdots,f_r) = X \\
U = Speck[x_1,\cdots,x_{n+r}]_g/(f_1,\cdots,f_{n+r})_g \arrow[u, "="] \arrow[r, hook] \arrow[dr] & Speck[x_1,\cdots,x_{n+r}]/(f_1,\cdots,f_{n+r}) = Y \arrow[u, hook, "closed"'] \arrow[d] \\
 & Speck
\end{tikzcd}$$

thus we are done. $\square$

2 Topics Related to the Notions Discussed Above

In this section, we will introduce notions that are not necessarily geometric in general, but uses or improves you understanding of the notions we discussed above. In the first part, we will introduce valuation criterion for properness and separatedness, which is useful in general, the reason we put this section here is just because it uses the valuation stuff we discussed before. Then we will discuss some facts about regular local rings that helps you to understand them better. Finally, we make a small detour to discuss Artin-Ress lemma, which generalize the geometric intuition that a function vanishes analytically at a point should vanish in a small neighborhood of that point.

2.1 Valuative Criteria for Separatedness and Properness

We begin with a criterion for separatedness that will suffice for most of the cases that are of interest:

Theorem 2.1. Suppose that $\pi : X \to Y$ is a morphism of finite type of locally Noetherian schemes, then π is separated if and only if the following condition holds: For any discrete valuation domain A, and any commutative diagram of the form:

$$\begin{tikzcd}
SpecK(A) \arrow[r] \arrow[d, hook, "open"] & X \arrow[d, "\pi"'] \\
SpecA \arrow[r] & Y
\end{tikzcd}$$

there is at most one morphism $SpecA \to X$ such that the following diagram commutes:

$$\begin{tikzcd}
SpecK(A) \arrow[r] \arrow[d, hook, "open"'] & X \arrow[d, "\pi"'] \\
SpecA \arrow[r] \arrow[ur, dashed, "\leqslant 1"] & Y
\end{tikzcd}$$

Before proving it, notice that the morphism $SpecK(A) \to SpecA$ is an open embedding since A is a DVR thus a regular local ring of dimension 1, thus $SpecA$ is only a space of two points, with one closed point, thus the generic point is open.

One direction of the theorem is easy:

Proposition 2.2. If π is separated, then the valuative criterion above holds.

Proof. Suppose that there are two morphisms $f, g : SpecA \to X$ making the diagram commute, then we know that since $SpecA$ is reduced, and X is separated over Y, we can apply the reduced to separated theorem, we only have to prove that $SpecK(A)$ is an dense open of $SpecA$ to conclude the proposition, but since A is a domain, then the (0) ideal is indeed dense. $\square$

As a consequence of this direction, we can prove:

Proposition 2.3. Suppose that X is a Noetherian separated curve, if p is a regular closed point, then $\mathcal{O}_{X,p}$ is a discrete valuation ring (Recall that it's regular local and of dimension 1 since the codimension of this point is at least 1 but it's bounded by the inequality that $codim(p) + 0 \leqslant 1$, thus the codimension is at most 1, thus it's 1). Then each one of the point yields a discrete valuation discrete valuation on $K(X)$. Then distinct points gives you distinct valuations.

Proof. Well, recall that the fraction ring of $\mathscr{O}_{X,p}$ is indeed $K(X)$ since X is integral.

Now suppose that $X \to Y$ is separated, then for a regular closed point p we can consider the following diagram:

$$\begin{tikzcd}
SpecK(X) \arrow[r] \arrow[d, hook, "open"'] & X \arrow[d, "\pi"] \\
Spec\mathcal{O}_{X,p} \arrow[r] \arrow[ur] & Y
\end{tikzcd}$$

where the morphism $Spec\mathcal{O}_{X,p} \to X$ is the canonical morphism and $Spec\mathcal{O}_{X,p} \to Y$ is the composition.

