Appendix 14 for Math 632: 2025 Winter Notes

Chunye Yang

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Finally, we have come to the part of the notes which is about differentials. We have waited so long before we bring this intuitive and basic notion you see so early in differential geometry onto our stage. This is another topic that could be talked about way before right now, but we discuss it here because it's not so easy to find the algebraic manifestation of differentials without the geometry we have seen. In this chapter, we will first discuss the theory and then go to examples.

1 Motivation and Game Plan

The goal of this section, as said above, is to understand the differentials, for example, if you have a "smooth" variety, you will then want to understand the tangent bundle and cotangent bundle as what we did for smooth manifolds. However, as what we have seen before, "cotangent objects" behave better in the algebraic setting contrary to our intuition that we understand the space via the tangent objects in smooth manifolds. This is because here we are trying to understand the space via the sheaf of functions living in the space, rather than the space itself. So let's try to define the cotangent sheaf first.

For the cotangent sheaf, what you will see is that the knowledge we had in calculus will show up here as well when available. This cotangent sheaf may not be a vector bundle, but at least it's a quasicoherent sheaf. Moreover, the right way to define it can work more generally for non-smooth spaces or even non finite type spaces. Finally, this construction will work in a relative setting in the sense that for any morphism $\pi : X \to Y$, we can define $\Omega_\pi = \Omega_{X/Y}$, the sheaf of relative differentials. When you take $Y = Speck$, you get the cotangent sheaf on X. Therefore, this relative version of the sheaf of differentials will be the core of this chapter.

A good geometric picture to have in mind is to think of a submersion of smooth manifolds and you will find that the tangent space of to the fibers of this map will fit together to form a vector bundle, called the relative tangent bundle of π whose dual is $\Omega_{X/Y}$.

2 Definitions and First Properties

We will first study the affine case. Say A is a B-algebra, which means we have a morphism of rings $\phi : B \to A$ and a corresponding morphism of schemes $SpecA \to SpecB$. We will define an A-module $\Omega_{A/B}$ in three ways, which is called the **module of relative differentials** or the **module of Kahler differentials**. The module of differentials will be defined to be this module together with a map $d : A \to \Omega_{A/B}$, satisfying three properties (you will find that this map is not necessarily A-linear, but B-linear, so in a priori, we only think of it as a map of abelian groups):

1. Additivity: $da + da' = d(a + a')$.
2. Leibniz: $d(aa') = ada' + a'da$.
3. Triviality on pullbacks: $db = 0$ for any $b \in \phi(B)$.

Except for the last property, the requirements are all familiar from calculus. The last requirement is the analogue of the relative differentials you will see for smooth manifolds as well. This could be less familiar, but let's don't dive in to the differential geometry that much.

Proposition 2.1. $d : A \to \Omega_{A/B}$ is B-linear.

Proof. We need to prove $d(ba) = b(da)$, but by Leibniz, we know $d(ba) = a(db) + b(da)$. Then from the pullback triviality, we know that $db = 0$, therefore we are done. And you notice that this map is not in general A-linear, again because of the Leibniz rule. □

Proposition 2.2. There is also a analogue of the quotient rule, but formulated in a different way. If $a' = as$ then for $a, a', s \in A$, then:

$$s^2 da = sda' - a'ds$$

Proof. $da' = ads + sda$ by Leibniz rule, i.e.:

$$sda = da' - ads \Rightarrow s^2 da = sda' - sads = sda' - a'ds$$

since we assumed $a' = as$. □

Proposition 2.3. Similarly, there is a version of the change rule. Suppose that f is a polynomial with B-coefficients and $g \in A$, say $f = \sum b_i x^i$, then:

$$d(f(g)) = f'(g)dg$$

where f' is the usual formal definition of the derivative of a polynomial.

Proof. Let's first prove a simple version of the chain rule. Suppose that $f = bx^n$, then $d(f(g)) = n \cdot bg^{n-1}dg$. This is because by B-linearity:

$$d(bg^n) = bd(g^n)$$

Then Leibniz rule applied to $d(g^n)$ gives you $g^{n-1}dg$. Then the general result is just the B-linearity again. □

We were essentially proving the properties without knowing what is $\Omega_{A/B}$. Let's now begin to understand what it is and the first attempt we have here is to give it a hands-on definition.

2.1 First definition: Explicit construction

We define $\Omega_{A/B}$ to be a specific quotient of the free A-module generated by symbols of the form da where $a \in A$ where the quotient is defined by:

$$A\{da : a \in A\}/A\{d(a + a') - da - da', d(aa') - ada' - a'da, db : a, a' \in A, b \in \phi(B)\}$$

and the submodule A' is generated by elements coming from the rules of the map $d : A \to \Omega_{A/B}$. And the map d is defined to be $a \mapsto da$. For example, if $A = k[x, y]$ and $B = k$, then a sample differential is given by: $3x^2dy + 4dx$ and we have identities $d(3xy^2) = 3y^2dx + 6xydy$

The following is a key fact: Suppose that A is generated as a B-algebra by $x_i \in A$ where i belongs to some index set I, possibly infinite, subject to some relations r_j where j is also in some index set J possibly infinite (here the relations are generators of the ideal, not just as generators of a B-module), each of which is a polynomial in finitely many x_i's, then I claim that the module $\Omega_{A/B}$ is generated by dx_i, subject to relations $dr_j = 0$. In short, we don't need to take every element in A to generate Ω and the submodule we quotient out doesn't need to be generated by all the relations, we only need the relations $dr_j = 0$. The following is the proof, which is a little bit confusing when you first look at it.

Proof. We need to give an isomorphism of A-modules between the two definitions of $\Omega_{A/B}$. Let's first define the morphism:

$$A\{da : a \in A\} \to A\{dx_i : i \in I\}/A\{dr_j : j \in J\}$$

By the universal property of free modules, we know that we need to specify where each da goes. Suppose that $a = f(x_{i1}, \cdots, x_{in})$ for finitely many x_i where f is a polynomial with coefficients in B, then I define the image to be the class of the differential you would normally expect. We want this to be a canonical map, therefore, suppose that $a = g(x_{k1}, \cdots, x_{km})$ as well, then I want to prove the image is the same. Recall that as a polynomial $f - g = \sum_j h_j r_j$ where h_j are some polynomials with coefficients in B. Then we need to compute the differential of $\sum_j h_j r_j$. Again, it's linear, thus we only need to care about $h_j r_j$, the differential is $r_j dh_j + h_j dr_j$, we know that the two r_j here have different meaning, I will let you figure out why $r_j = 0$ and dr_j is also 0. Therefore, we are good.

This morphism is obviously surjective. And it's also that all the relations we quotient out in the previous definition of the Kahler differential goes to 0, essentially by how we define this map. Therefore, you get a well defined morphism from the original definition to the new definition.

Conversely, let's define the inverse, and again, the definition is almost obvious. You first define it from the free module to $\Omega_{A/B}$ by dx_i maps to dx_i (the two x_i's have different meaning, one is just a formal symbol). Then you see that each dr_j is mapped to 0 as r_j is 0 in A. it's also easy to see that this is the inverse of the previous map, thus we have a canonical isomorphism between two definitions, we are done. □

In particular we have the following corollary:

Corollary 2.4. If A is a finitely generated B-algebra, then $\Omega_{A/B}$ is a finitely generated A-module. If A is a finitely presented B-algebra.

Proof. In the above proof, we can take dx_i to be finitely many and dr_j to be finitely many if A is finitely generated and finitely presented. □

If A is Noetherian, we know that finitely presented is the same as finitely generated.

Now, let's look at a few examples, and these examples are going to be the building blocks for all the ring maps we will care about in practice, i.e. adding free generators, localizing and taking quotients. The first two examples are going to be very trivial, based on what we have proved above.

Example 2.5. If $A = B/I$, i.e. the map $B \to A$ is surjective, then $\Omega_{B/A}$ is 0.

Example 2.6. If $A = B[x_1, \cdots, x_n]$ then $\Omega_{A/B}$ is $Adx_1 \oplus \cdots \oplus Adx_n$ and if we have infinitely many such generators, $\Omega_{A/B} = \oplus_{i \in I} Adx_i$ where I is the indexing set.

Now let's do an explicit nontrivial example:

Example 2.7. Consider the plane curve $y^2 = x^3 - x$ in $\mathbb{A}^2_k$ where the characteristic is not 2. Then let $A = k[x,y]/(y^2 - x^3 + x)$ and $B = k$. Then the above computation shows that the module of differential is generated as an A-module by dx, dy, but it's subject to the following relation:

$$d(y^2 - x^3 + x) = 2ydy - (3x^2 - 1)dx = 0 \Rightarrow 2ydy = (3x^2 - 1)dx$$

I claim that the associated sheaf $\widetilde{\Omega_{A/B}}$ is a line bundle, i.e. locally free of rank 1. To see why, if we localize at y, this sheaf over $D(y)$ is the associated sheaf of the following module:

$$A_y dx \oplus A_y dy/(2ydy = (3x^2 - 1)dx)$$

But we know that since y is invertible, we know that dy is represented as something times dx, i.e. this module is actually isomorphic to $A_y dx$. Therefore it's free over $D(y)$ of rank 1.

Similarly, if we localize at $3x^2 - 1$, we get a free module. Then recall that $D(3x^2 - 1)$ and $D(y)$ covers the curve because we know if y vanishes, then $x^3 - x$ vanishes, but this implies that $3x^2 - 1$ doesn't vanish, otherwise, we know that $2x^2$ vanish, i.e. x vanishes, but this implies that 1 vanishes, a contradiction. Therefore, we find a cover of the space over each of the cover, $\Omega_{A/B}$ is free of rank 1, i.e. a line bundle.

Remark 2.8. We can interpret dx and dy geometrically and it will give you great intuition for what comes later. Let's first do this for dx. Recall that dx vanishes iff it's image in the stalk is in the maximal ideal times the stalk. We have shown that on the locus where $y \neq 0$, dx is the free generator, thus not vanish. On the patch $D(3x^2 - 1)$, we see that $dx = \frac{2y}{3x^2-1}dy$, therefore we saw that dx vanish precisely at the locus where y vanish. This is very plausible geometrically, as if you look at a picture of this curve, the locus where $y = 0$ is the place where the tangent vector is vertical. And the fact that the tangent vector is vertical means that the valuation of dx on any elements at the tangent vector here is 0, therefore it's the zero element.

We can also do this for for dy and conclude that it vanishes at the locus $3x^2 - 1 = 0$ and this again has geometrically meaning, when $x = -\frac{1}{3}^{1/2}$, we find the tangent vector is horizontal.

The above discussion applies to any plane curve $k[x,y]/f(x,y)$. Suppose for the sake of convenience, $k = \overline{k}$, then you see that the module of Kahler differentials is generated by dx, dy subject to relation $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$

Then the module is free of rank 1 on $D(\frac{\partial f}{\partial x})$ and $D(\frac{\partial f}{\partial y})$. Recall that by the Jacobian criterion for regularity, the singular closed are given by equations $f = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$, then if the curve is regular, there are no singular closed point, therefore all the closed points are in $D(\frac{\partial f}{\partial x})$ and $D(\frac{\partial f}{\partial y})$, then since we work with finite type over k scheme, each closed subset contains a closed point, then the union of $D(\frac{\partial f}{\partial x})$ and $D(\frac{\partial f}{\partial y})$ is the whole space. Conversely, if these two covers the whole space, then again, the Jacobian criterion tells you that all the closed points are regular. Then the curve must be reduced, ortherwise, we know that nonreudced points are closed, then there is a closed point in this nonreduced subset, a contradiction to regularity at all closed point. Then reducedness at the generic point implies it's regular there, since we want to prove the local ring is regular, but the spec of the local ring consist of just a single point and it is reduced. This is because the only prime ideal is 0 as the intersection of all primes is the nilradical. The above discussion shows that the module is locally of rank 1 iff the curve is regular. Or in the negative formulation, the module is not locally free of rank 1 iff the curve is not regular. For example, consider the curve $A = k[x,y]/(y^2 - x^3)$, we know the module of Kahler differentials is:

$$Adx \oplus Ady/(3x^2dx = 2ydy)$$

Then we know that over the origin, the module is of rank 2 and the curve is not regular at the origin.

What is implicit in the above discussion is the following result:

Proposition 2.9. Suppose that $A = B[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$ then:

$$\Omega_{A/B} = (\oplus_i Adx_i)/(Adf_1 + \cdots + Adf_r)$$

It can be interpreted as the cokernel of the Jacobian matrix $J : A^{\oplus r} \to A^{\oplus n}$.

Proof. The stuff we quotient out can be interpreted as the submodule generated by $df_1, \cdots, df_r$, which is just the computation we had before about how to represent the module using generators and relations between the generators.

Recall the Jacobian matrix is $(df_1, \cdots, df_r)$ where each df_i is a column vector. Then the image of this Jacobian matrix is just $Adf_1 + \cdots + Adf_r$, we are done. $\square$

Remark 2.10. Recall that we define A to be smooth over B, in a preliminary definition by the Jacobian matrix to have full rank at every point. This implies that $\Omega_{A/B}$ is locally free. This can be proved by the following argument: recall that over a reduced locally Noetherian scheme, a sheaf is locally free iff the fiber dimension is the same at each point. The fiber of the module, by the above discussion, since tensor preserves cokernel, is the cokernel of the map $J \otimes A_p/pA_p$. However, recall that the rank of the Jacobian matrix is full at each point, and the rank at point p is just the rank of $J \otimes A_p/pA_p$, therefore the dimension of the cokernel is fixed at each point, thus it's locally free.

As you will see later, the module $\Omega_{A/B}$ will correspond to the cotangent sheaf and the following result is a good indicator of that, when you consider k-valued points:

Proposition 2.11. Suppose that B is a finite type k-algebra and m is a k-valued point of B, then there is an isomorphism of k-vector spaces:

$$\delta : m/m^2 \to \Omega_{B/k} \otimes_B k$$

where k is viewed as a B-module via the ring isomorphism $B/m \simeq k$.

Proof. Since we work with finite type k-schemes, let's assume $B = k[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$. Then we know that the cotanegnt space at a closed point is computed by the cokernel of Jacobian matrix J, but consider to be over the residue field B/m. But the above remark shows that $\Omega_{B/k} \otimes_B k$ is also the cokernel of the Jacobian matrix over B/m, then there is a canonical isomorphism of k-vector spaces making the two objects isomorphic. $\square$

Proposition 2.12. Suppose that X is a finite type scheme over $k = \overline{k}$, then the function from the closed points of X to $\mathbb{Z}^{\geqslant 0}$ defined by $p \mapsto \dim T_pX$ is upper semicontinuous in the Zariski topology.

Proof. The Zariski topology on the closed points can be interpreted as the subspace topology, or you can think of it as the topology cut out by equations since we are in a finite type scheme over k. The advantage of $k = \overline{k}$ is that all closed points are going to be k-valued. We want k-valued points because the previous result only tells you what happens for k-valued points, but there will be a result later to extend it.

Now to prove uppersemicontinuity, we need to show the inverse image of $(-\infty, x)$ is open, or $[x, \infty)$ is closed. We can assume that the space is just affine since we can cover and then prove the inverse image is open or closed in each affine open. Then the question just becomes a question for the rank of the Jacobian matrix, but we know that the rank of the matrix is lower semicontinuous as the rank drops on closed points (where various determinants of submatrices vanish), therefore, when you subscribe the rank, it becomes uppersemicontinuous. □

The final example, as promised, is localization:

Example 2.13. Suppose that $A = T^{-1}B$ where T is a multiplicative system in B, then I claim $\Omega_{A/B} = 0$. This is because if $a = b/t$ for $b, t \in B$, then we know that $at = b$, therefore we have the quotient rule:

$$t^2 da = tdb - bdt = 0$$

since t is invertible, we know $da = 0$, we are done.

This is also very convincing as A is just a open subset of B, therefore should not have any relative cotangent space.

In the following result, you will see that the notion of relative differentials is even interesting for finite extension of fields, in other words, even if you map a reduced point to a reduced point, something interesting with differentials can happen:

Proposition 2.14. Suppose that K/k is a separable algebraic extension then $\Omega_{K/k}$ is 0.

Proof. Let $\alpha \in K$, then since this is a separable algebraic extension, there is a polynomial $f(x)$ with coefficients in k such that $f(\alpha) = 0$ but $f'(\alpha) \neq 0$. Then we want to prove $d\alpha = 0$. Recall that $df(\alpha) = 0$, but by Lebinz rule

$$d(f(\alpha)) = f'(\alpha)d\alpha = 0$$

while $f'(\alpha) \neq 0$, therefore $d\alpha = 0$. □

The above shows that separable algebraic extensions have nonontrivial relative differential. This is not true for non-separable extensions:

Proposition 2.15. Let k be a field with $\mathrm{char} k = p$ and let $K = k(t^p)$ while $L = k(t)$, consider the obvious inclusion $K \hookrightarrow L$ I claim $\Omega_{L/K} \neq 0$.

Proof. I claim that the module of Kahler differentials is in this case L, i.e. the ground field. What we notice first is that L is actually the following field:

$$L = K[X]/X^p - t^p$$

Recall that k is of characteristic p, therefore $X^p - T^p = (X - t)^p$, the isomorphism is given by $X \mapsto t$, it's an isomorphism because the inverse is given by $L \to K[X]/X^p - t^p$ where t is mapped to x as a k-algebra morphism, and it's that the compositions are identities. But this means that L is not a separable extension of K, in this case we will show that there is indeed something interesting happening for the differential.

By our computation before, we know that $\Omega_{L/K}$ can be written as:

$$LdX/Ld(X^p - t^p) = LdX$$

as the characteristic is p, thus $d(X^p - t^p) = 0$. Therefore the differential is L. □

We now delve deeper and discuss two geometrically motivated exact sequences:

Theorem 2.16. Suppose that $C \to B \to A$ is a sequence of ring maps, then there is a natural exact sequence of A-modules:

$$A \otimes_B \Omega_{B/C} \to \Omega_{A/C} \to \Omega_{A/B} \to 0$$

Proof. The maps between each other should be clear. The first map, by the universal property of extension of scalars, is just a B-linear map between two differentials and it's defined to be:

$$\Omega_{B/C} \to \Omega_{A/C} \quad db \mapsto db$$

We need to show that this is well-defined, and we know that this is just a map:

$$B\{db : b \in B\}/ \cdots \to A\{da : a \in A\}/ \cdots$$

So we can think of the above definition as first a map coming out of a free module, then we know that the kernel of this map clearly contains the stuff we quotient out, therefore we are done.

The second map should be defined again by $da \mapsto da$ and you again think of it as a map first out of a free module and then show the kernel behaves as expected.

The proof of this, algebraically is very simple, so let's first prove it and you will later see in the remark that this has a fundamental geometric meaning. Surjectivity of the last map is obvious. Then we want to identify $\Omega_{A/B}$ as the cokernel, but we know that $\Omega_{A/B}$ and $\Omega_{A/C}$ are the same except for $\Omega_{A/B}$ have extra relations $db = 0$. But this is exactly the image of $A \otimes_B \Omega_{B/C}$. ☐

Remark 2.17. This remark is to comment on the geometric meaning of the above exact sequence, which is often called the **cotangent exact sequence**. We haven't prove any of these statement below, therefore what we will discuss below only serves as a good way to memorize the above exact sequence.

Let's again, think of smooth manifolds for example, as we haven't really explained how the cotangent sheaf is connected to smoothness. Think of a smooth map of smooth manifolds $\pi : X \to Y$ where X is a surface and Y is a curve for example. let p be a point of X, which you should think of as a closed point of X, the image $\pi(p)$ is also a smooth point. Consider the fiber of $\pi(p)$, which is a smooth curve in X that contains p. Then there are three tangent spaces you can think of, i.e. the tangent space of p of the fiber, the tangent space at p of X and the tangent space of $\pi(p)$ of Y. What is the relation of these three tangent spaces? First, the tangent space at p of the fiber is a subspace of the tangent space at p of X. It's now natural to ask what is the cokernel and it turns out that the cokernel is going to be a pullback of the tangent space at $\pi(p)$ of Y in the smooth case and more over, these relation patches together in the smooth case to for a exact sequence of relative tangent sheaves:

$$0 \to \mathcal{T}_{X/Y} \to \mathcal{T}_{X/Z} \to \pi^*\mathcal{T}_{Y/Z} \to 0$$

Dualizing it we get:

$$0 \to \pi^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

This is going to be the global version of the above affine exact sequence except we have left exactness. This is because we are working with "smooth" objects which enables the left exactness. The above theorem is going to be the most general statement.

Here is another exact sequence we need:

Theorem 2.18. Suppose that B is a C-algebra and I is an ideal of B, finally $A = B/I$ then there is a natural exact sequence of A-modules:

$$I/I^2 \xrightarrow{\delta} A \otimes_B \Omega_{B/C} \to \Omega_{A/C} \to 0$$

Proof. Again, we first give the algebraic proof and tell you how you should think about the geometry behind it.

Let's figure out what is the map δ, recall that I/I^2 can be identified with $B/I \otimes_B I$, then I claim that δ is:

$$1 \otimes d : B/I \otimes_B I \to B/I \otimes_B \Omega_{B/C}$$

We need to be careful about this map, recall that $I \to \Omega_{B/C}$ is not in general B-linear, as it is the restriction of the map $d : B \to \Omega_{B/C}$ which is C-linear, not B-linear. Then we can only say that this is a map of abelian groups, and we need to show that it's B/I-linear, i.e. we need to check:

$$b(1 \otimes b') = b \otimes b' = 1 \otimes bb' \mapsto d(bb') = bdb' + b'db = bdb'$$

the last equality holds because $b' \in I$ by assumption. In this definition we find that the class of i in I/I^2 is mapped to $1 \otimes di$.

Now we need to identity the cokernel of δ. Let's first consider the module $A \otimes \Omega_{B/C}$, as an A-module, it's generated by $db, b \in B$ with relations $dc = 0$ for all $c \in C$ and additivity and Leibniz rule (make sure you understand), but be careful that all of the relations are in B. Now think about $\Omega_{A/C}$, and as an A-module, it's generated by da, but subject to relations in A, i.e. $dc = 0$ and the additive and Leibniz rule in A. Here A is not just an arbitrary ring, but it's the ring B/I, therefore, you can think of $\Omega_{A/B}$ as again generated by $db, b \in B$, but there are more relations, i.e. $di = 0$ for $i \in I$ (this is the key here). But this is precisely what you get by quotient out the image of I/I^2 in $A \otimes_B \Omega_{B/C}$. I will let you figure out the detail, but the key isomorphism can be interpreted as follows:

$$B/I\{d\bar{b} : b \in B\}/\cdots \simeq B\{db : b \in B\}/\{\cdots, di\}$$

I will also let you think about how to define this isomorphism. □

Remark 2.19. The above exact seuqence is often called the **conormal exact sequence**. As promised, let's discuss what is the geometric meaning of this sequence.

To motivate the geometric meaning, again, let's look at what happens in the smooth manifold case. Suppose that X, Y are smooth manifold and X is of dimension 1, Y is of dimension 2. Say that $j : X \to Y$ is an embedding of the curve into the surface. Let Z be a point of X, then again, we have three spaces to think about. They are the tangent space of p in X, the pullback of the tangent space of p at Y and the normal space of the tangent space of p at X in the pullback of the tangent space at p in Y. The relations between these three spaces should also be obvious as you clearly have an injection and the cokernel is going to be the normal space. Again, when everything is smooth enough, we can patch these together to for a SES of sheaves:

$$0 \to \mathcal{T}_{X/Z} \to j^*\mathcal{T}_{Y/Z} \to \mathcal{N}_{X/Y} \to 0$$

Dualizing it gives:

$$0 \to \mathcal{N}_{X/Y}^{\vee} \to j^*\Omega_{Y/Z} \to \Omega_{X/Z} \to 0$$

This is the above sequence, and again, except the above sequence may not be left exact in algebraic geometry. We will see later that similar to the cotangent exact sequence, we will obtain left exactness in a smooth enough situation and the "conormal" bundle will somehow correspond to I/I^2.

Remark 2.20. Let's make another remark on the so called conormal module and conormal sheaf. Again, let's first deal with the affine case and then later we have a more general treatment.

Suppose that $SpecA \hookrightarrow SpecB$ is a closed embedding cut out by the ideal I, then I define the **conormal module** on A to be I/I^2 and the associated sheaf to be the **conormal sheaf**. There are several reasons on why we would like to make this definition. First, if $SpecA$ is just a closed point, a reasonable guess is that the conormal space is the whole cotangent space, and is indeed m/m^2. For example, the conormal space of $\mathbb{A}_k^2$ at the origin is the space of linear forms.

More generally, if you consider, for example the z-axis in $\mathbb{A}_k^3$, cut out by $I = (x, y)$. Elements in the module I/I^2 is $\alpha(z)x + \beta(z)y$ and this could reasonably be the conormal sheaf as you expect to first be 2-dimensional over $Speck[z]$ and at different points, the space at that point is spanned by x, y, which looks like you killed z and x, y are those that remains.

But the most important intuition still comes from the closed embedding between smooth manifolds. For example, $X \hookrightarrow Y$ is an embedding of a k-dimensional manifold inside a n-dimensional manifold, defined by the ideal sheaf $\mathcal{I}$ (that is right, embeddings of smooth manifolds are also defined by ideal sheaves), and there is a natural identification between $\mathcal{I}/\mathcal{I}^2$ with the conormal sheaf. For example if you check it locally, we know that such an embedding is analytically locally $R^k \to R^n$, then the cokernel of the embedding of the tangent space is going to be R^{n-k}, so is the dual.

Therefore, people expect that in algebraic geometry, when we have appropriate smooth conditions, we have similar conclusions. This prompts the following definition:

Definition 2.21. If $X \hookrightarrow Y$ is a closed embedding of schemes cut out by the quasicoherent ideal sheaf $\mathcal{I}$, then the **conormal sheaf for a closed embedding** is $\mathcal{I}/\mathcal{I}^2$, and denoted by $\mathscr{N}^\vee_{X/Y}$ again quasicoherent and can be viewed as a sheaf on X as it's annihilated by the ideal sheaf of X.

We can also define the normal sheaf $\mathscr{N}_{X/Y}$ to be the dual of the conormal sheaf:

$$\mathscr{N}_{X/Y} = \mathcal{H}om(\mathscr{N}^\vee_{X/Y}, \mathcal{O}_X)$$

This is not a perfect notation, as this might suggest that the dual of $\mathscr{N}_{X/Y}$ is always $\mathscr{N}^\vee_{X/Y}$, but this is obvious not true as the natural map from a module to it's double dual is not in general an isomorphism.

If the normal sheaf is locally free of finite rank, we call it the **normal bundle** of X in X.

Now what happens for locally closed embeddings, does it have a conormal sheaf and hence a normal sheaf. Suppose that $X \hookrightarrow Y$ is such a locally closed embedding, then it can be decomposded as $X \hookrightarrow Z \hookrightarrow Y$ where the first one is a closed embedding and the second one is an open embedding. We know that $X \hookrightarrow Z$ is cut out by a quasicoherent ideal sheaf, and we can define the conormal sheaf similarly. But we have to show that if you have a different decomposition, we get the same sheaf. But we can take the intersection of the two opens and use the intersection to show it's well defined and I will let you fill the details.

However, if in good situations, the conormal sheaf is locally free of finite rank, then indeed we have the double dual is itself, or in other words, the dual of $\mathcal{N}_{X/Y}$ is indeed $\mathscr{N}^\vee_{X/Y}$. And one important case is when $X \hookrightarrow Y$ is a regular embedding. Let's first begin with the easiest case, i.e. X is a Cartier divisor in Y:

Proposition 2.22. Suppose that $D \hookrightarrow X$ is an effective Cartier divisor then the conormal bundle is $\mathcal{O}(-D)|_D$, which is a line bundle hence it's dual is $\mathcal{O}(D)|_D$, justifying our preliminary definition of the normal bundle to an effective Cartier divisor as $\mathcal{O}(D)|_D$.

Proof. Recall that $\mathcal{O}(-D) = \mathcal{I}$ where $\mathcal{I}$ is the quasicoherent ideal sheaf cutting out D. We know that:

$$\mathcal{O}(-D)|_D$$

can be interpreted by both the pullback of $\mathcal{O}(-D)$ to D or the tensor product $\mathcal{O}(D) \otimes i_*\mathcal{O}_D$ depending whether you want to think of it as a sheaf on D or as a sheaf on X.

By definition, the conormal sheaf is $\mathcal{O}(-D)/\mathcal{O}(-2D)$. You can think of this as the tensor:

$$\mathcal{O}(-D) \otimes_{\mathcal{O}_X} (\mathcal{O}_X/\mathcal{O}(-D))$$

But we know that the tensor is isomorphic to $i_*\mathcal{O}_D$ because we have the SES of the closed embedding, thus we are done. □

Remark 2.23. Don't forget that Cartier divisor always implies a regular embedding of codimension 1, but a regular codimension 1 embedding may not be an Cartier divisor. We need to assume Noetherian condition for the later statement to hold.

Now let's treat the general situation, and in this case the above proof is not so helpful since we didn't talk about regular embedding that is not an effective Cartier divisor that much and we don't have a counter part of $\mathcal{O}(-D)$ yet. So to prove it, we just use definition and local algebra:

Proposition 2.24. If $X \hookrightarrow Y$ is a regular embedding of codimension r of locally Noetherian schemes, then the conormal sheaf is locally free of rank r. This is the consequence of of the following algebraic fact: If the ideal $I \subset B$ is generated by a regular sequence $x_1, \cdots, x_r$ and B is a Noetherian local ring, then the map:

$$\gamma : (B/I)^r \to I/I^2$$

defined by $(a_1, \cdots, a_r) \mapsto a_1x_1 + \cdots + a_rx_r$ describes I/I^2 as a free module of rank r over B/I with basis $x_1, \cdots, x_r$.

Proof. Let's prove the algebraic fact first. Notice that this map is surjective, thus we only need to show injectivity. For the illustration of the main idea, let's first consider $r = 1$, we need to show that the map:

$$\times x_1 : B/(x_1) \to (x_1)/(x_1^2)$$

is injective. Suppose that $bx_1 \in (x_1^2)$, i.e. $bx_1 = cx_1^2$, but since we know that x_1 is not a zerodivisor, therefore multiply by x_1 on B is injective, therefore $b = cx_1$, as desired.

For the general case, suppose that we have $b_1, \cdots, b_n \in B$ such that:

$$b_1x_1 + \cdots + b_rx_r \in I^2$$

we want to show $b_i \in I$. Now recall that the above relation, modulo $(x_1, \cdots, x_{r-1})$ implies that $b_rx_r \in (x_r^2)$, modulo $(x_1, \cdots, x_{r-1})$, therefore since x_r is not a zero divisor after modulo $(x_1, \cdots, x_{r-1})$, we know that b_r is a multiple of x_r modulo $(x_1, \cdots, x_{r-1})$, thus $b_r \in I$. Since we know that any reordering of $(x_1, \cdots, x_r)$ will be a regular sequence, the above argument applies to any b_i.

Finally, to prove the assertion on the regular embedding, suppose that $X \hookrightarrow Y$ is a regular embedding of codimension r and X, Y are locally Noetherian, then at each point of X, we know that the ideal is generated by a regular sequence of length r, then Noetherianity implies that the ideal sheaf is finite type and also finitely presented, then one corollary of the geometric Nakayama tells you that the sheaf is locally free iff the fiber is free for each point, but this is just the conclusion of the algebraic result. □

2.2 Second definition using universal property

Now we want to introduce the second definition of the module of Kahler differentials, which uses a universal property to characterize it.

Suppose that A is again a B-algebra and M is an A-module. A **B-linear derivation of A into M** is a map $d : A \to M$ of B-modules via the B-algebra structure (not necessarily a map of A-modules) satisfying the Leibniz rule $d(fg) = fdg + gdf$. For example, if $B = k, A = k[x]$ and $M = A$, then the derivative map $\frac{d}{dx}$ is a derivation, or if $B = k, A = k[x]$ and $M = k$, the compute the value of the derivative at 0 is a derivation as well. Finally, we know that by definition: $d : A \to \Omega_{A/B}$ is also a derivation. Moreover, it's the universal object in all of B-linear derivations in the sense that it satisfies the following universal property: any other B-linear derivation factors uniquely through d:

$$\begin{tikzcd} A \arrow[rr, "d'"] \arrow[dr, "d"'] & & M \\ & \Omega_{A/B} \arrow[ur, "f"'] & \end{tikzcd}$$

or you could say that $\Omega_{A/B}$ is the representing object of the functor from A-modules to sets where an A-module M is mapped to the set of B-linear derivations from A to M, often denoted by $\mathrm{Der}_B(A, M)$, this is more than just a set, and indeed is a B-module. You could directly see from the definition that this universal property is satisfied and it actually comes from the universal property of free A-modules and quotients.

Proposition 2.25. If we have a commutative diagram of rings:

$$\begin{tikzcd} A' & A \arrow[l] \\ B' \arrow[u] & B \arrow[l] \arrow[u] \end{tikzcd}$$

then there is a natural map of A'-modules $A' \otimes_A \Omega_{A/B} \to \Omega_{A'/B'}$, an important special case is when $B = B'$. This is usually called the p**ullback of differentials**. Furthermore, if the above morphism is a fibered coproduct diagram (or tensor diagram), the natural map is an isomorphism.

Proof. Since we just introduced the universal property, let's think a little bit of how to use the universal property to define this map. Recall that by the universal property of extension of scalars, we only have to define an A-linear map from $\Omega_{A/B}$ to $\Omega_{A'/B'}$. Consider the map $A \to A' \xrightarrow{d} \Omega_{A'/B'}$, I claim that it's a B-linear derivation. It's clear that it satisfies the Leibniz rule, so we only have to show it's B-linear. Notice that $A' \to \Omega_{A'/B'}$ is B'-linear, thus can be though as B-linear by restriction, moreover $A \to A'$ is A-linear, thus is also B-linear by restriction, we are done.

Therefore, in conclusion, the map is defined by $a' \otimes da \mapsto a' \cdot da$ where you should think of a as elements in A' or in A as needed.

To understand what happens when the diagram is "co-Cartesian". Let's first show that we have a natural bijection:

$$\mathrm{Der}_{B'}(A', N) \to \mathrm{Der}_B(A, N)$$

for an A'-module N. To see what is the map, suppose that $A' \to N$ is a B'-linear derivation, we know that compose it with $A \to A'$ gives you a B-linear derivation. This is an isomorphism because we can construct an inverse, suppose that we have a B-linear derivation $A \to N$, we want a B'-linear derivation $A' \to N$. Recall that by the universal property of tensor, we know that a B-linear map from A' to N is equivalent to a B-bilinear map from $A \times B'$ to N. It's clear how to define this and it should be $(a, b') \mapsto b' d_A(a)$. It's clearly B-linear. Then you can check that this is actually B'-linear as:

$$(1 \otimes b')(a \otimes b'') = (a \otimes b'b'') \mapsto b''b' d_A(a) = (b')(b'' d_A(a))$$

It also satisfies the Leibniz rule as

$$(a \otimes b)(a' \otimes b') = bb' d_A(aa') = bb' a d_A(a') + bb' a' d_A(a)$$

Finally, you can check that the above two constructions are inverses of each other thus an isomorphism of B-modules. How you should think of this morphism is that it's a natural isomorphism between two functors from the category of A'-modules to the category of B-modules.

