Appendix 15 for Math 632: 2025 Winter Notes

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In this part of the note series we will review spectral sequences and introduce derived functors in general. If you are also reading the etale cohomology note series, you should read this section before you read the chapter on group cohomology.

1 The idea behind spectral sequences

We won't introduce spectral sequences in general in this section as we won't need that generality to work with almost all of the stuff you will see. Here we will work with the category of A-modules where A is a commutative ring. The same argument in this section carries over to all abelian categories.

The goal of this section to do introduce enough of the idea behind spectral sequences (especially how it works) without going into the complicated constructions behind it so that people feel at ease with spectral sequences next time they see it. As some examples and practice, we will later use idea of spectral sequences to essentially prove results you are already familiar with (for example "the snake lemma"). By doing these practices, you will realize that spectral sequences is just another big abstract machine that helps you do all the abstract computations, and there is essentially nothing too fancy about them.

After this introduction section, if you don't want to get into the weeds of details, you can move forward to section 3 on derived functors directly. But if you are interested in seeing how this is formally defined, you can take a look at section 2 where we do have a formal treatment. However, you shoule always be aware that what matters is not spectral sequence, which is just tool, what matters is how to use it.

Definition 1.1. A double complex is a collection of A-modules $E^{p,q}$ with $p, q \in \mathbb{Z}$ and rightward morphisms $\mathrm{d}_{\rightarrow}^{p,q} : E^{p,q} \to E^{p+1,q}$ and upward morphisms $\mathrm{d}_{\uparrow}^{p,q} : E^{p,q} \to E^{p,q+1}$. In the superscript, p denotes the column coordinate (the x-coordinate if you like to think this way) and q is used to denote the row coordinate (the y-coordinate). This convention is the opposite of the convention for matrices. We sometimes will only write $d_{\rightarrow}, d_{\uparrow}$ and omit the superscripts.

In this double complex, we require $\mathrm{d}_{\rightarrow}^2 = 0, \mathrm{d}_{\uparrow}^2 = 0$ and one more condition is that we need $\mathrm{d}_{\rightarrow} \circ \mathrm{d}_{\uparrow} = \mathrm{d}_{\uparrow} \circ \mathrm{d}_{\rightarrow}$ (which is called the square commutes) or $\mathrm{d}_{\rightarrow} \circ \mathrm{d}_{\uparrow} + \mathrm{d}_{\uparrow} \circ d_{\rightarrow} = 0$ (which is called the square anticommutes). You can easily switch back and forth between the two conventions easily by replace $\mathrm{d}_{\uparrow}^{p,q}$ with $(-1)^p \mathrm{d}_{\uparrow}^{p,q}$. You will soon find that there is no difference in these two conventions when to comes to spectral sequences, basically because the image and kernel of the map f with be the same with that of $-f$. Therefore we will just use the "anticommute" convention.

Definition 1.2. Given a double complex $E^{p,q}$, we can define it's total complex $E^{\bullet}$, which is given by $E^k = \oplus_{i \in \mathbb{Z}} E^{i,k-i}$ and differential is given by $d = d_{\rightarrow} + d_{\uparrow}$.

You can imagine that this single complex is a direct sum of the anti-diagonal (recall that the index convention here is the opposite of the matrices, therefore the direction of the anti-diagonal in a double complex is the same as the direct of the diagonal in a matrix, i.e. from left-top to right-bottom.) The morphism is indeed a differential since we know that:

$$(d_{\rightarrow} + d_{\uparrow})^2 = d_{\rightarrow}^2 + d_{\uparrow}^2 + d_{\rightarrow} d_{\uparrow} + d_{\uparrow} d_{\rightarrow} = 0$$

You see this is where the anti-commute convention helps since otherwise we have to define the differential using index adjusting for example $d^{p,q} = (-1)^p d_{\uparrow}^{p,q} + d_{\rightarrow}^{p,q}$. The anti-commute convention simplifies the definition. This single complex $E^{\bullet}$ is always called the **total complex** of the double complex $E^{p,q}$. Notice that the arrows in this total complex goes from $E^k \to E^{k+1}$.

Definition 1.3. The cohomology of the total complex is sometimes called the **hypercohomology** of the double complex. But we will just call it the **cohomology of the double complex**.

The initial goal of spectral sequences is to compute the information of the cohomology of this double complex. But later we will see that is also has other goals.

Definition 1.4. A **spectral sequence associated to the double complex** $E^{p,q}$ with **rightward orientation** is a sequence of "table" or "**pages**":

$$ {}_{\to}E_0^{p,q}, {}_{\to}E_1^{p,q}, {}_{\to}E_2^{p,q}, \cdots, \quad p, q \in \mathbb{Z} $$

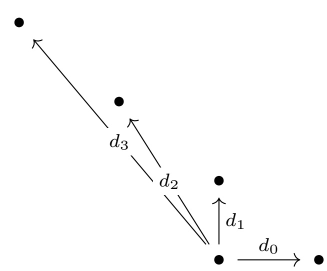
where ${}_{\to}E_0^{p,q} = E^{p,q}$ for the double complex $E^{p,q}$, along with a differential:

$$ {}_{\to}\mathrm{d}_r^{p,q} : E_r^{p,q} \to E^{p-r+1,q+r}, \quad r \in \mathbb{Z}^{\geqslant 0} $$

such that ${}_{\to}\mathrm{d}_r^{p,q} \circ {}_{\to}\mathrm{d}_r^{p+r-1,q-r} = 0$. Moreover, we have a prescribed isomorphism of the cohomology of ${}_{\to}\mathrm{d}_r$ at $E_r^{p,q}$ (i.e. $\ker {}_{\to}\mathrm{d}_r^{p,q} / \mathrm{im} {}_{\to}\mathrm{d}_r^{p+r-1,q-r}$) with $E_{r+1}^{p,q}$.

This **rightward orientation** is understood as when $r = 0$, ${}_{\to}\mathrm{d}_0^{p,q} : E_0^{p,q} \to E_0^{p+1,q}$ which is just the $d_{\to}$ morphism $E^{p,q} \to E^{p+1,q}$ in the double complex. Usually we will omit the left subscript $\to$.

Notice that if we fix r then these $d_r^{p,q}$ only operates within the "same page", i.e. within $E_r^{p,q}$ and it won't go into other pages. The direction of these morphism $d_r^{p,q}$ is best understood by the diagram:



But pleast notice that d with different subcript r means that they are in different pages, even though it looks like the same page in the above diagram.

Remark 1.5. Since in the definition, the spectral sequence is said to be associated to the double complex $E^{p,q}$ and indeed the 0-th page is $E^{p,q}$ with the differentials $\mathrm{d}_0^{p,q}$ going from $E^{p,q}$ to $E^{p+1,q}$ for each p, q, you will naturally ask that are all the later pages $E_r^{p,q}$ and differentials $d_r^{p,q}$ related to the double complex as well.

The fact is they are indeed related and in fact constructed using the information contained in the double complex, such that all the requirements in the above definition is satisfied. We currently postpone the construction but you should be aware of it and have the faith that all these above definition is not just coming from no where but rather very natural.

Remark 1.6. Note that via the isomorphism between $E_r^{p,q}$ and the cohomology of $d_{r-1}^{p,q}$, $E_r^{p,q}$ is always a subquotient of the object in the previous page. Recall that subquotient is transitive, therefore we know that $E_r^{p,q}$ is always a subquotient of $E_0^{p,q} = E^{p,q}$. Therefore if $E^{p,q} = 0$, then $E_r^{p,q} = 0$ for all $r \geqslant 0$.

Now suppose that $E^{p,q}$ is a **first quadrant double complex**, that is $E^{p,q} = 0$ for all $p, q < 0$. So according to the previous remark, $E_r^{p,q} = 0$ for all r unless $p, q \geqslant 0$. Therefore we know that for any fixed p, q once r gets sufficiently large $d_r^{p,q} : E_r^{p,q} \to E^{p-r+1,q+r}$ and $d_r^{p+r-1,q-r} : E_r^{p+r-1,q-r} \to E_r^{p,q}$ will 0 as their target and 0 as their source, therefore we conclude that:

$$ E_r^{p,q} \simeq E_{r+1}^{p,q} $$

$$
\begin{tikzcd}
0 & & \\
 & E_r^{p,q} \arrow[ul, "d_r^{p,q}"'] & \\
 & & 0 \arrow[ul, "d_r^{p+r-1,q-r}"']
\end{tikzcd}
$$

WE can keep going and in fact we have a sequence of isomorphisms:

$$E_r^{p,q} \simeq E_{r+1}^{p,q} \simeq E_{r+2}^{p,q} \simeq \cdots$$

As these objects are all isomorphic any r large enough, we will use $E_\infty^{p,q}$ to denote it in a very suggestive way. We are suggesting that as r goes to ∞, $E_r^{p,q}$ stabilizes and becomes "constant". Similar situation happens in other cases as well. For example, when the double complex $E^{p,q}$ is nonzero for a finite number of rows, or columns. This will come up later in the example of mapping cones and long exact sequence.

As explained in one of the previous remark, the spectral sequence associated with a double complex is totally constructed using the information of the double complex. That is to say that each page $E_r^{p,q}$ and the differentials $d_r^{p,q} : E_r^{p,q} \to E_r^{p-r+1,q+r}$ is constructed using the original double complex $E^{p,q}$. Though we will not give the full description in this introduction to spectral sequences, let's nevertheless describe how things look like in the first few pages, just to give you a sense that everything here is very natural. This will also help you understand the general machinery if you want to study it in full details later.