Now for the sake of contradiction, we suppose that p, q are two points such that they result in the same discrete valuation of $K(X)$, then we claim that these two points can't lie in the same affine open $SpecA$, otherwise we will have diagrams below:

$$\begin{tikzcd}
& K(A) & \\
\mathcal{O}_{X,p} \arrow[ur, hook] & A \arrow[l, hook'] \arrow[r, hook] \arrow[u, hook] & \mathcal{O}_{X,q} \arrow[ul, hook']
\end{tikzcd}$$

But this implies that $p = q$ since we know that their image in $K(A)$ is the same, thus inducing an isomorphism, which forces $p = q$.

Now suppose that they are in different affine opens, $SpecA$ and $SpecB$, then we have the following diagram:

$$\begin{tikzcd}
\mathcal{O}_{X,q} \arrow[rr, hook] & & K(X) & & \mathcal{O}_{X,p} \arrow[ll, hook'] \\
& B \arrow[ul, hook'] \arrow[ur, hook] & & A \arrow[ul, hook'] \arrow[ur, hook] &
\end{tikzcd}$$

Notice that they still have the same image in $K(A)$, thus we still have an isomorphism of the two local rings, which you can use to create the following diagram:

$$\begin{tikzcd}
& SpecK(X) \arrow[rr] \arrow[dl, hook'] \arrow[dr, hook, "g"] & & X \arrow[d, "\pi"] \\
Spec\mathcal{O}_{X,q} \arrow[urrr] \arrow[rr, "\simeq"] & & Spec\mathcal{O}_{X,p} \arrow[ur, "f"'] \arrow[r] & Y
\end{tikzcd}$$

Notice that $f : Spec\mathcal{O}_{X,p} \to X$ is not equal to $Spec\mathcal{O}_{X,p} \to Spec\mathcal{O}_{X,q} \to X$ since the maximal ideal p is mapped to p, q respectively.

Here, we have two strategies, the first one is to directly quote the reduced to separated theorem and notice that since $f : Spec\mathcal{O}_{X,p} \to X$ and $Spec\mathcal{O}_{X,p} \to Spec\mathcal{O}_{X,q} \to X$ agree on $SpecK(X)$, a dense open, thus they agree every where, which is a contradiction.

Or use the criterion, here we have to assume that Y is $Speck$, a space with one point, thus even though we have two distinct maps from $Spec\mathcal{O}_{X,p}$ to X, after we compose with $X \to Speck$, they agree, thus a contradiction to the valuative criterion. $\square$

Before we head into the proof, let's spend a few minutes explaining the intuition behind the theorem. We are basically trying to generalize how the line with the double origin goes wrong in this criterion. If you think about what this criterion is telling you, the *Spec* of a DVR is the germ of a curve, and the *Spec* of the fraction field can be thought of as the germ minus the closed point, or you can think of that as the origin. The criterion is saying that if you have a morphism from the germ of a curve to Y and a way to lift this map away from the origin, then there is at mots one way to lift this map from the entire germ.

This indeed generalizes the flaw of the line with a doubled origin. If X is a line with doubled origin and Y is just $Speck$, then if you take the DVR to be the germ of the line at the origin, then if you consider the lift from the germ minus the origin, then you can map the origin to either of the origin in X, which prevents it from being separated.

Proof. This is not a complete proof, for the reason that we will use a lot that is going to take us to far from our goal, and there is a more powerful version of this criterion which needs even more powerful tools in commutative algebra, thus we only tells you the idea behind the proof.