Then we have the following sequence of isomorphisms:

$$\mathrm{Hom}_{A'}(\Omega_{A'/B'}, N) \simeq \mathrm{Der}_{B'}(A', N) \simeq \mathrm{Der}_B(A, N) \simeq \mathrm{Hom}_A(\Omega_{A/B}, N) \simeq \mathrm{Hom}_{A'}(A \otimes_A \Omega_{A/B}, N)$$

Notice that each morphism is going to be functorial in N, therefore, the final isomorphism:

$$\mathrm{Hom}_{A'}(\Omega_{A'/B'}, N) \simeq \mathrm{Hom}_{A'}(A \otimes_A \Omega_{A/B}, N)$$

is induced by a unique isomorphism of A'-modules $A \otimes_A \Omega_{A/B} \to \Omega_{A'/B'}$. Notice that an A'-module map:

$$\Omega_{A'/B'} \to \quad da' \mapsto n_{a'}$$

is mapped to first the derivation where a' is mapped to $n_{a'}$ and then it's mapped to a is mapped to n_a, then it's mapped to da is mapped to n_a and finally it's mapped to $1 \otimes da \mapsto n_a$, therefore the unique morphism is defined by:

$$1 \mapsto da \mapsto da$$

which is exactly the morphism we started with. □

Next we prove a more general result about how localization affects the Kahler differentials:

Proposition 2.26. Suppose that $\phi : B \to A$ is a map of rings, S is a multiplicative subset of A and T is a multiplicative subset of B with $\phi(T) \subset S$, then by the universal property of localization, we have the following diagram:

$$\begin{array}{ccc} S^{-1}A & \longleftarrow & A \\ \uparrow & & \uparrow \\ T^{-1}B & \longleftarrow & B \end{array}$$

then the pullback of differentials $S^{-1}\Omega_{A/B} \to \Omega_{S^{-1}A/T^{-1}B}$ is an isomorphism. How you should think of this map geometrically is that restrict maps to opens doesn't change the relative differentials.

Proof. This is not just a special case of the above result and we indeed need to do something to show this is an isomorphism. We again use the same idea as the above result and we want to show that there is a natural isomorphism:

$$\mathrm{Der}_{T^{-1}B}(S^{-1}A, N) \to \mathrm{Der}_B(A, N)$$

for any $S^{-1}A$-module N. Again, the map is defined to be restriction, and we need to give is an inverse. Similarly, suppose that you have a B-linear derivation $d_A : A \to N$, we now need to give a $T^{-1}B$-linear map $S^{-1}A \to N$, which is a derivation. What we need to do is to notice that there is a unique B-linear derivation $d_{S^{-1}A} : S^{-1}A \to N$ extending d_A. To see why, if there is a derivation extending d_A, then $\frac{a}{s}$ by the quotient rule is mapped to $\frac{sda-ads}{s^2}$, then we only need to show that this is indeed well defined and is a derivation. If $\frac{a}{s} = \frac{a'}{s'}$ then we know that there is a t such that $as't = a'st$, then $\frac{sda-ads}{s^2} = \frac{s'da-ads'}{s'^2}$ since you can divide by ss' on both sides and then apply d. I claim that this is actually $T^{-1}B$-linear because consider:

$$\frac{b}{t} \cdot \frac{a}{s} = \frac{ab}{st} \mapsto \frac{std(ab) - abd(st)}{s^2t^2} = \frac{sadb + sbda - adtds - absdt}{s^2t^2}$$

Recall that $db = 0 = dt$, then I will leave the details to you. Then let's check that it satisfies the Leibniz rule:

$$\frac{aa'}{ss'} \mapsto \frac{ss'd(aa') - aa'd(ss')}{s^2s'^2}$$

as $d(aa') = ada' + a'da, d(ss') = sds' + s'ds$ and in the end we get:

$$\frac{a}{s}d\frac{a'}{s'} + \frac{a'}{s'}d\frac{a}{s} = \frac{a}{s} \cdot \frac{s'da' - a'ds'}{s'^2} + \frac{a'}{s'} \cdot \frac{sda - ads}{s^2} = \frac{ass'da' + a's'sda - saa'ds' - s'a'ads}{s^2s'^2}$$

This proves that $d_{s^{-1}A} : S^{-1}A \to N$ is actually a $T^{-1}B$-derivation.

Then we need to prove that the above two constructions are inverses of each other, but this is obvious as we proved above that there is a unique extension. And this clearly is a functorial map in N, thus we are done by essentially the same argument as the above result. Consider the sequence of isomorphisms:

$$\mathrm{Hom}_{S^{-1}A}(\Omega_{S^{-1}A/T^{-1}B}, N) \simeq \mathrm{Der}_{T^{-1}B}(S^{-1}A, N) \simeq \mathrm{Der}_B(A, N) \simeq \mathrm{Hom}_A(\Omega_{A/B}, N) \simeq \mathrm{Hom}_{S^{-1}A}(S^{-1}\Omega_{A/B}, N)$$

and you notice that $d\frac{a}{1} \mapsto n$ is mapped to $\frac{a}{1} \mapsto n$ which is mapped to $a \mapsto n$ which is mapped to $da \mapsto n$ which is mapped to $\frac{da}{s} \mapsto n/s$, therefore the unique isomorphism is induced by $\frac{da}{s} \mapsto \frac{1}{s} \cdot d\frac{a}{1}$, we are done as this is the pullback map. □

Using the above two results, we have the following result on field extensions:

Example 2.27. Suppose k is a field, we will compute in this example $\Omega_{k(t)/k}$ in this example.

Let's consider the following diagram:

$$\begin{tikzcd} k(t) & k[t] \arrow[l] \\ k \arrow[u] & k \arrow[l] \arrow[u] \end{tikzcd}$$

We know that we have an isomorphism $S^{-1}\Omega_{k[t]/k} \to \Omega_{k(t)/k}$ where $S \subset k[t]$ is all the nonzero elements and we know that $\Omega_{k[t]/k}$ is generated as a $k[t]$-module by dt, i.e. $\Omega_{k[t]/k} = k[t]dt$, therefore:

$$\Omega_{k(t)/k} = S^{-1}k[t]dt = k(t)dt$$

Therefore, $\Omega_{k(t)/k}$ is a one dimensional vector space over $k(t)$ generated by dt. The same argument shows that $\Omega_{k(x_1,\cdots,x_n)/k}$ is a $k(x_1, \cdots, x_n)$ vector space with basis $dx_1, \cdots, dx_n$.

Example 2.28. Suppose that K/k is separably generated by $t_1, \cdots, t_n \in K$, i.e. $t_1, \cdots, t_n$ forms transcendence basis and $K/k(t_1, \cdots, t_n)$ is algebraic and separable. We will show in this example that $\Omega_{K/k}$ is a K-vector space with a basis $dt_1, \cdots, dt_n$.

To do this, let's first show that it's generated by $dt_1, \cdots, dt_n$ and then show they are linearly independent. To show they generate the module, consider the sequence:

$$k \hookrightarrow k(t_1, \cdots, t_n) \hookrightarrow K$$

Then we have the cotangent exact sequence:

$$K \otimes_{k(t_1,\cdots,t_n)} \Omega_{k(t_1,\cdots,t_n)/k} \to \Omega_{K/k} \to \Omega_{K/k(t_1,\cdots,t_n)} \to 0$$

since $K/k(t_1, \cdots, t_n)$ is algebraic and separable we know that the last term is 0. Therefore we get a surjection:

$$K \otimes_{k(t_1,\cdots,t_n)} \Omega_{k(t_1,\cdots,t_n)/k} \to \Omega_{K/k}$$

But we know that $K \otimes_{k(t_1,\cdots,t_n)} \Omega_{k(t_1,\cdots,t_n)/k}$ is a K vector space generated by $dt_1, \cdots, dt_n$ and the map is defined by $f \otimes dt_1 \mapsto f \cdot dt_1$, therefore dt_1 is mapped to dt_1, therefore, this shows that $\Omega_{K/k}$ is generated as a K-vector space by $dt_1, \cdots, dt_n$.

To show they are linearly independent, I claim that there is a unique map of K-vector spaces $\Omega_{K/k} \to K$ sending dt_1 to 1 and all the other dt_i to 0 and similarly a map sending dt_2 to 1 and all the others to 0 and for all the other dt_j. Then by this, suppose that $\sum a_i dt_i = 0$, by applying each of these maps we now that $a_i = 0$ for all i, therefore they are independent. Thus our goal now is to construct these maps.

First, we know that by the universal property of the Kahler differentials, we need a derivation $K \to K$ such that $t_1 \mapsto 1$ and other $t_i \mapsto 0$. Let's first do this for $k(t_1, \cdots, t_n) \to k(t_1, \cdots, t_n)$. We know that there is a k-linear derivation on $k[t_1, \cdots, t_n] \to K$ defined by taking derivative with respect to t_1. This is the unique k-linear derivation that maps t_1 to 1 and all the other varaibles to 0 because this uniquely determined the derivation by Leibniz rule. Now, we can extend this uniquely to a k-linear derivation $k(t_1, \cdots, t_n) \to K$. This is done by the quotient rule since if you want the new derivation to extend the on on $k[t_1, \cdots, t_n]$, we know that we can only define it by:

$$\frac{f}{g} \mapsto \frac{gdf - fdg}{g^2}$$

This is well defined and it can be checked by definition and it's obviously k-linear, moreover you can check Leibniz rule by hand as well. Therefore this is indeed a k-linear derivation $k(t_1, \cdots, t_n) \to k(t_1, \cdots, t_n)$. Now let's assume that $K/k(t_1, \cdots, x_n)$ is generated by one element α, for simplicity, let's denote $k(t_1, \cdots, t_n) = F$ then we know that $K = F(\alpha)$ and let $f(x) \in F[x]$ be the monic minimal polynomial of α and since K/F is separable, we know that $f'(\alpha) \neq 0$. We want to have a k-linear derivation on $F(\alpha)$ extending that on F, let's assume we already have it and apply it to $D(f(\alpha)) = 0$, now you probably want to use the chain rule and conclude that $D(f(\alpha)) = f'(\alpha)D(\alpha)$, but this is not true, as D is only assumed to be k-linear, not F-linear, then we need to consider:

$$D(\alpha) = D(\alpha^n + \sum a_i \alpha^i) = n\alpha^{n-1}D(\alpha) + \sum i \cdot a_i \alpha^{i-1} D(\alpha) + \sum \alpha^i D(a_i) = f'(\alpha)D(\alpha) + g(\alpha) = 0$$

where $g(x) = \sum D(\alpha_i)x^i$. Now we know that we solve the above equation and obtain that:

$$D(\alpha) = -\frac{g(\alpha)}{f'(\alpha)}$$

since we already know $f'(\alpha) \neq 0$, this is well defined. Then recall that we have the following diagram:

$$\begin{tikzcd}
K & & F[x] \arrow[ll, "x \mapsto \alpha"'] \arrow[dl, hook] \\
& F(x) \arrow[ul, "\simeq"] &
\end{tikzcd}$$

Let's define a k-linear derivation on $F[\alpha] \hookrightarrow K$ extending that on F. The definition is uniquely determined if we want $D(\alpha) = \frac{g(\alpha)}{f'(\alpha)}$ as the image of a polynomial is determined by linearity and the Leibniz rule. First, let's show it satisfies the Leibniz rule, i.e. $D(h(\alpha) \cdot g(\alpha)) = D(h(\alpha)) \cdot g(\alpha) + h(\alpha) \cdot D(g(\alpha))$. For example we have $a_1\alpha^m \cdot a_2\alpha^n = a_1 \cdot a_2 \cdot \alpha^{m+n}$, by our definition, apply D to the product, we get:

$$(a_1D(a_2) + a_2D(a_1))\alpha^{m+n} + a_1a_2(m+n)\alpha^{m+n-1}D(\alpha)$$

and is indeed equal to $D(a_1)\alpha^m + a_1 m\alpha^{m-1}D(\alpha) \cdot a_2\alpha^n + a_1\alpha^m(D(a_2)\alpha^n + a_2 n\alpha^{n-1}D(\alpha))$. Then since this definition is obvious linear ,suppose we now compute:

$$D(h(\alpha) \cdot g(\alpha)) = D(\sum \cdot) = \sum D(\cdots)$$

where the $\cdots$ represents the multiplication of all the terms separately suppose that $h(\alpha) = h_0(\alpha) + h_1(\alpha) + \cdots + h_n(\alpha)$ and $g(\alpha) = g_0(\alpha) + \cdots + g_m(\alpha)$, then the $\cdots$ are $h_i(\alpha)g_j(\alpha)$, but we know that $D(h_i(\alpha)g_j(\alpha)) = D(h_i(\alpha))g_j(\alpha) + h_i D(g_j(\alpha))$, we get:

$$\sum_{ij} D(h_i(\alpha))g_j(\alpha) + h_i D(g_j(\alpha)) = D(h(\alpha))g(\alpha) + h(\alpha)D(g(\alpha))$$

again by linearity.

Then we show this is well defined. If $h(\alpha) = g(\alpha)$ then we know that $h(\alpha) - g(\alpha) = f(\alpha) \cdot k(\alpha)$, then by Leibniz rule we know that $D(f(\alpha)) \cdot k(\alpha) + f(\alpha) \cdot D(k(\alpha))$, recall that $D(f(\alpha)) = 0$ by our definition of D on α moreover $f(\alpha) = 0$, therefore we know that this is well defined. Now we need to extend this to $K \to K$ and we can extend by the quotient rule where:

$$\frac{h}{g} \mapsto \frac{Dh \cdot g - hDg}{g^2}$$

this is obvious k-linear but we need to check this is well defined and satisfies Leibniz rule. The Leibniz rule is easy to check as this is just a formal property of this quotient definition. To check it's well defined, we know that if $\frac{h_1}{g_1} = \frac{h_2}{g_2}$, then since we are in a field, $h_1 g_2 = h_2 g_1$, and we need to prove:

$$\frac{Dh_1 \cdot g_1 - h_1 Dg_1}{g_1^2} = \frac{Dh_2 \cdot g_2 - h_2 Dg_2}{g_2^2}$$

We know that $Dh_1 \cdot g_2 + h_1 Dg_2 = Dh_2 \cdot g_1 + h_2 Dg_1$, multiply both sides by $g_1 g_2$, we get:

$$Dh_1 \cdot g_1 g_2^2 + h_2 g_1^2 Dg_2 = Dh_2 g_1^2 g_2 + h_1 g_2^2 Dg_1$$

Finally, to show this is the unique k-linear derivation $K \to K$ such that t_1 is mapped to 1 and t_i is mapped to 0. Suppose that D' is another such k-linear derivation, then we know that it's restriction on F is the same as the restriction of D to F because we have already argued before that t_1 to 1 and other t_i to 0 uniquely gives a k-linear derivation $F \to K$, then we know that $D'(f(\alpha)) = f'(\alpha)D(\alpha) + \sum \alpha^i D'(a_i)$ we know that $\sum \alpha^i D'(a_i) = \sum \alpha^i D(a_i)$ because $a_i \in F$, therefore $D'(\alpha) = D(\alpha)$, but this again uniquely determined the derivation on $F[\alpha]$ and K because we defined this completely based on Leibniz rule.

Now, what if $K/k(t_1, \cdots, t_n)$ is finitely generated. We know that we can decompose the extension to a tower, i.e. $F = k(t_1, \cdots, t_n) \hookrightarrow K$ is:

$$F = K_0 \subset K_1 \subset \cdots \subset K_n = K$$

where each K_i/K_{i-1} is generated by a single element and is finite separable. Then we know that we can do the unique extension for K_i/K_{i-1} each step and then finally get the derivation $K \to K$. To show that this is unique, suppose that there is another such extension, we know that we can restrict it to each K_i, and by the uniqueness, it agrees with the extension we started with, thus the final extension also agrees.

Finally, what if $K/k(x_1, \cdots, x_n)$ is not finitely generated, we know that in this case:

$$K = \bigcup_i K_i$$

where $i \in I$ I is the set of all the finitely generated subextensions over $k(t_1, \cdots, t_n)$. This gives a derivation on all of K, because if K_i and K_j are two such subextensions, we know that the derivation on $K_i \cap K_j$ is the same with that on $K_i \cap K_j$ we directly define using the above process due to uniqueness, we are done.

2.3 Third definition; global version

Now, we begin to go beyond the affine description and try to generalize it to any morphisms of schemes $\pi : X \to Y$. As what most of the texts do, we can try to define this affine by affine, but we will need to make sure that on overlaps, the module of Kahler differentials behave well. Due to the fact that we have a universal property, we could go about this approach, however, there are still a lot of stuff that you need to check. Instead doing this, we first give a definition that is not so intuitive in the beginning and then tell you how to think about it and how to work locally with that definition.

Now let $\pi : X \to Y$ be any morphism of schemes, recall that $\delta : X \to X \times_Y X$ is always a locally closed embedding, let's define the **relative cotangent sheaf** $\Omega_{X/Y}$ or Ω_π as the conormal sheaf $\mathscr{N}^\vee_{X/X\times_Y X}$ of the diagonal. If π is separated, we don't even need to think about why the locally closed embedding issue. This is also a good place to define the **relative tangent sheaf** $\mathscr{T}_{X/Y}$ as the dual of the relative cotangent sheaf, i.e. $\mathcal{H}om(\Omega_{X/Y}, \mathcal{O}_X)$. If we work with the category of k-schemes, then $\Omega_{X/k}$ and $\mathscr{T}_{X/k}$ are often called the **cotangent** and **tangent sheaf** of X.

Now we also need a derivation morphism $d : \mathcal{O}_X \to \Omega_{X/Y}$. To define this, suppose that the two projections $X \times_Y X \to X$ are pr_1 and pr_2. Then on $U \subset X$, we define the morphism by:

$$df = pr_2^* f - pr_1^* f$$

We need to be careful about this definition as we know that $\Omega_{X/Y}$ is actually the conormal sheaf therefore we need to first interpret $pr_2^* f - pr_1^* f$ as a section of the conormal sheaf. We know that $pr_2^* f$ and $pr_1^* f$ live on $X \times U$ and $U \times X$ respectively, therefore, let's first restrict it to $U \times U$. Therefore, this is first a section of the structure sheaf on $U \times U$. Let's assume for simplicity that δ is a closed embedding just to avoid distractions first. If we assume that $\mathcal{I}$ is the ideal sheaf defining the closed embedding and $pr_2^* f - pr_1^* f$ lies in the ideal sheaf, then we can consider the canonical morphism $\mathcal{I} \to \mathcal{I}/\mathcal{I}^2$ then it's image in the sheaf is what we are looking for. We only have to show that it is annihilated by the morphism δ which can be checked by looking the pullback by δ, recall that the inverse image of $U \times U$ under δ is U, therefore we only have to show:

$$\delta^* pr_2^* f - \delta^* pr_1^* f = 0$$

But recall that the composition $pr_i \circ \delta = id$ is the identity, therefore we are done. For the general case where we only have a locally closed embedding, have the following diagram:

$$\begin{tikzcd} U \arrow[r, hook, "closed"] \arrow[d, hook, "open"] & U' \arrow[r, "open"] \arrow[d, hook, "open"] & U \times U \arrow[d, hook, "open"] \\ X \arrow[r, hook, "closed"'] & X' \arrow[r, hook, "open"'] & X \times_Y X \end{tikzcd}$$

where the composition of the morphisms below is δ and X' is the union of $SpecA \times_Y SpecA$ for affine opens $SpecA \subset X$. All the squares are Cartesian, now we first further restrict $pr_2^* f - pr_1^* f$ to U' and we wish to show that it's annihilated by $U \hookrightarrow U$. But recall that if $pr_2^* f - pr_1^* f$ is annihilated by $U \to U$, then it's the restriction of to U' is certainly annihilated by $U \hookrightarrow U'$, therefore we still only have to show that $pr_2^* f - pr_1^* f$ is annihilated by δ, we are done.

This is indeed a morphism of sheaves of abelian groups as if you consider first restrict f to V and then apply d, you get $pr_2^*(f|_V) - pr_1^*(f|_V)$ which is interpreted to be an element in $V \times V$, but on the other hand, if you restrict $pr_2^* f - pr_1^* f$ to $V \times V$ it's also the same as first restrict f to V and then pullback then take intersection because we have the following diagram:

$$\begin{tikzcd} U \times U \arrow[r, hook] & U \times X \arrow[r, "pr_1"] & U \\ V \times V \arrow[r, hook] \arrow[u, hook] & V \times X \arrow[r, "pr_1"] \arrow[u, hook] & V \arrow[u, hook] \end{tikzcd}$$

Just as the affine derivation, this morphism is usually not a morphism of $\mathcal{O}_X$-modules, but is a morphism of $\mathcal{O}_Y$-modules, satisfying a few properties. This is indeed true and we will discuss these shortly and you will find this agrees with our affine definition when we work locally, thus globalizing the affine version we had before. This map gives us a map on each U:

$$\Gamma(U, \mathcal{O}_X) \to \Gamma(U, \Omega_{X/Y})$$

which is usually referred to as taking derivatives.

Remark 2.29. Before we connect this to our other definitions and show this indeed globalize the affine version, let's spend a little bit of time to discuss why this definition is a reasonable thing to do.

Say for example X is a manifold and Y is a point, then recall that the tangent bundle $T_{X\times X}$ on X is $pr_1^* T_X \oplus pr_2^* T_X$ where pr_1 and pr_2 are the projections. There is a normal bundle exact sequence in differential topology which is:

$$0 \to T_\Delta \to T_{X\times X}|_\Delta \to N_{\Delta/X} \to 0$$

which is very intuitive as the tangent space of $X \times X$ along Δ that don't come from the tangent space on Δ is going to be the normal vectors. The morphism on the left at each point is $v \mapsto (v, v)$ so you can think of the cokernel as $(v, -v)$ therefore the normal bundle is isomorphic to the tangent bundle of Δ, we dualize it to obtain the corresponding conormal sheaf is isomorphic to the cotangent sheaf. Therefore in algebraic geometry, since we know how to define the conormal sheaf first, can turn the process above head on and use the conormal sheaf to define the cotangent sheaf.

Now let's try to test this out in the affine case:

Example 2.30. Consider the morphism $SpecA \to SpecB$ corresponding to a ring morphism. Then consider the diagonal, which is $\delta : SpecA \hookrightarrow Spec(A \otimes_B A)$ correspond to the ideal I of $A \otimes_B A$, which is the kernel of the map:

$$x \otimes y \mapsto xy$$

The first observation we have is that the ideal I is generated by elements of the form $1 \otimes a - a \otimes 1$. It's obvious that element of this form is in the kernel, and suppose that:

$$\sum x_i \otimes y_i \mapsto \sum x_i y_i = 0$$

Then we know that:

$$\sum x_i \otimes y_i = \sum x_i \otimes y_i - x_i y_i \otimes 1 = \sum (x_i \otimes 1)(1 \otimes y_i - y_i \otimes 1)$$

Then the derivation is given by:

$$d : A \to A \otimes_B A \quad a \mapsto (1 \otimes a - a \otimes 1)$$

which is indeed an element of the ideal I, and we then take the quotient by I^2 to get the corresponding element in I/I^2. Let check d is indeed a derivation (to be precise, it's not appropriate to call it d since d is used as the symbol for the derivation to $\Omega_{A/B}$, but we will prove it is indeed d later). Clearly it's B-linear since if it's clearly linear and we know that $b \cdot (x \otimes y) = bx \otimes y = x \otimes by$. Then we only need to check the Leibniz rule, i.e. we need to prove $d(aa') - a'da - ada' \in I^2$:

$$d(aa') - ada' - a'da = 1 \otimes aa' - aa' \otimes 1 - a \otimes a' + aa' \otimes 1 - a' \otimes a + aa' \otimes 1$$

$$= 1 \otimes aa' + aa' \otimes 1 - a \otimes a' - a' \otimes a$$

$$= (1 \otimes a - a \otimes 1)(1 \otimes a' - a' \otimes 1) \in I^2$$

This proves that d is a derivation, therefore by the universal property of $\Omega_{A/B}$, we get a A-linear morphism $\Omega_{A/B} \to I/I^2$:

Theorem 2.31. The natural morphism $f : \Omega_{A/B} \to I/I^2$ we constructed earlier is an isomorphism, thus I/I^2 is indeed the module of Kahller differentials.

Proof. We will first show that f is surjective, and then we will prove there is a map of A-module $g : I/I^2 \to \Omega_{A/B}$ such that $g \circ f = id$, which proves that f is injective, finishing the proof.

Recall that f sends da to $1 \otimes a - a \otimes 1$, and we know that I is generated by these elements therefore it's surjective. Then we will construct g. Let's first consider a map of B-modules:

$$A \otimes_B A \to \Omega_{A/B} \qquad x \otimes y \mapsto xdy$$

which is defined by the universal property of tensors, then we consider the restriction of this map to I, if I^2 is mapped to 0, then we get a well defined map of B-modules $I/I^2 \mapsto \Omega_{A/B}$. We need to prove that:

$$(\sum x_i \otimes y_i)(\sum x'_j \otimes y'_j) = \sum_{i,j} x_i x'_j \otimes y_i y'_j \mapsto 0$$

where $\sum x_i y_i = \sum x'_j y'_j = 0$. By definition, we know that the image is:

$$\sum_{ij} x_i x'_j d(y_i y'_j) = \sum_{ij} x_i x'_j y_i dy'_j + \sum_{ij} x_i x'_j y'_j dy_i = \sum_j \sum_i x_i y_i (x'_j du'_j) + \sum_i \sum_j x'_j y'_j (x_i dy_i) = 0$$

Then we need to prove that the composition of f with g is id. $da \mapsto (1 \otimes a - a \otimes 1) \mapsto 1da - ad1 = da$ since $d1 = 0$. □

Proposition 2.32. Suppose that $\pi : X \to Y$ is a morphism of schemes with open subschemes $SpecA \subset X$ and $SpecB \subset Y$ such that $SpecA \subset \pi^{-1}SpecB$. Then we can identity $\Omega_{X/Y}|_{SpecA}$ with the associated sheaf of $\Omega_{A/B}$ thus the idea of describing the cotangent sheaf locally naturally glues together to $\Omega_{X/Y}$.

Proof. The key argument here is to observe that we have a Cartesian diagram:

$$\begin{tikzcd}
SpecA \arrow[r, hook, "closed"] \arrow[d, hook, "open"'] & SpecA \times_B SpecA \arrow[r] \arrow[d, hook, "open"'] & SpecA \times_B SpecA \arrow[d] \\
X \arrow[r, hook, "closed"'] & X' \arrow[r, hook, "open"'] & X \times_Y X
\end{tikzcd}$$

This implies that the restriction of the conormal sheaf to $SpecA$ is first restrict to $SpecA \times_B SpecA$ and the restrict to $SpecA$, and if you recall how we construct this map $X \to X'$, on $SpecA \times_B SpecA$, it's of the form $A \otimes_B A \to A$ and once you restrict to affine, the quotient of quasicoherent sheaves are just module quotients, therefore it's just I/I^2 together with the derivation $A \to I/I^2$ defined by $a \mapsto 1 \otimes a - a \otimes 1$, but we have already seen that this is just $\Omega_{A/B}$ together with $d : A \to \Omega_{A/B}$, we are done. □

With the above result, we can now understand how $\Omega_{X/Y}$ behave locally, generalizing our previous affine local statement to the global case:

Proposition 2.33. Suppose that $U \subset X$ is an open, then the map induced by the derivation $\mathcal{O}_X \to \Omega_{X/Y}$ on U is also a derivation.

Proof. We need to think a little bit on what does this mean, as a "derivation" should be a some-linear derivation. Recall that a morphism of sheaves give you a morphism of sheaves of rings $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$. This is equivalent to a map $\pi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ by the adjointness property, then I claim that this derivation $\mathcal{O}_X \to \Omega_{X/Y}$ is $\pi^{-1}\mathcal{O}_Y$-linear via the morphism $\pi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. But in practice, what we will use is to think that on $U \subset X$, suppose that V is any open on Y such that U maps to V, then the morphism is $\mathcal{O}_Y(V)$-linear and I will let you think about why this implies $\pi^{-1}\mathcal{O}_Y$-linear.

To prove the linearity and Leibniz rule, it's actually quite hard to proceed directly without the help of our local understanding. For example, if you want to prove linearity for $g \in \mathcal{O}_Y(V)$ and $f \in \mathcal{O}_X(U)$ where U maps to V:

$$d(gf) = pr_2^*(gf) - pr_1^*(gf)$$

but you actually don't know what to do right now. So instead proceed directly with this definition, let's do it locally. For the linearity, let's first cover U with affine opens A such that it maps to affine opens in V.

Then we first restrict the map on U to $SpecA$, and here we see that the morphism is $A \otimes_B A \to A$ and the derivation is defined by:

$$f \mapsto 1 \otimes f - f \otimes 1$$

We know that it's B-linear, thus $\mathcal{O}_Y(V)$-linear and this proves that all the restrictions of the derivation on U is $\mathcal{O}_Y(V)$-linear, then the sheaf property tells you that the map on U is also $\mathcal{O}_Y(V)$-linear.

Then we prove Leibniz rule, which means we need to compute $d(f_1 f_2)$ and by definition it's:

$$pr_2^*(f_1 f_2) - pr_1^*(f_1 f_2)$$

Let's first restrict to $SpecA$, it becomes $1 \otimes f_1 f_2 - f_1 f_2 \otimes 1$, we already see that this is equal to:

$$f_1(1 \otimes f_2 - f_2 \otimes 1) + f_2(1 \otimes f_1 - f_1 \otimes 1) = (f_1 \otimes f_2 - f_1 f_2 \otimes 1) + (f_2 \otimes f_1 - f_1 f_2 \otimes 1)/I^2$$

But notice that this is the restriction of:

$$f_1(pr_2^* f_2 - pr_1^* f_2) + f_2(pr_2^* f_1 - pr_1^* f_1)$$

Here we need to be a little careful on how to this of multiplying f_1 and f_2 as technically speaking $pr_2^* f_2 - pr_1^* f_2$ and $pr_2^* f_1 - pr_1^* f_1$ are on $U \times U$. To avoid confusion, we mean that we first pullback $pr_2^* f_1 - pr_1^* f_1$ and $pr_2^* f_2 - pr_1^* f_2$ to U and then multiply by f_1. I will leave the details for you to figure out as there are indeed identification you need to think over. □

Proposition 2.34. Suppose that $\pi : X \to Y$ is locally of finite type and Y is locally Noetherian, then $\Omega_{X/Y}$ is coherent on X.

Proof. To prove $\Omega_{X/Y}$ is coherent on X, recall that we first need to prove it's a quasicoherent sheaf. We need to find an affine cover on X such that the restriction on each affine open is quasicoherent. But the above proof shows that we can first find an affine cover of Y and cover each inverse image by affine opens and on each affine open, the restriction is the associated sheaf of the module of Kahler differentials. Then to check coherence, suppose that $SpecA \to SpecB$ is a pair of opens of X and of Y, since Y is assumed to be locally Noetherian, we can assume B is a Noetherian ring and since we assume π is finite type, we can assume that A is a finitely generated B-algebra, therefore we know that A is Noetherian and $\Omega_{A/B}$ is finitely generated over A, therefore finitely presented and finite type and coherent since A is Noetherian. □

Proposition 2.35. Suppose that $\pi : X \to Y$ is smooth of relative dimension n then $\Omega_{X/Y}$ is locally free of rank n.

Proof. Let's first prove that if we have consider the canonical morphism:

$$W \hookrightarrow SpecA = SpecB[x_1, \cdots, x_{n+r}]/(f_1, \cdots, f_r) \to SpecB$$

where W is an open subscheme of $SpecB[x_1, \cdots, x_{n+r}]/(f_1, \cdots, f_r)$ such that the Jacobian matrix of f_i's with respect to the first r x_j's is an invertible function on W.

Now recall that the sheaf of differentials $\Omega_{A/B}$ is described to be the associated module of the cokernel of the Jacobian matrix, then suppose that we pick $W = SpecA_f$ to be a distinguished open, then we know that the relative differential of $\Omega_{W/Specb}$ can be computed by the following diagram:

$$\begin{tikzcd} A_f & A \arrow[l] \\ B \arrow[u] & B \arrow[l] \arrow[u] \end{tikzcd}$$

and a previous proposition shows that $\Omega_{A_f/B} = (\Omega_{A/B})_f$, therefore we know that the restriction on $\Omega_{A/B}$ to $SpecA_f$ is the relative differential. Then we know that the stalk of the relative differential on $SpecA_f$ can be computed by that on $SpecA$. Recall that the relative differential is computed by the cokernel of the whole Jacobian matrix. Therefore if we denoted J as the Jacobian matrix then the module of relative differentials is the cokernel of:

$$A^{\oplus r} \xrightarrow{J} A^{\oplus n+r}$$

Recall that our assumption implies that the $r \times r$-minor on the left top is invertible on W, then we know that is you localize at a prime p in W, you get a morphism $A_p^{\oplus r} \to A_p^{\oplus n+r}$, I claim that the cokernel is also free, i.e. is $A_p^{\oplus n}$. This is exactly like what happens in the field theory and matrix with elements coming from a field. This is exactly a linear algebra question and I will let you do this, but recall that for finitely presented modules, stalk being free is equivalent to locally free around the stalk, and we are done.

I will let you figure out how to reduce from smooth of relative dimension n to this situation, which is essentially by unpacking the definition. $\square$

Now we try to generalize the cotangent exact sequence and conormal exact sequence, let's first do the cotangent exact sequence;

Theorem 2.36. Suppose $X \xrightarrow{\pi} Y \xrightarrow{\rho} Z$ are morphisms of schemes, then there is an exact sequence of quasicoherent sheaves on X:

$$\pi^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

Suppose that $i : X \hookrightarrow Y$ is a closed embedding of schemes with conormal sheaf $\mathscr{N}_{X/Y}^\vee$ and $\rho : Y \to Z$ is a morphism of schemes, then there is an exact sequence of sheaves on X:

$$\mathscr{N}_{X/Y}^\vee \xrightarrow{\delta} i^*\Omega_{Y/Z} \to \Omega_{X/Y}$$

Proof. You should think of these two sequences are the global version of the two exact sequences on affine schemes.

For the first cotangent exact sequence, we first need to construct this sequence of morphisms. For example, let's first do this for $\Omega_{X/Z} \to \Omega_{X/Y}$. What we know ring now is only how this maps works on affine opens, therefore let's first choose an affine cover of Z, say $SpecC_i$ and we cover each inverse image in Y by affine, get another affine cover $SpecB_j$, finally we cover the inverse image in X again to get an affine cover $SpecA_k$. For simplicity, let's just use the notation $SpecA \to SpecB \to SpecC$, and on $SpecA$ we know that we need to define a morphism for:

$$\Omega_{A/C} \to \Omega_{A/B}$$

and we define this just by $da \mapsto da$. Now the question becomes do all of these definitions compatible on the intersection. We know that we can cover the intersection by distinguished opens of both larger affines. Therefore, our strategy becomes: for each distinguished open of $SpecA$ we first restrict $\Omega_{X/Y}$ and $\Omega_{X/Z}$ to that distinguished open, we construct a new map and show that it equal to the restriction of that map we constructed on $SpecA$.