The 0-th page: As given in the definition, the 0-th page is just with objects $E_0^{p,q}$ in the double complex $E^{p,q}$. And morphisms $d_0^{p,q} = d_\rightarrow^{p,q}$, therefore we know that the picture looks like:

$$
\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}
$$

The 1-th page: Then by our definition, the 1-th page is just the cohomology of the 0-th page, i.e. the cohomology of the rightward differentials in the double complex. Then recall that the upward morphisms $d_\rightarrow$ can be though as a map of complexes (here being the horizontal objects in the double complex) with a sign convention, therefore we know that $d_\uparrow$ induces vertical maps $d_1^{p,q} : E_1^{p,q} \to E_1^{p,q+1}$ on the cohomology of the row cohomology groups. The picture is now:

$$
\begin{tikzcd}
\bullet & \bullet & \bullet \\
\bullet \arrow[u] & \bullet \arrow[u] & \bullet \arrow[u] \\
\bullet \arrow[u] & \bullet \arrow[u] & \bullet \arrow[u]
\end{tikzcd}
$$

The 2-th page: Then we know that the 2-th page consists of the cohomology of the above vertical complexes. And we should be able to define natural morphisms d_2 from a cohomology of an entry to the cohomology of the entry that is 2 steps upward and 1 step left to it. That is illustrated in the following picture:

$$
\begin{tikzcd}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet \arrow[uul] & \bullet \arrow[uul]
\end{tikzcd}
$$

It's worth while to sit down and think a little bit how to define such natural morphisms as if you look at the next chapter, you will find that this correspond to the complicated machinery given there. For example, let's try to define $E_2^{p,q} \to E_2^{p-1,q+2}$. Notice that:

$$E_2^{p,q} = \frac{\ker(E_1^{p,q} \to E^{p,q+1})}{\mathrm{im}(E_1^{p,q-1} \to E_1^{p,q})}$$

Then we take a class $[x] \in E_2^{p,q}$, which is represented by some element in the kernel of $E_1^{p,q} \to E^{p,q+1}$, which is:

$$\ker\left(\frac{\ker E_0^{p,q} \to E_0^{p+1,q}}{\mathrm{im} E_0^{p-1,q} \to E_0^{p,q}} \xrightarrow{d_\uparrow} \frac{\ker E_0^{p,q+1} \to E_0^{p+1,q+1}}{\mathrm{im} E_0^{p-1,q+1} \to E_0^{p,q+1}}\right)$$

Then we know that x can be though as an element in $E_0^{p,q}$ thagt is mapped to an element in $\mathrm{im} E_0^{p-1,q+1} \to E_0^{p,q+1}$, i.e. if we consider the image of x under $d_\uparrow$, the image comes from some element y in $E_0^{p-1,q+1}$. Then notice that we will want to map $[x]$ to some element in the following object:

$$E_2^{p-1,q+2} = \frac{\ker(E_1^{p-1,q+2} \to E_1^{p-1,q+3})}{\mathrm{im}(E_1^{p-1,q+1} \to E_1^{p-1,q+2})}$$

Notice that $\ker(E_1^{p-1,q+2} \to E^{p-1,q+3})$ is:

$$\ker\left(\frac{\ker E_0^{p-1,q+2} \to E_0^{p,q+2}}{\mathrm{im} E_0^{p-2,q+2} \to E_0^{p-1,q+2}} \to \frac{\ker E_0^{p-1,q+3} \to E_0^{p,q+3}}{\mathrm{im} E_0^{p-2,q+3} \to E_0^{p-1,q+3}}\right)$$

This is to say that we would like to find something in $E_0^{p-1,q+2}$ such that x is mapped to naturally. Notice that if we consider $d_\uparrow x$ which is in $E_0^{p,q+1}$, we know that it comes from some y in $E_0^{p-1,q+1}$, then a natural though is to apply $d_\uparrow$ again and we know that we get an element in $E_0^{p-1,q+2}$. Therefore we will define the image of $[x]$ under the map $E_2^{p,q} \to E_2^{p-1,q+2}$ will be $[d_\uparrow(y)]$. A diagram is drawn below to help you understand the flow:

$$\begin{tikzcd}
E_0^{p-1,q+2} & \\
y \in E_0^{p-1,q+1} \arrow[u, dashed, "d_\uparrow"] \arrow[r, dashed, "d_\rightarrow"] & E_0^{p,q+1} \\
& x \in E_0^{p,q} \arrow[u, dashed, "d_\uparrow"]
\end{tikzcd}$$

Then we need to check this is well defined. First, we need to check that $[d_\uparrow y]$ is in $\ker E_1^{p-1,q+2} \to E_1^{p-1,q+3}$. This is clear since we know that $d_\uparrow d_\uparrow y = 0$. Then suppose that y, y' both maps to x, then we know that $y - y'$ is in the kernel of $d_\rightarrow$. We need to show that $d_\uparrow(y - y')$ is in $\mathrm{im}(E_1^{p-1,q+1} \to E_1^{p-1,q+2})$. But notice that since $y - y'$ is in the kernel of $d_\rightarrow$, then it's indeed in the image of $E_1^{p-1,q+1} \to E_1^{p-1,q+2}$, which is itself $\frac{\ker d_\rightarrow}{\cdots} \xrightarrow{d_\uparrow} \frac{\ker d_\rightarrow}{\cdots}$.

I strongly recommend you do this for one time, and you will find that this argument is very similar to what happens in the snake lemma construction or in the long exact sequence construction. This is not a coincidence as you will see later that we can indeed compute the snake lemma and the associated LES using the spectral sequence argument, so in some sense these results we saw before are just special cases of spectral sequences.

Then there is a theorem that states that once we continue to construct later pages and $d_r^{p,q}$ naturally way, then if the double complex is first quadrant or is nonzero only on finitely many rows or columns, there will be a filtration of the cohomology of the total complex $E^\bullet$ by $E_\infty^{p,q}$ such that $p + q = k$. Recall that there will always be some p, q such that $p + q = k$ and for all $p' < p$, $E_\infty^{p',q} = 0$, similarly there will always be p'' such that for all $p > p''$, we have $E_\infty^{p,q} = 0$. Then we know that we can always write the filtration by:

$$E_\infty^{p,k-p} \overset{E_\infty^{p+1,k-1-p}}{\hookrightarrow} ? \overset{E_\infty^{p+2,k-2-p}}{\hookrightarrow} ? \cdots ? \overset{E_\infty^{p'',k-p''}}{\hookrightarrow} H^k(E^\bullet)$$

where the successive quotients are indicated above the arrow. For example, when we have a first quadrant double complex, we have the quotient:

$$E_\infty^{0,k} \overset{E_\infty^{1,k-1}}{\hookrightarrow} ? \overset{E_\infty^{2,k-2}}{\hookrightarrow} ? \cdots ? \overset{E_\infty^{k,0}}{\hookrightarrow} H^k(E^\bullet)$$

Here is a tip for you to remember why the filtration starts with $E_\infty^{0,k}$ but not $E_\infty^{k,0}$. You can always remember that the subobjects is in the direction of the "arrowheads" of differentials on each pages. We know that the arrowhead goes in a direction where p keeps going smaller and q keeps getting larger. Then the subobjects should be $E_\infty^{0,k}$ which has a smaller p and larger q compared to $E_\infty^{k,0}$. There is not a formal reason why this works this way, but the advantage is that by remembering this way, you only have to keep track of the arrow in the spectral sequence, not the actually index. The spectral sequence we introduced above is just one version of it and different authors may have different preference on the index, but in the end, only the direction of the arrows matters. So you should remember in terms of the arrows, not the indices.

To give a term for the above filtration of $H^\bullet(E^\bullet)$, we will say that the spectral sequence $E_\bullet^{\bullet,\bullet}$ **converges** to $H^\bullet(E^\bullet)$ or the spectral sequence ${}_{\rightarrow}E_2^{\bullet,\bullet}$ **abuts** to $H^\bullet(E^\bullet)$.

Remark 1.7. By a filtration, we can only get partial information on $H^\bullet(E^\bullet)$, even if we know all these $E_\infty^{p,q}$. However, there are still some situations where we can get full information of $H^\bullet(E^\bullet)$ just by the filtration.

For example: if all $E_\infty^{i,k-i} = 0$, then we can conclude $H^k(E^\bullet)$ is 0. Or if only one of them, say $E^{i,k-i}$ is nonzero, then we know $H^k(E^\bullet) \simeq E^{i,k-i}$. Or if we work in the category of k-vector spaces, we can get information about the dimension using the filtration.

Notice that one special case is that at some pages r, $E_r^{p,q}$ has only one row or column that is nonzero. We say that the spectral sequences **collapses at page** r in this case.

The following proposition shows you that we can also extract some useful information from the second page:

Proposition 1.8. Suppose that we have a first quadrant double complex $E^{p,q}$, then $H^0(E^\bullet) = E_\infty^{0,0} = E_2^{0,0}$ and the following sequence is exact:

$$0 \to E_2^{0,1} \to H^1(E^\bullet) \to E_2^{1,0} \overset{d_2^{1,0}}{\to} E_2^{0,2} \to H^2(E^\bullet)$$

Proof. Notice that by the filtration we have, $H^0(E^\bullet)$ is just $E_\infty^{0,0}$ as we know that for $p+q=0$, only when $p=q=0$ are those two numbers not simultaneously negative. Then notice that the differential on $d_2^{0,0} : E_2^{0,0} \to E_2^{-1,2} = 0$ and $d_2^{1,-2} : 0 = E_2^{1,-2} \to E_2^{0,0}$ makes $E_2^{0,0} \simeq E_\infty^{0,0}$.