If $X \to Y$ is not separated, then the diagonal of $X \times_Y X$ is not a closed embedding, rather, only a locally closed embedding. Then we claim that there is a point p not in Δ and a point $q \in \Delta$ such that p is in the closure of q. This comes from the facts about closed constructible subsets, thus we will not prove it here. Also, we claim that there is no point r that lies between p, q in the sense that r is not p, q and p is in the closure of r and r is in the closure of q. Then if you consider Q to be the closed subscheme with the reduced structure on the closure of q, we know that if we define B to be the local ring at p, then B is Noetherian local and of dimension 1. If we can prove B is a DVR, then we are done, since we can consider the following diagram:

$$\begin{tikzcd}
SpecB \arrow[dr] & & & \\
Speck(q) \arrow[u] \arrow[r] & Q \arrow[r, hook] & X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"] & X \\
& & X &
\end{tikzcd}$$

since q is in the diagonal, then $Speck(q)$ to X via the two projections are equal, but since p is not in the diagonal, the morphism from $SpecB$ to X via the two projections are different. We omit some details here. ☐

If we are willing to invoke more commutative algebra involving valuations rings, not just DVRs, then you can prove the following more powerful version:

Theorem 2.4. Suppose that $\pi : X \to Y$ is a quasi-separated morphism, then π is separated iff for any valuation ring A, and any diagram of the form in the previous theorem, there is at most one $SpecA \to X$ such that that diagram commutes.

There are also criteria for properness, which combines the valuative criteria of separatedness with the criteria for universal closedness. We again begin with a DVR version:

Theorem 2.5. Suppose that $\pi : X \to Y$ is a morphism of finite type of locally Noetherian schemes, then π is universally closed iff for any discrete valuation ring A and any diagram of the form in the previous theorems, there is at least one morphism $SpecA \to X$ such that the diagram commutes.

Remark 2.6. Notice that instead of we require there is at most one, here we requrie there is at least 1 morphism.

Here we assume finite type since we require proper morphisms to be of finite type, and the criterion for universally closedness plus the cirterion for separatedness implies that the cirterion for properness is that there is exactly one morphism.

Using this we can reprove one result we proved before:

Proposition 2.7. If A is a Noetherian ring, then $\mathbb{P}^n_A \to SpecA$ is proper.

Proof. To use the criterion, suppose that we have a diagram:

$$\begin{tikzcd}
SpecK(B) \arrow[r] \arrow[d] & \mathbb{P}^n_A \arrow[d] \\
SpecB \arrow[r] & SpecA
\end{tikzcd}$$

Recall that to specify a morphism from $\mathbb{P}^n_A$ to $SpecK(B)$, we only have to specify where the homogeneous coordinates go, thus we can assume that $[x_0, \cdots, x_n]$ goes to $[b_0, \cdots, b_n]$ where $b_i \in B$, and this also specifies a morphism $SpecB \to \mathbb{P}^n_A$ that makes the diagram commute, we are done. □

Also, we can prove:

Proposition 2.8. If X is an irreducible regular Noetherian curve(which implies reducec), proper over k, then there is a bijection between discrete valuations on $K(X)$ for which the elements in k has value 0 and closed points of X.

Proof. Recall that we have proved that different closed points gives you different valuations with elements in k valuating to 0.

Conversely, suppose that we have a discrete valuation ring A coming from a discrete valuation on $K(X)$ with element in k valuating to 0. Consider the usual diagram, then being proper implies that there is exactly one morphism such that the diagram commutes, we only have to prove that the morphism maps the maximal ideal to a closed point.

This comes from the fact that X is an irreducible curve. Then we only have to show that the image of the maximal ideal of A is not the generic point of X. If so, you will have a commutative diagram:

$$\begin{tikzcd} K(X) & K(X) \arrow[l, "id"'] \arrow[dl] \\ A_m = A \arrow[u, hook] & \end{tikzcd}$$

but we know that $A_m = A \hookrightarrow K(X)$ is not $K(X)$, thus a contradiction, we are done. □

Finally, let's state the most fancy version of criterion for properness:

Theorem 2.9. Suppose that $\pi : X \to Y$ is quasi-separated and finite type, then π is universally closed iff the following condition holds: For any commutative diagram of the form in the previous theorems, there is at least one morphism $SpecA \to X$ such that the diagram commutes.

Again, since proper implies separatedness, the criterion for properness requires exactly one moprhism such that the diagram commutes.

References