We know that by our previous computation that on $SpecA_f$, the restrictions become $(\Omega_{A/C})_f$ and $(\Omega_{A/B})_f$, we know that the restriction of our morphism above is defined by $\frac{da}{f^n} \mapsto \frac{da}{f^n}$. But on the other hand, notice that this distinguished open also maps to $SpecB$ and $SpecC$, then we can define:

$$\Omega_{A_f/C} \to \Omega_{A_f/B}$$

where $d\frac{a}{f^n} \mapsto d\frac{a}{f^n}$, consider the diagram:

$$\begin{tikzcd}
\Omega_{A/C} \arrow[r] \arrow[d] & \Omega_{A/B} \arrow[d] \\
(\Omega_{A/C})_f \arrow[r] \arrow[d, "\simeq"'] & (\Omega_{A/B})_f \arrow[d, "\simeq"'] \\
\Omega_{A_f/C} \arrow[r] & \Omega_{A_f/B}
\end{tikzcd}$$

recall the isomorphism is given by $\frac{da}{f^n} \mapsto \frac{1}{f^n} d\frac{a}{1}$ then I will let you show that the bottom square is commutative. Is this enough to show the morphism is well defined? Well suppose that $SpecA'$ is another such affine opens in X, it maps to $SpecB'$ and $SpecC'$, then we need to consider the following issue, we can define the morphism on $SpecA'$ by $\Omega_{A'/C'} \to \Omega_{A'/C'}$. On the intersection with $SpecA$, say the distinguished open is $SpecA_f$ and $SpecA_g$, we need to prove that the following diagram is commutative:

$$\begin{tikzcd}
\Omega_{A_f/C} \arrow[r] \arrow[d, dashed, "\simeq"] & \Omega_{A_f/B} \arrow[d, dashed, "\simeq"] \\
\Omega_{A'_g/C'} \arrow[r] & \Omega_{A'_g/B'}
\end{tikzcd}$$

where the vertical isomorphism exists because they should be the restriction of the sheaf of relative differentials to that open subset so the question becomes what are these isomorphisms. This goes to the following statement, suppose that A are C and C'-algebras, and the ring morphisms $C \to A$ and $C' \to A$ factors through a ring B that is both the localization of C and localization of C', then $\Omega_{A/C} \simeq \Omega_{A/C'}$ via $da \mapsto da$. To see why, consider the following diagram:

$$\begin{tikzcd}
C \arrow[dr] & & \\
& B \arrow[r] & A \\
C' \arrow[ur] & &
\end{tikzcd}$$

then we know from the cotangent SES that and the fact that $\Omega_{B/C}$ and $\Omega_{B/C'}$ are 0, we get a canonical isomorphisms:

$$\Omega_{A/C} \overset{da \mapsto da}{\to} \Omega_{A/B} \overset{da \mapsto da}{\leftarrow} \Omega_{A/C'}$$

Now back to our question, we know that the distinguished open in the intersection of $SpecA$ and $SpecA'$ can be taken to be the inverse image of some distinguished open in the intersection of $SpecC$ and $SpecC'$(do you see why), therefore we are now in the above situation let's don't make the notation and argument too complicated (even though it's not hard in essence) so we won't write out all the details, but such a simple description where we know the isomorphism is given by $da \mapsto da$ makes it possible to prove the following diagram is commutative:

$$\begin{tikzcd}
& & \Omega_{A_f/B} \arrow[d] \arrow[dr] & \\
\Omega_{A_f/C} \arrow[urr] \arrow[d, "\simeq"] \arrow[dr, "\simeq"] & & \Omega_{A'_g/B'} \arrow[dr] & \Omega_{D/B} \arrow[d] \\
\Omega_{A'_g/C'} \arrow[urr] \arrow[dr, "\simeq"] & \Omega_{D/C} \arrow[urr] \arrow[d, "\simeq"] & & \Omega_{D/B'} \\
& \Omega_{D/C'} \arrow[urr] & &
\end{tikzcd}$$

that is all we need to prove the morphism is well defined.

Then we need to define the first morphism, we again do the same covering argument, and we know that on $SpecA$, the morphism should correspond to a morphism of A-modules $A \otimes_B \Omega_{B/C} \to \Omega_{A/C}$ and we know it's defined by $a \otimes db \mapsto a \cdot db$. The question is still about is this well defined or not. We still want to check on the distinguished opens we know that on a distinguished open, it's the same as the following morphism:

$$A_f \otimes_B \Omega_{B/C} \xrightarrow{\frac{a}{f^n} \otimes db \mapsto \frac{a}{f^n} \cdot d\frac{b}{1}} \Omega_{A_f/C}$$

It's also isomorphic to the following morphism:

$$A_f \otimes_{B_g} \Omega_{B_g/C} \xrightarrow{\frac{a}{f^n} d\frac{b}{g^m} \mapsto \frac{a}{f^n} \cdot d\frac{b}{g^m}} \Omega_{A_f/C}$$

Do you see what this means (for example what are the g and h) and can you see why this is true(think about localizations of module of differentials)?

Then following the same idea as the above discussion, we will need to prove the following diagram is commutative:

$$\begin{tikzcd}
 & & \Omega_{A_f/C} \arrow[d, dashed, "\simeq"] \arrow[dr] & \\
A_f \otimes_{B_g} \Omega_{B_g/C} \arrow[urr] \arrow[dr] \arrow[d, "\simeq"'] & & \Omega_{A'_{f'}/C'} \arrow[dr] & \Omega_{D/C} \arrow[d] \\
A'_{f'} \otimes_{B'_{g'}} \Omega_{B'_{g'}/C'} \arrow[urr] \arrow[dr] & D \otimes_E \Omega_{E/C} \arrow[urr] \arrow[d] & & \Omega_{D/C'} \\
 & D \otimes_E \Omega_{E/C'} \arrow[urr] & &
\end{tikzcd}$$

Again, it's easy to prove that besides the face we want it to be commutative, every face is commutative, this fact together with some isomorphisms, makes the face we are interested commutative, and we are done. In conclusion, the morphisms are well defined and on affines, are exactly what you expect them to be.

The makes it easier to check exactness, as exactness can be checked affine locally, but the affine version is exact by our previous results.

Then we deal with the conormal exact sequence, where we have already defined the map $i^*\Omega_{Y/Z} \to \Omega_{X/Y}$, therefore we only need to define the morphism $\delta : \mathcal{N}^\vee_{X/Y} \to i^*\Omega_{Y/Z}$. We again want to define this by working with affines as we don't know other ways to describe the sheaf of differentials. We again cover Z using affines $SpecC$ and then cover the preimages with $SpecB$, but what is different here is that the closed embedding is affine and we know that the inverse image of B is $SpecA$, we know that on each of these $SpecA$, this morphism is given by:

$$I/I^2 \to A \otimes_B \Omega_{B/C}$$

We still have to check that on distinguished opens, this definitions is compatible. And on $SpecA_f$ for example, we just get:

$$I_f/I_f^2 \to A_f \otimes_B \Omega_{B_f/C}$$

for simplicity of the following argument, we write it in an isomorphic form:

$$I_f/I_f^2 \to A_f \otimes_{B_f} \Omega_{B_f/C}$$

where $\frac{i}{f^n} \mapsto \frac{1}{f^n} \cdot d\frac{i}{1}$. Then again, this time we need to check that the following diagram:

$$\begin{tikzcd}
 & A_f \otimes_{B_f} \Omega_{B_f/C} \arrow[d, dashed, "\simeq"] \arrow[dr] & \\
I_f/I_f^2 \arrow[ur] \arrow[dr] \arrow[d, "\simeq"'] & A'_g \otimes_{B'_g} \Omega_{B'_g/C'} \arrow[dr] & D \otimes_E \Omega_{E/C} \arrow[d] \\
I'_g/I'^2_g \arrow[ur, "\simeq"] \arrow[dr, "\simeq"] & J/J^2 \arrow[ur] \arrow[d, dashed, "id"] & D \otimes_E \Omega_{E/C'} \\
 & J/J^2 \arrow[ur] &
\end{tikzcd}$$

where J is the kernel of the surjection $E \to D$, and the fact that the most left face commutes is because we know that the morphism schemes gives us the following diagram:

$$\begin{tikzcd}
B'_g \arrow[d, two heads] & D \arrow[l, dashed, "\simeq"'] \arrow[r, dashed, "\simeq"] \arrow[d, two heads] & B_f \arrow[d, two heads] \\
A'_g & E \arrow[l, dashed, "\simeq"'] \arrow[r, dashed, "\simeq"] & A_f
\end{tikzcd}$$

where I_f, J, I_g are the obvious kernels, I will let you fill out the rest details. Again, once the morphism is established, we know that exactness can be check on affine opens. $\square$

The above two exact sequence should be expected to be left exact as well if we have enough smoothness condition. We will see quite a lot of examples along this line later, here is the first example:

Proposition 2.37. The cotangent exact sequence is exact on the left if π and ρ are both smooth.

Proof. It's not hard to prove this directly. Suppose that π and ρ are smooth of relative dimension m, n respectively, if we want left exactness, we need to check that $\pi^*\Omega_{Y/Z} \to \Omega_{X/Z}$ is injective. We can check this on affine opens and this is given by:

$$A \otimes_B \Omega_{B/C} \to \Omega_{A/C}$$

By our assumption on the smoothness of ρ and π, we can pick $SpecB$ to be in the following diagram:

$$SpecB \hookrightarrow SpecC[x_1, \cdots, x_{n+r}]/(f_1, \cdots, f_r) \to SpecC$$

and we know that the first $r \times r$ submatrix of the Jacobian J_1 has invertible determinant on $SpecB$, then we know that we can identify $A \otimes_B \Omega_{B/C}$ as $A \otimes_B \mathrm{coker} J_1$.

We also prove before that the composition of π and ρ is again smooth of relative dimension $m + n$. Let's do it again here just as a reminder and it also helps to prove this result. Recall that smoothness is local both on the source and target, therefore, we can assume that we can assume that the composition take place between affines:

$$SpecA \to SpecB \to SpecC$$

We assume $SpecA \to SpecB$ is of the form:

$$\begin{tikzcd}
SpecA \arrow[r, hook] \arrow[dr] & SpecB[x_1, \cdots, x_n]/(f_1, \cdots, f_r) \arrow[d] \\
& SpecB
\end{tikzcd}$$

where $SpecA \hookrightarrow SpecB[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$ is an open embedding. We can assume that $SpecA$ is an distinguished open of $SpecB[x_1, \cdots, f_{n+r}]/(f_1, \cdots, f_r)$, i.e. it's $SpecB[x_1, \cdots, f_{n+r}]/(f_1, \cdots, f_r)_s$ for some polynomials s, then we take the product of all the coefficients t in the polynomial, then consider:

$$\begin{tikzcd}
Spec(B_t[x_1, \cdots, f_{n+r}]/(f_1, \cdots, f_r))_s \arrow[r, hook] \arrow[d, hook] & SpecB[x_1, \cdots, f_{n+r}]/(f_1, \cdots, f_r)_s \arrow[d, hook] \\
SpecB_t[x_1, \cdots, f_{n+r}]/(f_1, \cdots, f_r) \arrow[r, hook] \arrow[d] & SpecB[x_1, \cdots, f_{n+r}]/(f_1, \cdots, f_r) \arrow[d] \\
SpecB_t \arrow[r, hook] & SpecB
\end{tikzcd}$$

Therefore we can actually assume that in the definition of the smoothness, don't use the open condition, but just the isomorphism condition, then the rest of the proof is clear, you have the following diagram:

$$\begin{tikzcd}
SpecA \arrow[r, "="] \arrow[dr] & SpecB[x_1, \cdots, x_{n+r}]/(f_1, \cdots, f_r) \arrow[r, dashed, "="] \arrow[d] & SpecC[x_1, \cdots, y_{m+s}]/(f_1, \cdots, g_s) \arrow[d] \\
& SpecB \arrow[r, "="] \arrow[dr] & SpecC[y_1, \cdots, y_{m+s}]/(g_1, \cdots, g_s) \arrow[d] \\
& & SpecC
\end{tikzcd}$$

We arrange the variables in $SpecC[x_1, \cdots, y_{m+s}]/(f_1, \cdots, g_s)$ such that the first $r+s$ are $x_1, \cdots, x_r, y_1, \cdots, y_s$, then we know that in this case the first $r + s \times r + s$ submatrix is a block upper triangular matrix, thus the determinant is the product of two invertible functions, thus invertible on A as well.

Now back to our question, assume that A, B, C showing up in $A \otimes_B \Omega_{B/C} \to \Omega_{A/C}$ is as in the above discussion. Then let's assume that the Jacobian matrix of $SpecB \to SpecC$ is given by J_1 which is a $(m+s) \times s$ matrix and we know that we have an exact sequence:

$$B^{\oplus s} \xrightarrow{J_1} B^{\oplus n+s} \to \Omega_{B/C} \to 0$$

tensor this with A we get:

$$A^{\oplus s} \xrightarrow{J_1} A^{\oplus n+s} \to A \otimes_B \Omega_{B/C}$$

where in this time we view the elements of J_1 as their image in A. Similarly we have:

$$A^{\oplus r+s} \xrightarrow{J_2} A^{\oplus m+n+r+s} \to \Omega_{A/C}$$

The reason why we need smoothness is we can assume that, for example the top $s \times s$ submatrix of J_1 along the diagonal is invertible, say with inverse P. Moreover, we can assume that the top $(r+s) \times (r+s)$ submatrix of J_2 is invertible, with inverse $R = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$.

Then we have the following two sequences of isomorphisms:

$$\begin{tikzcd}
A^{\oplus r+s} \arrow[r, "J_2"] & A^{\oplus r+s+m+n} \arrow[r] & \Omega_{A/C} \arrow[r] & 0 \\
A^{\oplus r+s} \arrow[u, "R"] \arrow[r, "J_2 \circ R"] & A^{\oplus r+s+m+n} \arrow[u, "id"] \arrow[r] & \text{Coker} = A^{\oplus m+n} \arrow[u, "\simeq"] \arrow[r] & 0 \\
A^{\oplus s} \arrow[d, "P"] \arrow[r, "J_1 \circ P"] & A^{\oplus s+m} \arrow[d, "id"] \arrow[r] & \text{Coker} = A^{\oplus m} \arrow[u, hook] \arrow[d, "\simeq"] \arrow[r] & 0 \\
A^{\oplus s} \arrow[r, "J_1"] & A^{\oplus s+r} \arrow[r] & A \otimes_B \Omega_{B/C} \arrow[r] & 0
\end{tikzcd}$$

the two isomorphisms between the cokernel and the module of differentials can be described as making $\Omega_{A/C}$ as a free A-module, generated by dx_i, dy_j for $i > r$ and $j > s$, similarly describes $A \otimes_B \Omega_{B/C}$ as a free A-module generated by dy_j for $j > s$. Then you can check that the vertical composition of the last column is exactly the desired map $A \otimes_B \Omega_{B/C} \to \Omega_{A/C}$, therefore we know that it's injective as it's the composition of an isomorphism with something injective and again with an isomorphism. $\square$

Then, let's globalize the pullback morphism of relative differential sheaves:

Proposition 2.38. If the following diagram is a commutative diagram of schemes, we have a natural morphism:

$$\mu^* \Omega_{X/Y} \to \Omega_{X'/Y'}$$

If the diagram is Cartesian, the morphism is an isomorphism.

$$\begin{tikzcd}
X' \arrow[r, "\mu"] \arrow[d] & X \arrow[d] \\
Y' \arrow[r] & Y
\end{tikzcd}$$

Proof. We again need to first define this diagram, what we do is again cover Y by $SpecB$ and cover the inverse images in X and Y' by $SpecA$ and $SpecB'$ respectively, finally in X', we cover X' by affine opens $SpecA'$ in the intersections of the inverse image in X and Y'.

Now on each $SpecA'$, we define it by $A' \otimes_A \Omega_{A/B} \to \Omega_{A'/B'}$, just as what we did for the affine case, the key question is why this definition glues well. Recall how we defined this morphism in the affine case. It's defined by two universal properties, first it's defined by that of extension of scalars, which reduces to an A-linear map from $\Omega_{A/B} \to \Omega_{A'/B'}$, and this is defined by the universal property of $\Omega_{A/B}$ and the A-linear derivation $A \to A' \to \Omega_{A'/B'}$. Now let's think about what is the restriction on a distinguished open which is the inverse image of some distinguished open in $SpecB$ thus in $SpecA$, $SpecB'$, it's the map:

$$A'_f \otimes_{A_f} \Omega_{A_f/B_f} \to \Omega_{A'_f/B'_f}$$

defined by universal properties, which is first an A_f-linear morphism $\Omega_{A_f/B} \to \Omega_{A'_f/B'}$, which is then given by $A_f \to A'_f \to \Omega_{A'_f/B'}$.

Now if you start with something other rings you get:

$$P' \otimes_P \Omega_{P/Q} \to \Omega_{P'/Q'}$$

and the restriction that common distinguished affine open is:

$$P'_g \otimes_{P_g} \Omega_{P_g/Q_g} \to \Omega_{P'_g/Q'_g}$$

which is given by $P_g \to P'_g \to \Omega_{P'_g/Q'}$. Then recall that we have discussed how $\Omega_{A_f/B}, \Omega_{P_g/Q}$ and how $\Omega_{A'_f/B}, \Omega_{P'_g/Q'}$ are related in the sense that there is an explicit canonical isomorphism make each pair isomorphic. Therefore, we know that the above two morphism can be identified with:

$$D \otimes_E \Omega_{E/F} \to \Omega_{D/G}$$

where the morphism is given by universal properties, thus they agree.

After we constructed this morphism, the assertion that the morphism is an isomorphism when the diagram is Cartesian is just the affine version result. ☐

Remark 2.39. Moreover, a particular example of the above result is to consider fibers, which shows that the fiber of the sheaf of differentials is the sheaf of differentials on the fiber, therefore we know that the sheaf of differentials indeed glues well on fibers.

Proposition 2.40. Suppose that $\alpha : X \to Z$ and $\beta : Y \to Z$ are two morphisms, consider the fibered product and the induced morphisms $\beta' : X \times_Z Y \to X$ $\alpha' : X \times_Z Y \to Y$, there is an isomorphism:

$$\Omega_{X\times_Z Y/Z} \simeq (\beta')^*\Omega_{X/Z} \oplus (\alpha')^*\Omega_{Y/Z} = \Omega_{X/Z} \boxplus \Omega_{Y/Z}$$

Proof.

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\alpha'"] \arrow[d, "\beta'"'] & Y \arrow[d, "\beta"] \\ X \arrow[r, "\alpha"'] & Z \end{tikzcd}$$

This proposition doesn't follow from the previous theorem and we indeed need to do some constructions here. The affine version of the story is to consider:

$$\begin{tikzcd} SpecR = SpecA \otimes_C B \arrow[r] \arrow[d] & SpecB \arrow[d] \\ SpecA \arrow[r] & SpecC \end{tikzcd}$$

and we want to describe an isomorphism:

$$\Omega_{R/C} \simeq (\Omega_{A/C} \otimes_A R) \oplus (\Omega_{B/C} \otimes_B R)$$

By the universal property of $\Omega_{R/C}$, this is defined to be an C-linear derivation:

$$R \to (\Omega_{A/C} \otimes_A R) \oplus (\Omega_{B/C} \otimes_B R)$$

To define this, it's first of all an C-linear morphism, then we know from the universal property of tensor, we define it to be:

$$a \otimes b \mapsto (da \otimes (1 \otimes b)), db \otimes (a \otimes 1))$$

We need to show that this is indeed a derivation, i.e. this definition satisfies the Leibniz property and we only need to check:

$$(a_1a_2 \otimes b_1b_2) \mapsto (a_1da_2 + a_2da_1 \otimes (1 \otimes b_1b_2), b_1db_2 + b_2db_1 \otimes a_1a_2 \otimes 1)$$

On the other hand, we can compute:

$$a_1 \otimes b_1(da_2 \otimes (1 \otimes b_2), db_2 \otimes (a_2 \otimes 1)) + a_2 \otimes b_2(da_1 \otimes (1 \otimes b_1), db_1 \otimes (a_1 \otimes 1))$$

and you can check that they are equal, therefore we are done, and the morphism is defined by:

$$d(a \otimes b) \mapsto (da \otimes (1 \otimes b), db \otimes (a \otimes 1))$$

It's not obvious that this is an isomorphism, so we do this by construct an inverse. By the universal property of direct sums, let's first do this for $\Omega_{A/C} \otimes_A R \to \Omega_{R/C}$. This correspond to an A-linear map $\Omega_{A/C} \to \Omega_{R/C}$, which is an C-linear derivation given by $A \to R \to \Omega_{R/C}$. Then we know that this linear map is given by:

$$(1 \otimes 1) \otimes da \mapsto da$$

Then you do the same for $\Omega_{B/C} \otimes_B R \to \Omega_{R/C}$, then check these two constructions are indeed inverses of each other.

Then our old friend comes here forcing use to consider the question of whether this glues well or not. Recall that since the two maps we constructed in the affine case is an isomorphism, then no matter you show which one glues well, we are done. Using the property of direct sum, we want to show that the map:

$$\Omega_{A/C} \otimes_A R \to \Omega_{R/C}$$

glues well, notice that this is not quite the map we saw in the previous proposition as the base here is C not B. But this is exactly the left-most map we have in the cotangent exact sequence, thus we are done. $\square$

2.4 Smoothness of varieties revisited

Finally, we will be discussing an important topic, which is to reinterpret the smoothness of varieties using the sheaf of differentials. Suppose that k is a field, we have long before defined the smoothness of k-varieties using some awkward definition involving affine covers and the corank of Jacobian matrix. Here, we are about to make our definition more robust using the sheaf of differentials.

Definition 2.41. A k-scheme X is said to be k-smooth of dimension n or smooth of dimension n over k if it's locally of finite type of pure dimension n and the sheaf $\Omega_{X/k}$ is locally free of rank n. Sometimes, we omit the dimension n.

Proposition 2.42. This new definition agrees with our previous definition on smooth k-schemes.

Proof. First, we already proved before that if a morphism is smooth of relative dimension n, then $\Omega_{X/Y}$ is locally free of rank n, moreover, we know that the definition of smooth over k of dimension d means that (by permuting the variables and the equations) there is a affine cover of X making the structure morphism $X \to Speck$ smooth of relative dimension d, which shows that the sheaf $\Omega_{X/k}$ is locally free of rank d. The condition of locally finite type and pure dimension d is in the original definition.

Conversely, let's assume our new definition. Since $\Omega_{X/k}$ is locally free of rank d, we can choose an affine cover of X say by $SpecA$ where A is a finitely generated k-algebra, say by $Speck[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$, where $\Omega_{X/k}|_{SpecA}$ is free of rank d. Recall that on $SpecA \to Speck$, the module of differential is computed by the cokernel of the Jacobian matrix:

$$A^{\oplus r} \xrightarrow{J} A^{\oplus n} \to \text{coker} \to 0$$

since we know that the module is free of rank d, we know that the cokernel is also free of rank d. Then we compute the corresponding map at the residue point of each p, we get the conclusion that the Jacobian matrix has corank d at each point, we are done. $\square$

Remark 2.43. We will later also revisit the definition of smooth morphisms.

As dicsussed before, the conormal exact sequence should also be expected to be exact on the left given appropriate smoothness condition, and it turns out being a smooth variety over a field is one of the conditions:

Theorem 2.44. Suppose that $i : X \hookrightarrow Y$ is a closed embedding of smooth varieties over k with conormal sheaf $\mathscr{N}^\vee_{X/Y}$:

$$X \hookrightarrow Y \to Speck$$

then the conormal sheaf is locally free with the conormal exact sequence exact on the left.

$$0 \to \mathscr{N}^\vee_{X/Y} \to i^*\Omega_{Y/k} \to \Omega_X \to 0$$

Moreover, recall that when dualizing a SES of locally free sheaves, the result is still exact and daulizing commutes with pullback, therefore we obtain:

$$0 \to \mathscr{T}_{X/k} \to \mathscr{T}_{Y/k}|_X \to \mathscr{N}_{X/Y} \to 0$$

Remark 2.45. Before we prove this result, it's worth mentioning that conormal exact sequence is exact on the left even if in more general circumstances, but since we don't need them, we only prove the above version.

Proof. The most important part is to prove that the assertion in the theorem is true iff it's true in the base change to the algebraic closure, i.e. we consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, hook] & Y \arrow[r] & Speck \\
X_{\overline{k}} \arrow[r, hook] \arrow[u] & Y_{\overline{k}} \arrow[r] \arrow[u] & Spec\overline{k} \arrow[u]
\end{tikzcd}$$

Recall that when we have a SES of quasiocherent sheaves:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

and $\mathcal{F}, \mathcal{F}'$ are locally free of finite rank, we can conclude that $\mathcal{F}$ is also locally free of finite rank. Also recall that we already proved that smoothness over a field is stable under extension of fields, thus $X_{\overline{k}} \hookrightarrow Y_{\overline{k}}$ is a closed embedding of smooth varieties over $\overline{k}$, thus our assumption doesn't change. Then we know that the terms involving relative differentials are all locally free of finite rank, therefore the locally free assertion follows from exactness, therefore, we only need to prove exactness on the left, therefore we need to prove that the sequence before extension is exact iff the extension after the extension is exact.

Recall that we can cover Y by $SpecB$ and we will have the below local diagram:

$$\begin{tikzcd}
SpecA \arrow[r, hook] & SpecB \arrow[r] & Speck \\
SpecA \otimes_k \overline{k} \arrow[r, hook] \arrow[u] & SpecB \otimes_k \overline{k} \arrow[r] \arrow[u] & Spec\overline{k} \arrow[u]
\end{tikzcd}$$

We only that the conormal exact sequence is given by:

$$I/I^2 \to A \otimes_B \Omega_{B/k} \to \Omega_{A/k} \to 0$$

Recall that tensor over k with $\overline{k}$ is exact, therefore we know that the kernel of $B \otimes_k \overline{k} \to A \otimes_k \overline{k}$ is $I \otimes_k \overline{k}$. On the other hand, we know that the sheaf of differentials of $SpecA \otimes_k \overline{k}$ over $Spec\overline{k}$ is the pullback of that of $SpecA$ over $Speck$, so is that of $SpecB \otimes_k \overline{k}$, then we know that the conormal exact sequence of the lower row in the above diagram is just the tensor of that of the first row, therefore we will expect the diagram below:

$$\begin{tikzcd}
(I/I^2) \otimes_k \overline{k} \arrow[r] \arrow[d, "\simeq"] & (A \otimes_B \Omega_{B/k}) \otimes_k \overline{k} \arrow[r] \arrow[d, "\simeq"] & (\Omega_{A/k}) \otimes_k \overline{k} \arrow[r] \arrow[d, "\simeq"] & 0 \\
I_{\overline{k}}/I^2_{\overline{k}} \arrow[r] & A_{\overline{k}} \otimes_{B_{\overline{k}}} \Omega_{B_{\overline{k}}/\overline{k}} \arrow[r] & \Omega_{A_{\overline{k}}/\overline{k}} \arrow[r] & 0
\end{tikzcd}$$

If the above diagram really exists, recall that tensor with $\overline{k}$ over k is faithfully flat, i.e. the sequence after tensoring is exact iff the sequence before tensoring, thus we are done with the reduction. Therefore our job is to explain the diagram where the maps on the second row is still given by the conormal exact sequence. The map of the first column is clear, it's just the class of $i \otimes k$ is mapped to the class of $i \otimes k$ as we know that the isomorphism of $I \otimes_k \overline{k}$ with $I_{\overline{k}}$ is given by the inclusion. What about the second vertical map. Recall that we have a canonical isomorphism:

$$\Omega_{B/k} \otimes_B B_{\overline{k}} \simeq \Omega_{B_{\overline{k}}/\overline{k}}$$

Therefore we know that $A_{\overline{k}} \otimes_{B_{\overline{k}}} \Omega_{B_{\overline{k}}/k} \simeq A_{\overline{k}} \otimes_{B_{\overline{k}}} B_{\overline{k}} \otimes_B \Omega_{B/k} \simeq A_{\overline{k}} \otimes_B \Omega_{B/k} \simeq (A \otimes_B \Omega_{B/k}) \otimes_k \overline{k}$, and we know that the isomorphism is given by:

$$k \otimes a \otimes db \mapsto (k \otimes a) \otimes (1 \otimes 1) \otimes db \mapsto (k \otimes a) \otimes d(1 \otimes b)$$

Finally the last isomorphism is given by:

$$\Omega_{A_{\overline{k}}/\overline{k}} \simeq \Omega_{A/k} \otimes_A (A \otimes_k \overline{k}) \simeq \Omega_{A/k} \otimes_k \overline{k}$$

where: $da \otimes k \mapsto da \otimes (1 \otimes k) \mapsto kd(a \otimes 1)$, then we can easily check this diagram is indeed commutative for example in the first square, $i \otimes k$ is mapped to $(1 \otimes di) \otimes k \mapsto (k \otimes 1) \otimes d(i \otimes 1)$ and on the other hand, it's mapped to $i \otimes k \mapsto 1 \otimes d(i \otimes k)$. But notice that $(k \otimes 1) \otimes d(i \otimes 1) = (1 \otimes 1) \otimes (k \otimes 1 d(i \otimes 1))$, we know that d is $\overline{k}$-linear, thus they agree.

Then for the second square, we know that $(a \otimes db) \otimes k \mapsto (adb) \otimes k \mapsto akd(b \otimes 1)$, on the other hand, we know that $(a \otimes db) \otimes k \mapsto (k \otimes a) \otimes d(1 \otimes b) \mapsto ka \otimes d(1 \otimes b)$. There are a few identifications going on in the argument, but I will let you figure it out. But either way, the conclusion is that the diagram is indeed commutative and thus we can assume we work with $k = \overline{k}$, and in particular, k is an infinite field. Also, we are working with locally finite type schemes over k, thus in particular with locally Noetherian schemes, all of the properties associated points behave well in this setting.

This implies that closed points of X, Y are regular, then if we consider the ideal sheaf $\mathcal{I}$ of $i : X \hookrightarrow Y$, I claim that this is a regular embedding at each closed point of X, i.e. $\mathcal{I}$ is generated by a regular sequence at each closed point. To see this, recall that we can consider the closed embedding $p \hookrightarrow p$ induced by $X \hookrightarrow Y$, whose ideal sheaf is just $\mathcal{I}_p$, and we know this is generated by a regular sequence. Suppose that X is of dimension m and Y is of dimension n, then I claim that this regular embedding is of codimension r, this is because we know that the dimension of the local ring at p is m in X and is n in Y then we have also showed in a quite early proposition that the regular embedding is of codimension $n - m$, and we also know regular embedding is a local, also closed points are dense in locally finite type k-schemes, therefore we know that $X \hookrightarrow Y$ is a regular embedding of codimension $n - m$.

Recall that the conormal sheaf of a regular embedding of codimension r is locally free of finite rank r. To check that δ is injective, we need to show that $\mathrm{Ker}\delta$ is the 0 sheaf. Recall that every non zero quasicoherent sheaf has non empty associated points, therefore to prove it's 0, we need to show that the associated points is empty. Moreover, recall that since $\mathrm{Ker}\delta$ is a subsheaf of $\mathcal{I}/\mathcal{I}^2$, the set of associated points is a subset of the associated points of $\mathcal{I}/\mathcal{I}^2$. However, recall that this is a locally free sheaf, therefore the associated points are just the associated points of X.

X is smooth, therefore, reduced, therefore it has no embedded points, and we know that for a reduced scheme, the stalk at a generic point is a field, therefore to check the generic points is not an associated point of ker, we only have to show that it's stalk vanishes there. However, since the stalk is a field, we work with vector spaces with the dimension of $(\pi^*\Omega_{Y/k})_p$ being n and the dimension of $(\Omega_{X/k})_p$ being m the dimension of $(\mathcal{I}/\mathcal{I}^2)_p$ being $r = n - m$, we know the stalk of the kernel is of dimension 0, thus 0. We are done. $\square$

The following is an important corollary of the above theorem:

Corollary 2.46. Suppose that Y is a smooth k-variety and $q \in Y$ is a closed point whose residue field $k(q)$ is separable over k, then the conormal exact sequence for $q \hookrightarrow Y$ yields an isomorphism of the Zariski cotangent space of Y at q with the fiber of $\Omega_{Y/k}$ at q.

Proof. The fact that $k(q)$ is separable over k means that $k(q)/k$ is smooth of dimension 0, then the closed embedding $q \hookrightarrow Y$ is an embedding of smooth k-varieties. Then the conormal exact sequence gives:

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{Y/k}|_q \to \Omega_{q/k} \to 0$$

Recall that $\Omega_{q/k}$ is 0 since we know that it's locally free of rank 0, then we know that this is just 0. Then take the stalk at q, we get the desired isomorphism. ∎

Remark 2.47. This result fulfills our promise earlier that we have the Jacobian description of the Zariski cotangent space not just at k-valued points, but over those closed points whose residue field is finite separable over k if Y is smooth.

Our previous result of using the cokernel to describe the cotangent space only works at a k-valued point because we will need the maximal ideal to look like $(x_1 - k_1, \cdots, x_n - k_n)$ so that our computation works. Therefore it only works for k-valued points, where our improvement here works more general, especially when $\mathrm{char}k = 0$ or when k is a finite field.

Remark 2.48. In fact the smooth of Y is not necessary but to prove the general case will take us to far. The reason behind this is that in order for the conormal exact sequence to be exact on the left for the closed embedding $i : X \hookrightarrow Y$, Y doesn't have to be smooth.

Even though we will not prove the general case, the following proposition deals with the case when the closed embedding is just a closed point.

Proposition 2.49. *Suppose p is a closed point of a k-variety X with residue field $k(p)$ that is separable over k. We consider the base change to the algebraic closure $\overline{k}$, and $\pi : X_{\overline{k}} \to X$ is the morphism in the base change. Suppose that $q \in \pi^{-1}p$, then $X_{\overline{k}}$ is regular at q iff X is regular at p.*

Proof. First of all, we can just assume that X is affine because this statement is local. Then we have the following diagram:

$$\begin{tikzcd} q \arrow[r, hook] \arrow[d] & SpecA \otimes_k \overline{k} \arrow[r] \arrow[d, "\pi"] & \overline{k} \arrow[d] \\ p \arrow[r, hook] & SpecA \arrow[r] & k \end{tikzcd}$$

I claim that if p is a regular closed point, there is an affine open $SpecA$ near p such that $SpecA \to Speck$ is smooth. This is because we can compute the Zariski cotangent space of p using Jacobian criterion, suppose $SpecA = Speck[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$, we know that if the Zariski cotangent space at p is given by the cokernel of the Jacobian matrix J, then since this can be translated to the nonvanishing determinant condition and this is local, then there will be an affine open neighborhood $SpecA'$ of p such that $SpecA' \to Speck$ is smooth of some relative dimension. Therefore what we really need is that the previous corollary holds, i.e. the conormal exact sequence is exact on the left. Let's prove it directly.

Recall that what we want to prove is that the following sequence of modules is exact on the left:

$$m/m^2 \to \Omega_{A/k} \otimes_A k(p) \to \Omega_{k(p)/k} \to 0$$

Here we can take A to be the coordinate ring of $SpecA$ which is an affine open of p. However, recall that $k(p)$ is also A_p/pA_p, therefore we know that A can also be thought as A_p, i.e. a local ring and m is just pA_p, the unique maximal ideal of that lcoal ring, i.e. we have the following isomorphism:

$$\begin{tikzcd} pA_p/p^2A_p \arrow[r] & \Omega_{A_p/k} \otimes_{A_p} k(p) & & \\ m/m^2 \arrow[r] \arrow[u, "\simeq"] & \Omega_{A/k} \otimes_A k(p) \arrow[r] \arrow[u, "\simeq"] & \Omega_{k(p)/k} \arrow[r] & 0 \end{tikzcd}$$

I will let you check this indeed commutes, therefore now let's assume A is just a local ring.