To see why this sequence is exact. First notice that $E_2^{0,1}$ is also $E_\infty^{0,1}$ for the same reason as why $E_\infty^{0,0} = E_2^{0,0}$. Then we know that we have the following filtration:

$$E_\infty^{0,1} \hookrightarrow H^1(E^\bullet)$$

with cokernel $E_\infty^{1,0}$, this gives the exact sequence:

$$0 \to E_2^{0,1} \to H^1(E^\bullet) \to E_\infty^{1,0} \to 0$$

Notice that the image of $d_2^{2,-2} : 0 = E_2^{2,-2} \to E_2^{1,0}$ has 0 image thus kernel of $E_2^{1,0} \overset{d_2^{1,0}}{\to} E_2^{0,2}$ is $E_3^{1,0}$, which is $E_\infty^{1,0}$. Then we have the exact sequence:

$$0 \to E_2^{0,1} \to H^1(E^\bullet) \to E_2^{1,0} \overset{d_2^{1,0}}{\to} E_2^{0,2}$$

where the map $H^1(E^\bullet) \to E_2^{1,0}$ is given by:

$$H^1(E^\bullet) \to E_\infty^{1,0} \hookrightarrow E_2^{1,0}$$

Finally, note we have a filtration of $H^2(E^\bullet)$ by:

$$E_\infty^{0,2} \overset{E_\infty^{1,1}}{\hookrightarrow} ? \overset{E_\infty^{2,0}}{\hookrightarrow} H^2(E^\bullet)$$

Notice that $d_2^{0,2} : E_2^{0,2} \to E_2^{-1,4} = 0$, therefore we know that the cokernel of $d_2^{1,0}$ is $E_3^{0,2} = E_\infty^{0,2}$, therefore we can extend the exact sequence to:

$$0 \to E_2^{0,1} \to H^1(E^\bullet) \to E_2^{1,0} \xrightarrow{d_2^{1,0}} E_2^{0,2} \to E_\infty^{0,2}$$

Notice that since $E_2^{0,2} \to E_\infty^{0,2}$ is the cokernel of $d_2^{1,0}$, we know that the composition is 0, therefore has image 0, then we have the following exact sequence:

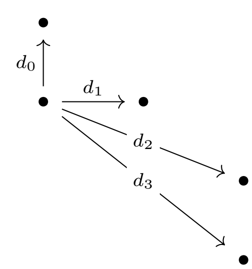
$$0 \to E_2^{0,1} \to H^1(E^\bullet) \to E_2^{1,0} \xrightarrow{d_2^{1,0}} E_2^{0,2} \to H^2(E^\bullet) \to 0$$

where $E_2^{0,2} \to H^2(E^\bullet)$ is defined to be:

$$E_2^{0,2} \to E_\infty^{0,2} \to H^2(E^\bullet)$$

Therefore the kernel of $E_2^{0,2} \to H^2(E^\bullet)$ is the kernel of $E_2^{0,2} \to E_\infty^{0,2}$, thus is indeed exact. $\square$

You may have found that what we discussed above could all be done in the opposite direction, i.e. with the role of rows and columns reversed. Then we will get spectral sequences that looks like the following picture:



Here we will require the 0-th page to be again $E^{p,q}$, i.e. the double complex we started with. We also require the differentials in the 0-th page to be $d_\uparrow$ of the double complex $E^{p,q}$. This is called a spectral sequence with **upward orientation**. We would again get pieces of filtration of $H^\bullet(E^\bullet)$, with the order of the filtration given by the direction of the arrow as explained above.

Remark 1.9. In fact this observation that we can get a filtration of $H^\bullet(E^\bullet)$ is our secret goal. No matter we start with the horizontal differentials, or the vertical differential, we can compute the information about $H^\bullet(E^\bullet)$. Sometimes we will care about the information that is produced in one way, but we will often get it by doing the spectral sequence in another way!

Now we are ready to see some examples and see why this complicated algorithm is useful.

Example 1.10. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & D \arrow[r] & E \arrow[r] & F \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, "\alpha"] & B \arrow[r] \arrow[u, "\beta"] & C \arrow[r] \arrow[u, "\gamma"] & 0
\end{tikzcd}$$

where both rows are exact, then we want to show that there is an exact sequence:

$$0 \to \ker\alpha \to \ker\beta \to \ker\gamma \to \operatorname{coker}\alpha \to \operatorname{coker}\beta \to \operatorname{coker}\gamma \to 0$$

You probably have almost definitely seen this before, but here we want to use the spectral sequence machinery to get the result.

First, notice that by appending 0's, we can get a complex that is only in the first quadrant. Then we first compute the spectral sequence in the rightward orientation. However, we notice that since the rows are exact, then the cohomology are all 0! Therefore $E_\infty^{p,q} = 0$ for all p, q. Then we will compute this 0 in the upward orientation.

Notice that in the upward orientation the first page is:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{coker}\alpha \arrow[r] & \mathrm{coker}\beta \arrow[r] & \mathrm{coker}\gamma \arrow[r] & 0 \\
0 \arrow[r] & \ker\alpha \arrow[r] & \ker\beta \arrow[r] & \ker\gamma \arrow[r] & 0
\end{tikzcd}$$

Then what is the 2-th page, it looks like the following:

$$\begin{tikzcd}
0 \arrow[drr] & 0 \arrow[drr] & & & & & \\
& 0 \arrow[r] \arrow[drr] & ?? \arrow[r] \arrow[drr] & ? \arrow[r] \arrow[drr] & ? \arrow[r] & 0 & \\
& 0 \arrow[r] & ? \arrow[r] & ? \arrow[r] \arrow[drr] & ?? \arrow[r] \arrow[drr] & 0 & \\
& & & & & 0 & 0
\end{tikzcd}$$

We use the ? and ?? to indicate the cohomology of the 1-th page which we don't really care. Then we find that all the ? already stabilizes in the 2-page, therefore all the ? are 0 since we know that in the end the cohomology of the double complex will be 0, by the result we computed in the rightward orientation. We know that all $_\uparrow E_\infty^{p,q}$ should also all vanish. Moreover, we know that the ?? will stabilize on the 3-page, therefore the cohomology will be 0, thus ?? $\to$?? will be an isomorphism, therefore we have the following sequence:

$$0 \to \ker\alpha \to \ker\beta \to \ker\gamma \to \mathrm{coker}(\ker\beta \to \ker\gamma) \simeq \ker(\mathrm{coker}\alpha \to \mathrm{coker}\beta) \to \mathrm{coker}\alpha \to \mathrm{coker}\beta \to \mathrm{coker}\gamma \to 0$$

This gives us the desired exact sequence. Therefore, we find that the spectral sequence helps us do all the computations we used to do, for example, lifting and checking it's well-defined. Now we only need to apply the spectral sequence machinery and we automatically get the result.

One interesting aspect of getting this result using spectral sequences is that the spectral sequence machine already tells you how to modify the result, for example when $A \to B$ is not injective, or when $E \to F$ is not surjective. For example:

Example 1.11. Suppose we have a diagram:

$$\begin{tikzcd}
& 0 \arrow[r] & X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & W' \arrow[r] & \cdots \\
\cdots \arrow[r] & W \arrow[r] \arrow[u] & X \arrow[r] \arrow[u, dashed, "\alpha"] & Y \arrow[r] \arrow[u, dashed, "\beta"] & Z \arrow[r] \arrow[u, dashed, "\gamma"] & 0 \arrow[u] &
\end{tikzcd}$$

where the two rows are exact, that is we don't necessarily know that if $X \to Y$ is injective and $Y' \to Z'$ is surjective. We will see that we again have an exact sequence:

$$\begin{tikzcd}
\cdots \arrow[r] & W \arrow[r] & \ker\alpha \arrow[r] & \ker\beta \arrow[r] & \ker\gamma \\
\arrow[r] & \mathrm{coker}\alpha \arrow[r] & \mathrm{coker}\beta \arrow[r] & \mathrm{coker}\gamma \arrow[r] & W' \arrow[r] & \cdots
\end{tikzcd}$$

We again append 0's to make a double complex and then we compute the spectral sequence in the rightward orientation. Notice that since the row are exact, then we again get all the $E_\infty^{p,q} = 0$ and the cohomology of the double complex is also 0.

Then we compute in the upward orientation, here we again have the 1-th page:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{coker}\alpha \arrow[r] & \mathrm{coker}\beta \arrow[r] & \mathrm{coker}\gamma \arrow[r] & W' \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & W \arrow[r] & \ker\alpha \arrow[r] & \ker\beta \arrow[r] & \ker\gamma \arrow[r] & 0
\end{tikzcd}$$

Notice that the 2-th page looks like:

$$\begin{tikzcd}
0 \arrow[drr] & 0 \arrow[drr] & & & & & \\
& 0 \arrow[drr] & ?? \arrow[drr] & ? \arrow[drr] & ? & ? & \\
& ? & ? & ? \arrow[drr] & ?? \arrow[drr] & 0 & \\
& & & & & 0 & 0
\end{tikzcd}$$

Again we know that all the ? stabilizes in the second page, thus they are all 0. Then ?? stabilizes on the 3-th page, therefore we know that everything is the same as in the previous exercise and we indeed get the above exact sequence we want. The only difference is that in the case where $W = W' = 0$, the map $\ker\alpha \to \ker\beta$ is injective and $\mathrm{coker}\beta \to \mathrm{coker}\gamma$ is surjective. But now due to the obstruction of W, W', they are not right now. But besides this nothing changes.

Remark 1.12. You probably will notice that in the above example, the squares don't anticommute but commute. As what we explained above, aside the differentials in the total complex changes, nothing really changes in the spectral sequences machinery and everything works just like in the anti-commute case.

Example 1.13. This example is about the five lemma you also probably have proved before using elementary methods. Suppose that we have the following diagram:

$$\begin{tikzcd}
F \arrow[r] & G \arrow[r] & H \arrow[r] & I \arrow[r] & J \\
A \arrow[r] \arrow[u, "\alpha"] & B \arrow[r] \arrow[u, "\beta"] & C \arrow[r] \arrow[u, "\gamma"] & D \arrow[r] \arrow[u, "\delta"] & E \arrow[u, "\epsilon"]
\end{tikzcd}$$

We require the rows to be exact and the squares to commute. Suppose that $\alpha, \beta, \delta, \epsilon$ are isomorphism, then the claim is that γ is also an isomorphism.

To show this, let's again first get a double complex by appending 0's. Then we compute the spectral sequence in the rightward direction. We know that the 1-th page is:

$$\begin{tikzcd}
? & 0 & 0 & 0 & ? \\
? \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & ? \arrow[u]
\end{tikzcd}$$

Then in the 2-th page, we notice that there besides the arrows that are between 0's, we know that the arrows that start from ? or end in ? has the other end a 0. Therefore we know that the spectral sequences in the second pages. We can not conclude all the cohomology of the double complex vanishes, but what we can is that the cohomology in the degree that contains C and H vanish.