In this case, I claim that $m/m^2 \to \Omega_{A/k} \otimes_A k(p)$ is also injective. The key here is that we have a k-algebra map $k(p) \to A$ such that the following diagram is commutative:

$$\begin{tikzcd} k(p) \arrow[r] & A \arrow[r] & k(p) \\ & k \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

such that the composition of the horizontal two maps is the identity. If this is true consider the map:

$$A \to m/m^2, \qquad a \mapsto \overline{x}$$

where $a \in A$ is uniquely written as $\overline{a} + x$ where $x \in m$ and $\overline{a} \in k(p)$. This is a k-linear derivation. To see why, first we know that it's obviously k-linear (even $k(p)$-linear) and if $a = \overline{a} + x$ and $b = \overline{b} + y$, we know that ab is mapped to the desired element using the fact that this map is $k(p)$-linear. Then this induces a $k(p)$-linear map:

$$\Omega_{A/k} \otimes_A k(p) \to m/m^2$$

We know that $\overline{x}$ is mapped to $dx \otimes 1$ and it's again mapped to $\overline{x}$, thus the composition is the identity, thus $m/m^2 \to \Omega_{A/k} \otimes_A k(p)$ is injective, as desired. The reason why $k(p) \to A$ exists is just a purely algebraic question and you can look at **Stacks project tag 00TU**.

Conversely, if q is regular, we want to show that p is regular. In the previous part of discussion, we already know that the cotangent space at p could be computed by the fiber of $\Omega_{X/k}$ at p. Consider the following diagram:

$$\begin{tikzcd} X_{\overline{k}} \arrow[r, "\pi"'] \arrow[d] & X \arrow[d] \\ Spec\overline{k} \arrow[r] & Speck \end{tikzcd}$$

We know that $\pi^*\Omega_{X/k} \simeq \Omega_{X_{\overline{k}}/\overline{k}}$, from our assumption, we know that the dimension of $(\Omega_{X_{\overline{k}}/\overline{k}})_q \otimes \overline{k}$ is n. We know that:

$$(\Omega_{X_{\overline{k}}/\overline{k}})_q \otimes_{Y,q} \overline{k} \simeq (\Omega_{X/k})_p \otimes_{\mathcal{O}_{X,p}} \otimes \overline{k}$$

Notice that:

$$(\Omega_{X/k})_p \otimes_{\mathcal{O}_{X,p}} \otimes \overline{k} \simeq (\Omega_{X/k})_p \otimes_{\mathcal{O}_{X,p}} k(p) \otimes_{k(p)} \overline{k} = \Omega_{X/k}|_p \otimes_{k(p)} \overline{k}$$

This implies that as a $k(p)$-vector space, $\Omega_{X/k}|_p$ is also of dimension n. Recall that the point of p is regular iff the dimension of $\Omega_{X/k}|_p$ is n (as p, q are both closed points of codimension n), therefore we are done. $\square$

3 Examples

In this part, we give some important examples of the sheaf of differentials which involves a lot of geometric notions that we will use later. The first example is the geometric genus:

Definition 3.1. For a regular irreducible projective curve C over a field k, we define it's **geometric genus** to be $h^0(C, \Omega_{C/k})$, since we work with a Noetherian scheme, we know that $\Omega_{C/k}$ is coherent, therefore the Grothendieck's coherence theorem tells us that $h^0(C, \Omega_{C/k})$ is finite.

Sadly, this is not really a new invariant, as we have the following result:

Proposition 3.2. Let's assume Serre duality which states that $\Omega_{C/k}$ is the dualizing sheaf, then the geometric genus is equal to the arithematic genus.

Proof. Recall that by Serre duality:

$$h^0(C, \Omega_{C/k}) \simeq h^1(C, \Omega_{C/k} \otimes \Omega_{C/k}^\vee) = h^1(C, \mathcal{O}_C)$$

$\square$

Before we discuss this notion further, let's try to compute some differentials:

Example 3.3. As an important first example, consider $\mathbb{P}^1_k$ and we will compute the sheaf of differentials on it. Let's give it the usual projective coordinate x_0, x_1.

As usual, we know that we can cover the projective line by two affine opens where $x_0 \neq 0$ and $x_1 \neq 0$. Moreover, if we restrict it to each of the affine open $\mathbb{A}^1_k$, we know that the restriction is $\Omega_{k[x]/k}$, which is $k[x]dx$, i.e. we know that the sheaf of differential is a line bundle on $\mathbb{P}^1$. We know that the line bundles on projective space is $\mathcal{O}(d)$ for some d, therefore we need to find out this d.

Let's take the section $dx_{1/0}$ on $x_0 \neq 0$, it has no zeros or poles there as it's the free generator. Let's check on the patch $x_1 \neq 0$, or on the intersection of two patches. We know that on the intersection, the restriction of $x_{1/0}$ and $x_{0/1}$ satisfies:

$$x_{1/0} = \frac{1}{x_{0/1}}$$

Moreover, we know that we have a derivation from the ring on the intersection to the restriction of the differential sheaf there, therefore we know from the quotient rule:

$$dx_{1/0} = -\frac{1}{x_{0/1}^2} dx_{0/1}$$

This tells you that the the line bundle is $\mathcal{O}(-2)$, therefore the geometric genus of $\mathbb{P}^1 = 0$, which agrees with the arithmetic genus.

Now if we consider $\mathbb{P}^1_{\mathbb{Z}}$ instead of over k. First, recall how we concluded the line bundles on $\mathbb{P}^n_k$ is $\mathcal{O}(d)$. First, we know $k[x_1, \cdots, x_n]$ is a UFD, therefore we know that $\mathrm{Cl}\mathbb{A}^n_k = \mathrm{Pic}\mathbb{A}^n_k = 0$, this also shows that $\mathbb{P}^n_k$ is factorial. Then we use the excision exact sequence to conclude that $Cl\mathbb{P}^n_k$ is generated by $\mathcal{O}(1)$, since we know that on $\mathbb{P}^n_k$, $\mathcal{O}(m)$ is not trivial for $m \neq 0$, therefore the Picard group is really free with generator $\mathcal{O}(1)$. The same argument applies to the base $\mathbb{Z}$ as well since it's also a UFD, therefore $\mathbb{Z}[x_1, \cdots, x_n]$ is a UFD by Gauss' theorem, we want it to be a UFD such that our knowledge on the class group and Picard group can be applied. I will leave you to figure out other details but in conclusion we proved that:

$$\Omega_{\mathbb{P}^1_{\mathbb{Z}}/\mathbb{Z}} = \mathcal{O}(-2)$$

Now recall that we have the following base change diagram:

$$\begin{tikzcd} \mathbb{P}^1_A \arrow[r] \arrow[d] & \mathbb{P}^1_{\mathbb{Z}} \arrow[d] \\ SpecA \arrow[r] & Spec\mathbb{Z} \end{tikzcd}$$

It's easy to prove using transition function that the pullback of $\mathcal{O}(-2)$ is still $\mathcal{O}(-2)$, then our previous result of differentials behaves well with pullback tells that for any ring A:

$$\Omega_{\mathbb{P}^n_A/A} = \mathcal{O}(-2)$$

Remark 3.4. There is an interesting interpretation on this fact that the sheaf of differentials is $\mathcal{O}(-2)$. it shows that the sheaf of tangent vectors is $\mathcal{O}(2)$, which is of degree 2 say if we work with $\mathbb{P}^1_k$, and it has degree 2, and this is related to the "Hairy Ball Theorem" in topology.

Now let's think a little bit about hyperelliptic curves. In this discussion let's assume that $k = \overline{k}$ and char$k \neq 2$ so we can apply what we proved when we discussed hyperelliptic curves. Consider a double cover $\pi : C \to \mathbb{P}^1_k$ of a regular projective curve C, which branch over $2g+2$ distinct points. Recall that we had a very explicit coordinate description of the two affine covers of the hyperelliptic map where we assume that $0, \infty$ are not the branch points. And by Riemann-Hurwitz Theorem for hyperelliptic curves, we know that C is of genus g.

Example 3.5. In this example, we will compute the degree of the sheaf of differentials on a genus g hyperelliptic curve C. Recall that C is regular, therefore the degree of $\Omega_{C/k}$ can be computed by computing the zeros and poles of any nonzero rational section (first we need to make sure that $\Omega_{C/k}$ is indeed a line bundle, which means that we need to show that C is smooth over k, but this follows from the fact that we assumed C is regular and $k = \overline{k}$, or you can just use the fact about regular plane curves we discussed in the beginning of this section). As what is said above, we will need the explicit coordinate description of the hyperelliptic curve:

$$\begin{array}{ccc} Speck[x,y]/(y^2 - f(x)) & \longrightarrow & Speck[u,z](z^2 - u^r f(\frac{1}{u})) \\ \downarrow & & \downarrow \\ Speck[x] & \longrightarrow & Speck[u] \end{array}$$

For example, we know that the sheaf of differentials on $Speck[x,y]/(y^2 - f(x)) = SpecA$ is generated by dx an dy, subject to relations and we have a pullback map:

$$A \otimes_{k[x]} k[x]dx \to \Omega_{A/k}$$

and we know that $dx \mapsto dx$. Let's compute the zeros and poles of this section. I claim that it has $2g+2$ zeros and has 4 poles counted with multiplicity, therefore the degree is $2g-2$. For the zeros, recall that we know that dx is in $\Omega_{A/k}$ is subject to the relation:

$$f'(x)dx = 2ydy$$

This is also a reason why we don't want char$k = 2$. Then let's first compute where dx vanish on $Speck[x,y]/(y^2 - f(x))$. I claim that the zeros of dx are the branch points of π, each of which is a simple zero thus giving you $2g+2$ in total. To see why, let's first examine $f'(x)$. Recall that the Jacobian matrix is $(-f'(x), 2y)$, since our assumption is that $Speck[x,y]/y^2 - f(x)$ is regular and $k = \overline{k}$, we can assume this is smooth, since we know that y vanishes somewhere, the only possible case is $f'(x)$ is invertible on $Speck[x,y]/(y^2 - f'(x))$. Therefore we know that:

$$dx = \frac{2y}{f'(x)}dy$$

therefore we know that dx vanish iff y vanishes, i.e. it vanishes at the branch points in $Speck[x,y]/(y^2 - f(x))$. We know that $V(y)$ is only a finite collection of points as y is not a zero divisor, therefore the points are all closed points, i.e. the points are where $f(x)$ vanishes, i.e. they are the branch points in $Speck[x,y]/(y^2 - f(x))$. To compute the degree of vanishing, we know that in the point where y vanish, y generates the maximal ideal, therefore $\frac{2y}{f'(x)}$ has valuation 1, therefore the order of vanishing is 1 at each of these points. What about the zeros in another cover, first we know that dx on $Speck[x]$ goes to $-\frac{1}{u^2}du$ on the intersection with $Speck[u]$ which pulls back to $-\frac{1}{u^2}du$ and we know that the zeros still correspond to the zeros of du, let $u^r f(1/u)$ be $g(u)$ (this is indeed a polynomial in u because we know that f has degree r) and the same argument shows that the zeros are the branch points, each with degree 1, therefore we are done.

Then let's think about poles, obvious there are no poles in $Speck[x,y]/(y^2 - f(x))$, then we know that since on $Speck[z,u]/(z^2 - g(u))$ we have $\frac{1}{u^2}du$ since we know that again $du = \frac{2z}{g'(u)}dz$, we know that we have:

$$\frac{2z}{u^2 g'(u)}dz$$

again since $g'(u)$ is invertible on $Speck[z,u]/(z^2 - g(u))$, we only have to consider $\frac{2z}{u^2}dz$. Now, we know that a pole of this is where u vanishes and each of the locus give you a vanishing order of 2. Recall that the locus where u vanish, i.e. $V(u)$ is again a finite colletion of closed points, and we know that the closed points are of the form $(z-a, u)$, i.e. all k-valued points, by assumption, $u = 0$ is not a branch point, therefore we know that the inverse image of (u) are two closed points, each give you a order 2, summing up to 4. We are done.

Remark 3.6. Technically speaking, we also need to explicitly give the pullback map over $Speck[u,z]/(z^2 - g(u)) = SpecB$, which is given by:

$$B \otimes_{k[u]} k[u]du \mapsto \Omega_{B/k}$$

which is given by $q \otimes du \mapsto du$. Then we know that the map over the locus where u is invertible is given by:

$$(B \otimes_{k[u]} k[u]du)_u \to (\Omega_{B/k})_u$$

But it's still given by $1 \otimes du \mapsto du$ and it's B_u linear, therefore we concluded above that $\frac{1}{u^2}du$ is mapped to $\frac{1}{u^2}du$.

The above computation is an example of the general Riemann-Hurwitz Theorem we will discuss later.

Now our goal is to show that the geometric genus of C, i.e. $h^0(C, \Omega_{C/k})$ is g, i.e. equal to the arithmetic genus of C. We will do it by a series of results:

Proposition 3.7. Any regular differential ω on $Speck[x,y]/(y^2 - f(x))$, i.e. an element in $\Omega_{k[x,y]/(y^2-f(x))/k}$, that is preserved by the involution $y \mapsto -y$ is pulled back from a differential on $k[x]$.

Proof. First, let's be specific that the involution we talked about in the statement is just the hyperelliptic involution. Recall that when we describe the coordinate description of hyperelliptic curves, we conclude that the gluing is unique if we mod out the fact that $z = \pm y/x^{r/2}$, therefore the hyperelliptic curve is a automorphism over $\mathbb{P}^1$ that affects only the gluing, i.e. as in the diagram below:

$$\begin{tikzcd} C \arrow[rr, "i"] \arrow[dr] & & C \arrow[dl] \\ & \mathbb{P}^1 & \end{tikzcd}$$

and this automorphism i, i.e. the involution is characterized by $y \mapsto -y$ on the gluing part. What we need to be careful about is how this involution act on the sheaf of relative differentials which is going to be important in the computation below. To give a clear picture, consider the following diagram:

$$\begin{tikzcd} C \arrow[r, "i"] \arrow[d, "\pi'"] & C \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r, "id"] & \mathbb{P}^1 \end{tikzcd}$$

this is a Cartesian diagram, which gives you the pullback morphism from the sheaf of differential on C to C.

Then as what we observed in the previous result, $\Omega_{k[x,y]/(y^2-f(x))/k}$ is free and generated by dy. Recall that we also see that the pullback of dx is given by:

$$\frac{2y}{f'(x)}dy$$

Now suppose that w is such a regular differential that is preserved by the involution, say it's $g(x,y)dy$, then since it's preserved by the involution, we know since it's mapped to $g(x,-y)dy$, then:

$$g(x,-y)dy = g(x,y) - dy = -g(x,y)dy$$

I first claim that the involution preserves dx on $\Omega_{k[x,y]/(y^2-f(x))/k}$. To see why, the intuition should be the involution only reverse y but doesn't do anything on x. And you should be able to describe it using the diagram above. One thing that might be slightly confusing is that we can also write $dx = \frac{2y}{f'(x)}dy$, but we know that under the involution:

$$\frac{2y}{f'(x)}dy \mapsto \frac{-2y}{f'(x)} - dy = \frac{2y}{f'(x)}dy$$

thus it's still preserved by i^*.

Then let's consider the rational function corresponding to ω/dx, i.e.:

$$\frac{g(x,y)f'(x)}{2y}$$

By the relation we had above, the involution also preserves this element since:

$$\frac{g(x,y)f'(x)}{2y} \mapsto \frac{-g(x,y)f'(x)}{2-y} = \frac{g(x,y)f'(x)}{2y}$$

Now I claim that this is a rational element that only involves x. To see why, we know that the fractional field is generated by x, y, i.e. $g(x,y)$ is $yh(x)$ for some polynomial in x. To see why we know that we can assume that $g(x,y)$ only have linear terms of y, as higher terms can be written as $f(x)$ using the relation $y^2 = f(x)$. Then assume that;

$$g(x,y) = g_1(x) + g_2(x)y \Rightarrow g_1(x) - g_2(x)y = -g_1(x) - g_2(x)y \Rightarrow 2g_1(x) = 0$$

Therefore we are done showing the rational function $w/dx = h(x)$.

Then we know that this is not just a rational function on $Speck[x,y]/(y^2 - f(x))$, it's also a regular function. We know that in the end $w = h(x)dx$ where $h(x)$ is some polynomial in x. Then we know that w is the pullback of $h(x)dx$ on $Speck[x]$. □

Proposition 3.8. Any $\omega \in H^0(C, \Omega_{C/k})$ preserved by the involution $i : y \mapsto -y$ must be pulled back from a differential on $\mathbb{P}^1$ by π, thus must be 0. Moreover, for any $\omega \in H^0(C, \Omega_{C/k})$, we have $i^*\omega = -\omega$.

Proof. First consider the restriction of ω to $Speck[x,y]/(y^2 - f(x))$, we know that this is the pullback of something $h(x)dx$ on $Speck[x]$. Let's also consider the restriction on $Speck[u,z]/(z^2 - g(u))$, and we know that it's some $p(u,z)dz$ as we know that on this affine open, the differential is free with generator dz with $du = \frac{2z}{g'(u)}dz$

Recall that $h(x)dx$ on $Speck[x]$ in $\mathbb{P}^1$ is $-h(\frac{1}{u})/u^2 du$ and we know that it's pulled backed to $\frac{2zh(\frac{1}{u})}{g'(u)u^2}dz$ on the intersection, this is good as we know that if h is not the 0 polynomial, we know that $\frac{2zh(\frac{1}{u})}{g'(u)u^2}dz$ have poles, therefore we know that it can not be a global section, a contradiction.

Now I claim that this implies that for all $\omega \in H^0(C, \Omega_{C/k})$, $i^*\omega = \omega$. First, notice that $(i^*)^2 = Id$, i.e. i^* is idemnpotent. Then we know that possible eigenvalues of i^* are ± 1. Also we know that $(i^* - 1)(i^* + 1) = 0$, if we denote $m(x)$ by the minimal polynomial of i^*, we know that it splits as distinct linear factors, then the linear algebra argument shows that i^* is diagonalizable with eigenvalue either 1 or -1. However, we know that only 0 is fixed, therefore i^* is diagonalizable with two eigenvalues -1, i.e. $i^*\omega = -\omega$. □

Proposition 3.9. $\frac{dx}{y}$ is a regular differential on $Speck[x,y]/(y^2 - f(x))$ and for $0 \leqslant i < g$ $x^i \cdot (dx/y)$ extend to a global differential ω_i on X.

Proof. We already saw that $dx = \frac{2y}{f'(x)}dy$ where $f'(x)$ is invertible on $Speck[x,y]/(y^2 - f(x))$, therefore we know that $dx/y = \frac{2}{f'(x)}dy$, of course a regular differential on $Speck[x,y]/(y^2 - f(x))$.

To prove that $x^i \cdot (dx/y)$ extend to a global section, we know that C is regular thus normal, therefore we only need to show that it has no poles. We know that on another affine open, we know this section is:

$$-\frac{1}{u^2}du \cdot \frac{1}{u^i} \cdot \frac{u^{r/2}}{z}$$

where we know that $r = 2g + 2$. Recall that we have $du = \frac{2z}{g'(u)}dz$, thereofore on the intersection we have:

$$\frac{2u^{r/2}dz}{u^{i+2}g'(u)}$$

Recall that $g'(u)$ is invertible on $Speck[u,z]/(z^2 - g(u))$, therefore we only need to prove the following rational function has no poles:

$$\frac{2u^{g+1}}{u^{i+2}}$$

For $0 \leqslant i < g$, we know that $i + 1 \leqslant g + 1$, we are done. □

Proposition 3.10. I claim that w_i are linearly independent differentials, i.e. we know that the dimension of $H^0(C, \Omega_{C/k})$ is at least g.

Proof. Let p, q be the two closed points in $\pi^{-1}0$ as 0 is not a branch point of π. We know that 0 is the point $[0:1]$, therefore we know that we can only consider the cover $Speck[x,y]/(y^2 - f(x)))$, we know that w_i in this chart is given by the rational function:

$$\frac{2x^i}{f'(x)}$$

since $f'(x)$ is invertible, then we know that the valuation is determined by x_i. Moreover, we know that the two points are $(y - y_1, x)$ and $(y - y_2, x)$, then the uniformizer is given by x we know that it's regular at the point, and any thing in m/m^2 is a uniformizer. But this tells you that the valuation is i.

Suppose we consider a nontrivial linear combination of w_j written as:

$$\omega = \sum_{j=s}^{g-1} a_j w_j \quad a_s \neq 0$$

I claim that the valuation at p is s. There are two options you can choose to prove this, though they are essentially using the same ideal, for example what you can do first is to rewrite ω:

$$\omega = \sum_{j=s}^{g-1} a_j \frac{2x^j}{f'(x)} dy = \frac{x^s}{f'(x)} \sum_{j=s}^{g-1} a_j \frac{x^{j-s}}{1} dy$$

To show the valuation is s, we must show that the stuff in the summation is of valuation 0, i.e. invertible in the local ring, recall that in a local ring, invertible elements are those not in the maximal ideal, but notice that in the summation, it's one invertible element plus something in the maximal ideal, therefore it's also invertible. Using the same idea, you can show that even though we usually only have $\mathrm{val}(a+b) \geqslant \min(\mathrm{val}a, \mathrm{val}b)$, if a, b doesn't cancel each other, you get the equality, and then you can use the conclusion directly since the one with the smallest valuation appears only once. Therefore, the valuation is not ∞, thus not 0. $\square$

Proposition 3.11. Finally, we will show that these ω_i spans $H^0(C, \Omega_{C/k})$.

Proof. If $\omega \in H^0(C, \Omega_{C/k})$, consider:

$$\omega' = \omega - \sum_{i=0}^{g-1} a_i \omega_i$$

Let's assume that the valuation of ω is j, then I claim that we can choose unique $a_i \in k$ such that:

$$\mathrm{val}(\omega - \sum_{i=0}^{g-1} a_i \omega_i) \geqslant g$$

This is because if j is already at least g, then we can just choose the summation to be 0. If j is smaller than g, then let's assume that:

$$\omega = a_j x^j$$

where a_j is a invertible element, therefore it's easy to pick a_j such that the summation is 0 at that point, therefore the valuation is infinity.

To show uniqueness, suppose that we pick a_j and b_j, both making the valuation of the difference at least g, then we know that:

$$\sum (a_j - b_j)\omega_j$$

has valuation at least g at p, which is a contradiction if not every $a_j = b_j$. This shows uniqueness.

Finally, let's consider $i^*\omega'$, recall that under the involution $p \mapsto q$ and $q \mapsto p$, then I claim that:

$$\mathrm{val}_p(\omega') = \mathrm{val}_q(i^*\omega') = \mathrm{val}_q(-\omega') = \mathrm{val}_q(\omega')$$

The last two equalities are obvious, only the first one needs some clarification. it can be checked affine locally and further reduces to the question where: If $f : (A, m) \to (B, n)$ is an isomorphism of DVR mapping the uniformizer of m to the uniformizer of n, then $\mathrm{val}_A(t) = \mathrm{val}_B(f(t))$. Then you only need to check i^* is f on the local ring, which are DVRs.

Please be careful because you might be in a hurry and try to conclude that the valuation of ω' at q is the same as the valuation of ω' at p without the help of the involution just like what you did in the previous part. This argument is not true because in the previous part, ω_i on $Speck[x,y]/(y^2 - f(x))$ is of the form $\frac{2x^i}{f'(x)}dy$, thus we know that restrict it to p, q still gives you the power x^i. However, for ω', we don't know what it looks like after we restrict it to the affine open. I will let you give an example of a differential on $Speck[x,y]/(y^2 - f(x))$ such that it has differential valuation at p and at q. This is the reason why we need to use i^*.

Finally, all the above arguments show that the degree of ω' is at least $2g$ since it's a global section with at least $2g$ zeros. If it's not 0, we know that it can be used to compute the degree of $\Omega_{C/k}$, which is a contradiction, since we proved that the degree is $2g-2$. Therefore we are done. $\square$

The following two results illustrate a special case of Serre duality on hyperelliptic curves and when we prove Serre duality, you will appreciate this computation more:

Proposition 3.12. $h^1(C, \Omega_{C/k}) = 1$ and the generator can be interpreted as $x^{-1}dx$. In particular, we have an isomorphism induced by the pullback map $H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1/k}) \to H^1(C, \Omega_{C/k})$

Proof. Recall that for integral projective curves over an algebraically closed field, we have the Riemann Roch formula:

$$h^0(C, \Omega_{C/k}) - h^1(C, \Omega_{C/k}) = \deg\Omega_{C/k} - g + 1$$

We already proved that $h^0(C, \Omega_{C/k})$ is g and $\deg\Omega_{C/k}$ is $2g-2$, then we know that:

$$h^1(C, \Omega_{C/k}) = g + g - 1 - 2g + 2 = 1$$

You might want to prove this by using Cech complex to compute H^1 which also gives you the generator, but the problem here is that we don't fully understand what is the isomorphism connecting the sheaf of differentials on the intersection.

However, We can give a generator of $H^1(C, \Omega_{C/k})$. To see how, notice that we know $\Omega_{\mathbb{P}^1/k}$ quite well and we have a pullback map $H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1/k}) \to H^1(C, \Omega_{C/k})$, we know that:

$$H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1/k}) = H^1(\mathbb{P}^1, \mathcal{O}(-2))$$

which is rank 1 over k and generated by $x_0^{-1}x_1^{-1}$. If we use the usual $k[x], k[u]$ description of the projective space, we know we can interpret it as $k[x, x^{-1}]$ quotient all the rational functions that are not degree -1, i.e. it's generated by x^{-1}. Now we need to figure what is the pullback map. Let's think about the generator as the class of $\frac{dx}{x}$ in the intersection, then we know that the pullback is $\frac{1}{x}$ maps to $\frac{1}{x}$ and dx pullback to dx, therefore the result is $\frac{dx}{x}$, which is viewed as a class of $\Omega_{C/k}$ in the intersection. I claim that this is not the zero element in H^1. To see why, recall that this pullback is defined by the following morphism of complexes: **** why the pullback is not 0 still need to be checked* $\square$

Proposition 3.13. There is a natural perfect pairing:

$$H^0(C, \Omega_{C/k}) \times H^1(C, \mathcal{O}_C) \to H^1(C, \Omega_{C/k})$$

Proof. Using what we proved above which shows that $H^1(C, \Omega_{C/k})$ is 1 dimensional with generator $\frac{dx}{x^{-1}}$, the natural pairing is easy.

I will let you show that $H^1(C, \mathcal{O}_C)$ can be interpreted by the vector space spanned by the basis $\frac{y}{x}, \frac{y}{x^2}, \cdots, \frac{y}{x^g}$, then the pairing:

$$\langle \frac{dx}{y}, \frac{xdx}{y}, \cdots, \frac{x^{g-1}dx}{y} \rangle \times \langle \frac{y}{x}, \frac{y}{x^2}, \cdots, \frac{y}{x^g} \rangle \to \langle \frac{dx}{x^{-1}} \rangle$$

is given by formally multiply the basis and read of the coefficients of $\frac{dx}{x}$. $\square$

The next few results deals with how differentials play with DVRs, which is something we will use later in the proof of the general Riemann-Hurwitz Theorem: We will assume (A, m, k) is a DVR that is a localization of a finitely generated k-algebra, here we assume that the map $k \hookrightarrow A \to A/m$ is the identification of A/m with k,let t be a uniformizer of A, then we will show that $\Omega_{A/k} = Adt$.

Proposition 3.14. $\Omega_{A/k}$ is a finitely generated A-module.

Proof. Suppose that $A = S^{-1}B$ where B is a finitely generated k-algebra, then we have:

$$k \to B \to A = S^{-1}B$$

The cotangent exact sequence gives you:

$$A \otimes_B \Omega_{B/k} \to \Omega_{A/k} \to \Omega_{A/B} \to 0$$

We know that $A \otimes_B \Omega_{B/k}$ is a finitely generated A-module since $\Omega_{B/k}$ is a finitely generated B-module and tensor is right exact. Moreover, $\Omega_{A/B}$ is 0 since A is a localization of B, therefore we know $\Omega_{A/k}$ is also finitely generated because you have a surjection from something finitely generated to $\Omega_{A/k}$. $\square$

Proposition 3.15. I claim that $\Omega_{A/k}$ is generated by a single element dt, i.e. the A-module map:

$$\times dt : A \to \Omega_{A/k}$$

is surjective.

Proof. Let $\pi : A \to A/m = k$ be the projection. Then we know that if you consider:

$$k \hookrightarrow A \to A/m \simeq k \hookrightarrow A$$

we can view $\pi(a)$ as some element in A using the k-algebra structure. Then I claim that $a - \pi(a) \in m$. This is because we can further extend the above sequence:

$$k \hookrightarrow A \to A/m \simeq k \hookrightarrow A \to A/m \simeq k$$

Then given $a \in A$, it's mapped to $\pi(a)$ in k, it's again mapped to $\pi(a)$ in A then we consider $a - \pi(a)$, we apply π but notice that $\pi(\pi(a)) = \pi(a)$, therefore we are done.

Let's define $\sigma(a) = (a - \pi(a))/t$, this is well defined since $a - \pi(a)$ is indeed in m, thus is ut^i for some $i \geqslant 1$. Then apply d to $\sigma(a)$, we know that by the quotient rule, it's:

$$t^2 d\sigma(a) = td(a - \pi(a)) - (a - \pi(a))dt$$

Recall that d is linear, then we get:

$$t^2 d\sigma(a) = tda - td\pi(a) - t\sigma(a)dt$$

Recall that $d\pi(a) = 0$ as $\pi(a)$ is an element in k, and differentials of constants are 0, this shows:

$$da = td\sigma(a) + \sigma(a)dt$$

Therefore we know that:

$$\Omega_{A/k} \subset (dt) + m\Omega_{A/k}$$

Then recall that for any finitely generated module over a local ring, we can apply Nakayama's lemma, and the above result tells you:

$$\Omega_{A/k} \subset (dt)$$

We are done. $\square$

Proposition 3.16. By the above result, we know $\Omega_{A/k}$ is a principal A-module, I claim that $\times dt : A \to \Omega_{A/k}$ is also injective.

Proof. Recall that every DVR is a PID, therefore we have the classification of finitely generated modules over a PID. We know $\Omega_{A/k}$ is a direct sum of cyclic modules, and it's also principal generated by dt, to prove the following map is injective:

$$\times dt : A \to \Omega_{A/k}$$

We need to show that for each element x in A, xdt is not 0, since for every $x \in A$, it's either in m or invertible, for an invertible element, we need to show that $dt \neq -$. If this holds, we know that each element in m is of the form ut^m for some positive integer m and invertible element u. Therefore we need to show that $t^m dt \neq 0$ for all $m \geqslant 0$. This will be our goal.

Now consider the surjection $A \to A/(t^N)$, this induces a map $\Omega_{A/k} \to \Omega_{A/(t^N)/k}$ where $da \mapsto da$, so it suffice to show that the imgae of $t^m dt$ in $\Omega_{A/(t^N)/k}$ is nonzero in $\Omega_{A/(t^N)/k}$.

Next important point is to notice that $A/(t^N) \simeq k[t]/(t^N)$. First, we have a map:

$$k[t] \to A/(t^N)$$

where $t \mapsto t$. It's clear that this is surjective and the kernel of this map is (t^N). To see this, consider $f(t)$:

$$f(t) = a_n t^n + \cdots + a_1 t + a_0 \mapsto 0 = a_n t^n + \cdots + a_1 t + a_0$$

I claim we only need to consider polynomials with degree less than N, and a_0 has to be 0 (I will let you figure out why). Then keep going we know that if f is 0. Therefore the universal property of quotients tells you that $k[t]/(t^N) \to A/(t^N)$ is indeed an isomorphism. We are done.

Then consider the map:

$$d : k[t]/(t^N) \to k[t]/(t^{N-1})$$

which is given by mapping a polynomial to it's derivative. This is a derivation over k, moreover t^m is not mapped to 0 for large enough N. Then the universal property of $\Omega_{A/(t^N)/k}$ gives a $A/(t^N)$-linear map:

$$\Omega_{A/(t^N)/k} \to k[t]/(t^{N-1})$$

Since we know that t^{m+1} in $A/(t^N)$ is mapped to $(m+1)t^m dt$, we know that $t^m dt$ is mapped to $\frac{(m+1)t^m}{(m+1)} = t^m$, and for large enough N, t^m is not 0, then $(m+1)t^m dt \neq 0$, we are done. $\square$

Next, we examine the differentials of projective space $\mathbb{P}^n_k$, or more generally if $\mathbb{P}^n_A$ over an arbitrary base ring A. We know that projective space is covered by $n+1$ $\mathbb{A}^n_A$, and on each of the affine opens, the differentials are free of rank n, we know that over the projective space, the sheaf of differentials is a rank n locally free sheaf. The key to understand the sheaf of differentials is via the following theorem:

Theorem 3.17. The sheaf of differentials $\Omega_{\mathbb{P}^n_A/A}$ satisfies the following **Euler exact sequence**:

$$0 \to \Omega_{\mathbb{P}^n_A/A} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n_A}(-1)^{\oplus(n+1)} \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}^n_A/A} \to 0$$

Moreover, since everything here locally free, we can dualize this sequence can still preserve exactness:

$$0 \to \mathcal{O}_{\mathbb{P}^n_A} \xrightarrow{\phi^\vee} \mathcal{O}_{\mathbb{P}^n_A}(1)^{\oplus n+1} \to \mathscr{T}_{\mathbb{P}^n_A/A} \to 0$$

Remark 3.18. Recall that we proved earlier that all finite rank locally free sheaves on $\mathbb{P}^1_k$ splits as a direct sum of $\mathcal{O}(d)$, possibly for different d. But if we work with $\mathbb{P}^n_k$ for a general n, this is not true anymore. Later we will be able to give an example using $\Omega_{\mathbb{P}^n_k/k}$ fulfilling our promise earlier.

Proof. We will first describe a morphism:

$$\mathcal{O}_{\mathbb{P}^n_A}(-1)^{\oplus(n+1)} \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}^n_A/A}$$

which is surjective, and then identify the kernel with $\Omega_{\mathbb{P}^n_A/A}$. To see what is this map, recall that we only need to define $\mathcal{O}(-1) \to \mathcal{O}$ for each component. For the i-th component, I define this morphism to be that corresponding to the closed embedding that embeds the i-th coordinate plane to $\mathbb{P}^n_A$. Or if you want to explicitly describe this map, on each of the standard affine opens, the map is defined to be multiply by the restriction of x_i to that affine open (you can also by hand check this is well defined). Then suggestively, let's write this morphism as:

$$\phi : (s_0, \cdots, s_n) \mapsto x_0 s_0 + \cdots + x_n s_n$$

Then we check ϕ is surjective. We do this by checking everything on $D(x_i)$. Then notice that the i-th factor on $D(x_i)$ is given by:

$$k[\frac{x_j}{x_i}] \mapsto k[\frac{x_j}{x_i}] \quad f \mapsto 1 \cdot f$$

obviously surjective, then since one of the factor is already surjective, we don't have to consider the other factors, we are dobe.

Now let's identify the kernel with the differentials. The first question is how can we do it. As we don't understand the differential on arbitrary affine opens, except for the standard affines, we naturally want to check this on each $D(x_i)$. However, working locally always has the problem of compatibility. Therefore, we have to be extra careful. Let's first do it on $D(x_0)$, with the usual coordinates $\frac{x_j}{x_0}$ with $j \geqslant 1$. On this affine open, we know that differentials are free with generators $d\frac{x_j}{x_0}$ and we can assume that we have a differential:

$$f_1(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0})dx_{1/0} + \cdots + f_n(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0})dx_{n/0}$$

We need to produce $n+1$ sections on $\mathcal{O}(1)$ in a way that this differential becomes the kernel.

As a motivation, let's first consider $f_1 dx_{1/0}$. If we interpret $x_{1/0}$ as $\frac{x_1}{x_0}$, then we know from the quotient rule that:

$$dx_{1/0} = f_1 \frac{x_0 dx_1 - x_1 dx_0}{x_0^2} = f_1 \frac{x_0 dx_1}{x_0^2} - f_1 \frac{x_1 dx_0}{x_0^2}$$

Notice that x_1 times the first terms plus x_0 times the second term is 0. This is very suggestive as we know that $\times x_1$ and $\times x_0$ is just the map ϕ we defined earlier. To match this notation, we can think of $\mathcal{O}(-1)$ on $D(x_0)$ as $\frac{1}{x_0}k[\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}]$ to match this sort of degree -1 notation. And on other patches, we do the same, this doesn't change the transition function just to use the notation that is most suggestive.

Then we define the map α on $D(x_0)$ as:

$$f_1 dx_{1/0} + \cdots + f_n dx_{n/0} \mapsto (-f_1 \frac{x_1}{x_0^2} - \cdots - f_n \frac{x_n}{x_0^2}, \frac{f_1}{x_0}, \cdots, \frac{f_n}{x_0})$$

Then $\frac{f_i}{x_0}$ components shows that this map is injective, and indeed it maps to the kernel. Now if $(g_0, \cdots, g_n)$ is in the kernel, we know that it implies $\sum g_i x_i = 0$, then we take $f_i = x_0 g_i$ for $i > 0$. I will let you figure out this satisfies our need and thus indeed, at least on $D(x_0)$, the differential as the kernel. Know we can similarly define such maps on each $D(x_i)$ and essentially the same argument shows that the kernel is the differential. For example on $D(x_1)$, we define:

$$g_0 dx_{0/1} + g_2 dx_{2/1} + \cdots + g_n dx_{n/1} \mapsto (\frac{g_0}{x_1}, -g_0 \frac{x_0}{x_1^2} - \cdots - g_n \frac{x_n}{x_1^2}, \frac{g_2}{x_1}, \cdots, \frac{g_n}{x_1})$$

The only issue now is does this assignment glues well on intersections. For example, let's check on the intersection of $U_0 = D(x_0)$ and $U_1 = D(x_1)$. This checking is quite tedious, thus I will not do it here. But what is more important is to know how to check it. For example, how do you check locally defined morphism between quasicoherent sheaves is well defined. What you do is essentially doing identifications to reduced something which involves too many to be checked to some algebra question (which is essentially why in so many cases we care about equivalence of categories a lot). And in this case, all the identifications can be summarized to consider the following:

$$\begin{tikzcd}
k[\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}] \arrow[dr] & & k[\frac{x_0}{x_1}, \cdots, \frac{x_n}{x_1}] \arrow[dl] \\
& k[\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}, \frac{x_0}{x_1}, \cdots, \frac{x_n}{x_1}] &
\end{tikzcd}$$

and how the differentials behave with this ring maps. Essentially everything happens here is related to the fact that $S^{-1}\Omega_{A/B} \simeq \Omega_{S^{-1}A/T^{-1}B}$, and I will let you see why. $\square$

Now we use this Euler exact sequence to give the counterexample we promised:

Proposition 3.19. $h^1(\mathbb{P}^2_k, \Omega_{\mathbb{P}^2_k/k}) > 0$, and in particular, $\Omega_{\mathbb{P}^2_k/k}$ is not a direct sum of line bundles and can't even be filtered by line bundles.

Proof. Recall that the Euler exact sequence is:

$$0 \to \Omega_{\mathbb{P}^2_k/k} \to \mathcal{O}(-1)^{\oplus 3} \to \mathcal{O} \to 0$$

Then we get an associated LES of cohomology:

$$0 \to H^0(\Omega_{\mathbb{P}^2_k/k}) \to H^0(\mathcal{O}(-1))^{\oplus 3} \to H^0(\mathcal{O}) \to H^1(\Omega_{\mathbb{P}^2_k/k}) \to H^1(\mathcal{O}(-1))^{\oplus 3} \to \cdots$$

Recall that $H^0(\mathcal{O}(-1)) = H^1(\mathcal{O}(-1)) = 0$, therefore we have an isomorphism:

$$H^0(\mathcal{O}) \simeq H^1(\Omega_{\mathbb{P}^2_k/k})$$

Recall that $h^0(\mathcal{O}) = 1$, therefore $h^1(\Omega_{\mathbb{P}^2_k/k}) = 1 > 0$.

This implies that this is never a direct sum of line bundles as we know that line bundles on $\mathbb{P}^2_k$ are all $\mathcal{O}(n)$ and we know that $H^1(\mathcal{O}(n))$ on $\mathbb{P}^2_k$ are all 0.

I claim that we can even show that $\Omega_{\mathbb{P}^2_k/k}$ can not be filtered into line bundles. Suppose that it's possible, since we know that the rank of $\Omega_{\mathbb{P}^2_k/k}$ is 2, then suppose that the filtration is:

$$0 \to \mathcal{O}(m) \to \Omega_{\mathbb{P}^2_k/k} \to \mathcal{O}(n) \to 0$$

we know that part of the LES of cohomology is:

$$0 = H^1(\mathcal{O}(m) \to H^1(\Omega_{\mathbb{P}^2_k/k}) \to H^1(\Omega(n)) = 0$$

Therefore we know that $H^1(\Omega) = 0$ as well, which is a contradiction. $\square$

4 The Riemann-Hurwitz Formula

Definition 4.1. A finite morphism between integral schemes $X \to Y$ is said to be separable if it's dominant and the induced extension of function fields $K(X)/K(Y)$ is separable.

Similarly, a generically finite morphism is generically separable if it's dominant and the induced extension of function fields is a finite separable extension. Note that in characteristic 0 every dominant finite morphism is automatically separable.

Proposition 4.2. If $\pi : X \to Y$ is a generically finite morphism of irreducible varieties of the same dimension n and X, Y are smooth then the cotangent exact sequence is exact on the left:

$$0 \to \pi^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0$$

Proof. We need to check this first arrow is injective, or in other words, the kernel is 0. Recall that since Y is smooth of dimension n, $\Omega_{Y/k}$ is locally free of rank n, then we know that $\pi^*\Omega_{Y/k}$ is also locally free of rank n on X.

Recall that on an integral scheme, any locally free sheaf $\mathcal{F}$ is torsion free, i.e. $\mathcal{F}_p$ is a torsion free $\mathcal{O}_{X,p}$-module since we know that $\mathcal{O}_{X,p}$ is a domain and $\mathcal{F}_p$ is free. Moreover, given any point p say the generic point is η, I claim that we have an injective map:

$$\mathcal{F}_p \to \mathcal{F}_\eta$$

Knowing this, if we have a subsheaf $\mathcal{G}$ of $\mathcal{F}$ (such that the kernel of a morphism), we have a diagram:

$$\begin{tikzcd} \mathcal{F}_p \arrow[r, hook] & \mathcal{F}_\eta \\ \mathcal{G}_p \arrow[u, hook] \arrow[r] & \mathcal{G}_\eta \arrow[u, hook] \end{tikzcd}$$

Then this shows that $\mathcal{G}_p \to \mathcal{G}_\eta$ is also injective, therefore we know that $\mathcal{G}$ is not zero iff $\mathcal{G}_\eta$ is not 0.

Therefore to show injectivity, we only need to show that the morphism is injective at the generic point η. Therefore we can first assume that this morphism of schemes is $SpecA \to SpecB$ and $\Omega_{X/Y}$ is just $\Omega_{A/B}$. We localize this at the generic point, and recall that the localization is isomorphic to $\Omega_{K(A)/K(B)}$, recall that $K(A)/K(B)$ is finite separable by assumption, then $\Omega_{K(A)/K(B)}$ is 0. Moreover, we know that the stalk of $\pi^*\Omega_{Y/k} \to \Omega_{X/k}$ are both n-dimensional $\mathcal{O}_\eta$-spaces, then after we take the stalk, the sequence is still exact and becomes:

$$\mathcal{O}_\eta^{\oplus n} \to \mathcal{O}_\eta^{\oplus n} \to 0 \to 0$$

Then since it's a surjective map between spaces of the same dimension, it's also injective, therefore we are done. $\square$

Remark 4.3. Let's take a look at what could go wrong if we work with a field of characteristic p. If you examine the above proof, the kep where we used separability is to show that $\Omega_{X/Y}$ is 0 after we take the stalk at η, which is automatically true if the characteristic is 0.

Now consider the morphism $\pi : \mathbb{A}_k^1 = Speck[t] \to \mathbb{A}_k^1 = Speck[u]$ defined by $u \mapsto t^p$, notice this defines a finite morphims. What is the module of differentials in this case, i.e. what is $\Omega_{k[t]/k[u]}$. I claim that $\Omega_{k[t]/k[u]}$ is a free $k[t]$-module generated by dt. To see why, let's directly use:

$$k[t] \otimes \Omega_{k[u]/k} \to \Omega_{k[t]/k} \to \Omega_{k[t]/k[u]} \to 0$$

Notice that this first arrow is defined by mapping $f \otimes du \mapsto f \cdot d(t^p) = pf \cdot t^{p-1}dt = 0$, therefore the first morphism is 0. Therefore the second morphism is an isomorphism, i.e. $\Omega_{k[t]/k[u]} = k[t]dt$.

Proposition 4.4. If X, Y are irreducible smooth varieties of dimension n and $\pi : X \to Y$ is a generically separable (in particular generically finite), then the locus of non-smoothness is pure codimension 1 and has a natural interpretation as an effective Cartier divisor.

Proof. Recall that since X, Y are both smooth of relative dimension n, therefore we know that $\Omega_{X/k}$ and $\Omega_{Y/k}$ are locally free of rank n. Let $x \in X$, let the image in Y be y. Then we can assume that locally the map π looks like:

$$Speck[x_1, \cdots, x_N]/(f_1, \cdots, f_r) \to Speck[y_1, \cdots, y_M]/(g_1, \cdots, g_s)$$

Let's assume that by a reordering of the variables, the module of relative differentials are free with basis $dx_1, \cdots, dx_n$ and $dy_1, \cdots, dy_n$. Then we know that locally this map ϕ looks likes:

$$\oplus Ady_i \to \oplus Adx_i$$

where $dy_i \mapsto d(\pi(y_i))$. If we denote $\pi(y_i)$ by $h_i(x_1, \cdots, x_N)$, we know that

$$d(\pi(y_i)) = d(h_i) = \sum_{j=1}^{n} \frac{\partial h_i}{\partial x_j} dx_j$$

Then we know that locally ϕ is represented by this Jacobian matrix:

$$J = (\frac{\partial h_i}{\partial x_j})_{i,j}$$

Now to fully justify what we will do afterwards, we will have to use something we haven't proved, which will be discussed later when we discuss flatness of morphisms. First, since $Y \to Speck$ is smooth, it's unramified, then since $X \to Y \to Speck$ is smooth, it's flat, then $X \to Y$ is flat. Then to prove $X \to Y$ is smooth at a point x, since X, Y are both of dimension n, we only need to show that it's unramified at x(as it's smooth iff etale iff flat and unramified), which is equivalent to $(\Omega_{X/Y})_x = 0$. Therefore we know that $X \to Y$ is smooth at x iff $(\Omega_{X/Y})_x = 0$.

Now let's consider the determinant of the Jacobian matrix $\det J$, we know that after we localize at a point x, π is smooth at x iff $(\Omega_{X/Y})_x$ is 0 iff ϕ_x is surjective. If ϕ_x is surjective, we know that after we quotient out the maximal ideal, it's again surjective, but this times since we work with vector spaces, it's an isomorphism, and it's an isomorphism implies that the determinant is not 0, i.e. the determinant doesn't vanish at x iff it's a unit in $\mathcal{O}_{X,x}$. Conversely, if the determinant is a unit, the morphism is an isomorphism at x then π is smooth at x. Therefore we know that π is smooth at x iff the determinant doesn't vanish at that point. Therefore we know that the nonsmooth locus is the vanishing subscheme of the determinant, at least locally.

Now we will need the assumption that π is generically separable, which means that at least at the generic point $\Omega_{X/Y}$ is 0, thus the nonsmooth locus could not be the whole variety X. Now consider the determinant of the Jacobian matrix, we already know that it's not 0, otherwise there will be no smooth locus, which is a contradiction to our discussion above. Therefore by Krull's principal ideal theorem we know that the vanishing subscheme is of pure codimension 1.

This argument works for every $x \in X$ therefore the nonsmooth locus is indeed of pure codimension 1. Technically speaking, what we showed is that locally the nonsmooth locus is an effective Cartier divisor, but we didn't show that these local descriptions patch together to give a global effective Cartier divisor. Let's first put this away, we will discuss this question later again as needed. $\square$

Definition 4.5. Let $\pi : X \to Y$ be a generically finite morphism between irreducible k-varieties of dimension n. We will call the locus of non-smoothness in X the **ramification locus** and it's image in Y the **branch locus**.

Due to the above proposition, we will also use the term **ramification divisor** and **branch divisor** (assume the image is still a divisor).

As you will guess, our main interest lies in the case where X, Y are of dimension 1, i.e. are curves. But before we do this, let's consider the following case: Suppose that X, Y are irreducible smooth k-varieties and $\pi : X \to Y$ is a surjective generically separable morphism such that there is an open U of X where π is an isomorphism over that open, if the complement of U has codimension at least 2, then we know that Y can not be smooth because we know that π the locus where π is not smooth at least have codimension 2, but if Y is smooth, the previous example shows that the non-smooth locus is of codimension 1, a contradiction.

Now let X, Y be smooth irreducible k-varieties dimension 1, and π is still generically separable. Then we know that the non-smooth locus, aka the ramification locus, is of codimension 1, i.e. dimension 0, i.e. a finite collection of points, therefore we know that the branch locus is also a finite collection of points. From now on, let's assume $k = \overline{k}$ and let $p \in X$ be an arbitrary point, we will examine $\Omega_{X/Y}$ near p, i.e. we want to understand the stalk of $\Omega_{X/Y}$ at p.

Let $q = \pi(p) \in Y$, recall that $\Omega_{X/Y}$ respects base change, then consider the canonical morphism $Spec\mathcal{O}_{Y,q} \to Y$ and the base change, consider the following Cartesian diagram:

$$\begin{tikzcd} X' \arrow[r, "\pi'"] \arrow[d, "f'"'] & Spec\mathcal{O}_{Y,q} \arrow[d, "f"] \\ X \arrow[r, "\pi"'] & Y \end{tikzcd}$$

Then we know that $(f')^*\Omega_{X/Y} \simeq \Omega_{X'/Spec\mathcal{O}_{Y,q}}$. Recall that if $p' \in X'$ maps to p by f', then we know that $(\Omega_{X'/Spec\mathcal{O}_{Y,q}})_{p'} = (\Omega_{X/Y})_p$, therefore the stalk at p is the same as computing the stalk of $\Omega_{X'/Spec\mathcal{O}_{Y,q}}$ at p'. Let's denote $Y' = Spec\mathcal{O}_{Y,q}$, then we know from Y is smooth that $\mathcal{O}_{Y,q}$ is a discrete valuation ring. Then if we consider the sheaf $\Omega_{X'/Y'}$ near p', we can first assume that X' is affine by restricting to an affine open of that point, then we know that the image of p' in Y' is the maximal ideal. Since the module of differentials is again compatible with localizations, we know that $(\Omega_{X'/Y'})_{p'}$ is the same as the module of differentials $\Omega_{\mathcal{O}_{X,p}/\mathcal{O}_{Y,q}}$. I claim that $\mathcal{O}_{X,p}$ is also a discrete valuation ring. This is because we have $\mathcal{O}_{X',p'} = \mathcal{O}_{X,p}$ canonically. Therefore we know that π correspond to a ring map $B \to A$ where A, B are discrete valuation rings $B \to A$ induces an isomorphism on residue fields $\overline{k}$.

Suppose that the uniformizer of A is s and the uniformizer of B is t, then let's assume that $t \mapsto us^n$ where u is invertible. Then we know that:

$$dt = d(us^n) = uns^{n-1}ds + s^n du$$

Notice that this differential on $SpecA$ vanishes to the order at least $n-1$ at the maximal ideal of A. If n is not divisible by the order p, we know that it vanishes to the order of precisely $n-1$.

Definition 4.6. If the vanishing order is $n-1$, we call it **tame ramification** and if the order is greater than $n-1$, we call it **wild ramification**.

We also define this order plus 1 to be the **ramification index** of π at this point of X.

Proposition 4.7. The degree of $\Omega_{X/Y}$ at $p \in X$ is precisely the ramification index of π at p minus 1, where X is a projective curve over $k = \overline{k}$.

Proof. Again, we need to assume that $k = \overline{k}$ in this proposition.

Let's first figure out what does this degree at a point mean. Recall that on an irreducible projective curve X, we can define the degree of a coherent sheaf $\mathcal{F}$:

$$\deg\mathcal{F} = \chi(X, \mathcal{F}) - \operatorname{rank}\mathcal{F} \cdot \chi(X, \mathcal{O}_X)$$

Now assume further that $\mathcal{F}$ is a torsion sheaf, which means that $\operatorname{rank}\mathcal{F} = 0$ and it only supports on a finite collection of closed points. We first know that $\deg\mathcal{F} = \chi(X, \mathcal{F})$. Moreover, if we compute $\chi(X, \mathcal{F})$, we know that since $H^1(X, \mathcal{F})$ vanishes, we know that $\chi(X, \mathcal{F})$ is just $h^0(X, \mathcal{F})$. Recall that we know that $\mathcal{F}$ is supported on a finite collection of closed points, say $p_1, \cdots, p_n$, we can choose an affine open U that contains all these points (do you see why), then let's assume $U = SpecA$ and $\mathcal{F}|_{SpecA} = \widetilde{M}$ for a finitely A-module M and $\chi(X, \mathcal{F}) = \dim_k M$. First, we know that since A is Noetherian and M is finitely generated supported on a finite collection of points, M is finite length, and we can assume that the composition series of M is:

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

We know that each $M_i/M_{i-1} \simeq A/m_i = \overline{k}$ where m_i is some maximal ideal which M supports at. Then we know that the dimension of M as a k-vector space is indeed it's length of M. Finally, suppose m_i is a maximal ideal M supports at, we know that we can localize the composition series above to get a composition series of M_{m_i} and we know that:

$$\operatorname{length}M = \sum_{m \in \operatorname{Supp}M} \operatorname{length}M_m$$

Therefore we get the desired formula:

$$\deg\mathcal{F} = \chi(X, \mathcal{F}) = \operatorname{length}M = \sum_{m \in \operatorname{Supp}M} \operatorname{length}M_m$$

Then we can naturally define the degree of $\mathcal{F}$ at m to be the length of $\mathcal{F}_m$.

Therefore we know that the degree of $\Omega_{X/Y}$ at p is just the length of $(\Omega_{X/Y})_p$ as an $\mathcal{O}_{X,p}$-module. We know from the above discussion that this is the module of differentials of the map:

$$\mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$$

where q is the image of p under π. Let say this is $B \to A$ where the uniformizer of B is t and the uniformizer of A is s and $t \mapsto us^n$. Consider $k \to B \to A$, we have a exact sequence:

$$A \otimes_B \Omega_{B/k} \to \Omega_{A/k} \to \Omega_{A/B} \to 0$$

Therefore we know that $\Omega_{A/B}$ is the cokernel of $A \otimes_B \Omega_{B/k} \to \Omega_{A/k}$ which is a map:

$$Adt \mapsto Ads$$

where dt is mapped to $uns^{n-1}ds + s^n du$. Assume that the ramification index at this point is i, then we know that $\Omega_{A/B} \simeq A/(s^{i-1})$ which is of dimension $i-1$ over $k = \overline{k}$ and we know that the length of $A/(s^{i-1})$ at the unique maximal ideal of A generated by s is equal to the dimension, we are done. ☐

Proposition 4.8. Let's still take $k = \overline{k}$ to be an algebraically closed field, $\pi : X \to Y$ is a genericially separable morphism between two smooth irreducible projective curves over k (then I will let you check that this is necessarily finite). Recall that in this case we define the degree of π to be the rank of the locally free sheaf $\pi_*\mathcal{O}_X$.

Then suppose that $q \in Y$ is a point such that all the ramification points above q is tame, which is always true in characteristic 0, then the degree of the branch divisor at q is $\deg\pi - |\pi^{-1}q|$. Therefore, the multiplicity of the branch divisor at q counts the extend to which the number of preimages is less than the degree.

Proof. Let's first compute the degree of the branch divisor. But first of all what is the branch divisor, to answer this question, let's first answer the question of what is the ramification divisor. Recall that we have seen that the ramification locus is locally described by the vanishing closed subscheme of the determinant of some Jacobian matrix coming from the sequence:

$$\pi^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0$$

We know that this vanishsing subscheme is just a single point, with some multiplicities when X, Y are irreducible smooth of dimension 1. How to compute the multiplicity? I claim that it's just the ramification index minus 1, because we know that the multiplicity is just the order of that determinant in $m_p\mathcal{O}_{X,p}$. I will let you check that this is the ramification index minus 1. Then I will let you see that the branch divisor could be defined as:

$$\sum (e_p - 1)p$$

where p ranges over all ramification points and e_p is the ramification index at p.

Then we can define the branch divisor to be the pushforward of the ramification divisor, i.e. it's defined by:

$$\sum_{q=\pi(p)} (\sum_{p\in\pi^{-1}q} (e_p - 1))q$$

Then we just want to show that $\sum_{p\in\pi^{-1}q}(e_p - 1)$ is $\deg\pi - |\pi^{-1}q|$. Let's recall that one way to compute the degree of π over algebraically closed fields is to use:

$$\deg\pi = \sum_{p\in\pi^{-1}q} \mathrm{val}_p\pi^*t$$

where t is the uniformizer of $\mathcal{O}_{Y,q}$. Recall that $\mathrm{val}\pi^*t = e_p$, then I will let you see the desired equality right now is a triviality. □

Remark 4.9. The general definition of the pushforward of a divisor by a finite morphism is discussed before and our definition of the branch divisor is just a direct application of that.

Now we can finally discuss the Riemann-Hurwitz formula:

Theorem 4.10. Let's again assume that we work with a field $k = \overline{k}$. Suppose that $\pi : X \to Y$ is a finite separable morphism of irreducible projective regular curves, of pure degree n. Let R be the ramification divisor then:

$$2g(X) - 2 = n(2g(Y) - 2) + \deg R$$

Proof. Recall that we have the cotangent SES:

$$0 \to \pi^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0$$

Then we know from the fact that the degree is additive in SES that:

$$\deg\Omega_{X/k} = \deg\pi^*\Omega_{Y/k} + \deg\Omega_{X/Y}$$

Recall that $\Omega_{X/Y}$ is a torsion sheaf, therefore we know that it's degree is just $h^0(\Omega_{X/Y})$. Moreover, we know that since π is of degree n, then $\deg\pi^*\mathcal{O}_{Y/k} = n\deg\Omega_{Y/k}$.

Recall that

$$\deg\Omega_{X/k} = \chi(X, \Omega_{X/k}) - \chi(X, \mathcal{O}_X) = h^0(\Omega_{X/k}) - h^1(\Omega_{X/k}) - h^0(\mathcal{O}_X) + h^1(\mathcal{O}_X)$$

Also recall that for an irreducible smooth projective curve X $H^1(\Omega_{X/k}) \simeq H^0(\mathcal{O}_X)$ via Serre duality and we know that the dimension of $H^0(\mathcal{O}_X) = 1$. Similarly you argue that $h^0(\Omega_{X/k}) = h^1(\mathcal{O}_X) = g(X)$, therefore:

$$\deg\Omega_{X/k} = 2g(X) - 2$$

Similarly, you can argue $\deg\pi^*\mathcal{O}_{Y/k} = n(2g(Y) - 2)$.

Finally we only need to show that $h^0(\Omega_{X/Y})$ is $\deg R$, the degree of R which is defined to be the degree of the corresponding line bundle. Recall that $R = \sum_p (e_p - 1)p$ where p runs over all the ramification points, therefore the degree of R is just $\sum_p (e_p - 1)$. Now what is $h^0(\Omega_{X/Y})$, it's just the sum of the dimension of $(\Omega_{X/Y})_p$ when p ranges over all the ramification points. However, the dimension of $(\Omega_{X/Y})_p$ is just $(e_p - 1)$, therefore $\deg R = h^0(\Omega_{X/Y})$, which completes the proof. □

Now we can see some applications of this formula:

Example 4.11. Notice that by Riemann-Hurwitz formula, the degree of the ramification divisor R is always an even number, that is to say, counted with multiplicity, any cover of a curve (a surjective finite morphism of degree n) must always ramify at an even number of points.

Proposition 4.12. There is no nonconstant morphism from a smooth projective irreducible genus 2 curve to a smooth projective irreducible genus 3 curve, over an algebraically closed field of characteristic 0.

Remark 4.13. Recall that the degree of the ramification divisor is always at least nonnegative. However, if there is such a nonconstant map, it must be dominant, and therefore surjective, therefore finite. Since we work with fields of characteristic 0, this morphism is also finite separable. Therefore we can apply the Riemann-Hurwitz formula and obtain:

$$\deg R = (2g(X) - 2) - n(2g(Y) - 2) = 2 - 4n$$

where $n \geqslant 1$ is the degree of the morphism, therefore this is a negative number a contradiction.

Definition 4.14. Let X be a connected scheme, then if the only finite etale surjective morphisms from a connected scheme to X are isomorphisms, we say that X is **simply connected in the etale topology**.

We won't use this term later besides the following example, but if you will to read the etale cohomology note series, you will run into this term there.

Example 4.15. Let $k = \overline{k}$, then the only connected unbranched finite separable cover of $\mathbb{P}^1_k$ is an isomorphism. The reason is as follows:

In the future, we will see that if $X \to \mathbb{P}^1_k$ is a morphism with no ramification, then if X is connected, it will be a curve and smooth. Then we can apply the Riemann-Hurwitz formula while assuming the ramification divisor is 0, so:

$$2g(X) - 2 = -2n + 0$$

This shows that $2n = 2 - 2g(X)$ with $n \geqslant 1$, therefore we know that $n = 1$ and $g(X) = 0$. We have already seen that the only smooth projective curve with genus 0 is the projective line $\mathbb{P}^1$.

Example 4.16. This is an example that is somewhat similar to the previous one. We can show that in characteristic 0 and when $k = \overline{k}$, the only connected unbranched cover of $\mathbb{A}^1_k$ is itself.

Say $X \to \mathbb{A}^1_k$ is a finite surjective etale morphism where X is smooth and connected. Recall that for smooth curves (actually for regular curves) connectedness is equivalent to irreducibility. Therefore X is also an integral curve. Then consider the morphism $X \to \mathbb{A}^1_k$, which induces a finite separable field extension:

$$k(t) \hookrightarrow k(X)$$

Then consider the morphism:

$$X \to \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$$

This also gives you the field extension $k(t) \hookrightarrow k(X)$, then we can consider the normalization of $\mathbb{P}^1_k$ in $k(X)$, which gives you a diagram:

$$\begin{tikzcd} & \widetilde{X} \arrow[dr] & \\ X \arrow[ur, dashed] \arrow[rr] & & \mathbb{P}^1_k \end{tikzcd}$$

where $X \to \widetilde{X}$ is induced by the universal property of the normalization. Recall that $\widetilde{X}$ is a projective integral normal curve. However, for curves normal implies regular and when $k = \overline{k}$ regular is equivalent to smooth. Then if you recall how $\widetilde{X}$ is constructed, we find that $\widetilde{X} \to \mathbb{P}^1_k$ agrees with $X \to \mathbb{A}^1_k$ over $\mathbb{A}^1_k$. Therefore we know that $\widetilde{X} \to \mathbb{P}^1_k$ can only ramify over the point ∞. Then we can apply the Riemann-Hurwitz formula:

$$2g(\widetilde{X}) - 2 = d(2g(\mathbb{P}^1_k) - 2) + \deg R$$

We know that $2g(\widetilde{X}) - 2 \geqslant -2$ and on the right hand side we have $-2d + \deg R$ where d is the degree of the morphism. Recall that in the case of characteristic 0 all ramification points are tame, therefore we know that the degree of the branch divisor is $d - n$ where n is the number of the ramification points. Then since we know the degree of the branch divisor is the same as the degree of the ramification divisor when $k = \overline{k}$, we know that $\deg R = d - n$, therefore we have:

$$2g(\widetilde{X}) - 2 = -2d + -n = -d - n$$

where $d, n \geqslant 1$, therefore we know that the only possibility is $d = n = 1$ and $g(\widetilde{X}) = 0$, therefore we know $\widetilde{X} = \mathbb{P}^1_k$ and we know that by composing with automorphisms of $\mathbb{P}^1_k$ we can assume $\widetilde{X} \to \mathbb{P}^1_k$ is the identity, therefore $X \to \mathbb{A}^1_k$ is also the identity, then we are done.

Corollary 4.17. Luroth's Theorem Let's keep the notion in the Riemann-Hurwitz formula and assume further that $g(X) = 0$. This implies that $g(Y) = 0$ as well. Informally, the only maps from a genus 0 curve to a curve of positive genus are the constant maps. (In our case we assumed $k = \overline{k}$ and the characteristic is 0).

This has nonobvious algebraic consequences, i.e. any subfield of $k(x)$ which contains k is of the form $k(f(x))$ where $f(x)$ is a polynomial in $k(x)$, i.e. any subfield of $k(x)$ containing k is generated by a single element in $k(x)$. To see why, suppose that $k \subset L \subset k(x)$ is a subfield, then we have a corresponding nonconstant morphism:

$$\mathbb{P}^1_k \to C$$

where C is also a smooth projective curve, then using the above discussion, C is also $\mathbb{P}^1_k$, therefore the function field of C is $k(y)$ for some $y \in k(x)$.

Example 4.18. Using Luroth's Theorem, we can show the following result: There is no nonconstant morphism from $\mathbb{P}^1_{\mathbb{C}}$ to the Fermat curve $x^n + y^n = z^n$ in $\mathbb{P}^1_n$ for $n > 2$. Recall that by the Jacobian criterion, this is a smooth curve and it's obvious projective, irreducible.

Recall that the genus of a degree d hypersurface in $\mathbb{P}^1_k$ is computed by:

$$g = (d-1)(d-2)/2$$

Then when $d > 2$, say $d = 3$, then we know that $g > 0$, but Luroth theorem just say that there is no nonconstant morphism from $\mathbb{P}^1$ to it.

5 Understanding Smooth Varieties using Cotangent Bundles

Definition 5.1. Suppose that X is a projective scheme over k, then we know that $h^0(X, \Omega_{X/k})$ is an invariant of X.

We define the sheaf of relative i-forms by:

$$\Omega^i_{X/Y} = \bigwedge^i \Omega_{X/Y}$$

Sections of $\Omega^i_{X/Y}$ are called relative i-forms (over that open set). Obviously, $h^0(X, \Omega^i_{X/k})$ are also invariants of a projective k-scheme X.

In fact, what we can prove is that $h^0(X, \Omega^i_{X/k})$ are birational invariants when X is smooth:

Proposition 5.2. Let X, X' be birational irreducible smooth projective k-varieties, then for each i, $H^0(X, \Omega^i_{X/k}) \simeq H^0(X, \Omega^i_{X'/k})$.

Proof. Let $\phi : X \dashrightarrow X'$ be a fixed birational morphism. Then since X is Noetherian regular and X' is projective, we know that the complement of the domain of definition is at least of codimension 2. Let U be the domain of definition and we know that we can pullback the differentials on X' to get a morphism:

$$\phi^* : H^0(X', \Omega^i_{X'/k}) \to H^0(U, \Omega^i_{X/k})$$

I.e. we can first pullback $\Omega^i_{X'/k}$ to $\pi^*\Omega^i_{X'/k}$, then we have the morphism $\pi^*\Omega_{X'/k} \to \Omega_{U/k}$, which induces a morphism:

$$\pi^*\Omega^i_{X'/k} \simeq \bigwedge^i(\pi^*\Omega_{X'/k}) \to \bigwedge^i \Omega_{X/k}$$

Here we use the fact that there is a canonical isomorphism for a quasicoherent sheaf $\mathcal{F}$:

$$\pi^*(\bigwedge^i \mathcal{F}) \to \bigwedge^i \pi^*\mathcal{F}$$

where on affine opens can be identitfied with the canonical isomorphism:

$$B \otimes_A \bigwedge^i M \to \bigwedge^i B \otimes_A M$$

These together gives the morphism ϕ^*. Now given a section in $H^0(U, \Omega^i_{X/k})$, recall that since X is smooth $\Omega_{X/k}$ is locally free, therefore $\Omega^i_{X/k}$ is also locally free, then we know from the fact that X is normal and algebraic Hartog's lemma that such a sections extends to a section in $H^0(X, \Omega^i_{X/k})$, therefore we get a map $\phi^* : H^0(X', \Omega^i_{X'/k}) \to H^0(X, \Omega^i_{X/k})$.

Similarly, suppose that the inverse of the birational map is $\psi : X' \to X$, we get a morphism $\psi^* : H^0(X, \Omega^i_{X/k}) \to H^0(X', \Omega^i_{X'/k})$. I claim that these two are inverses of each other. Consider the composition $\psi^* \circ \phi^*$ for example. Suppose that $U' \subset X'$ is the image of $\phi : U \to X'$ then we know that ϕ^* actually fit in such a diagram below:

$$\begin{tikzcd}
H^0(X', \Omega^i_{X'/k}) \arrow[r, "\phi^*"] \arrow[d, "\mathrm{res}"'] & H^0(X, \Omega^i_{X/k}) \arrow[d, "\mathrm{res}"] \\
H^0(U', \Omega^i_{X'/k}) \arrow[r, "\phi^*"'] & H^0(U, \Omega^i_{U/k})
\end{tikzcd}$$

where the ϕ^* below is again defined by first pullbacks sections to $\pi^*\Omega^i_{X'/k}$ then consider the morphism $\pi^*\Omega^i_{X'/k} \to \Omega^i_{X/k}$. Then we can consider:

$$\begin{tikzcd}
H^0(X', \Omega^i_{X'/k}) \arrow[r, "\phi^*"] \arrow[d, "\mathrm{res}"'] & H^0(X, \Omega^i_{X/k}) \arrow[r, "\psi^*"'] \arrow[d, "\mathrm{res}"] & H^0(X', \Omega^i_{X'/k}) \arrow[d, dashed, "\mathrm{res}"] \\
H^0(U', \Omega^i_{X'/k}) \arrow[r, "\phi^*"'] & H^0(U, \Omega^i_{U/k}) \arrow[r, "\psi^*"'] & H^0(U', \Omega^i_{X'/k})
\end{tikzcd}$$

I claim that the composition of the two maps below is the identity. To see this, we first know that both ϕ^* and ψ^* commutes with restrictions, it's easy to see that $\pi^*\Omega^i_{X'/k} \to \Omega^i_{X/k}$ commutes with restrictions as it's defined locally. The key is to show that the first pullback maps commutes with restrictions. After you are done with it, you also need to show that on affines it can be identified by:

$$M \to M \otimes_B A, \qquad m \mapsto m \otimes 1$$

Then we can cover U' by affines and then conclude that the map is given by:

$$dt_1 \wedge \cdots \wedge dt_i \mapsto d\phi(t_1) \wedge \cdots \wedge d\phi(t_i) \mapsto d\psi\phi(t_1) \wedge \cdots \wedge d\psi\phi(t_i) = dt_1 \wedge \cdots \wedge dt_i$$

thus is the identity. I claim that this implies that the composition on the top is also the identity. To see why, the result we are using is that if X is an integral scheme and U is a dense open of X, then the restriction map $H^0(X, \mathcal{E}) \to H^0(U, \mathcal{E})$ is injective for any vector bundle $\mathcal{E}$. We know this holds for $\mathcal{O}_X$ and the proof for vector bundles is almost the same to $\mathcal{O}_X$. This is because we can use the sheaf property to reduce to the case X is affine and $\mathcal{E} = \mathcal{O}_X^{\oplus n}$. ☐

Definition 5.3. If X is a dimension n smooth k-variety, we know that $\Omega_{X/k}$ is a vector bundle of rank n, then we can define the canonical sheaf or the canonical bundle to be:

$$\mathscr{K}_{X/k} = \det\Omega_{X/k} = \bigwedge^n \Omega_{X/k}$$

After we prove Serre duality, we will see that when X is projective, this will be the dualizing sheaf.