Then we do the computation in the upward orientation. Notice that the 1-th page is:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & ? \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & ? \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Then we know that the sequence already stabilizes in the 1-page, then we know that the two ? must be the cohomology of the double complex, thus must be 0. But notice that the two ? are the cokernel and kernel, therefore γ is indeed an isomorphism.

Let's do a few more exercises to get more confidence in the fact that spectral sequence argument is not as obscure you might though.

Example 1.14. Let's do another version of the five lemma. In fact in order to conclude that γ is an isomorphism, we don't need all $\alpha, \beta, \delta, \epsilon$ to be isomorphism.

Notice that as long as we still require the row to be exact, the result of the computation done in the rightward orientation will be the same. Then we notice that as long as we assume that α is surjective and ϵ is injective, then again the spectral sequence stabilizes in the 1-th page and we get:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & ? \arrow[r] & 0 \arrow[r] & ? \\
? \arrow[r] & 0 \arrow[r] & ? \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Then the same argument as the above shows the desired result.

Remark 1.15. The key in the above argument is that it should stabilizes at the 1-page. Notice that if we instead get the following 1-page, then the spectral sequence doesn't stabilize in the 1-th page:

$$\begin{tikzcd}
? \arrow[r] & 0 \arrow[r] & ? \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r] & ? \arrow[r] & 0 \arrow[r] & ?
\end{tikzcd}$$

Then reason is that even though if you compute the 2-th page, you get the same object as in the 1-page, the differentials in the 2-page is:

$$\begin{tikzcd}
? \arrow[r] \arrow[drr] & 0 \arrow[r] & ? \arrow[r] \arrow[drr] & 0 \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r] & ? \arrow[r] & 0 \arrow[r] & ?
\end{tikzcd}$$

Therefore there are still nontrivial stuff happening between the nonzero objects, therefore we get that 3-th page looks different, therefore it stabilizes at 3-th page, instead at 1-th page.

Example 1.16. Here is another subtle version of the five lemma. Again we consider the diagram in the usual five lemma with rows exact, then I claim that if β, δ are injective, α is surjective, then γ is injective. Dually, if β, δ are surjective and ϵ is injective, then γ is surjective.

Let's first prove the first result, since we didn't assume anything about ϵ, then we can just delete the last vertical column and we know that the rightward computation becomes:

$$\begin{tikzcd}
? & 0 & 0 & ? \\
? \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & ? \arrow[u]
\end{tikzcd}$$

Then we know that the spectral sequence again stabilizes at 1-th page and then we know even though we can not conclude all cohomology of the double complex is 0, we know that the degree that contains the source of γ is 0.

Then we can do the computation in the upward direction, which gives you:

$$\begin{tikzcd}
0 \arrow[r] & ? \arrow[r] & ? \arrow[r] & ? \\
? \arrow[r] & 0 \arrow[r] & ? \arrow[r] & 0
\end{tikzcd}$$

Notice that this doesn't stabilizes at the 1-th page, but obviously it stabilize in the 2-page. Then we know that in the second page, we get the fact that in 2-th page:

$$\begin{tikzcd}
0 \arrow[dr] & & & & & \\
& 0 \arrow[r] \arrow[drr] & ? \arrow[r] & ? \arrow[r] & ? & \\
& ? \arrow[r] & 0 \arrow[r] & ? \arrow[r] \arrow[drr] & 0 & \\
& & & & & 0
\end{tikzcd}$$

Then we know that The fact that it should stabilize at 2-page and the fact that the limit at the second ? on the second row is 0 tells you that ? = 0. But notice that ? on the 2-page is the same as the one on the 1-page, which is the kernel of γ, we are done.

The dual statement is proved similarly.

The following two examples doesn't involve first quadrant double complexes anymore, but as what you will see in doing them, nothing is essentially different.

Example 1.17. Let $\mu : A^\bullet \to B^\bullet$ be a morphism of complexes, then let $C^\bullet$ be the total complex of this double complex (still by appending 0's). This is called the **mapping cone** of the morphism of complex μ. If you have seen the definition of a mapping cone before, you will notice that here we re indeed using the sign convention to make sure that the total complex is indeed a complex. Then I claim that we will have a long exact sequence:

$$\cdots \to H^{i-1}(C^\bullet) \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to \cdots$$

Let's first be clear that the i-th degree of the mapping cone will be $A^{i+1} \bigoplus B^i$, then notice that μ induces $H^i(A^\bullet) \to H^i(B^\bullet)$ for all i.

Then let's compute the spectral sequence of this double complex $\mu : A^\bullet \to B^\bullet$ in the rightward direction:

$$\begin{tikzcd}
\cdots & H^{i-1}(B^\bullet) & H^i(B^\bullet) & H^{i+1}(A^\bullet) & \cdots \\
\cdots \arrow[u] & H^{i-1}(A^\bullet) \arrow[u] & H^i(A^\bullet) \arrow[u] & H^{i+1}(A^\bullet) \arrow[u] & \cdots \arrow[u]
\end{tikzcd}$$

In order to make our indexing more convenient, let's assume that the row of $A^\bullet$ is the -1-th row and the row of $B^\bullet$ is the 0-th row, and we will call the double complex $E^{p,q}$ for example $A^0 = E^{-1,0}$, $B^1 = E^{0,1}$.

Then we know that $H^i(C^\bullet)$ will have filtration:

$$E^{i,0}_\infty \overset{E^{i+1,-1}_\infty}{\hookrightarrow} H^i(C^\bullet)$$

That is we have an exact sequence:

$$0 \to E^{i,0}_\infty \to H^i(C^\bullet) \to E^{i+1,-1}_\infty \to 0$$

Notice that if you compute what is $E^{i+1,-1}_\infty$, we get it's the kernel of $H^{i+1}(A^\bullet) \to H^{i+1}(B^\bullet)$ and $E^{i,0}_\infty$ is the cokernel of $H^i(A^\bullet) \to H^i(B^\bullet)$. Then if we define: $H^i(C^\bullet) \to H^{i+1}(A^\bullet)$ by

$$H^i(C^\bullet) \to E^{i+1,-1}_\infty \hookrightarrow H^{i+1}(A^\bullet)$$

and we define $H^i(B^\bullet) \to H^i(C^\bullet)$ to be

$$H^i(B^\bullet) \to E^{i,0} \hookrightarrow H^i(C^\bullet)$$

In this way, we get a long sequence:

$$\cdots \to H^{i-1}(C^\bullet) \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to \cdots$$

and we only need to show it's exact. I will let you do this last step.

Remark 1.18. As a corollary, we can prove that μ induces an isomorphism on each cohomology iff the mapping cone is exact!

Example 1.19. As the last example, let's prove the classical result: suppose that $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 9$ is a SES of complexes, then cohomology induces a LES:

$$\cdots \to H^{i-1}(C^\bullet) \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to \cdots$$

Let's compute the complex first in the upward orientation. We know that since the morphism of complexes are exact, we know that the first page is already all 0, therefore we know that the cohomology of the double complexes vanishes at each degree. Then we compute the spectral sequence in the rightward direction, we know that the 1-th page is:

$$\begin{tikzcd}
\cdots & H^{i-1}(C^\bullet) & H^i(C^\bullet) & H^{i+1}(C^\bullet) & \cdots \\
\cdots \arrow[u] & H^{i-1}(B^\bullet) \arrow[u] & H^i(B^\bullet) \arrow[u] & H^{i+1}(B^\bullet) \arrow[u] & \cdots \arrow[u] \\
\cdots \arrow[u] & H^{i-1}(A^\bullet) \arrow[u] & H^i(A^\bullet) \arrow[u] & H^{i+1}(A^\bullet) \arrow[u] & \cdots \arrow[u]
\end{tikzcd}$$

We know that obviously this stabilizes in the 3-page and if you look at the 2-th page, there will be morphism of the form:

$$\ker(H^i(A^\bullet) \to H^i(B^\bullet)) \to \operatorname{coker}(H^{i-1}(B^\bullet) \to H^{i-1}(C^\bullet))$$

Notice that since the cohomology of the total complex vanishes, this morphism musty be an isomorphism, therefore we conclude that we have the following morphism:

$$H^{i-1}(C^\bullet) \to \operatorname{coker}(H^{i-1}(B^\bullet) \to H^{i-1}(C^\bullet)) \simeq \ker(H^i(A^\bullet) \to H^i(B^\bullet)) \hookrightarrow H^i(A^\bullet)$$

This will be the connecting morphism we use. Then we will need to prove that this long sequence is indeed exact. It's easy to show that this sequence is exact at $H^\bullet(A^\bullet)$ by the definition of $H^{i-1}(C^\bullet) \to H^i(A^\bullet)$. Similarly it's easy to prove it's exact at $H^\bullet(C^\bullet)$. Finally to prove that it's exact at $H^\bullet(B^\bullet)$, we consider the 2-page of the spectral sequence and we find that we have a sequence:

$$0 \to \frac{\ker(H^i(B^\bullet) \to H^i(C^\bullet))}{\operatorname{im}(H^i(A^\bullet) \to H^i(B^\bullet))} \to 0$$

Recall that the sequence stabilizes on the 3-th page, therefore we know that $\frac{\ker(H^i(B^\bullet) \to H^i(C^\bullet))}{\operatorname{im}(H^i(A^\bullet) \to H^i(B^\bullet))} = 0$, that is the sequence is exact at $H^\bullet(B^\bullet)$.

Remark 1.20. Recall that the connecting homomorphism given above should be "natural", in the sense that if we have a map of SES between $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ and $0 \to D^\bullet \to E^\bullet \to F^\bullet \to 0$, then this induces a map between the two associated LES, then the connecting homomorphisms should make this diagram commute.