The following result is sometimes called the **adjunction formula** for $\mathscr{K}_X$:

Proposition 5.4. Suppose that X is a smooth variety and $Z \subset X$ is a smooth subvariety of X, then:

$$\mathscr{K}_Z = \mathscr{K}_X|_Z \otimes \det\mathscr{N}_{Z/X}$$

Proof. Consider the closed embedding $i : Z \hookrightarrow X$, since both X and Z are smooth, we obtain the following conormal exact sequence:

$$0 \to \mathscr{N}^\vee_{Z/X} \to i^*\Omega_{X/k} \to \Omega_{Z/k} \to 0$$

Since we know that both $\Omega_{Z/k}$ and $i^*\Omega_{X/k}$ are locally free of finite rank, we know that $\mathscr{N}^\vee_{X/Y}$ is also locally free of finite rank. Then recall that if we have a SES of locally free sheaves of finite rank on a scheme X:

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

Then we know that we have:

$$\det\mathcal{E}' \otimes \det\mathcal{E}^\vee \otimes \det\mathcal{E}'' \simeq \mathcal{O}_X$$

Apply this to the conormal exact sequence on Z:

$$\det\mathscr{N}^\vee_{Z/X} \otimes \det(i^*\Omega_{X/k})^\vee \otimes \det\Omega_{Z/k} \simeq \mathcal{O}_Z$$

This implies that:

$$\det\mathscr{N}_{Z/X} \otimes \det i^*\Omega_{X/k} \simeq \det\Omega_{Z/k} = \mathscr{K}_Z$$

Recall that we have a canonical isomorphism:

$$\det i^*\Omega_{X/k} \simeq i^*\det\Omega_{X/k} = \mathscr{K}_X|_Z$$

since we know that $i^*\Omega_{X/k}$ on Z and $\Omega_{X/k}$ are of the same rank. This concludes the proof. $\square$

Remark 5.5. Recall that if Z is an effective Cartier divisor, then we have an isomorphism:

$$\mathscr{N}_{Z/X} \simeq \mathcal{O}_X(Z)|_Z$$

Then the above adjunction formula becomes:

$$\mathscr{K}_Z \simeq \mathscr{K}_X|_Z \otimes \mathcal{O}(Z)|_Z \simeq (\mathscr{K}_X \otimes \mathcal{O}_X(Z))|_Z$$

Definition 5.6. Suppose that X is a projective smooth k-variety, the birational invariant $h^0(X, \mathscr{K}_X)$ is called the **geometric genus** of X, denoted by $p_g(X)$. This agrees with our definition earlier for curves.

Suppose that X is not smooth, then if there is a smooth projective k-variety X' that is birational to X, we define:

$$p_g(X) = p_g(X')$$

For example, if X is an irreducible projective curve over k, then it's geometric genus can be defined to be the geometric genus of it's normalization $\tilde{X}$. Since this is a birational invariant, this definition is independent of choices.

Example 5.7. As an example, let's suppose $k = \overline{k}$ and suppose that Z is a regular degree d surface in $\mathbb{P}^3_k$, we will compute the geometric genus of Z.

We don't have to use the adjunction formula above and instead we will do some direct computation. First I claim that Z is irreducible, and I will let you check it (suppose it's not then the defining polynomial is the product, then use the fact that the subschemes defined by these to factors always intersect to draw a contradiction).

Then we can use Serre duality to conclude that $\mathscr{K}_Z$ is the dualizing sheaf and:

$$h^0(Z, \mathscr{K}_Z) = h^2(Z, \mathcal{O}(Z)) = h^2(X, i_*\mathcal{O}_Z)$$

where $i : Z \hookrightarrow X = \mathbb{P}^3_k$ is the closed embedding of the surface. Recall that we have a SES:

$$0 \to \mathcal{O}_X(-d) \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$$

Then consider the induced LES, part of that is given by:

$$H^2(X, \mathcal{O}_X(-d)) = 0 \to H^2(X, \mathcal{O}_X) = 0 \to H^2(X, i_*\mathcal{O}_Z) \to H^3(X, \mathcal{O}_X(-d)) \to H^3(X, \mathcal{O}_X) = 0$$

Therefore we have a $p_g(Z) = h^3(X, \mathcal{O}_X(-d))$, and we know that when $d \leqslant -4$, it's of dimension $\binom{-d-1}{-d-4}$ and if not $p_g(Z) = 0$. In particular when $d = 4$, $p_g(Z) = 1$.

Let's also compute the geometric genus of $\mathbb{P}^2_k$ which is $h^2(\mathbb{P}^2_k, \mathcal{O}_X)$ and we know that it's 0. Therefore I claim that there is no regular quartic surface in $\mathbb{P}^3_k$ that is rational (i.e. birational to $\mathbb{P}^2_k$) since they have different geometric genus, one being 0 and one being 1.

Now let's introduce some properties of smooth k varieties that is related to it's canonical bundle $\mathscr{K}_X$:

Definition 5.8. Suppose that X is a smooth projective k-variety, then X is said to be **Fano** if $\mathscr{K}_X^\vee$ is ample and it's called **Calabi-Yau** if $\mathscr{K}_X \simeq \mathcal{O}_X$.

Definition 5.9. Again let X be a smooth projective k-variety, then we define it's ***j*-th plurigenus** by:

$$h^0(X, \mathscr{K}_X^{\otimes j})$$

Proposition 5.10. If $j \geqslant 0$, then the j-th plurigenus is a birational invariant, i.e. if X' is a smooth projective variety that is birational to X, then they have the same j-th plurigenus.

Proof. Let $\phi : X \dashrightarrow X'$ be a birational morphism, then if the domain of definition is U whose image denoted by U', we know that the complement of U is at most codimension 2. Therefore we get a map:

$$H^0(X', \mathscr{K}_{X'}^{\otimes j}) \to H^0(X, \mathscr{K}_X^{\otimes j})$$

by first pulling back a section of $\mathscr{K}_{X'}^{\otimes j}$ to U', then to $\phi^*\mathscr{K}_{U'}^{\otimes j}$. Recall that we have a morphism $\phi^*\mathscr{K}_{U'} \to \mathscr{K}_U$ which is constructed before. This induces a morphism between the tensor products, i.e. we have a morphism:

$$\phi^*\mathscr{K}_{U'}^{\otimes j} \to \mathscr{K}_U^{\otimes j}$$

thus we get a section of $\mathscr{K}_X^{\otimes j}$ on U. Again, since X is smooth, $\mathscr{K}_X^{\otimes j}$ is locally free, therefore Algebraic Hartog's lemma gives the extension of that section to X, thus we obtain a section of $\mathscr{K}_X^{\otimes j}$ on X. Let's denote this morphism as:

$$\phi^* : H^0(X', \mathscr{K}_{X'}^{\otimes j}) \to H^0(X, \mathscr{K}_X^{\otimes j})$$

By definition, it fits into the following diagram:

$$\begin{tikzcd}
H^0(X', \mathscr{K}_{X'}^{\otimes j}) \arrow[r, "\phi^*"] \arrow[d, "\mathrm{Res}"] & H^0(X, \mathscr{K}_X^{\otimes j}) \arrow[d, "\mathrm{Res}"] \\
H^0(U', \mathscr{K}_{X'}^{\otimes j}) \arrow[r, "\phi^*"] & H^0(U, \mathscr{K}_X^{\otimes j})
\end{tikzcd}$$

Then similar to what we proved when we showed the geometric genus is a birational invariant, say $\psi : X' \to X$ is the inverse of the birational morphism, we know that the composition of ϕ^* and ψ^* on U, U' is the identity, you prove it basically by the same way like how you proved it's the identity on $\mathscr{K}_{X'} \to \mathscr{K}_X$ since our definition here is compatible with the tensors. Then we again use the restrictions to U, U' are injective to conclude it's the identity on X, X'. $\square$

Definition 5.11. Let X be a smooth projective k-variety, we define the **Kodaira dimension** of X, denoted by $\kappa(X)$ by the smallest k such that $h^0(X, \mathscr{K}_X^{\otimes j})/j^k$ is bounded for all j.

If $h^0(X, \mathscr{K}_X^{\otimes j})$ are all 0, we define $\kappa(X) = -1$. The previous proposition shows that the Kodaira dimension is also a birational invariant. Finally, if $\kappa(X) = \dim X$, we say that X is of **general type**.

Remark 5.12. It's a nontrivial fact that Kodaira dimension for smooth projective k-varieties always exists and is always between -1 and $\dim X$ inclusive. Therefore, a general type X is such a variety that it's Kodaira dimension is as large as possible.

Proposition 5.13. If $\mathscr{K}_X$ is ample, then X is general type.

Proof. Let the dimension of X be n. We need to estimate the growth of $h^0(X, \mathscr{K}_X^{\otimes j})/j^k$. First, we know that since $\mathscr{K}_X$ is ample and X is projective, we know that for j very large, all $H^i(X, \mathscr{K}_X^{\otimes j}) = 0$ for $i > 0$.

Therefore we know that for large j:

$$h^0(X, \mathscr{K}_X^{\otimes j}) = \chi(X, \mathscr{K}_X^{\otimes j})$$

Recall that for a smooth projective k-variety, let $\mathcal{F}$ be a coherent sheaf on X, then if $\dim\mathrm{Supp}\mathcal{F} = n$, then if $\mathcal{L}$ is a line bundle on X, we have:

$$q(m) = \chi(X, \mathcal{L}^{\otimes m} \otimes \mathcal{F}) = (\mathcal{L}^n \cdot \mathcal{F})/n! \cdot m^n + \cdots$$

where the $\cdots$ are some lower degree terms. Apply this to $\mathcal{L} = \mathscr{K}_X$ and $\mathcal{F} = \mathcal{O}_X$, then we know from the asymptotic Riemann-Roch that that for large m:

$$q(m) = h^0(X, \mathscr{K}_X^{\otimes m}) = (\mathscr{K}_X^n)/n! \cdot m^n + \cdots$$

Recall that since $\mathscr{K}_X$ is ample with support dimension n, we have $(\mathscr{K}_X^n) = (\mathscr{K}_X^n \cdot X) > 0$, therefore $q(m)$ is indeed a polynomial of degree n, and we know that for $k < n$, we can roughly estimate the growth rate of $q(m)/m^k$ by a polynomial of $n-k$, which is obviously not bounded, and you can see that $k = n = \dim X$ is exactly the smallest integer such that $q(m)/m^k$ is bounded. ☐

For curves, we have the following useful result:

Proposition 5.14. Let X be a geometrically irreducible smooth projecctive curve over k. Then it's Fano (resp. Calabi-Yau, general type) if it's of genus 0 (resp. 1 and > 1).

Proof. If the genus is 0, we want to prove that $\mathscr{K}_X^\vee$ is ample. To prove this, we only need to show that for large m $\mathscr{K}_X^{\otimes -m}$ is very ample. I claim that the degree of $\mathscr{K}_X^\vee$ is positive, in fact 2 in case, which then implies that for large m the degree of $\mathscr{K}_X^{\otimes -m}$ is arbitrarily large.

Now recall that smooth is stable under any base change of field extensions of k, therefore X is geometrically regular. Since it's also geometrically irreducible, we know that in this case $\deg\mathcal{L} \geqslant 2g+1$ implies that $\mathcal{L}$ is very ample, where g is the genus of X. Therefore we can conclude that $\mathscr{K}_X^\vee$ in this case is actually very ample and we don't have to raise it to higher powers.

Therefore our job is to now compute the degree of $\mathscr{K}_X^\vee$. Let's compute the degree of $\mathscr{K}_X$ first, by definition, it's:

$$\chi(X, \mathscr{K}_X) - \chi(X, \mathscr{O}_X)$$

Recall that $\chi(X, \mathscr{K}_X) = h^0(X, \mathscr{K}_X) - h^1(X, \mathscr{K}_X)$, now we can use Serre duality to conclude that it's actually $-\chi(X, \mathcal{O}_X)$, therefore we know that:

$$\deg\mathscr{K}_X = -2\chi(X, \mathcal{O}_X) = -2(h^0(X, \mathcal{O}_X) - g(X)) = -2$$

Then:

$$\deg\mathscr{K}_X^\vee = -\deg\mathscr{K}_X = 2$$

Now, let's prove if the genus is 1, then X is Calabi-Yau, i.e. we need to prove that $\mathscr{K}_X \simeq \mathcal{O}_X$. Recall that in this case, we can compute the degree of $\mathscr{K}_X$, which is again $-2\chi(X, \mathcal{O}_X)$. But in this case we know that $\chi(X, \mathcal{O}_X) = 0$, therefore the degree of the canonical bundle is 0. In this case, it's isomorphic to the structure sheaf iff it has a nonzero global section, but we know that:

$$h^0(X, \mathscr{K}_X) = h^1(X, \mathcal{O}_X) = g = 1$$

We are done.

Finally, if the genus is larger than 1, let's again compute the degree of $\mathscr{K}_X$, which is $-2\chi(X, \mathcal{O}_X) = -2(1-g) = 2g-2 > 0$. Recall that we already discussed that this implies that $\mathscr{K}_X$ is ample, which again implies X is general type. $\square$

Remark 5.15. This above trichotomy could be thought morally as the reason why curves behave differently depending on which of these three classes they lie in. In some sense, this classification extends to higher dimensions, which is why these definitions are important. The following result is one example of this.

Proposition 5.16. Let k be an algebraically closed field. Suppose that X is a smooth complete intersection in $\mathbb{P}^N_k$ of hypersurfaces of degree $d_1, \cdots, d_n$, then $\mathscr{K}_X \simeq \mathcal{O}(-N-1+d_1+\cdots+d_n)|_X$.

Proof. Let these hypersurfaces be $H_1, \cdots, H_n$ of degree $d_1, \cdots, d_n$, cut out by homogeneous polynomials $f_1, \cdots, f_n$ respectively. Recall that on feature of complete intersection in $\mathbb{P}^n_k$ of hypersurfaces is that H_1 is an effective Cartier divisor in $\mathbb{P}^N_k$ and $H_1 \cap H_2$ is an effective Cartier divisor of H_1 and you can keep going. The dimension of this complete intersection is $N-n$.

You probably would like to use the adjunction formula for effective Cartier divisors to conclude the result inductively. However, if you want to use it, we need to ensure that each of the intermediate intersections $H_1 \cap \cdots \cap H_i$ are all smooth, which is typically not true. Therefore, let's try to use the adjunction directly for $Z \hookrightarrow \mathbb{P}^N_k = X$. Therefore we know that:

$$\mathscr{K}_Z \simeq \mathscr{K}_X|_Z \otimes \det \mathscr{N}_{Z/X}$$

Recall that the canonical bundle on $\mathbb{P}^N_k$ is $\mathcal{O}(-N-1)$. Therefore we only need to figure out what is $\mathscr{N}_{Z/X}$. We can first figure out what is $\mathscr{N}^\vee_{Z/X}$, which is isomorphic to the pullback of $\mathcal{I}/\mathcal{I}^2$ where $\mathcal{I}$ is the ideal sheaf determining Z. I claim that this is isomorphic to $\oplus_{i=1}^n \mathcal{O}(-d_i)$ and to see why. Recall that $\mathscr{I} = \widetilde{(f_1, \cdots, f_r)}$ where $(f_1, \cdots, f_r)$ is a graded ideal whose grading comes from the grading on $S = k[x_0, \cdots, x_N]$. Then recall that we have a morphism of quasicoherent sheaves:

$$\widetilde{S(-d_i)} \to \widetilde{(f_1, \cdots, f_n)}$$

defined by the map of graded S-modules:

$$S(-d_i) \to (f_1, \cdots, f_n), \qquad 1 \mapsto f_i$$

Then we get a morphism:

$$\oplus_{i=1}^n \mathcal{O}(-d_i) \to \mathcal{I}/\mathcal{I}^2$$

and I claim that this is an isomorphism. To see why, recall that both sides are rank n vector bundles, and we only need to prove surjectivity in order to prove injectivity. To prove surjectivity, let's look at what happens on each $D_+(x_j)$. Recall that on $D_+(x_j)$, $\mathcal{I}$ is determined by the $k[\frac{x_0}{x_j}, \cdots, \frac{x_N}{x_j}]$-module generated by $(\frac{f_1}{x_j^{d_1}}, \cdots, \frac{f_n}{x_j^{d_n}})$ as an ideal of $k[\frac{x_0}{x_j}, \cdots, \frac{x_N}{x_j}]$, therefore we know that on $D_+(f)$, the sheaf $\mathcal{I}/\mathcal{I}^2$ is determined by the module $(\frac{f_1}{x_j^{d_1}}, \cdots, \frac{f_n}{x_j^{d_n}})/(\frac{f_1}{x_j^{d_1}}, \cdots, \frac{f_n}{x_j^{d_n}})^2$. Similarly, we know that $\mathscr{O}(-d_i)$ on each $D_+(x_j)$ is determined by the $k[\frac{x_0}{x_j}, \cdots, \frac{x_N}{x_j}]$-module $k[\frac{x_0}{x_j^{d_i+1}}, \cdots, \frac{x_N}{x_j^{d_i+1}}] = k[\frac{x_0}{x_j}, \cdots, \frac{x_N}{x_j}] \cdot \frac{1}{x_j^{d_i}}$. Then we know that on $D_+(x_j)$, the morphism $\mathcal{O}(-d_i) \to \mathcal{I}/\mathcal{I}^2$ correspond to the module map that maps $\frac{1}{x_j^{d_i}}$ to $\frac{f_i}{x_j^{d_i}}$. Then consider the map obtained via the direct sum, we know that it's already surjective to $(\frac{f_1}{x_j^{d_1}}, \cdots, \frac{f_n}{x_j^{d_n}})$, so of course surjective to $(\frac{f_1}{x_j^{d_1}}, \cdots, \frac{f_n}{x_j^{d_n}})/(\frac{f_1}{x_j^{d_1}}, \cdots, \frac{f_n}{x_j^{d_n}})^2$. Finally, we know that pullbacks preserves surjectivity, therefore we know that on X we have an isomorphism:

$$\oplus_{i=1}^n \mathcal{O}_X(-d_i) \simeq \mathcal{I}/\mathcal{I}^2$$

This proves that $\mathscr{N}_{Z/X} = \oplus_{i=1}^n \mathcal{O}_X(d_i)$. Then what is the determinant? I claim that if $\mathcal{E}$ and $\mathcal{F}$ are locally free sheaves of rank r, s, then the determinant of $\mathcal{E} \oplus \mathcal{F}$ is:

$$\bigwedge^{r+s}(\mathcal{E} \oplus \mathcal{F}) \simeq \bigwedge^{r} \mathcal{E} \otimes \bigwedge^{s} \mathcal{F}$$

To see why, recall that the right hand side has transition functions defined by product of determinant of the transition matrix of $\mathcal{E}$ and $\mathcal{F}$, so is the left, therefore they are isomorphic.

This implies that:

$$\det \oplus_{i=1}^n \mathcal{O}_X(d_i) = \otimes_{i=1}^n \det \mathcal{O}_X(d_i) = \mathcal{O}_X(\sum_{i=1}^n d_i)$$

Finally, we conclude that we have:

$$\mathscr{K}_Z \simeq \mathcal{O}_X(-N - 1 + d_1 + \cdots + d_n)$$

□

After we computed the canonical bundle, here are some applications:

Proposition 5.17. If $-N - 1 + d_1 + \cdots + d_n$ is negative (resp. zero, positive), then X is Fano (resp. Calabi-Yau, general type).

Proof. First let's assume that it's negative. We want to prove that $\mathscr{K}_Z^\vee \simeq \mathcal{O}_X(N + 1 - d_1 - \cdots - d_n)$ is ample. We know that $N + 1 - d_1 - \cdots - d_n$ is positive, therefore we know that it's the pullback of something ample on $\mathbb{P}_k^N$ by a closed embedding, thus still ample.

Then it's it's 0, obviously $\mathscr{K}_Z \simeq \mathcal{O}_Z$.

Finally, if it's positive, this time $\mathscr{K}_Z$ is ample, and we know this implies that the variety is general type. □

Example 5.18. Another example related to our discussion above is that we can find a list of all possible values of $N, d_1, \cdots, d_n$ such that X is Calabi-Yau of dimension at most three.

First, we know that the dimension of X is $N - n \leqslant 3$, i.e. we know that $N \leqslant n + 3$, and we know that if it's Calabi-Yau, we have:

$$d_1 + \cdots + d_n = N + 1$$

This implies that $d_1 + \cdots + d_n - n \leqslant 4$. Notice that each d_i is at least 1, therefore we know that for a fixed N, we know that $N - 3 \leqslant n \leqslant N$, and for each such n, we can choose $d_1, \cdots, d_n$ such that they are at least 1 and the $d_1 + \cdots + d_n - n \leqslant 4$.

Definition 5.19. Suppose that X is a projective k-variety of pure dimension 2, then it's called a **del Pezzo surface** if it's smooth and Fano.

Example 5.20. The following surfaces are del Pezzo: $\mathbb{P}_k^2, \mathbb{P}_k^1 \times \mathbb{P}_k^1$ and smooth surfaces in $\mathbb{P}_k^3$ of degree at most 3.

The first example is obvious, as we know that the canonical bundle on $\mathbb{P}_k^2$ is $\mathcal{O}(-3)$, therefore it's dual is $\mathcal{O}(3)$ ample and obviously the projective surface is smooth.

Recall that we know that there is an isomorphism $\Omega_{\mathbb{P}_k^1 \times_k \mathbb{P}_k^1/k} \simeq p_1^* \Omega_{\mathbb{P}_k^1} \oplus p_2^* \Omega_{\mathbb{P}_k^1/k}$, then we know that the determinant is:

$$p_1^* \mathcal{O}(-2) \otimes p_2^* \mathcal{O}(-2) = \mathcal{O}(-2, -2)$$

Therefore we know that the dual is $\mathcal{O}(2, 2)$ and we know that this is ample on $\mathbb{P}_k^1 \times_k \mathbb{P}_k^1$.

For the last example, we know that the canonical bundle is:

$$\mathcal{O}_X(-4 + d)$$

where $d \leqslant 3$, therefore the dual is ample.

Definition 5.21. A **K3 surface** over a field k is a proper smooth, geometrically connected Calabi-Yau surface X such that $H^1(X, \mathcal{O}_X) = 0$.

Example 5.22. As an example, let's show that smooth complete intersections in $\mathbb{P}^N_k$ of dimension 2 that are Calabi-Yau are all K3 surfaces.

First, we need to prove that it's geometrically connected, but we have already seen this before where we prove that all complete intersections X in $\mathbb{P}^N_k$ of positive degree are geometrically connected as $h^0(X, \mathcal{O}_X) = 1$. Then we need to show that $H^1(X, \mathcal{O}_X) = 0$. Recall that we have already showed that if X is a dimension m complete intersection in $\mathbb{P}^N_k$, then $h^i(X, \mathcal{O}_X(n)) = 0$ for all $0 < i < m$ and all n, then here $m = 2$ $i = 1$, thus we are done.

Even though the definitions of Fano, Calabi-Yau and general type are useful and we know that for curves all curves fall into this trichotomy. However, for surfaces, not every surface is of the kind of the above three:

Example 5.23. Let C be a smooth projective irreducible complex curve of genus greater than 1, then $C \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$ is neither Fano, Calabi-Yau nor general type.

To show this, recall that the canonical bundle on $X = C \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$ is just $p_1^*\mathscr{K}_C \otimes p_2^*\mathcal{O}(-2)$, then we know that it's dual is $p_1^*\mathcal{K}_C^\vee \otimes p_2^*\mathcal{O}(2)$ where p_1, p_2 are the two projections to C and $\mathbb{P}^1_{\mathbb{C}}$. Let's prove that this is not ample, thus X is not Fano.

Consider the closed embedding in the following diagram:

$$\begin{tikzcd}
\{t\} \times C \arrow[r, hook] \arrow[d] & \mathbb{P}^1_{\mathbb{C}} \times C \arrow[r] \arrow[d] & C \arrow[d] \\
\{t\} \arrow[r, hook] & \mathbb{P}^1_{\mathbb{C}} \arrow[r] & Spec\mathbb{C}
\end{tikzcd}$$

Let's consider what is the pullback of $p_1^*\mathcal{K}_C^\vee \otimes p_2^*\mathcal{O}(2)$ to $\{t\} \times C$. We know that $\{t\} \times C \to C$ is an isomorphism, and we know that $\{t\} \times C \to \{t\} \to \mathbb{P}^1_{\mathbb{C}}$ is the constant morphism, then since we know that pullback commute with tensors, I will let you show that the pullback is isomorphic to $\mathscr{K}_C$. Recall that we work with complex numbers, therefore we know that ample is equivalent to having positive degree, but we know that $\mathcal{K}_C$ is of positive degree, therefore it's dual is of negative degree, thus not ample. However, we know that ample pulls back to ample via closed embeddings, thus $p_1^*\mathcal{K}_C^\vee \otimes p_2^*\mathcal{O}(2)$ is not ample.

Then we will prove that $p_1^*\mathcal{K}_C \otimes p_2^*\mathcal{O}(-2)$ is not isomorphic to the structure sheaf. We can again use the pullback by closed embeddings argument since we know that structure sheaf pulls back to structure sheaf.

Finally, we need to prove that it's not general type, which means that we need to estimate the growth rate of $h^0(p_1^*\mathcal{K}_C^{\otimes m} \otimes p_2^*\mathcal{O}(-2m))$. I claim that it's actually 0 for all $m \geqslant 1$, thus the Kodaira dimension is -1, thus not general type. To see this, we can either use the Kunneth Formula which we will prove later or we can do it directly. Suppose that s is a section s in $H^0(p_1^*\mathcal{K}_C^{\otimes m} \otimes p_2^*\mathcal{O}(-2m))$, if we want to prove $s = 0$, all we need to do is to prove that for all closed points of the form (x, y) where x, y are closed points in C and $\mathbb{P}^1_{\mathbb{C}}$, $s|_{(x,y)} = 0$. Consider the following diagram:

$$\begin{tikzcd}
\{(x,y)\} \arrow[r, hook] \arrow[d, hook] & \mathbb{P}^1_{\mathbb{C}} \times \{y\} \arrow[r] \arrow[d, hook] & \{y\} \arrow[d, hook] \\
\{x\} \times C \arrow[r, hook] \arrow[d] & \mathbb{P}^1_{\mathbb{C}} \times C \arrow[r] \arrow[d] & C \arrow[d] \\
\{x\} \arrow[r, hook] & \mathbb{P}^1_{\mathbb{C}} \arrow[r] & Spec\mathbb{C}
\end{tikzcd}$$

We know that $s|_{(x,y)}$ vanishes iff the pullback to first $\mathbb{P}^1_{\mathbb{C}} \times \{y\}$ and then to $\{(x, y)\}$ is 0. However, we already know that the pullback to $\mathbb{P}^1_{\mathbb{C}} \times \{y\}$ is isomorphic to $\mathcal{O}(-2)$, which has no nonzero global section, therefore we are done.

Another important result about the canonical bundle $\mathcal{K}_X$ is the **Kodaira vanishing theorem**. However, since we will not use it later, we only breifly mention it's statement without proving:

Theorem 5.24. Suppose that k is a field of characteristic 0 and X is a smooth projective k variety. Then for any ample invertible sheaf $\mathcal{L}$, we know that $H^i(X, \mathscr{K}_X \otimes \mathcal{L}) = 0$ for all $i > 0$.

Even though we will not prove it, let's mention something about the proof. Firstly, the assumption that char$k = 0$ is necessarily and we indeed have an example where Kodaira vanishing fails in positive characteristic.

The original proof of this is to reduce the proof to the case of complex analytic version and then use Serre's GAGA theorem to turn it into a complex algebraic statement. The general characteristic case can be reduced to $\mathbb{C}$ using the so called Lefschetz principle. But there is also an algebraic proof available later. You will eventually run into the terms listed here if you continue the study.

Finally, before we end this section, let's talk a little bit about **Hodge theory** and discuss some basics of that.

Definition 5.25. Suppose that X is an irreducible smooth k-variety, then the invariant $h^j(X, \Omega_{X/k}^{\otimes i})$ is called the Hodge number $h^{i,j}(X)$ of X.

We have already saw that $h^{i,0}(X)$ are birational invariants of X. However, we will later see an example that for a general j, $h^{i,j}(X)$ is not a birational invariant.

Proposition 5.26. Let X be an n dimensional smooth projective k-variety which is geometrically irreducible (this implies that the canonical bundle is Serre dualizing), then we have the Hodge symmetry, i.e. $h^{p,q} = h^{n-p,n-q}$.

Proof. We want to show that:

$$h^q(X, \Omega^p_{X/k}) = h^{n-q}(X, \Omega^{n-p}_{X/k})$$

Here is a good place to review the notion of pairing of vector spaces and vector bundles. This is also a place to review an important isomorphism, listed below. (Recall that here $\Omega^i_{X/k} = \bigwedge^i \Omega_{X/k}$.)

Suppose that U, V, W are three finite dimensional k-vector spaces, a pairing is a k-bilinear map:

$$U \times V \to W$$

This is the same as a k-linear map $U \otimes V \to W$. We say that this is a perfect pairing if the induced k-linear map is an isomorphism:

$$U \to \mathrm{Hom}_k(V, W) \simeq V^\vee \otimes W$$

The isomorphism above $V^\vee \otimes W \to \mathrm{Hom}_k(V, W)$ is canonically given by $\varphi \otimes w \mapsto (v \mapsto \varphi(v)w)$. This is indeed an isomorphism as you can choose basis of V, W say $v_1, \cdots, v_n, w_1, \cdots, w_m$, then we know that a basis of $V^\vee \otimes w$ is given by $v_i^\vee \otimes w_j$, and a basis of $\mathrm{Hom}_k(V, W)$ is given by $m \times n$ matrices. Then we know that the map defined above maps $v_i^\vee \otimes w_j$ to the matrix whose (i, j) entry is 1 and other entries being 0. One special case is when V is a vector space of dimension n, then we have a perfect pairing:

$$\bigwedge^r V \times \bigwedge^{n-r} V \to \bigwedge^n V$$

defined by $(v_1 \wedge \cdots \wedge v_r, v_{r+1} \wedge \cdots \wedge v_n) \mapsto v_1 \wedge \cdots \wedge v_r \wedge v_{r+1} \wedge \cdots \wedge v_n$. The reason this is a perfect pairing is because if you look at the induced map:

$$\bigwedge^r V \to \mathrm{Hom}_k(\bigwedge^{n-r} V, \bigwedge^n V), \qquad v_{r+1} \wedge \cdots \wedge v_n \mapsto v_1 \wedge \cdots \wedge v_r \wedge v_{r+1} \wedge \cdots \wedge v_n$$

it could be described it by choosing a basis of V.

The above discussion can be easily extended to free A-modules of finite rank over any ring A and now we want to extend this to vector bundles on a scheme X, say. Suppose that $\mathcal{D}, \mathcal{E}, \mathcal{F}$ are vector bundles of rank l, m, n. Recall that we have seen that there is a natural isomorphism:

$$\mathcal{H}om(\mathcal{E}, \mathcal{O}_X) \otimes \mathcal{F} \to \mathcal{H}om(\mathcal{E}, \mathcal{F})$$

It's defined by the universal property of tensor products and then on affine trivializing opens, identified with the above isomorphism of modules. Then we define a pairing of $\mathcal{D}, \mathcal{E}$ to $\mathcal{F}$ is a $\mathcal{O}_X$ bilinear morphism:

$$\mathcal{D} \times \mathcal{E} \to \mathcal{F}$$

which is again the same as a $\mathcal{O}_X$-module morphism $\mathcal{D} \otimes \mathcal{E} \to \mathcal{F}$. We will call it a perfect pairing iff the induced morphism is an isomorphism:

$$\mathcal{D} \to \mathcal{H}om(\mathcal{E}, \mathcal{F}) \simeq \mathcal{H}om(\mathcal{E}, \mathcal{O}_X) \otimes \mathcal{F}$$

Similar to our discussion above, if $\mathcal{E}$ is a vector bundle of rank n, we have a perfect pairing:

$$\wedge^r \mathcal{E} \times \wedge^{n-r} \mathcal{E} \to \wedge^n \mathcal{E}$$

You check it's a perfect pairing locally on trivializing affine opens.

Back to our question, recall that $\Omega_{X/k}$ is a locally free sheaf on X of rank n. Recall that by Serre duality, we have an isomorphism:

$$H^q(X, \Omega^p_{X/k}) \simeq H^{n-q}(X, \mathcal{K}_X \otimes (\Omega^p_{X/k})^\vee)$$

Recall that $\mathcal{K}_X = \bigwedge^n \Omega_{X/k}$, and we have an isomorphism:

$$\bigwedge^{n-p} \Omega_{X/k} \to (\bigwedge^{p} \Omega_{X/k})^\vee \otimes \mathscr{K}_X$$

Plug this back in the cohomology, we get the desired isomorphism. $\square$

Let's try to compute the Hodge numbers for the projective space:

Example 5.27. Consider the projective n-space $X = \mathbb{P}^n_k$ let's compute the Hodge numbers $h^{p,q}(X)$ in this example. I claim that $h^{p,q}(X) = 1$ if $0 \leqslant p, q \leqslant n$ and $h^{p,q}(X) = 0$ otherwise.

Recall that we have the Euler exact sequence:

$$0 \to \Omega_{X/k} \to \mathcal{O}_X(-1)^{\oplus n+1} \to \mathcal{O}_X \to 0$$

Then we know that using the associated LES to conclude the desired assertion for $\Omega_{X/k}$.

Recall that for each $r \geqslant 1$, we have a filtration:

$$\bigwedge^r \mathcal{O}_X(-1)^{\oplus n+1} = \mathcal{G}^0 \supset \mathcal{G}^1 \supset \cdots \mathcal{G}^{r-1} \supset \mathcal{G}^r \supset \mathcal{G}^{r+1} = 0$$

with subquotients:

$$\mathcal{G}^p/\mathcal{G}^{p+1} \simeq (\bigwedge^p \Omega_{X/k}) \otimes (\bigwedge^{r-p} \mathcal{O}_X)$$

Notice that we know that when $r - p \geqslant 2$, the wedge is 0, therefore we know that $\mathscr{G}^0 = \cdots = \mathscr{G}^{p-2}$, therefore the filtration is actually just:

$$\bigwedge^r \mathcal{O}_X(-1)^{\oplus n+1} \supset \mathcal{G} = \Omega^r_{X/k}$$

with the quotient being $\Omega^{r-1}_{X/k}$. Therefore we have a SES:

$$0 \to \Omega^r_{X/k} \to \bigwedge^r \mathcal{O}_X(-1)^{\oplus n+1} \to \Omega^{r-1}_{X/k} \to 0$$

By assumption, we know that we know the cohomology of $\Omega^{r-1}_{X/k}$, and we always have an isomorphism:

$$\bigwedge^r \mathcal{O}_X(-1)^{\oplus n+1} \simeq \mathcal{O}_X(-r)^{\oplus \binom{n+1}{r}}$$

Therefore we also know it's cohomology, then we can use the associated LES to compute the cohomology of $\Omega^r_{X/k}$. I will let you show that we indeed have the desired assertion.

Remark 5.28. In this remark, we introduce the so called **Hodge diamond** where we work with $k = \mathbb{C}$. This notion of Hodge diamond works for any field k, but it will have extra symmetry for $k = \mathbb{C}$.

Let X be an irreducible smooth projective complex variety, then it turns out that there is a direct sum decomposition:

$$H^m(X, \mathbb{C}) = \oplus_{i+j=m} H^j(X, \Omega^i_{X/\mathbb{C}})$$

where the left hand side is the singular cohomology with coefficients in $\mathbb{C}$. From this we know that $h^m(X, \mathbb{C}) = \sum_{i+j=m} h^{i,j}(X)$ where we obtain a purely topological numerical data from something that is completely algebraic (i.e. the Hodge numbers).

Moreover, if we work with complex numbers, we have an additional symmetry coming from complex conjugation:

$$H^j(, \Omega^i_{X/k}) \simeq H^i(X, \Omega^j_{X/k})$$

From which we conclude that they have the same dimension, i.e. $h^{i,j}(X) = h^{j,i}(X)$. Now if we write the Hodge numbers in a diamond, where $h^{i,j}(X)$ is the i-th element in the $i+j$-th row. For example, if we consider a smooth projective surface, we have something that looks like the following:

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{0,1} & & h^{1,0} & \\ h^{0,2} & & h^{1,1} & & h^{2,0} \\ & h^{1,2} & & h^{2,1} & \\ & & h^{2,2} & & \end{array}$$

Notice that we know that the maximal wedge power is 2 therefore for $h^{i,j}$ we must have $i \leqslant 2$, similarly, we know that for $j > 2$, all cohomology vanishes, thus $j \leqslant 2$ as well. These are all the elements in the above diamond. We know that the symmetry that comes from Serre duality makes elements that is symmetric with respect to $h^{1,1}$ equal, the complex conjugation symmetry make elements that are symmetric with respect to the row in the middle equal. Therefore, we actually have something like:

$$\begin{array}{ccccc} & & 1 & & \\ & q & & q & \\ p_g & & h^{1,1} & & p_g \\ & q & & q & \\ & & 1 & & \end{array}$$

where p_g are the geometric genus of X.

All the above discussion can be applied to the case of an arbitrary field k, but the conjugation symmetry and the decomposition may not hold there.

Now let's actually try to compute some diamonds. We don't need the decomposition result and the conjugation result to do the following computations:

Example 5.29. Consider a geometrically irreducible smooth projective curve of genus g, then I claim that the diamond looks like:

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$$

To see why, we know that $h^{0,0}$ is always 1. $h^{1,0}$ is the dimension of $H^0(X, \Omega_{X/k})$ which is the geometric genus of X, equal to the arithmetic genus g due to the Serre duality. We know that $h^{0,1}$ is the dimension of $H^1(X, \mathcal{O}_X)$, which is the arithmetic genus g. Finally $h^{1,1} = h^{0,0}$ due to the Serre duality symmetry.

Example 5.30. We can also compute the diamond for $X = \mathbb{P}^1_k \times \mathbb{P}^1_k$.

I claim that it looks like the following:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

To see why, $h^{0,0}$ is always 1 as it's irreducible. I claim that the sheaf of differentials is $\Omega_{X/k} = p_1^*\Omega_{\mathbb{P}^1_k/k} \oplus p_2^*\Omega_{\mathbb{P}^1_k/k}$, therefore we know that the canonical bundle is $\bigwedge^2 p_1^*\Omega_{\mathbb{P}^1_k/k} \oplus p_2^*\Omega_{\mathbb{P}^1_k/k}$, which is $\mathcal{O}(-2,-2)$.

Then consider $h^{0,1}, h^{1,0}$ which are the dimensions of $H^1(X, \mathcal{O}_X)$ and the dimension of $H^0(X, \Omega_{X/k})$. We know that $\Omega_{X/k}$ is the direct sum of $p_1^*\mathcal{O}(-2)$, and we know that it has no global sections, so is the direct sum therefore it's 0. Then we consider $H^1(X, \mathcal{O}_X)$, I claim that it's also 0. To see why, the easiest way is to use the Kunneth formula, which we will prove later, here we know that $\mathcal{O}_X = p_1^*\mathcal{O}_{\mathbb{P}^1_k} \otimes p_2^*\mathcal{O}_{\mathbb{P}^1_k}$, therefore by Kunneth formula, we have:

$$H^1(X, \mathcal{O}_X) = (H^0(\mathbb{P}^1_k, \mathcal{O}_X) \otimes H^1(\mathbb{P}^1_k, \mathcal{O}_X)) \oplus (H^1(\mathbb{P}^1_k, \mathcal{O}_X) \otimes H^0(\mathbb{P}^1_k, \mathcal{O}_X)) = 0 \oplus 0 = 0$$

Then going further, we need to compute $H^1(X, \Omega_X) = H^1(X, p_1^*\mathcal{O}(-2) \oplus p_2^*\mathcal{O}(-2)) = H^1(X, p_1^*\mathcal{O}(-2)) \oplus H^1(X, p_2^*\mathcal{O}(-2))$. We again use Kunneth formula and you get the dimension is 2.

We again need to compute $H^2(X, \mathcal{O}_X)$, which is $H^0(X, \Omega_{X/k})$ and the dimension is just $h^{1,0} = 0$. We need to compute $H^0(X, \mathscr{K}_X) = H^0(\mathcal{O}(-2,-2))$ and the dimension is 0 again using Kunneth formula. Then we are done as the rest can be computed using the Serre duality symmetry.

Remark 5.31. If you compare this with the Hodge diamond of $\mathbb{P}^2_k$, you will find $h^{i,j}$ in general is not a birational invariant as all the $h^{i,j}$ of $\mathbb{P}^2_k$ are either 0 or 1.

6 Generic Smoothness and Consequences

Let's begin directly with the main theorem:

Theorem 6.1. If X is an integral finite type k-scheme over a perfect field k of dimension n, then there is a dense open subset U of X such that U is smooth over k of dimension n.

Remark 6.2. Recall that we mentioned before that if (A, m) is a regular local ring, then the localizations of A at any prime is still a regular local ring. We will not prove this as the general case requires homological methods, which we don't want to do here. However, later we will prove this for varieties over a perfect field, i.e. we will prove that if X is a k-variety over a perfect field k and it's regular at all closed points, then it's regular.

Again, recall that if k is perfect, and X is a finite type k-scheme, then it's smooth iff it's regular at all closed points. The reason is as follows: We know that X is smooth iff $X_{\overline{k}}$ is smooth iff it's regular at all closed points. Now since k is perfect, which means that all algebraic extensions of k is separable, in particular if p is a closed point of X then $k(p)$ is separable over k. Then we know that in this case a closed point $p \in X$ is regular iff closed points in it's inverse images are all regular in $X_{\overline{k}}$. This implies that $X_{\overline{k}}$ is regular at all closed points iff X is regular at all closed points.

Combine these discussions, we obtain the fact that U is smooth iff it's regular at all closed points iff it's regular, thus U is regular.

Now let's prove this result:

Proof. When $dim X = 0$, it's just a single point, which is a dimension 0 integral domain, which is a field, which is finite over k. Since k is perfect, this is separable, therefore this point is smooth.

Recall that on a reduced scheme, a finite type quasicoherent sheaf is a vector bundle iff it has constant rank. Also recall that the rank of a finite type quasicoherent sheaf on a scheme is upper semi-continuous, then if we can prove that the rank of $\Omega_{X/k}$ at the generic point is n, we know that upper semicontinuous gives us two consequences: First, any point $x \in X$ I claim that the rank at x is at least n, since it's it's $n-1$, upper semicontinuous gives us that the rank at the generic point is at most $n-1$, which is a contradiction. Secondly, since we know that the rank at the generic point is n, there is an dense open on which the rank is at most n, then we know that on this dense open, the rank is constant and is n. Therefore, our job is to show that the rank of $\Omega_{X/k}$ at η is n, where η is the generic point.

Let K be the function field of X, we know that $(\Omega_{X/k})_\eta \simeq \Omega_{K/k}$. Recall that K is firstly separable over k, then we know that K/k is a field extension of transcendence degree n. Recall that K/k is also finitely generated, then we know that it can be factored into:

$$k \to k(t_1, \cdots, t_n) \to K$$

where the first is purely transcendental and the second one is finite and separable, or in other words, separably generated by $(t-1, \cdots, t_n)$. Then we have proved before that in this case $\Omega_{K/k}$ is free over K with dimension n and basis $dt_1, \cdots, dt_n$. Therefore, we are done. $\square$

The above generic smoothness result for varieties allows us to prove:

Theorem 6.3. If X is a finite type scheme over a perfect field and it's regular at all closed points, then it's regular.

Proof. Suppose that X is a finite type k-scheme that is regular at all closed points, we know that $X_{\overline{k}}$ is regular at all closed points, which implies that $X_{\overline{k}}$ is smooth which implies X is smooth., let η be an arbitrary point of X, let's consider the canonical reduced closed subscheme $Y = \overline{\eta}$, and we have the closed embedding:

$$i : Y \hookrightarrow X$$

Recall that X is also a finite type integral k-scheme therefore we know that there is a dense open of Y such that it's smooth over k. Say that open comes from an open of X, we replace X by that open and assume that Y is smooth. Recall that we proved the conormal exact sequence is also exact on the left for such a closed embedding (our statement used k-varieties there, but we find that all we need is just finite type over k and reduced, which are all true here). Therefore we have:

$$0 \to \mathcal{I}/\mathcal{I}^2 \to i^*\Omega_{X/k} \to \Omega_{Y/k} \to 0$$

In this case we know that $\mathcal{I}/\mathcal{I}^2$ on Y is locally free of rank $\mathrm{codim}_X Y = \dim\mathcal{O}_{X,\eta}$.

If we want to prove that η is regular, we need to show that the dimension of m/m^2 is $\dim\mathcal{O}_{X,\eta}$, where m is the maximal ideal of $\mathcal{O}_{X,\eta}$. We know that the rank of $\mathcal{I}/\mathcal{I}^2$ is $\dim\mathcal{O}_{X,\eta}$, therefore we want to show that the dimension of m/m^2 is the rank of $\mathcal{I}/\mathcal{I}^2$. We know that the stalk of $\mathcal{I}/\mathcal{I}^2$ at η is just m/m^2, therefore the dimension of m/m^2 is the rank of $\mathcal{I}/\mathcal{I}^2$, we are done. $\square$

As we have promised earlier, we also have the following:

Proposition 6.4. Suppose that (A, m) is a regular local ring that is the localization of a finitely generated k-algebra over a perfect field k. Then the localization of A at any prime is regular.

Before we prove it, notice that this is not a direct consequence of the previous theorem as we know that $SpecA \to Speck$ may not express $SpecA$ as a finite type k-scheme over k. This is true if S is finitely generated, but it could went wrong if S is not finitely generated. Therefore, we will need something else to prove the result:

Proof. To give you a concrete understanding of what happens here, suppose that $B = k[x, y]$ and $A = B_p$ where p is a prime ideal of B. Then A is a local ring, and if we assume that it's regular local, what we are assuming is that p is a regular point of B and we want to prove that for all primes q that lies in p, q is also a prime ideal of B.

Now assume that $A = S^{-1}B$ where B is a finitely generated k-algebra. First, I claim that B could be taken as a domain. To see why, firstly I claim that $S^{-1}B$ can be taken as B_p for a prime ideal p in B. To see why, suppose that the unique maximal ideal m in (A, m) correspond to the prime p in B, we know that $A \simeq A_m$, but we know that there is a canonical isomorphism $A_m = B_p$. Consider the morphism:

$$B \to B_p = A$$

Recall that any regular local ring are domains, therefore we know that the kernel of this map is a prime ideal q, then consider $C = B/q$, recall that $q \subset p$ since if $x \in B$ is mapped to $\frac{x}{1} = 0$, we know that there is a element y not in p such that $xy \in p$, therefore x is in p. Then we consider $C_p = B_p/qB_p$, by our assumption, we know that $qB_p = 0$, therefore we know that $B_p = C_p$ and C is a domain and is again finitely generated over k.

Then in our case, we know that the prime in p C is regular. Now consider $SpecC$, we know that by the generic smoothness, there is a dense open U of $SpecC$ which is smooth, and in this case $U_{\overline{k}}$ is smooth and all it's closed points are regular and all the closed points of U is regular. I claim that U can be taken to be again finite type over k since we can just take U to be a distinguished open. Then we know that this implies that U is regular. Then we know that if p is in U, we are done as we know that for a prime lying under p, we know that since U is open and contains p, it contains that prime as well, then we already know that U is regular thus regular at that prime, therefore the localization of A at that prime is also regular.

To show this, recall that if $SpecC = X$ has dimension n, then the smooth locus is where the rank of $\Omega_{X/k}$ is n. Recall that we know that the rank at the generic point is n, which implies that the rank of $\Omega_{X/k}$ at any point is at least n, therefore we know that the rank at a point is n, it's the rank is at most n. Then consider the rank of $\Omega_{X/k}$ at p, recall that we consider the morphism:

$$\overline{p} = SpecC/p \hookrightarrow X = SpecC$$

which gives the conormal exact sequence:

$$p/p^2 \to \Omega_{C/k} \otimes_C C/p \to \Omega_{C/p/k} \to 0$$

Consider the localization at p, we get the exact sequence of $k(p)$ vector spaces:

$$m/m^2 \to \Omega_{C_p/k} \otimes k(p) \to \Omega_{k(p)/k} \to 0$$

where m is the unique maximal ideal of C_p, we know that since p is regular, the dimension of m/m^p is the dimension of $\mathcal{O}_{X,p}$ which is at most n. Recall that $\Omega_{k(p)/k}$ is 0, then the first arrow is surjective, therefore the rank of $\Omega_{C/k}$ at p is at most n, we are done. □

Now let's discuss generic smoothness for general morphisms, not necessarily for morphism to $Speck$. We will get somewhat similar conclusions, at the cost of working with fields of characteristic 0:

Theorem 6.5.
1. Suppose that $\pi : X \to Y$ is a dominant finite type morphism of integral Noetherian schemes such that char$K(Y) = 0$. Then there is a nonempty open $U \subset X$ such that $\pi|_U : U \to Y$ is smooth of relative dimension trdeg$K(X)/K(Y)$.
2. In particular, let k be of characteristic 0, let $\pi : X \to Y$ be a dominant morphism of integral k-varieties, then there is a nonmepty open $U \subset X$ such that $\pi|_U : U \to Y$ is smooth of relative dimension dim$X -$ dim$Y$

Before we prove this theorem, let's see an application:

Proposition 6.6. If we have polynomials $f_1(x_1, \cdots, x_n), \cdots, f_m(x_1, \cdots, x_n)$ in $k[x_1, \cdots, x_n]$, then if $m > n$, then these polynomials are algebraically dependent, if further we assume $k = 0$ and $m = n$, then these polynomials are algebraically dependent iff the determinant of the Jacobian matrix is the 0 polynomial.

Proof. Recall that $f_1, \cdots, f_m$ are algebraically dependent iff the k-algebra morphism:

$$k[T_1, \cdots, T_m] \to k[x_1, \cdots, x_n], \qquad T_i \mapsto f_i$$

has a nonzero kernel.

Then when $m > n$, we know that if this kernel is 0, then the induced morphism of k-varieties $\mathbb{A}^n_k \to \mathbb{A}^m_k$ is dominant, this implies that the dimension of $\mathbb{A}^n_k$ is at least the dimension of $\mathbb{A}^m_k$, but we know that $m > n$, a contradiction.

Now if $m = n$, again consider the induced morphism of irredcuible k-varieties:

$$\mathbb{A}^n_k \to \mathbb{A}^n_k$$

since we assumed that k is of characteristic 0, we can apply the above theorem and we know that the morphism is dominant iff these polynomials are not algebraically dependent. Or conversely, these polynomials are algebraically dependent iff this morphism is not dominant.

Consider the following diagram:

$$\begin{tikzcd}
k[T_1, \cdots, T_n] \arrow[rr, "T_i \mapsto f_i"] \arrow[dr, "T_i \mapsto T_i"'] & & k[x_1, \cdots, x_n] \\
& k[T_1, \cdots, T_n, x_1, \cdots, x_n]/(T_i - f_i) \arrow[ur, "T_i \mapsto f_i"'] &
\end{tikzcd}$$

This diagram induces an diagram:

$$\begin{tikzcd}
Speck[T_1, \cdots, T_n] & & Speck[x_1, \cdots, x_n] \arrow[ll] \arrow[dl, "\simeq"] \\
& Speck[T_1, \cdots, T_n][x_1, \cdots, x_n]/(T_i - f_i) \arrow[ul] &
\end{tikzcd}$$

Consider the Jacobian matrix:

$$J = (\frac{\partial f_i}{\partial x_j})$$

We know that it's not identically 0 iff there is an open subset of $Speck[x_1, \cdots, x_n]$ such that $U \to Speck[T_1, \cdots, T_n]$ is smooth of relative dimension 0 iff the morphism $\mathbb{A}_k^n \to \mathbb{A}_k^n$ is dominant iff these polynomials are algebraically independent. it's clear that dominant implies there is such an open from the above generic smoothness result, to see why being smooth on U implies being dominant. This is because any open in $\mathbb{A}_k^n$ is dense, which means that it contains the 0 prime of the ring, and we know that the inverse image of 0 in $Speck[T_1, \cdots, T_n]$ is the 0 prime as well, i.e. it's dominant. This implies if the determinant is not identically 0 iff these polynomials are independent. $\square$

Here are two examples where we prove that characteristic 0 is necessary:

Example 6.7. Consider the purely inseparable extension $\mathbb{F}_p(t)/\mathbb{F}_p(t^p)$, it induces:

$$Spec\mathbb{F}_p(t) \to Spec\mathbb{F}_p(t^p)$$

I claim that it's not smooth. To see why, I claim that if K/L is an finite field extension, then K/L is separable iff $SpecK \to SpecL$ is smooth. It's easy to show from finite separable to smoothness as we always have the primitive element theorem. Now let's prove the converse. Suppose that we have a finite extension K/L, we can always assume that it's finitely generated. For example, let's first pick $\alpha_1 \in K$ and consider $L_1 = L(\alpha)$, then we pick something $\alpha_2 \in K$ but not in L_1 and keep going, since this extension is finite dimension, this comes to an end, and suppose that the chain of sequence is:

$$L = L_0 \subset L_1 \subset \cdots \subset L_n = K$$

Then let's f_i be the minimal polynomial of α_i in L_i, then we know that for example:

$$L_1 = L_0[x]/f_1, L_2 = L_1[x]/f_2, \cdots, L_n = L_{n-1}[x]/f_n$$

This implies that:

$$K = L[x_1, \cdots, x_n]/(f_1(x_1), f_2(x_1, x_2), \cdots, f_n(x_1, \cdots, x_n))$$

Recall that to show that K/L is separable, it is equivalent to show that each of the intermediate extension L_i/L_{i-1} is separable. I claim that this is equivalent to the fact that the Jacobian matrix $(\frac{\partial f_i}{\partial x_j})$ is invertible. To see why, we know that the determinant of the matrix is $\prod_i \frac{\partial f_i}{\partial x_i}$, it's invertible iff it's image in K is not zero, but recall that the isomorphism is defined by:

$$L[x_1, \cdots, x_n]/(f_1(x_1), f_2(x_1, x_2), \cdots, f_n(x_1, \cdots, x_n)) \to K, \qquad x_i \mapsto \alpha_i$$

Then this implies that:

$$\frac{\partial f_i}{\partial x_i}(\alpha_1, \cdots, \alpha_i) \neq 0$$

This is exactly what we need for α_i to be separable, conversely, we can argue the same way. However, we notice that this matrix is invertible iff $SpecK \to SpecL$ is smooth of relative dimension 0. Therefore, if it's smooth, then K/L is separable.

Then we know that since $\mathbb{F}_p(t)/\mathbb{F}_p(t^p)$ is purely inseparable ($(t)^p = t^p \in \mathbb{F}_p(t^p)$), therefore we know that $Spec\mathbb{F}_p(t) \to Spec\mathbb{F}_p(t^p)$ is not smooth.

We even have an example when k is algebraically closed but of characteristic p, therefore, you see the essence of the problem actually is the fact that the characteristic is p, not something else. Consider the morphism:

$$Speck[t] \to Speck[u], \qquad u \mapsto t^p$$

We know that in this case we have the diagram:

$$\begin{tikzcd} Speck[t] \arrow[rr, "u\mapsto t^p"] \arrow[dr, "\simeq"'] & & Speck[u] \\ & Speck[u][t]/(u-t^p) \arrow[ur] & \end{tikzcd}$$

Then we know that the Jacobian matrix is $pt^{p-1} = 0$, i.e. is the 0 matrix. This implies that $\Omega_{k[t]/k[u]} = Adt/d(u-t^p)$. Recall that $du = 0$ as $u \in k[u]$, since $dt^p = pt^{p-1}dt = 0$ we know that $\Omega_{k[t]/k[u]}$ is not 0, i.e. the morphism can't be smooth. We are done.

Now let's prove part 1 of the theorem:

Proof. Let $n = \text{trdeg} K(X)/K(Y)$, recall that our goal is to find a nonempty open subset U of X such that $\pi|_U : U \to Y$ is smooth. Notice that if the image of U is contained in an open, say V, then we know that this morphism $\pi|_U$ factors as:

$$\begin{tikzcd} U \arrow[rr, "\pi|_U"] \arrow[dr] & & Y \\ & V \arrow[ur, hook] & \end{tikzcd}$$

Notice that the open embedding $V \hookrightarrow Y$ is always sooth of relative dimension 0. Then if we want to prove that $\pi|_U$ is smooth of relative dimension n, it suffices to prove that $U \to V$ is smooth of relative dimension.

This discussion allows us to replace Y by an affine open and X also by an affine open and the morphism is just $SpecA \to SpecB$ where A, B are Noetherian domains and π correspond to $B \to A$. We can assume that $K(A)/K(B)$ has a transcendence basis that lies in A, not in $K(A)$. To see why, we know that A is a finitely generated B-algebra, i.e. we have $A = B[x_1, \cdots, x_N]/(f_1, \cdots, f_n)$, then we know that $K(A)/K(B)$ is generated by $x_1, \cdots, x_N$, then we could pick n elements in $x_1, \cdots, x_n$ such that they form a transcendence basis. Then we can assume that we add $x_{n+1}, \cdots, x_N$ such that $x_1, \cdots, x_N$ generates A as an B-algebra.

Now for each $i = 0, \cdots, N$, let F_i be the intermediate field extension generated by $x_1, \cdots, x_i$ over $K(B)$. For example $F_i = K(B)$ and $F_n = K(B)(x_1, \cdots, x_n)$ and $F_N = K(A)$. For $i \leqslant n$, F_i over $K(B)$ is a purely transcendental extension. By our choice of $x_1, \cdots, x_n$, we can assume that for $i = n+1, \cdots, N$, the element x_i is algebraic over F_{i-1}, therefore we can denote the minimal polynomial by $m_i \in F_{i-1}[t]$. Recall that $F_{i-1} = K(B)(x_1, \cdots, x_{i-1})$, then we can assume that m_i is of the form $m_i(x_1, \cdots, x_n, t)$ for $m_i \in K(B)[y_1, \cdots, y_{i-1}, t]$.

Now multiply $m_i(x_1, \cdots, x_n, t)$ by the product of the denominators of it's coefficient, we obtain $M_i \in B[y_1, \cdots, y_{i-1}, t]$ so that $M_i(x_1, \cdots, x_{i-1}, t) \in F_{i-1}[t]$ is also a minimal polynomial of $x_i \in K(A)$ over F_{i-1}. We have to be specific here that here our saying that M_i's are still minimal polynomials are in the general sense of up to scalars as typically we will require the leading coefficient of the minimal polynomial to be 1. However, if we allow minimal polynomials to be up to scalar, this doesn't affect the ideal it generates, thus as you will see it will not affect our algebraic discussion later.

Now by our discussion, we know that for example $F_{n+1} = K(B)(x_1, \cdots, x_n)[t]/(M_{n+1})$ and $F_{n+2} = F_{n+1}[t]/(M_{n+2})$. Notice that:

$$F_{n+1} = (K(B)(x_1, \cdots, x_n)[t]/(M_{n+1}))[t]/(M_{n+2})$$

By using different variables for these two t, we know that:

$$F_{n+2} = (K(B)(x_1, \cdots, x_n)[y_1]/(M_{n+1}))[y_2]/(M_{n+2}) \simeq K(B)(x_1, \cdots, x_n)[y_1, y_2](M_{n+1}, M_{n+2})$$

Here M_{n+1} can be written as in $K(B)(x_1, \cdots, x_n)[y_1]$ and $M_2 \in K(B)(x_1, \cdots, x_n)[y_1, y_2]$. Keep going, we have:

$$K(A) = K(B)(x_1, \cdots, x_n)[y_{n+1}, \cdots, y_N]/I, \qquad I = (M_{n+1}, \cdots, M_N)$$

where each $M_{n+i} \in K(B)(x_1, \cdots, x_n)[y_1, \cdots, y_i]$.

Recall that by our construction of M_{n+i}, we can assume that it's actually in $B[x_1, \cdots, x_n][y_1, \cdots, y_i]$ Now consider:

$$X' = SpecA', \qquad A' = B[y_1, \cdots, y_N]/(M_{n+1}(y_1, \cdots, y_{n+1}), \cdots, M_N())$$

Now, consider the following morphism:

$$X' \to \mathbb{A}^n_B, \qquad y_i \mapsto y_i$$

Then we know that if you consider the generic fiber of this, we know that it's:

$$A' \otimes_{B[y_1, \cdots, y_n]} K(B)(y_1, \cdots, y_n) \simeq K(A)$$

This implies that there is exactly one irreducible component of X' that dominants $\mathbb{A}^n_B$. Let's say that this component is Z_1 and other components are $Z_2, \cdots, Z_s$. We will define $U' = Z_1 - (Z_1 \cap (Z_2 \cup \cdots \cup Z_s)) \subset X'$.

Recall that U' is still irreducible, then let η' be the generic point of U', I claim that there is an open neighborhood U'' of η' such that $U' \to \mathbb{A}^n_B$ is smooth of relative dimension 0. Consider the following diagram:

$$\begin{tikzcd}
U' \arrow[r, hook] \arrow[d] & SpecB[y_1, \cdots, y_n][y_{n+1}, \cdots, y_N]/(M_{n+1}, \cdots, M_N) \arrow[dl] \\
SpecB[y_1, \cdots, y_n] &
\end{tikzcd}$$

Recall that the Jacobian matrix here we care is:

$$\begin{bmatrix} \frac{\partial M_{n+1}}{\partial y_{n+1}} & \cdots & \frac{\partial M_{n+1}}{\partial y_N} \\ \cdots & \cdots & \cdots \\ \frac{\partial M_N}{\partial y_{n+1}} & \cdots & \frac{\partial M_N}{\partial y_N} \end{bmatrix}$$

Consider the determinant and I claim that it doesn't vanish at the generic point of U'. To see why, recall that by our construction, M_{n+1} is a polynomial in $y_1, \cdots, y_{n+1}$, and M_{n+2} is a polynomial in $y_1, \cdots, y_{n+2}$, therefore we know that the Jacobian matrix is actually lower triangular, i.e. the determinant is actually (up to a sign):

$$\prod_i \frac{\partial M_i}{\partial y_i}$$

Then we only need to show that each of these factors doesn't vanish at the generic point. Recall that what we know is that since $M_i(x_1, \cdots, x_{i-1}, t)$ is the minimal polynomial of $x_i \in K(A)$ over F_{i-1}, therefore we know that:

$$\frac{\partial M_i}{\partial t}(x_i) \neq 0 \in K(A) = K(B)(x_1, \cdots, x_n)[y_{n+1}, \cdots, y_N]/(M_{n+1}, \cdots, M_n)$$

On the other hand we know that if we want to evaluate $\frac{\partial M_i}{\partial y_i}$ at the generic point, this is the same as plug in the image of $y_1, \cdots, y_i$ in the residue field of η', i.e. $k(\eta')$, which is exactly:

$$K(B)(y_1, \cdots, y_n)[y_{n+1}, \cdots, y_N]/(M_{n+1}, \cdots, M_N)$$

because we have a natural injective morphism:

$$K(B)(y_1, \cdots, y_n)[y_{n+1}, \cdots, y_N]/(M_{n+1}, \cdots, M_N) \to k(\eta')$$

and this $K(B)(y_1, \cdots, y_n)[y_{n+1}, \cdots, y_N]/(M_{n+1}, \cdots, M_N)$ is already a field, then by our characterization of the fraction field as the smallest field that contains the original domain such that all nonzero elements of the domain is invertible, we know that this is an isomorphism. Essentially, this is the following diagram:

$$\begin{tikzcd}
Speck(p) \arrow[dr, dashed] \arrow[drr] \arrow[ddr] & & \\
& SpecA \otimes_B K(B) \arrow[r] \arrow[d] & SpecK(B) \arrow[d] \\
& SpecA \arrow[r] & SpecB
\end{tikzcd}$$

Then by replacing $y_1, \cdots, y_n$ by $x_1, \cdots, x_n$, we obtain the desired relation that the partials doesn't vanish at the generic point, thus we can indeed find an irreducible open $U'' \subset U$ as the nonvanishing condition is open.

This implies that $U'' \to SpecB$ is smooth of relative dimension n as $\mathbb{A}^n_B \to SpecB$ is smooth of relative dimension n. Then I claim that we have the following diagram:

$$\begin{tikzcd} U'' \arrow[rr] \arrow[dr, dashed] & & SpecB \\ & SpecA \arrow[ur] & \end{tikzcd}$$

where the dotted arrow is a birational morphism over $SpecB$. To see why this rational map exists, why not we assume that U'' is also affine $SpecC$ such that C is also finitely generated over B, then we consider the image of the generators of A in the fraction field of C, which are not 0 therefore each of which doesn't vanish on a neighborhood of the generic point, we take the intersection and assume it's still affine, this defines the desired morphism over B as the isomorphism of function fields is over B. Then by replacing $SpecA$ with that open, we can assume that we have:

$$\begin{tikzcd} U'' \arrow[rr] \arrow[dr, "\simeq"'] & & SpecB \\ & SpecA \arrow[ur] & \end{tikzcd}$$

such that $U'' \to SpecB$ is smooth of relative dimension n, therefore $SpecA \to SpecB$ is also smooth of relative dimension n. ☐

Once 1 is proved, 2 is very easy:

Proof. First recall that if $\pi : X \to Y$ is a dominant morphism of integral k-varieties, then $\dim X \geqslant \dim Y$ thus smooth of $\dim X - \dim Y$ makes sense.

Then suppose that $\dim X = m, \dim Y = n$. We know that the induced extension of function fields is $K(Y) \hookrightarrow K(X)$ over k where $K(Y)$ is of transcendence degree n over k and $K(X)$ is of transcendence degree m over k. From this we know that $\mathrm{trdeg} K(X)/K(Y) = m - n$, then we can directly apply part 1. ☐

If further more, X is a smooth k variety in part 2 of the above theorem, we can even conclude more:

Theorem 6.8. Suppose that $\pi : X \to Y$ is a morphism of k-varieties where $\mathrm{char} k = 0$. If X is smooth over k, then there is a dense open U of Y such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}U \to U$ is smooth.

Remark 6.9. Notice that we don't have to require π to be dominant in this case. If it's not dominant, as you will see in the proof later, $\pi^{-1}U$ will be empty and the assertion that π restricted to $\pi^{-1}U$ is smooth is just empty.

To prove this theorem, we will use a neat trick: Suppose that $\pi : X \to Y$ is a morphism between finite type k-schemes where $\mathrm{char} k = 0$, then we define:

$$X_r = \{p \in X : \mathrm{rank}(\pi^* \Omega_{Y/k} \to \Omega_{X/k})|_p \leqslant r\}$$

Proposition 6.10. X_r is a constructible subset of X.

Proof. Let's consider the following exact sequence:

$$\pi^* \Omega_{Y/k} \to \Omega_{X/k} \to \mathrm{coker}\phi \to 0$$

Since tensor is always right exact, we have the following exact sequence:

$$\pi^* \Omega_{Y/k}|_p \to \Omega_{X/k}|_p \to \mathrm{coker}\phi|_p \to 0$$

We know that the rank at p is:

$$\text{rank}|_p = \dim\Omega_{X/k}|_p - \dim\text{coker}\phi|_p \leqslant r$$

For notational simplicity, let's denote $\dim\Omega_{X/k}|_p$ by $o(p)$ and $\dim\text{coker}\phi|_p = c(p)$, therefore we know that we want:

$$o(p) \leqslant r + c(p)$$

Recall that coker is also a coherent sheaf, therefore we know that $c(p)$ is an upper semicontinuous function, i.e. we know that the set where $c(p) \leqslant n$ is open, then we know that the set where $c(p) = n$ is the difference of two opens, therefore is constructible, in fact even locally closed.

Now we know that:

$$\{p \in X : o(p) \leqslant r + c(p)\} = \bigcup_n \{p \in X : o(p) \leqslant r + c(p), c(p) = n\}$$

In other words, we know that $\{p \in X : o(p) \leqslant r + c(p)\}$ is a finite union of:

$$\{p \in X : o(p) \leqslant r + n\} \cap \{p \in X : c(p) = n\}$$

The first set is open, the second is locally closed, therefore we know that the intersection is constructible, and this holds for each n. Notice that we only need to consider finitely many such n as we know that we can cover X by finitely many $SpecA$ where A are Noetherian rings, and cokerϕ is given by a finitely generated A-module, say generated by m elements, then we know that on $SpecA$, $c(p)$ is at most m. Then we only have to take n to be the maximum among such finitely many m's.

Finally, we know that a finite union of constructible set is constructible, we are done. □

Recall that by Chevalley's theorem, $\pi(X_r)$ is again constructible, thus we can talk about it's dimension in a reasonable way. This is not saying that we can't define the Krull dimension for an arbitrary subset, it's just saying that if we work with arbitrary subsets, the notion of definition can be pathologic. Recall that by our definition, the Krull dimension of a topological space is just the supremum of the length of irreducible closed chains. In this case, I claim that the dimension of a constructible subset Y of X is the maximum of the maximum of the length of irreducible closed chains such that they have nonempty intersection Y, and in particular the dimension of Y is at most the dimension of X.