Actually the above argument helps us show this result, consider the following diagram:

$$\begin{tikzcd}
H^{i-1}(C^\bullet) \arrow[r] \arrow[d] & \operatorname{coker}(H^{i-1}(B^\bullet) \to H^{i-1}(C^\bullet)) \arrow[r, "\simeq"] \arrow[d] & \ker(H^i(A^\bullet) \to H^i(B^\bullet)) \arrow[r, hook] \arrow[d] & H^i(A^\bullet) \arrow[d] \\
H^{i-1}(F^\bullet) \arrow[r] & \operatorname{coker}(H^{i-1}(E^\bullet) \to H^{i-1}(F^\bullet)) \arrow[r, "\simeq"] & \ker(H^i(D^\bullet) \to H^i(E^\bullet)) \arrow[r, hook] & H^i(D^\bullet)
\end{tikzcd}$$

It's easy to see that the first square and the last sqaure commute, then if the middle square commute, we are done. Recall that the isomorphism is the map on the 2-th page, therefore we have given a clear definition of what this map should be, I will let you check that this indeed commutes, which should not be hard to check.

The above examples should make you familiar enough with how to work with spectral sequences. And you can directly go to section 3 and further sections to see more examples of spectral sequences. For example, the Grothendieck composition of functors spectral sequence, which will be very important and has a lot variants.

However, we still didn't prove anything about why spectral sequences works like what we have discussed, mainly because we didn't even give you the construction recipe of the pages $E_r^{p,q}$ and $d_r^{p,q}$. Moreover, though spectral sequence associated to double complexes is the most common spectral sequence we see, there is indeed a more general way to define spectral sequences using filtration. We include all these discussion in the following section. Nevertheless, I don't recommend you read the next section right after you finished this section, you should definitely first for example, read about the Grothendieck composition of functors spectral sequence and have some practice before you spend sometime reading the formalisms and figure out there is really nothing fancy.

2 The Tor functors

Before we introduce the derived functors, we begin with a warm-up example, i.e. the Tor functor. This will be a useful tool to prove facts about flatness and here Tor is short for torsion.

The ideal behind Tor is simple, recall that $M \otimes_A -$ is a right exact functor, the fact that it's not exact on the left is annoying and we want to do something to fix it. The thing we hope for is that if we have the following SES:

$$0 \to N' \to N \to N'' \to 0$$

We hope that after we apply the tensor functor $M \otimes_A -$, the sequence we get:

$$M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

can be the end of a LES, i.e. we hope to extend this to the following exact sequence:

$$\begin{gathered} \cdots \to \mathrm{Tor}_i^A(M, N') \to \mathrm{Tor}_i^A(M, N) \to \mathrm{Tor}_i^A(M, N'') \to \cdots \\ \to \mathrm{Tor}_1^A(M, N') \to \mathrm{Tor}_1^A(M, N) \to \mathrm{Tor}_1^A(M, N'') \\ \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0 \end{gathered}$$

Here for a fixed i, we hope that morphisms in $\mathrm{Tor}_i^A(M, N') \to \mathrm{Tor}_i^A(M, N) \to \mathrm{Tor}_i^A(M, N'')$ are induced by the morphisms in the SES. And we need to have natural connecting morphisms:

$$\delta : \mathrm{Tor}_{i+1}^A(M, N'') \to \mathrm{Tor}_i^A(M, N')$$

So to make the above idea precise, we want to have covariant functors $\mathrm{Tor}_i(M, -)$ for each i with $\mathrm{Tor}_0(M, -) = M \otimes_A -$ and the natural connected homomorphisms δ such that for every SES, we get a LES like above. Here δ being "natural" means that if we have a map of SES, then we get two LES and the map of the SES induces a map between the two LES, then we require that the square involving the connecting morphism δ should commute. (Other squares automatically commute from the fact that $\mathrm{Tor}_i(M, -)$ is a functor).

Turns out it's not hard to make this work, which will motivate the construction for all derived functors. Recall that given any A-module N, we have a free resulotion $\mathscr{R}$:

$$\cdots \to A^{\oplus n_2} \to A^{\oplus n_1} \to A^{\oplus n_0} \to N \to 0$$

Here notice that we don't need n_i to be finite. Let's truncate this resolution by taking away the last term N, i.e. we consider:

$$\cdots \to A^{\oplus n_2} \to A^{\oplus n_1} \to A^{\oplus n_0} \to 0$$

Then we apply $M \otimes_A$ and we get a complex with terms $M^{\oplus n_i}$ as tensor commutes with direct sums. Then we will let $\mathrm{Tor}_i^A(M, N)_{\mathscr{R}}$ be the homology at the i-th stage for all $i \geqslant 0$. At this point, we are not sure if this indeed is well-defined (since it seems that we are choosing a resolution $\mathscr{R}$) and if it's well-defined, does it really define a functor. And most importantly, why does it assembles to a LES for any given SES.

Let's make some quick observations:

1. We notice that we have a canonical isomorphism:

$$\mathrm{Tor}_0^A(M,N)_{\mathscr{R}} \simeq M \otimes_A N$$

   This is because tensor is right exact, therefore we have an exact sequence:

$$M^{\oplus n_1} \to M^{\oplus n_0} \to M \otimes_A N \to 0$$

   Then we have a canonical isomorphism induced by $M^{\oplus n_0} \to M \otimes_A N$:

$$\frac{M^{\oplus n_0}}{\mathrm{im}(M^{\oplus n_1} \to M^{\oplus n_0})} \simeq M \otimes_A N$$

   Then we notice that the homology of the complex $\cdots \to M^{\oplus n_1} \to M^{\oplus n_0} \to 0$, we know that the 0-th homology is just $\frac{M^{\oplus n_0}}{\mathrm{im}(M^{\oplus n_1} \to M^{\oplus n_0})}$.

2. If $M \otimes_A -$ is exact, i.e. when M is exact, then we know that $\mathrm{Tor}_i^A(M,N)_{\mathscr{R}}$ is 0 for all $i > 0$.

Then suppose that we are given a morphism of A-modules $N \to N'$, and we choose free resolutions $\mathscr{R}, \mathscr{R}'$ for both objects, we get:

$$\begin{tikzcd} \cdots \arrow[r] & A^{\oplus n_2} \arrow[r] & A^{\oplus n_1} \arrow[r] & A^{\oplus n_0} \arrow[r] & N \arrow[d] \\ \cdots \arrow[r] & A^{\oplus n'_2} \arrow[r] & A^{\oplus n'_1} \arrow[r] & A^{\oplus n'_0} \arrow[r] & N' \end{tikzcd}$$

On important observation is that we can always lift the map $N \to N'$ to $A^{\oplus n_i} \to A^{\oplus n'_i}$ such that we get a commutative diagram:

$$\begin{tikzcd} \cdots \arrow[r] & A^{\oplus n_2} \arrow[r] \arrow[d, dashed] & A^{\oplus n_1} \arrow[r] \arrow[d, dashed] & A^{\oplus n_0} \arrow[r] \arrow[d, dashed] & N \arrow[d] \\ \cdots \arrow[r] & A^{\oplus n'_2} \arrow[r] & A^{\oplus n'_1} \arrow[r] & A^{\oplus n'_0} \arrow[r] & N' \end{tikzcd}$$

To see why, we do it inductively for i. First I will let you see that since $A^{\oplus n_0}$ is free, then it's easy to define a map $A^{\oplus n_0} \to A^{\oplus n'_0}$ such that the first square commutes. Then suppose that we have already defined the maps for $i-1$, then consider $A^{\oplus n_i}$, we need to lift the map $b : A^{\oplus n_i} \to A^{\oplus n_{i-1}} \to A^{\oplus n'_{i-1}}$. We can again arbitrarily lift and conclude we are done using freeness. (Here do you see why there is always a lift? Once we know that there is a lift, we can use axiom of choice.)

Let's call this lift by $\mathscr{R} \to \mathscr{R}'$. Now again, truncate both complexes of the term N, N' and then apply $M \otimes_A -$, we get a map of complexes which induces maps on homology, therefore we have:

$$\mathrm{Tor}_i^A(M,N)_{\mathscr{R}} \to \mathrm{Tor}_i^A(M,N')_{\mathscr{R}'}$$

Now obviously that this map has something to do with the choice of the lift. Therefore, in order to make this canonical, we have to prove that this is not dependent on the choice of the lifting. Recall that any two maps of complexes $f, f' : C_\bullet \to C'_\bullet$ induce the same map on homology if they are homotopic, i.e. there are sequence of maps $w : C_i \to C'_{i+1}$ such that $f - f' = dw + wd$.

Then one observation is that any two such liftings will be homotopic. To see why, we need to define $w : A^{\oplus n_i} \to A^{\oplus n'_{i+1}}$. Let's first define it for $A^{\oplus n_0} \to A^{\oplus n'_1}$. Notice that since both $A^{\oplus n_0} \to A^{\oplus n'_0}$ are liftings, we know that after we compose with $A^{\oplus n'_0} \to N'$, their difference is 0, therefore we know that the difference of the image in $A^{\oplus n'_0}$ will be in the kernel of $A^{\oplus n'_0} \to N'$, therefore comes from something in $A^{\oplus n'_1}$. Then we will define $e_i \in A^{\oplus n_0}$ to be mapped to an arbitrary lifting of it's image in $A^{\oplus n'_0}$ in $A^{\oplus n'_1}$. Then we will do induction. Suppose that we have defined for $A^{\oplus n_{i-1}} \to A^{\oplus n'_i}$. The following is a diagram that helps you understand how this induction works and I wil let you figure it out:

$$\begin{tikzcd}
& \bullet \arrow[r, "d"] \arrow[dl, "?"'] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & \bullet \arrow[r] \arrow[dl, "w"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & \bullet \arrow[d] \arrow[dl] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

$$f - g - wd(x) \longrightarrow d(f - g - wd(x))$$

(We want $f - g - wd(x)$ to be in the kernel of d, thus we can use eaxctness to find the lift, so you apply d to $f - g - wd(x)$, then you should check why it's 0.)

Now let's make all our observations together:

1. We know that once we choose resolutions $\mathscr{R}, \mathscr{R}'$ for N, N', then we get a unique map $\mathrm{Tor}_i^A(M,N)_{\mathscr{R}} \to \mathrm{Tor}_i^A(M,N')$ independent of lifts $\mathscr{R} \to \mathscr{R}'$.
2. Therefore if we have two resolutions $\mathscr{R}_1, \mathscr{R}_2$ for N, we get a canonical isomorphism between $\mathscr{T}_i^A(M,N)_{\mathscr{R}_1}$ and $\mathscr{T}_i^A(M,N)_{\mathscr{R}_1}$ obtained by lifting $\mathrm{Id}_N : N \to N$.

So how do we define the functor $\mathrm{Tor}_i^A(M,-)$? We choose a free resolution $\mathscr{R}_N$ of N for each N, and we use it to define $\mathrm{Tor}_i^A(M,-)$. For the temporary notation, we will call this $\mathrm{Tor}_i^A(M,-)_{\{\mathscr{R}_N\}}$. Then for different choices of $\{\mathscr{R}_N\}$, we get different $\mathscr{T}$ functors. However, the above discussions shows that we have the following diagram for all $N \to N'$:

$$\begin{tikzcd}
\mathrm{Tor}_i^A(M,N)_{\{\mathscr{R}_N\}} \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Tor}_i^A(M,N')_{\{\mathscr{R}_N\}} \arrow[d, "\simeq"] \\
\mathrm{Tor}_i^A(M,N)_{\{\mathscr{R}'_N\}} \arrow[r] & \mathrm{Tor}_i^A(M,N')_{\{\mathscr{R}'_N\}}
\end{tikzcd}$$

Notice that the maps in the diagram are uniquely defined and are canonical, and the diagram indeed commute. I will let you see why, but this implies that we have canonical isomorphism of functors for $\mathrm{Tor}_i^A(M,-)_{\{\mathscr{R}_N\}}$ and $\mathrm{Tor}_i^A(M,-)_{\{\mathscr{R}'_N\}}$, so there is no reason to distinguish them. Then we will use $\mathrm{Tor}_i^A(M,-)$ to just mean that the Tor functor defined by an arbitrary collection of resolutions $\{\mathscr{R}_N\}$. This will be our definition.

Remark 2.1. If N is itself free, we have a resolution $0 \to N \to N \to 0$, this implies that $\mathrm{Tor}_i^A(M,N) = 0$ for $i > 0$.

Now let's think about why $\mathrm{Tor}_i^A(M,-)$ gives LES for any SES:

Proposition 2.2. For any short exact sequence $0 \to N' \to N \to N'' \to 0$ as above, we get a long exact sequence for Tor as above.

Proof. Let's choose resolutions of N', N'', it looks like:

$$\begin{tikzcd}
& & & 0 \arrow[d] & \\
\cdots \arrow[r] & A^{\oplus n'_1} \arrow[r] & A^{\oplus n'_0} \arrow[r] & N' \arrow[r] \arrow[d] & 0 \\
& & & N \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & A^{\oplus n''_1} \arrow[r] & A^{\oplus n''_0} \arrow[r] & N'' \arrow[r] \arrow[d] & 0 \\
& & & 0 &
\end{tikzcd}$$

I claim that we can use the two resolutions, we can get a resolution of N and a diagram like the one below:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A^{\oplus n_1'} \arrow[r] \arrow[d] & A^{\oplus n_0'} \arrow[r] \arrow[d] & N' \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & A^{\oplus n_1'+n_1''} \arrow[r] \arrow[d] & A^{\oplus n_0'+n_0''} \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & A^{\oplus n_1''} \arrow[r] \arrow[d] & A^{\oplus n_0''} \arrow[r] \arrow[d] & N'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

such that each vertical column is exact. Clearly, we can define the vertical maps to be the inclusion and projection, which indeed makes it exact. To figure out the horizontal map, I define:

$$A^{\oplus(n_{i+1}'+n_{i+1}'')} \to A^{\oplus(n_i'+n_i'')}$$

to be the map defined by $A^{\oplus n_{i+1}''} \to A^{\oplus n_i'} \to A^{\oplus(n_i'+n_i'')}$, and a lift of $A^{\oplus n_{i+1}''} \to A^{\oplus n_i''}$ to $A^{\oplus(n_i'+n_i'')}$ such that the middle row is a complex.

Let's see why it's possible to choose such a lift. Notice that we can define the component $A^{\oplus n_{i+1}'}$ in $A^{\oplus(n_{i+1}'+n_{i+1}'')}$ to be mapped to 0. Then we only need to consider $A^{\oplus n_{i+1}''}$. Then notice that since the map $A^{\oplus(n_i'+n_i'')} \to A^{\oplus n_i''}$ is surjective, then it's possible to choose a lift due to freeness.

Then we know that automatically the row in the middle is exact since we know that if we have a SES of complexes, cohomology gives a LES. Since the first row and last row are exact, then the cohomology of the middle row automatically vanishes, we are done. Therefore the middle row is also a resolution. It may be helpful to notice that the "matrix" for the maps in the middle row is an upper triangular one. This procedure is often called the **horseshoe construction** as the picture of it seems like we are trying to fill in a horseshoe.

Then remove the rightmost column tensor with M, notice that the vertical direction remains exact since tensor commutes with direct sums, therefore we can apply the associated LES for cohomology to get the desired LES of Tor. ☐

Proposition 2.3. The following conditions are equivalent:

1. M is flat.
2. $\mathrm{Tor}_i^A(M,N) = 0$ for all $i > 0$ and all A-modules N.
3. $\mathrm{Tor}_1^A(M,N) = 0$ for all A-modules N.

Proof. We already saw $1 \Rightarrow 2$ and clearly $2 \Rightarrow 3$. Moreover, the associated LES tells us that $2 \Rightarrow 1$. Then we only need to show that $3 \Rightarrow 1$.

Notice that if $\mathrm{Tor}_1^A(M,N) = 0$ for all N, then for any SES, the LES tells that $M \otimes_A -$ is exact since $\mathrm{Tor}_1^A(M,N) = 0$. ☐

Remark 2.4. Recall that a free module is flat, then if we know that the Tor functor is symmetric, i.e.:

$$\mathrm{Tor}_i^A(M,N) \simeq \mathrm{Tor}_i^A(N,M)$$

then the above proposition implies the observation we had before about if N if free then $\mathrm{Tor}_i^A(M,N)$ vanishes for all $i < 0$ and all M. We will prove this symmetry later.

There is one last issue we need to solve before we can confidently say that we are done with the construction of Tor. Notice that in the LES, even though we know that maps $\mathrm{Tor}_i^A(M, N') \to \mathrm{Tor}_i^A(M, N) \to \mathrm{Tor}_i^A(M, N'')$ are defined uniquely. We don't know if δ, the connecting homomorphism is independent of choice of resolutions. But it turns out that the connecting homomorphism is also independent of the choice. Let's first be clear about what we mean by this independent of choices of resolutions. Notice that to be fair, if we use different resolutions, we indeed get different Tor, it's just that they are canonically isomorphic, therefore by the connecting homomorphism is independent of resolutions, we actually mean that the connecting homomorphisms commute with the canonical isomorphisms between different definitions of Tor functors.

Let's prove it now, suppose that we have two set of resolutions of N', N'', then we can build the horseshoe-like figure twice, say one on the top of another. Imagine we first lift the identity $N' \to N'$ to a map of complexes. Then we can also lift $N'' \to N''$ similarly. We want to ask is that a lifting of $N \to N$ such that the diagram commutes. Recall that the resolution of N is a direct sum of the resolution of N' and N'' with upper triangular differentials, then we can define the lift component-wise and I will let you check that it commutes. That is to say we have the following map of complexes

$$\begin{tikzcd}
0 \arrow[d] & \\
(N')_2^\bullet \arrow[d] \arrow[dr] & 0 \arrow[d] \\
N_2^\bullet \arrow[d] \arrow[dr] & (N')_1^\bullet \arrow[d] \\
(N'')_2^\bullet \arrow[d] \arrow[dr] & N_1^\bullet \arrow[d] \\
0 & (N'')_1^\bullet \arrow[d] \\
 & 0
\end{tikzcd}$$

where the subscript is implying that the resolutions are different. Then we consider the two LES induced by taking cohomology, we have a commutative diagram by the naturality of the connecting homomorphism:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Tor}_{i-1}^A(M, N'') \arrow[r, "\delta_1"] & \mathrm{Tor}_i^A(M, N') \arrow[r] & \mathrm{Tor}_i^A(M, N) \arrow[r] & \mathrm{Tor}_i^A(M, N'') \arrow[r, "\delta_1"] & \mathrm{Tor}_{i+1}^A(M, N') \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{Tor}_{i-1}^A(M, N'') \arrow[r, "\delta_2"] \arrow[u] & \mathrm{Tor}_i^A(M, N') \arrow[r] \arrow[u] & \mathrm{Tor}_i^A(M, N) \arrow[r] \arrow[u] & \mathrm{Tor}_i^A(M, N'') \arrow[r, "\delta_2"] \arrow[u] & \mathrm{Tor}_{i+1}^A(M, N') \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

Therefore we are done.

Using similar method, you can also show that if you have a map of SES, you get an induced map between the two induced LES. I will let you see why this holds.

3 Derived functors in general

This example of the functor Tor is just a special case for derived functors. We will introduce the general machinery in this section. The part of (universal) δ functors will be useful in the proof of Serre duality. (If you also are reading the etale cohomology note series, this notion is also used in several places there, for example when we discuss group cohomology.)

Notice that if you go over all the proof in the previous section again, you find that the only property we used is the lifting property. There are a class of objects that have these kind of lifting property, i.e. the **projective modules**. Recall that P is projective if given any map $P \to N$ and any surjective map $M \to N$, it's possible to lift $M \to N$ to $P \to M$. Equivalently, this is saying that $\mathrm{Hom}(P.-)$ is exact. More generally, this gives the definition of a projective object in an abelian category.

It's clear that if P is projective, then given any SES $0r \to Q \to R \to P \to 0$, then it splits. (we have a section $P \to R$.) Therefore we know that P is a direct sum of a free module. Moreover if any SES $0r \to Q \to R \to P \to 0$, then we know that P is also a direct summand of a free module, say $P \oplus Q$ is free, then you can prove that P has the lifting property since $P \oplus Q$ has the lifting property, thus P is projective. Therefore P is projective iff P is a direct summand of a free module.

Proposition 3.1. Projective modules are flat.