To see why this holds, recall that we have the following map:

$$\text{irreducible closed subset of } Y \to \text{irreducible closed subsets of } X \text{ that intersects } Y, \qquad C \mapsto \overline{C}$$

I claim that this is in general not a bijection, as surjectivity is not always obtained. For example, consider the point 0 in $\mathbb{A}^1_k = X$, which is closed, thus constructible, we know the only irreducible closed subset if 0, but we know that irreducible closed subset of X that intersects with 0 is at least $0, X$, and we know that $\overline{\{0\}} = \{0\}$, thus this is not always surjective. This is always injective however, recall if C is closed in Y, we know that $C = Y \cap \overline{C}$, then we know that if $\overline{C_1} = \overline{C_2}$, this implies $C_1 = C_2$. This implies that there should be a subset of irreducible closed subsets of X that intersects Y such that the map above is a bijection, and it turns out that the subset is:

$$\text{irreducible closed subsets } Z \text{ of } X \text{ that intersects } Y\text{, and } Y \cap Z \text{ is dense in } Z$$

In the case of schemes, this is just saying that $Y \cap Z$ contains the generic point of Z. In particular, this implies that if Y is open, surjectivity holds in the first proposed map and thus is a bijection. Let's prove surjectivity here, say $Z \cap Y$ is dense in Z, then we know that $Z \cap Y$ is indeed Z, therefore we only need to show that $Z \cap Y$ is irreducible closed in Y. It's obviously closed we can now assume that if:

$$(F_1 \cup F_2) \cap Y = Z \cap Y$$

then we know that $F_1 \cup F_2$ is dense in Z, then we know that $F_1 \cup F_2 = Z$ since $F_2 \cup F_1$ is closed. Notice that all the above discussion doesn't require Y to be constructible. The reason we want constructibility is because if Y is not constructible, the set of irreducible closed subsets Z of X that intersects Y, and $Y \cap Z$ is dense in Z can be quite nonreasonable. For example consider $\mathbb{A}_k^1 = X$ over an algebraically closed field k and Y is the set of all closed points in X. Notice that this is a dense subset, however, we know that any irreducible closed subset Z of X such that $Z \cap Y$ is dense are just closed points, therefore by definition, the dimension of Y is 0, which is weird since this is a dense subset of a dimension 1 space. Another example is consider the generic point of X, and you find the dimension is also 0, however, if you think of the generic point as a dense point, which should be really large, this is also weird. Such weird cases won't happen for constructible subsets, since any dense constructible set of an irreducible space contains a dense open.

All of the above discussion are very general and it works for any schemes that is Noetherian. However, what we will work with is going to be schemes that are finite type over k, which are very special since here we know that closed points are dense, i.e. every open contains at least one closed subset. Then in this case what we can prove first is that a locally closed subset $F \cap U$ has the same dimension as it's closure. To see why, notice that suppose $F = F_1 \cup \cdots \cup F_n$ where F_i are all the irreducible components of F, then we know that the closure is just $F_1 \cup \cdots F_n$, therefore we only need to show that the dimension of $U \cap F_i$ is the dimension of F_i. it's obvious that the dimension of F_i is larger, then suppose that the dimension of F_i is r, we know that since F_i can also be equipped with a scheme structure that is integral and finite type over k, then we know that the dimension of F_i is the codimension of a closed point, and we can pick that closed point in $U \cap F_i$, this finishes our proof.

Once we know this, I claim that for an contructible subset Y of X, if X is finite type over k, the dimension of Y is the dimension of it's closure. Again, we know that the dimension of the closure is clearly larger, then we only need to show the converse. Notice that if Y is the union of $Y_1, \cdots, Y_r$, each of which locally closed, we know that the closure is the union of $\overline{Y_i}$ and we know that in this case the dimension is the maximum of the dimension of $\overline{Y_i}$, then we know that:

$$\dim\overline{Y} = \dim\overline{Y_i} = \dim Y_i \leqslant \dim Y$$

We will use this in the following discussion. So let's summarize what we proved in the following lemma:

Lemma 6.11. Suppose that X is a finite type over k scheme, then the dimension of a constructible subset Y of X is equal to the dimension of it's closure.

Now let's go back to prove the theorem, which needs the following lemma:

Proposition 6.12. Let $\pi : X \to Y$ be a morphism of schemes that are finite type over k where $\mathrm{char}k = 0$, then $\dim\pi(X_r) \leqslant r$.

Proof. Recall that since k is of characteristic 0, it's automatically perfect, then what we know is that for closed points of X, the Zariski cotangent space is the fiber of the cotangent sheaf.

We are interested in the dimension of $\pi(X_r)$, now recall that we can assume that $X_r = \cup_j U_j \cap F_j$ where U_j are open and F_j are irreducible closed, then we know that:

$$\pi(X_r) = \cup_j \pi(U_i \cap F_j)$$

By our discussion above, we know that the dimension of $\pi(X_r)$ is the dimension of the closure, which is the maximum of the dimension of $\overline{\pi(U_i \cap F_j)}$ which is the maximum of the dimension of $\pi(U_i \cap F_j)$ since $\pi(U_i \cap F_j)$ is also constructible. Then suppose that the maximum is obtained at i, then we know that we can consider the following sequence of morphisms:

$$F_i \cap U_i \hookrightarrow U_i \hookrightarrow X \to Y$$

where the first morphism is a closed embedding and the second one is an open embedding. Recall that $F_i \cap U_i$ is also irreducible therefore we know that it's image is also irreducible (which is a purely topological fact). Then we consider the closure of the image, which is irreducible closed in Y, let's say it's Y', then consider the following diagram:

$$\begin{tikzcd} F_i \cap U_i \arrow[r] \arrow[d, hook] & Y' \arrow[d, hook] \\ X \arrow[r, "\pi"] & Y \end{tikzcd}$$

where the top morphism is induced by the universal property of closed embeddings.

Recall that $F_i \cap U_i$ is a locally closed subset of a Noetherian space, therefore Noetherian, therefore quaiscompact, therefore $F_i \cap U_i$ is still finite type over k, and Y' is still finite type over k as well, therefore we know that $F_i \cap U_i \to Y'$ is also a morphism of finite type schemes over k, moreover, by assumption we can assume that both $F_i \cap U_i$ and Y' are integral. Our goal is not to show that Y is of dimension at most r.

Recall that for $\pi' : F_i \cap U_i = X' \to Y'$, we again have a morphism:

$$(\pi')^* \Omega_{Y'/k} \to \Omega_{X'/k}$$

I claim that the rank of this map at each point of X' is at most r. To see why, we know that this can be computed locally, by working locally with $X \to Y$, we can assume that the map is $SpecA \to SpecB$ ad the induced morphism is:

$$SpecA/p \to SpecB/q$$

where p, q are primes of A, B and the morphism is again induced by $B \to A$. Then we know that in this case, if the morphism $SpecA \to SpecB$ gives a morphism:

$$\Omega_{B/k} \otimes_B A \to \Omega_{A/k}$$

we know that the above morphism is given by:

$$\Omega_{(B/q)/k} \otimes_{B/q} A/p \to \Omega_{(A/p)/k}$$

I claim that we have a commutative diagram like below:

$$\begin{tikzcd}
\Omega_{B/k} \otimes_B k(x) \arrow[r] \arrow[d] & \Omega_{A/k} \otimes_A k(x) \arrow[d] \\
\Omega_{(B/q)/k} \otimes_{B/q} k(x) \arrow[r] & \Omega_{(A/p)/k} \otimes_{A/p} k(x)
\end{tikzcd}$$

where x is a point in F_i (the key point here is that the residue field at x in $U_i \cap F_i$ is the same as the residue field of x in X). For example the left vertical morphism is defined by $\Omega_{B/k} \times k(x) \to \Omega_{(B/q)/k} \times k(x)$ where $\Omega_{B/k} \to \Omega_{(B/q)/k}$ is given by the universal property:

$$B \to B/q \to \Omega_{(B/q)/k}$$

and the second morphism is $k(x) \xrightarrow{\mathrm{Id}} k(x)$, it's clear that these two maps are B-linear, therefore we indeed get a morphism and you can then check that this is $k(x)$-linear and is surjective. Similarly we get the right morphism. The top vertical morphism is given by $B \to A$ and the below morphism is given by $B/q \to A/p$ and I will let you check this is commutative. But after this, the key of the proof is to use the following linear algebra fact:

Suppose that we have a diagram of vector spaces:

$$\begin{tikzcd}
V_1 \arrow[r, "\rho"] \arrow[d, two heads] & V_2 \arrow[d, two heads] \\
V_1' \arrow[r, "\rho'"'] & V_2'
\end{tikzcd}$$

Suppose that ρ has rank r, then I claim that ρ' has rank at most r. I will let you prove it, which should be quite simple.

After all the above discussion, our job is now, suppose that we have a morphism of integral scheme finite type over k $\pi : X \to Y$, if $\pi^* \Omega_{Y/k} \to \Omega_{X/k}$ has rank at most r for all $x \in X$, then the dimension of Y is at most r. Recall that by the generic smoothness, Y has a dense open where Y is smooth, then let's consider:

$$\begin{tikzcd}
\pi^{-1}U \arrow[r] \arrow[d, hook] & U \arrow[d, hook] \\
X \arrow[r, "\pi"] & Y
\end{tikzcd}$$

Recall that since Y is integral finite type over k, to prove the dimension of Y is at most r, it's equivalent to prove the dimension of U is at most r. Now, I claim that it's enough to consider the case where k is algebraically closed. To see why, we know that we have the morphism:

$$X_{\overline{k}} \to \Omega_{Y/k}$$

Recall that we have proved that Y is of dimension r iff $Y_{\overline{k}}$ is of dimension r. Moreover, if we look at the morphism of sheaves, if locally $X \to Y$ is $SpecA \to SpecB$, then this $X_{\overline{k}} \to Y_{\overline{k}}$ locally looks like:

$$SpecA \otimes_k \overline{k} \to SpecB \otimes_k \overline{k}$$

We know that in this case, the map between the sheave of differentials is:

$$\Omega_{B\otimes_k \overline{k}/\overline{k}} \otimes_{B\otimes_k \overline{k}} A \otimes_k \overline{k} \to \Omega_{A\otimes_k \overline{k}/\overline{k}}$$

Recall that $\Omega_{B\otimes_k \overline{k}/\overline{k}} = \Omega_{B/k} \otimes_k \overline{k}$ since we have the Cartesian diagram:

$$\begin{tikzcd}
SpecB \otimes_k \overline{k} \arrow[r] \arrow[d] & SpecB \arrow[d] \\
Spec\overline{k} \arrow[r] & Speck
\end{tikzcd}$$

Then I will let you check if we denote the map $\Omega_{B/k} \otimes_B k \to \Omega_{A/k}$ by φ, then the new map is identified as $\varphi \otimes_A A_{\overline{k}}$ where $A_{\overline{k}} = A \otimes_k \overline{k}$. Then suppose that $q \in SpecA \otimes_k \overline{k}$ has inverse image p in A, we know that the rank of the new map at q is:

$$\varphi \otimes_A A_{\overline{k}} \otimes_{A_{\overline{k}}} k(q) = \varphi \otimes_A k(q) = \varphi \otimes_A k(p) \otimes_{k(p)} k(q)$$

which is just the scalar of the fiber map at p to $k(q)$, therefore the rank stays the same. So now let's just assume that we work over $k = \overline{k}$.

Recall that locally the morphism of sheaves of differentials on the top looks exactly the same as the morphism induced by the morphisms of schemes at the bottom. Therefore, the rank is still at most r and we can assume Y is smooth. Recall that there is a dense open U of X such that $U \to Y$ is smooth, this means that if we consider the diagram:

$$\begin{tikzcd}
U \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\
& Speck &
\end{tikzcd}$$

We know that the cotangent exact sequence is also left exact, therefore we have:

$$0 \to \pi^*\Omega_{Y/k} \to \Omega_{U/k} \to \Omega_{U/Y} \to 0$$

Therefore for each $p \in U$, we know that we have:

$$0 \to (\pi^*\Omega_{Y/k})_p \to (\Omega_{U/k})_p \to (\Omega_{U/Y})_p \to 0$$

If you consider the first injective morphism:

$$(\Omega_{Y/p})_{\pi(p)} \otimes_{\mathcal{O}_{Y,\pi(p)}} \mathcal{O}_{X,p} \to (\Omega_{U/k})_p$$

Suppose that now p is a closed point, let's tensor with $k = k(p)$ as k is algebraically closed, which gives:

$$\Omega_{Y/k}|_{\pi(p)} \to \Omega_{U/k}|_p$$

which has rank at most r. If this is injective, we are done since this implies that the dimension of $\Omega_{Y/k}|_{\pi(p)}$ as a k-vector space is at most r, which implies that the dimension of $m_{\pi(p)}/m_{\pi(p)}^2$ is r, moreover, since we know that U is smooth, it's regular as k is of characteristic 0 and k is algebraically closed, therefore we know that the dimension of $m_{\pi(p)}/m_{\pi(p)}^2$ is the dimension of $\Omega_{Y,\pi(p)}$, which is the codimension of $\pi(p)$ in Y which is the dimension of Y as $\pi(p)$ is closed, which is at most r. We are done. $\square$

Now we have everything we need to show the theorem, except one characterization of smoothness which will be proved later:

Proof. First, we can reduce to a dense open of Y such that we can now assume Y is smooth over k. Say $\dim Y = n$, now we know that $\dim\overline{\pi(X_{n-1})} = \dim\pi(X_{n-1}) \leqslant n-1$, let's remove $\overline{\pi(X_{n-1})}$ from Y and we consider the inverse image as X. This way, we can assume that $\pi^*\Omega_{Y/k} \to \Omega_{X/k}$ is of rank n at each point of X. However, since Y is smooth, we know that $\Omega_{Y/k}$ is locally free of rank n, then so is $\pi^*\Omega_{Y/k}$, therefore we know from the map:

$$\pi^*\Omega_{Y/k}|_p \to \Omega_{X/k}|_p$$

is of rank at least n for each $p \in X$ implies that it's the morphism $\pi^*\Omega_{Y/k} \to \Omega_{X/k}$ is injective. To see why, we need to show that it's injective on the stalk of each point.

This boils down to the following algebra fact. Suppose that we have a local ring (R, m, k), and we have a morphism:

$$\varphi : R^n \to R^m$$

that is injective after we tensor with k, then I claim that $R^n \to R^m$ is injective itself. To see why, first recall that the map φ is represented by a $m \times n$ matrix with entries in R. After we tensor with k, the map is represented by the same matrix, but with elements thought as elements in k, then it's injective means that there is a $n \times n$ minor with invertible deterninant, then we know that by a change of order of coordinates on R^n and R^m, we know that:

$$c_2 \circ \varphi \circ c_1 : R^n \to R^m$$

has that $n \times n$ minor on the upper left corner of the matrix. Then consider the projection p of R^m to the first R^n, we know that in this case:

$$p \circ c_2 \circ \varphi \circ c_1 : R^n \to R^n$$

is going to be invertible, then we know that since c_1 is also invertible from R^n to R^n, let's assume that $p' \circ \varphi$ is invertible, therefore, φ has a left inverse, therefore φ is injective.

Now I claim that this is not only just a injective morphism of locally free sheaves, it's an injection of vector bundles. To see why, we know that the rank of the cokernel is always $m-n$ therefore cokernel is also locally free as we work with reduced schemes. (If you think about the famous example where an injective morphism of locally free sheaves that is not an injection of vector bundles which is given by the inclusion of a closed point to $\mathbb{A}^1_k$, you find that the rank of the cokernel is not constant.)

Finally, this implies that $\Omega_{X/Y}$ is locally free of rank $m-n$, which implies $X \to Y$ is smooth due to the characterization we will prove later. □

7 Unramified Morphisms

Definition 7.1. Suppose that $\pi : X \to Y$ is a morphism of schemes, we will call the support of the quasicoherent sheaf $\Omega_\pi = \Omega_{X/Y}$ the **ramification locus** of π and we will call it's image $\pi(\mathrm{Supp}\Omega_{X/Y})$ is called the **branch locus**.

If the ramification locus is empty, or equivalently the quasicoherent sheaf Ω_π is 0, we will say that π is **formally unramified**. If π is also locally of finite type, we say that π is **unramified**.

Here are some easy examples we can do right away:

Example 7.2. The first example we see is that all locally closed embeddings are unramified. First, we know that all locally closed embeddings are locally of finite type, therefore we only need to prove that $\Omega_\pi = 0$. We know that locally to compute Ω_π, we can reduce to the case where we have:

$$Spec A/I \to Spec A$$

and we already proved that in this case $\Omega_{(A/I)/A} = 0$, therefore we are done.

Now if S is a multiplicative subset of B, then I claim that $Spec S^{-1}B \to Spec B$ is unramifed. This is clear since we know that the sheaf of differentials associated to a localization is 0. In particular, this say that if we have an integral scheme Y, then the morphism $Spec\Omega_{Y,\eta} \to Y$ where η is the generic point is formally unramified.

Suppose that K/L is a finite separable extension, then I claim that $SpecK \to SpecL$ is unramified. First it's finite, thus locally of finite type. It's also formally unramified because we proved that if K/L is algebraic and separable, then $\Omega_{K/L}$ is 0.

Finally, I will let you show that smooth morphisms of relative dimension 0, i.e. etale morphisms are unramified.

Here are some more interesting examples:

Example 7.3. Recall the normalization of the node:

$$Speck[t] \to Speck[x,y]/(y^2 - x^2(x+1))$$

which is given by $(x,y) \mapsto (t^2-1, t(t^2-1))$. I claim that this is unramifed. Clearly, this is finite type as we know that $k[t]$ is a finitely generated algebra over $k[x,y]/(y^2 - x^2(x+1))$, which is generated by t for example. We need to show that the sheaf of differential is 0, i.e. $\Omega_{k[t]/k[x,y]/(y^2-x^2(x+1))} = 0$. To see why, recall that we know that dx is mapped to $2tdt$ and dy is mapped to $3t^2dt - dt$, then we know that the linear combination can be mapped to dt, therefore for simplicity, let's assume that the morphism is $SpecA \to SpecB$, then in the cotangent exact sequence:

$$A \otimes_B \Omega_{B/k} \to \Omega_{A/k} \to \Omega_{A/B} \to 0$$

The first morphism $A \otimes_B \Omega_{B/k} \to \Omega_{A/k}$ is surjective, therefore Ω_π as the cokernel, should be 0, we are done.

We can also give a morphism that is not unramified, which is the normalization of the cusp:

$$Speck[t] \to k[x,y]/(y^2 - x^3)$$

where $(x,y) \mapsto (t^2, t^3)$. In this case, we find that dx is mapped to $2tdt$ and dy is mapped to $3t^2dt$, which means that only using $k[t]$ combination, we can not obtain dt, therefore in this case $A \otimes_B \Omega_{B/k} \to \Omega_{A/k}$ is not surjective, therefore the cokernel is not 0.

Below are properties that makes unramified morphisms a nice class of morphisms:

Proposition 7.4. Unramified morphisms are preserved by composition.

Proof. Recall that locally of finite type is preserved by composition, therefore we only need to check the sheaf of differentials. Suppose that $X \xrightarrow{\pi} Y \xrightarrow{\rho} Z$ are two morphisms that are both unramifed, then consider the cotangent exact sequence:

$$\pi^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

By our assumption, $\Omega_{Y/Z} = 0, \Omega_{X/Y} = 0$, therefore we have:

$$0 \to \Omega_{X/Z} \to 0 \to 0$$

This implies that $\Omega_{X/Z} = 0$. ∎

Proposition 7.5. Unramified morphisms are preserved by base change.

Proof. Again, we only need to prove that formally unramified morphisms are preserved by base change.

If you consider the following Cartesian diagram:

$$\begin{tikzcd} X' \arrow[r, "\pi'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "\pi"] & Y \end{tikzcd}$$

One important property of the sheaf of differential we proved is that:

$$(\pi')^*\Omega_{Y'/Y} \simeq \Omega_{X'/X}$$

Then we only need to know that the pullback of 0 is 0. ∎

Finally, we need to show that it's local on the target:

Proposition 7.6. Unramifed morphisms are local on the target.

Proof. Again, we only need to check this for formally unramifed morphisms. The fact that if $\pi : X \to Y$ is formally unramified then $\pi^{-1}U \to U$ is is just the fact that formally unramifed morphism is preserved under base change.

Conversely, suppose that there is an cover of Y such that $\pi^{-1}U \to U$ is formally unramifed. Recall that $\Omega_{X/Y}$ can be computed locally by $\pi^{-1}U \to U$ basically by how we defined it, thus we are done. ☐

One interesting property of unramified morphisms is that we can check it on fibers. This is the first instance we see that fibers can be useful when we want to check some morphisms are of certain properties, we will see this later for our characterization of smooth morphisms:

Proposition 7.7. Suppose that $\pi : X \to Y$ is locally of finite type, then:

1. π is unramified iff for each $q \in Y$, $\pi^{-1}q$ is the scheme theoretic disjoint of the form $SpecK$ where K is a finite separable extension of $k(q)$.

2. π is unramified iff for each **geometric point** $\overline{q}$, $\pi^{-1}\overline{q}$ is the disjoint union of copies of $\overline{q}$.

Proof. Let's first do 1. Suppose that $X \to Y$ is unramified, consider:

$$\begin{tikzcd} \pi^{-1}q \arrow[r, "\pi'"] \arrow[d] & Speck(q) \arrow[d] \\ X \arrow[r, "\pi"] & Y \end{tikzcd}$$

We know that $\Omega_{\pi^{-1}q/q}$ is the pullback of $\Omega_{X/Y}$ to $\pi^{-1}q$, which is 0. Recall that if we want to show that $\pi^{-1}q$ is disjoint we only need to work locally, therefore we can now assume that X, Y are affine, say:

$$SpecA \to SpecB$$

where A is a finitely generated B-algebra. Then we know that locally the fiber looks like $SpecA \otimes_B k(q)$. We know that $A \otimes_B k(q)$ is a finitely generated $k(q)$-algebra. So our job now is the following: suppose that $k \to A$ makes A a finitely generated k-algebra and $\Omega_{A/k}$ is 0, then we want to show that A is a product of finite separable extension of k.

Let's say $A = k[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$. Recall that we can compute the sheaf of differentials by:

$$\Omega_{A/k} = \oplus_{i=1}^n Adx_i/(Adf_1, \cdots, Adf_r)$$

Recall that if r is less than n, we know that the sheaf of differentials can not be 0 (for example you can go to the residue field of a point and argue there since we are in a field). This implies that r is at least as large as n. Let's say that we compute the Jacobian matrix:

$$(\frac{\partial f_i}{\partial x_j})$$

which is a $n \times r$ matrix, we know that from the fact that we assumed that $\Omega_{A/k}$ is 0, we know that the corank of this Jacobian matrix is 0, or in other words, the rank is n at each point of $SpecA$. Therefore we know that by definition, this $SpecA$ is smooth of relative dimension 0 over k(recall that we had an argument before of why we automatically have dimension 0).

This implies that every point in $SpecA$ is a maximal ideal, moreover, since we know that we only have finitely many maximal ideals, then from the fact that A is Noetherian, we know that it has finite length, therefore we know that let's say these maximal ideals are $m_1, \cdots, m_s$, then we know that:

$$A = A/m_1^{l_1} \times \cdots \times A/m_s^{l_s}$$

Then we know that if we localize at m_i, we get $A_{m_i}/m_i^{l_i}A_{m_i}$, then we know that if this is a field, then $l_i = 1$, so our goal now is to show that the localization of A at m_i is a field. Recall that the localization at m_i is a

Noetherian ring with only one prime ideal, then if it's reduced, we are done, but this comes from the fact that it's smooth, thus reduced. Therefore, we know that the localization at m_i is just $k(m_i)$ we know that it's finite over k since m_i is a closed point, finally we only need to prove that $k(m_i)/k$ is a separable, or in other words, we need to prove $\Omega_{k(q)/k}$ is 0. To see this is true, we need to consider:

$$Speck(q) \hookrightarrow SpecA \to Speck$$

We consider the cotangent exact seqeunce:

$$k(q) \otimes_A \Omega_{A/k} \to \Omega_{k(q)/k} \to \Omega_{k(q)/A} \to 0$$

By our assumption, we know that $\Omega_{A/k}$ is 0, on the other hand, we know that $\Omega_{k(q)/A} = 0$ since $A \to k(q)$ is surjective. This implies $\Omega_{k(q)/k} = 0$, we are done.

Now let's assume that $\pi^{-1}q$ is the scheme theoretic disjoint of $SpecK$ where K is a finite separable extension of k. Our goal is to prove that $(\Omega_{X/Y})_p = 0$ for all $p \in X$. Recall that if $x \in \pi^{-1}q$ is a point such that it's mapped to $p \in X$ via $f : \pi^{-1}q \to X$, we know that we have the following relation:

$$(f^*\Omega_{X/Y})_x \simeq (\Omega_{X/Y})_p \otimes_{\mathcal{O}_{X.p}} \mathcal{O}_{\pi^{-1}q,x}$$

By our assumption, we know that $f^*\Omega_{X/Y} = 0$, therefore we know that $(\Omega_{X/Y})_p \otimes_{\mathcal{O}_{X.p}} \mathcal{O}_{\pi^{-1}q,x} = 0$. Now, we know that $\mathcal{O}_{\pi^{-1}q,x} = K$ where K is a finite separable extension of $k(q)$. Then we know that it's also a finite extension of $k(p)$ since we have a morphism $k(q) \to k(p) \to k(x) = K$, then we know that:

$$(\Omega_{X/Y})_p \otimes_{\mathcal{O}_{X.p}} \mathcal{O}_{\pi^{-1}q,x} = (\Omega_{X/Y})_p \otimes_{\mathcal{O}_{X.p}} k(p) \otimes_{k(p)} K$$

Recall that K is flat over $k(p)$, then we know that it's zero iff $(\Omega_{X/Y})_p \otimes_{\mathcal{O}_{X.p}} k(p)$ is 0. However, we know that since π is locally of finite type $\Omega_{X/Y}$ is coherent, therefore we know that by Nakayama, $(\Omega_{X/Y})_p \otimes_{\mathcal{O}_{X.p}} k(p) = 0$ implies that $(\Omega_{X/Y})_p = 0$. This works for every $p \in X$, we are done. □

Then, let's prove part 2. As you will see, for the geometric point, we only need it to be separable closed:

Proof. Again, let's consider the diagram of the geometric point:

$$\begin{array}{ccc} \overline{q} \times_Y X & \longrightarrow & \overline{q} = Spec\overline{k} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

In the diagram, we assume that $\overline{k}$ is a extension of k such that it's separably closed. Again, if $\pi : X \to Y$ is unramifed, let's again assume that $\pi : X \to Y$ is affine and it's $SpecA \to SpecB$, and we just need to consider $SpecA \otimes_B \overline{k} \to Spec\overline{k}$, whose sheaf of differentials is 0. Then again, we use exactly the same argument as in part 1 to conclude that $pecA \otimes_B \overline{k}$ is actually a bunch of $SpecK$ where K is a finite separable extension of $\overline{k}$, but we know that $\overline{k}$ has no nontrivial finite separable extensions, therefore $\overline{k} = K$, we are done.

I will leave the converse for you, which is again basically identical to how we proved part 1. □

If we work in a very geometric setting, i.e. with an algebraically closed field k and finite type schemes over k, being unramified has a very geometric meaning as well:

Proposition 7.8. Is a morphism $\pi : X \to Y$ is a morphism between finite type over k schemes where k is algebraically closed, then it's unramifed iff π is injective on tangent vectors at all closed point.

Proof. Well, recall that for every closed point x of X, let's denote it's image, which is again a closed point, by y. Then we know that we have a induced morphism:

$$m_y/m_y^2 \to m_x/m_x^2$$

which are k-vector space, then we can take the dual, which gives us an injective map by assumption. This implies that $m_y^2/m_y^2 \to m_x/m_x^2$ is surjective. Recall that if we work with algebraically closed field, we always have an isomorphism between $(\Omega_{X/k})(x) \simeq m_x/m_x^2$ and $(\Omega_{Y/k})(y) \simeq m_y/m_x^y$. Let's consider the cotangent exact sequence:

$$\pi^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0$$

Let's take the fiber at 0, and since we know that tensor is right exact, we still get an exact sequence:

$$\Omega_{Y/k}(y) \to \Omega_{X/k}(x) \to \Omega_{X/Y}(x) \to 0$$

I claim that the morphism $\Omega_{Y/k}(y) \to \Omega_{X/k}(x)$ is just the morphism $m_y/m_y^2 \to m_x/m_x^2$, to see this, recall that the isomorphism between $\Omega_{X/k}(x)$ and m_x/m_x^2 for example is given by the Jacobian matrix, as illustrated in the diagram below:

$$\begin{tikzcd}
\oplus_i k dx_i/(kdf_1 + \cdots + kdf_r) \arrow[d, "dx_i \mapsto x_i - l_i"'] & \oplus_j k dy_j/(kdg_1 + \cdots + kdg_s) \arrow[l, "dy_i \mapsto d\pi(y_i)"'] \arrow[d, "dy_j \mapsto y_j - k_j"] \\
m_x/m_x^2 & m_y/m_y^2 \arrow[l, "y_j - k_j \mapsto \overline{\pi(y_j) - k_j}"]
\end{tikzcd}$$

I will let you figure out why this commutes. But this implies that, $\Omega_{X/Y}(x) = 0$ at all closed points, therefore the rank of $\Omega_{X/Y}$ at each closed point is 0, therefore since each closed subset of X contains a closed point, we know that this implies that the rank of $\Omega_{X/Y}$ on X is 0, thus it's 0, thus unramified. $\square$

For later use, we prove another characterization of unramified morphisms. Let's first start with the locally Noetherian setting and then remove it:

Proposition 7.9. Suppose that $\pi : X \to Y$ is a locally finite type morphism of locally Noetherian schemes. Then $\pi : X \to Y$ is unramifed if and only if $\delta_\pi : X \to X \times_Y X$ is an open embedding.

Proof. To show this result, we use the following result: If $\pi : X \to Y$ is a closed embedding of Noetherian schemes, such that the ideal sheaf of ϕ, denoted by $\mathcal{I}$, satisfies $\mathcal{I} = \mathcal{I}^2$, then ϕ is also an open embedding.

To see this, we let's consider the stalk of $\mathcal{I}$ at a point $x \in X$, we know that $\mathcal{I}_x$ is a proper ideal of the local ring $\mathcal{O}_{Y,x}$. Let's denote it by I, then by definition, we know that $I = I^2$ and I is inside the maximal ideal m, therefore we know that $mI \subset I \subset mI$, therefore $mI = I$. Since $\mathcal{O}_{Y,p}$ is a Noetherian ring, I is finitely generated, therefore we know that by Nakayama's lemma $I = 0$. Then we know that the support of $\mathcal{I}$, since $\mathcal{I}$ is coherent (again due to locally Noetherianity), then it's support is closed, therefore the complement is open. Therefore we know that X is actually an open and closed subset of X. Then we know that this morphism factors as:

$$\begin{tikzcd}
& Y & \\
X \arrow[ur, "\pi"] \arrow[rr, "\pi"'] & & X \arrow[ul, hook]
\end{tikzcd}$$

Our job is to show that $X \to X$ is an isomorphism, which is local on the target. Suppose that $SpecA$ is an affine open on X, then we know that the morphism is defined to be:

$$SpecA/I \to SpecA, \qquad I = 0$$

Therefore is an isomorphism, we are done. Recall that we need Noetherianity here to conclude $I = 0$ and the complement is open.

Now back to our question, we know that $\delta_\pi : X \to X \times_Y X$ is always a locally closed embedding, and we can assume that it's:

$$X \hookrightarrow U \hookrightarrow X \times_Y X$$

We know that both X and U are locally Noetherian. Let's first assume that π is unramifed, which means that $\Omega_{X/Y} = 0$. Let's consider the ideal sheaf $\mathcal{I}$ of $X \hookrightarrow U$. We know that by definition, $\Omega_{X/Y} = \mathcal{I}/\mathcal{I}^\bullet$ which is 0 by assumption, therefore $\mathcal{I} = \mathcal{I}^2$, therefore $X \hookrightarrow U$ is an open embedding, therefore $X \hookrightarrow X \times_Y X$ is an open embedding. On the other hand, if this is an open embedding, $X \hookrightarrow U$ must also be an open embedding, therefore we know that again the $\mathcal{I} = 0$, thus $\mathcal{I}/\mathcal{I}^2 = 0$, therefore $\Omega_{X/Y} = 0$, we are done. $\square$

Now let's try to drop the Noetherian hypothesis.

Proof. Recall that the only thing that requires Noetherianity is to show that $\mathcal{I}$ is coherent. Therefore, if we don't assume Noetherianity, we can instead assume that:

$$X \hookrightarrow Y$$

is a closed embedding that is locally finitely presented. Then we know that for each $SpecA$ in Y, we know that $A \to A/I$ is finitely generated, i.e. we know that I is finitely generated. Therefore, now our job is to show that if $\pi : X \to Y$ is locally of finite type, then in $\delta_\pi : X \hookrightarrow U \hookrightarrow X \times_Y X$, $X \hookrightarrow U$ is locally finitely presented, we can just show that $X \hookrightarrow U$ is locally finitely presented since open embedding are always locally finitely presented.

To show this, recall that U is covered by $SpecA \times_B A$ where the morphism locally looks like:

$$SpecA \to SpecA \otimes_B A$$

which is given by $A \otimes_B A \to A, \qquad a \otimes a' \mapsto aa'$. We only need to show that the kernel is finitely generated. Recall that the ideal is generated by elements of the form $1 \otimes a - a \otimes 1$. Now if we assume that A is finitely generated over B by $a_1, \cdots, a_n$, then I claim that the kernel is generated by $a_i \otimes 1 - 1 - \otimes a_i$ where $1 \leqslant i \leqslant n$. To see this, suppose that $a = f(a_1, \cdots, a_n)$ where f is a polynomial with coefficients in B. Then we know that $1 \otimes a - a \otimes 1 = f(1 \otimes a_1, \cdots, 1 \otimes a_n) - f(a_1 \otimes 1, \cdots, a_n \otimes 1)$. Therefore we are done. □

Once we know this open embedding characterization of unramifed morphisms:

Proposition 7.10. Suppose that $\pi : X \to Y$ and $\rho : Y \to Z$ is locally of finite type, let $\tau = \rho \circ \pi$, then:

1. If τ is unramifed, then so is π.
2. If ρ is unramifed, and τ is smooth of relative dimension n, then π is smooth of relative dimension n.

Proof. The first one just comes from the any locally closed embedding is unramified, and the cancellation theorem.

For the second assertion, if π, notice that we know that if ρ is unramified, we know that the diagonal is smooth of relative dimension 0 as it's an open emebedding. Then we know that since smooth morphisms are closed under composition and under base change, cancellation theorem tells you that $X \to Y$ is also smooth.

We need to show the relative dimension is n. Recall that if $X \to Y$ smooth of relative dimension n, we can easily prove that the sheaf of differentials $\Omega_{X/Y}$ has constant rank n. Then let's consider the cotangent exact sequence:

$$X \xrightarrow{\pi} Y \to Z$$

which is:

$$\pi^* \Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

By assumption, $\Omega_{Y/Z} = 0$, therefore $\Omega_{X/Z} \to \Omega_{X/Y}$ is an isomorphism, which implies that the rank at a point are the same. Since $X \to Z$ is smooth of relative dimension n, the rank is n, therefore $\Omega_{X/Y}$ has rank n, we are done. □

Proposition 7.11. Suppose that $\pi : X \to Y$ is locally of finite type, then the locus in X where π is unramified is open.

Proof. The locus where π is unramified is the locus where $(\Omega_{X/Y})_p = 0$. Since π is locally of finite type, we know that $\Omega_{X/Y}$ is finite type, therefore we know that $(\Omega_{X/Y})_p$ is finitely generated over $\mathcal{O}_{X,p}$, therefore we know that it's rank is 0 iff it's zero by Nakayama's lemma. But recall that the ranks for finite type quasicoherent sheaves, is it's rank at p is 0, then there is a neighborhood of p such that the rank is 0, which is equivalent to there is a neighborhood where $\Omega_{X/Y} = 0$. □

References