Proof. Say $P \oplus Q$ is free, then since tensor with free modules is exact, it implies each component, i.e. tensoring with both P and Q are exact. $\square$

In general, if F is a right exact functor from the category of A-modules to any abelian category, we can redo all the constructions of the previous section and get a sequence of functors L_iF with $L_0F = F$ canonically and L_iF give LES to each SES with natural connecting homomorphisms (naturality not in the category of A-modules can not be justified by the remark in the previous section, we have to use the general functoriality of spectral sequences mentioned in the formalisms of spectral sequences). In fact if an abelian category $\mathcal{A}$ has enough projective objects, i.e. for any object N, there is a surjection from a projective object to N. We can again apply the construction of the previous section for any right exact covariant functor F from $\mathcal{A}$ to any other abelian category. What we get will be called the **left-derived functors** of F, denoted by L_iF with $L_0F = F$ canonically. (You can check that even if the argument that two liftings are homotopic are again completed only using the lifting properties.)

Remark 3.2. We can also define an injective object in an abelian category. We say that an object I is injective, iff for any injective map $N \to M$ and any morphism $N \to I$, we have a lifting $M \to I$. This is equivalent to the fact that the contravariant functor $\mathrm{Hom}(-, I)$ is exact. (Recall that without any assumption on I, this functor is only right exact, i.e. turns right exact sequences into left exact sequences) The diagrams for projective and injectives objects are the following:

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d] \\ M \arrow[r, two heads] & N \end{tikzcd} \qquad\qquad \begin{tikzcd} & I \\ M \arrow[ur, dashed] & N \arrow[l, hook'] \arrow[u] \end{tikzcd}$$

Since this is the dual notion of the projective object, again defined by the lifting property. You can check that essentially by the same idea, if you have a covariant left exact functor F and the category has enough injectives, you can define the right derived functor of F to be first embedding the object in an injective resolution and apply the functor and then take the cohomology. You can prove that this is independent of the resolution and it gives LES to SES essentially in the same way.

These are for covariant functors. Similarly, we can use the same definition for contravariant functors. For example, if F is a contravariant right exact functor, to define $F(N)$, we give a projective resolution of N, truncate N, apply F, take cohomology, we are done. If F is a contravariant left exact functor, to define $F(N)$, we embed N into a injective resolution, truncate N, apply F, we are done. Notice that whether we choose an injective or projective resolution depends on our goal to have a canonical isomorphism $L_0F = F$ or $R^0F = F$. You find that as long as the functor is right exact, no matter it's covariant or contravariant, we will use projective resolution. Similarly, if the functor is left exact, no matter if it's covariant or contravariant, we will use injective resolution.

For the notation we use, for a right exact functor F, we will use L_iF to denote the left derived functors. Here the subscript is used to indicate the fact that in the projective resolution, the index is descending. For a left exact functor, we will use R^iF to denote the right derived functors, here the superscript is used to indicate the fact that the index in the injective resolution is ascending.

Example 3.3. Recall that $\mathrm{Hom}(-, N)$ is a right exact contravariant functor, then we can define it's right derived functor. We will use $\mathrm{Ext}^i_A(M, N)$ as the i-th right derived functor of $\mathrm{Hom}(-, N)$ applied to M.

Suppose that we have a SES:

$$0 \to M' \to M \to M'' \to 0$$

we get an induced LES:

$$0 \to \mathrm{Hom}(M'', N) \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(M', N) \to \mathrm{Ext}^1_A(M'', N) \to \mathrm{Ext}^1_A(M, N) \to \mathrm{Ext}^1_A(M', N) \to \mathrm{Ext}^2_A(M'', N) \to \cdots$$

Recall that for any commutative ring A, the category of A-modules has enough injectives, therefore we can also define the following:

Example 3.4. Let M be an A-module, then we know that $\mathrm{Hom}(M, -)$ is a covariant left exact functor, then we can define the right derived.

We will define $\mathrm{Ext}_A^i(M, N)$ to be the i-th derived functor of $\mathrm{Hom}(M, -)$ applied to N. If we have a SES:

$$0 \to N' \to N \to N'' \to 0$$

we will have a LES:

$$0 \to \mathrm{Hom}(M, N') \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(M, N'') \to \mathrm{Ext}_A^1(M, N') \to \mathrm{Ext}_A^1(M, N) \to \mathrm{Ext}_A^1(M, N'') \to \mathrm{Ext}_A^2(M, N') \to \cdots$$

Remark 3.5. You will naturally have the question that since we define $\mathrm{Ext}_A^i(M, N)$ twice. How do we know that these two definitions agree. We will answer this sort of questions in the next section using spectral sequences.

Moreover, you will see in the next sections that the two LES we get here is better than any single one of them.

Proposition 3.6. If A is a Noetherian ring and M, N are finitely generated, then $\mathrm{Ext}_A^i(M, N)$ is finitely generated for all i.

Proof. Let's use the definition, where we compute the Ext-groups using the projective resolution. Notice that then we are computing the derived functor of $\mathrm{Hom}(-, N)$, which is right exact. Then since we know that M is finitely generated and A is Noetherian, we know that each term in the resolution can be chosen to be finitely generated. Then we know that after we apply $\mathrm{Hom}(-, N)$, we know that each term looks like $\mathrm{Hom}(X, N)$ where X, N are finitely generated. Notice that we have a exact sequence:

$$R^m \to R^n \to X \to 0$$

Then apply $\mathrm{Hom}(-, N)$, we get another exact sequence:

$$0 \to \mathrm{Hom}(X, N) \hookrightarrow \mathrm{Hom}(R^n, N) \to \mathrm{Hom}(R^m, N)$$

Therefore we only need to prove $\mathrm{Hom}(R, N) \simeq N$ is finitely generated, which is by assumption.. $\square$

We will use the Ext functor for sheaves later when we prove Serre duality.

You probably can tell from the way we introduce derived functors that we don't want to involve all the abelian category stuff to make the treatment too technical (though in essence, it's not different from the case in A-modules, just more teadious). Though we try to make the treatment as simple as possible, we still want to introduce (universal) δ-functors, as this will be quite useful in proving results for derived functors. This is useful also in our treatment of Serre duality later.

These so called (universal) δ functor, down to earth, is just a more general variant of derived functors. The advantage of the notion is that we can apply it even if the category doesn't have enough injectives.

Definition 3.7. A cohomological δ functor, from an abelian category $\mathcal{A}$ to another abelian category $\mathcal{B}$ is a collection of additive functors $T^i : \mathcal{A} \to \mathcal{B}$ for all $i \geqslant 0$ (if $i = 0$, we think of $T^i = 0$), along with morphisms $\delta^i : T^i(A'') \to T^{i+1}(A)$ for each SES in $\mathcal{A}$:

$$0 \to A' \to A \to A'' \to 0$$

such that:

1. For each SES like above, we have a LES:

$$\cdots \to T^{i-1}(A'') \xrightarrow{\delta^{i-1}} T^i(A') \to T^i(A) \to T^i(A'') \xrightarrow{\delta^i} T^{i+1}(A') \to \cdots$$

   In particular, T^0 is left exact.

2. For each morphism of SES in $\mathcal{A}$:

$$\begin{tikzcd} 0 \arrow[r] & a' \arrow[r] \arrow[d] & a \arrow[r] \arrow[d] & a'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & A \arrow[r] & A'' \arrow[r] & 0 \end{tikzcd}$$

δ^i makes the following diagram commute:

$$\begin{tikzcd} T^i(a'') \arrow[r, "\delta^i"] \arrow[d] & T^{i+1}(a') \arrow[d] \\ T^i(A'') \arrow[r, "\delta^i"] & T^{i+1}(A') \end{tikzcd}$$

Notice that the the connecting homomorphism in cohomology is natural, then we know that right derived functor of a left exact functor is an example of a cohomological δ-functor. If you recall what we did for the Cech cohomology for quasicoherent sheaves on a qcqs scheme, it's another example of a cohomological δ functor.

Remark 3.8. We also have homological δ-functors. It's again defined to be a sequence of functors T^i for $i \geqslant 0$ and $T^i = 0$ for $i < 0$ along with connecting homomorphisms for each SES. Such that for each SES, we have a LES in the opposite direction of the cohomological δ functors. In particular T^0 is right exact. The connecting homomorphisms should also be compatible with maps of SES.

All the above definitions are for covariant functors, we can also have the definition for contravariant δ functors. If you need to use the definition, you can define it yourself.

We can also define morphisms of δ functors. For example, let's do it for covariant cohomological δ functors. Suppose T^i, S^i are two collections of cohomological δ functors from $\mathcal{A}$ to $\mathcal{B}$. Then of course a morphism is first a collection of morphisms of functors $T^i \to S^i$. However, these morphisms of functors should be compatible with the connecting morphisms, i.e. for a SES $0 \to A' \to A \to A'' \to 0$, we need to have the following diagram commute for all i:

$$\begin{tikzcd} T^i(A'') \arrow[r, "\delta_T^i"] \arrow[d] & T^{i+1}(A') \arrow[d] \\ S^i(A'') \arrow[r, "\delta_T^i"] & S^{i+1}(A') \end{tikzcd}$$

Here the subscript of δ^i indicates which cohomological δ it belongs to. Then you can check that cohomological δ-functors forms a category. You can also define it for homological δ functors and etc.

Definition 3.9. A (cohomological) δ functor (T^i, δ^i) is said to be **universal** if for any other cohomological δ functor (T'^i, δ'^i) and any natural transformation $\alpha : T^0 \to T'^0$, there is a unique morphism of cohomological δ functors extending α.

Suppose F is a covariant left exact functor from $\mathcal{A}$ to $\mathcal{B}$. I claim that there is at most 1 universal cohomological δ functor extending F up to isomorphism (i.e. with an isomorphism $T^0 \simeq F$). This is because if we have two universal cohomological δ functors (T^i, δ^i) and (T'^i, δ'^i) such that there is an isomorphsim of functors between T^0 and T'^0, then there is a unique morphism of δ functors from (T^i, δ^i) to (T'^i, δ'^i) and universal property nonsense tells us this is an isomorphism. The key fact about universal cohomological functors is the following theorem:

Theorem 3.10. Suppose that (T^i, δ^i) is a covariant cohomological δ-functor from $\mathcal{A}$ to $\mathcal{B}$. If for all $A \in \mathcal{A}$, there is a monomorphism $A \to J$ with $T^i J = 0$ for all $i > 0$, then (T^i, δ^i) is universal.

This theorem is in fact the consequence of a more general version. To state it, we need the following notion:

Definition 3.11. A covariant additive functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is called **effaceable** if for every $A \in \mathcal{A}$, there is a monomorphism $A \to J$ such that $F(J) = 0$.

Then the requirement in the above theorem can be weakened to only requiring T^i to be effaceable for all $i > 0$. Notice that in the above theorem, we require a common J, but here to prove each $T^i, i > 0$ is effaceable, we could use different J, therefore this is indeed a generalization. Before we prove this result:

Corollary 3.12. If $\mathcal{A}$ is an abelian category with enough injectives and F is a covariant left exact functor, then R^iF with the accompanying δ^i form a universal δ-functor.

Proof. Notice that higher derived functors of an injective object I is 0 siunce we can always take the resolution $0 \to I \to I \to 0$. Then since this category have enough injectives, we are done. □

Then let's prove the theorem in it's general form:

Theorem 3.13. Let $F = \{F^n, \delta_F^n\}$ be a covariant cohomological δ functor from $\mathcal{A} \to \mathcal{B}$. If F^n is effaceable for all $n > 0$, then F is universal.

Proof. Since F^n is effaceable for $n > 0$, given an object A, there will be a monomorphism $\varphi : A \to M$ such that we have a SES:

$$0 \to A \xrightarrow{\varphi} M \to X \to 0$$

Consider the LES induced by F, it partially looks like:

$$0 \to F^0A \to F^0X \to F^0M \to F^1A \to 0 = F^1M \to \cdots$$

Let G be another cohomological δ functor with $f_0 : F^0 \to G^0$, we have a diagram:

$$\begin{tikzcd}
F^0(M) \arrow[r] \arrow[d, "f_0"] & F^0(X) \arrow[r, "\delta_F^0"] \arrow[d, "f_0"] & F^1(A) \arrow[r] & 0 \\
G^0(M) \arrow[r] & G^0(X) \arrow[r, "\delta_G^0"] & G^1(A)
\end{tikzcd}$$

our question is that is there a morphism f_1 from $F^1(A)$ to $G^1(A)$ and is it unique.

I claim that the kernel of δ_F^0 is contained in the kernel of $\delta_G^0 \circ f_0$. To see why, notice that the kernel of δ_F^0 is the image of $F^0(M) \to F^0(X)$, then we know that the kernel is mapped under $\delta_G^0 \circ f_0$, which is the same map as $\delta_G^0 \circ (G^0(M) \to G^0(X)) \circ f_0 = 0$.

Then since $F^1(A)$ is the cokernel of $F^0(M) \to F^0(X)$, we know that there is a unique homomorphism f_1 from $F^1(A) \to G^1(A)$ such that the square involving δ_F^9, δ_G^0 commute using the universal property of the cokernel. If we can show that f_1 satisfies the desired property, we know that we can construct other f^n by induction.

Let's first prove that f_1 is functorial. Let $u : A \to B$ be a morphism/ Recall that if we work with an abelian category, we always have the push-out of $\varphi : A \to M$ and and $u : A \to B$:

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d, "u"'] & M \arrow[d] \\
B \arrow[r] & P = \operatorname{coker}(A \xrightarrow{(-\varphi, u)} M \oplus B)
\end{tikzcd}$$

You can check that this indeed satisfies the push-out property. Since φ is a monomorphism, I claim that $B \to P$ is also a monomorphism. We can easily see this in the category of modules since we know that if $b \in B$ is mapped to 0, we know that it's mapped to the class of $(0, b)$, if this is 0, then it's in the image of $(-\varphi, u)$, which means that $(0, b)$ is equal to $(-\varphi(a), u(a))$. Since φ is monomorphism, then $\varphi(a) = 0$. The idea to prove this in any abelian category is just a generalization of this so omiited here.

Let $P \to N$ be a monomorphism that makes F^1 effaceable, then we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\varphi"] \arrow[d, "u"'] & M \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & B \arrow[r] & N \arrow[r] & Y \arrow[r] & 0
\end{tikzcd}$$

where $B \to N$ is the composition of $B \to P \to N$, which is still a monomorphism, then we take Y to be the cokernel, the dotted morphism is induced by the universal property of cokernel.

We know that functoriality of f_1 means that we need the following diagram to commute:

$$\begin{tikzcd}
F^1(A) \arrow[r, "F^1(u)"] \arrow[d, "f_1"'] & F^1(B) \arrow[d, "f_1"] \\
G^1(A) \arrow[r, "G^1(u)"'] & G^1(B)
\end{tikzcd}$$

To see it, recall that since we have a map of SES above, then F, G being cohomological delta functors, gives LES which gives us:

$$\begin{tikzcd}
& F^0(X) \arrow[rr] \arrow[dl] \arrow[dd] & & F^1(A) \arrow[dl] \arrow[dd] \\
F^0(Y) \arrow[rr, crossing over] \arrow[dd] & & F^1(B) & \\
& G^0(X) \arrow[rr, "\delta_G^0"] \arrow[dl] & & G^1(A) \arrow[dl] \\
G^0(Y) \arrow[rr] & & G^1(B) \arrow[from=uu, crossing over] &
\end{tikzcd}$$

We find that the diagram we are interested in is the right-hand side of the cube and in the cube, we know that all the morphisms on the other sides commutes. Notice that by our asumption that the two maps $F^0(X) \to F^1(A)$ and $F^0(Y) \to F^1(B)$ are epimorphisms (since the next term in the LES is 0). Then I will let you prove this implies the right-hand side commutes(actually we only need the fact that $F^0(X) \to F^1(A)$ is an epimorphism).

Then we need to show that f_1 commutes with δ_F^1. Let $0 \to A' \to A \to A'' \to 0$ be a SES, the same push-out diagram shows that there is the following diagram of SES:

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d, "\mathrm{id}"] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & M \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

such that $F^1(M) = 0$. Again, the two LES induced by F, G gives:

$$\begin{tikzcd}
& F^0(A'') \arrow[rr] \arrow[dl] \arrow[dd] & & F^1(A') \arrow[dl] \arrow[dd] \\
F^0(X) \arrow[rr, crossing over] \arrow[dd] & & F^1(A') & \\
& G^0(A'') \arrow[rr, "\delta_G^0"] \arrow[dl] & & G^1(A') \arrow[dl] \\
G^0(X) \arrow[rr] & & G^1(A') \arrow[from=uu, crossing over] &
\end{tikzcd}$$

This time we need to show that the side on the back is comutative (we know it commutes in the previous diagram because it's we defined it to comjmute there, but here we didn't). The top and bottom side commutes because F, G are δ functors. The left side commutes since we have a natural transformation $F^0 \to G^0$. The front and right side commute since we just proved we have a natural transformation $F^1 \to G^1$. Then I will let you use the diagram to prove the desired commutativity. Then we can use induction to conclude the unique extension. $\square$

Remark 3.14. What about covariant homological δ functors. It's easy to mimic and give the definition of a universal covariant homological functor in terms of unique extensions. But here the extension should be the dual version in the sense that if (T_i, δ_i^T) is a universal homological δ functor iff for any other homological δ functor (S_i, δ_i^S) and a morphism $s_0 \to T_0$, then it uniquely extend to $S \to T$. Then we will have a similar criterion to determine if a homological functor is universal or not.

We say that an additive functor F from an abelian category $\mathcal{A}$ to another abelian category $\mathcal{B}$ is **coeffaceable** if for any object $A \in \mathcal{A}$ there is an epimorphism $M \to A$ such that $F(M) = 0$. Then I will also let you show that in an abelian category push-ou diagrams also preserves epimorphisms.

Then I claim that if $F = (F_n, \delta_n^F)$ is a homological δ functor such that F_n is coeffecable for all $n > 0$, then F is a universal homological functor.

This is essentially proved using similar argument. Assume G is another homological δ-functor and we have a natural transformation $F^0 \to G^0$ Given A, we consider a SES:

$$0 \to X \to M \to A \to 0$$

Then consider the diagram:

$$\begin{tikzcd}
0 \arrow[r] & F^1(A) \arrow[r, "\delta_1^F"] & F^0(X) \arrow[r] & F^0(M) \\
 & G^1(A) \arrow[r, "\delta_1^G"'] \arrow[u, dashed] & G^0(X) \arrow[r] \arrow[u] & G^0(M) \arrow[u]
\end{tikzcd}$$

this gives the unique extension, and the way to check it's functorial is similar to the above argument.

Then as a consequence, we can prove that if a category has enough projectives, then the left derived functor of a right exact functor is a universal homological δ functor.

Remark 3.15. All the above discussions are for covariant δ functors.

We say that a family of contravariant additive functors (F^n, δ_F^n) is a contravariant cohomological δ functor F if for each SES:

$$0 \to M' \to M \to M'' \to 0$$

we have a LES connected by:

$$\delta^n : F^n(M') \to F^{n+1}(M'')$$

such that these connecting functors is functorial in morphisms of SES. We say that F is universal if for any other contravariant cohomological δ G and a natural transformation $F^0 \to G^0$, it uniquely extend to a morphism of δ functors. You can prove that F is universal if $F^n, n > 0$ if coeffaceable, i.e. for any A, there is an epimorphism $M \to A$ such that $F(M) = 0$. Similarly, you can define a contravariant homological functor F see that it's universal if $F^n, n > 0$ are effaceable.

4 Spectral sequences and derived functors

There are quite a lot of properties of derived functors that can be proved using spectral sequences arguments. In this section, you will have a lot of practices using spectral sequences in a real setting. We will also introduce the Grothendieck composition of functors spectral sequence, which is very important later.

Firstly, let's fulfill our promise earlier that the Tor functor is symmetric.

Proposition 4.1. There is an isomorphism between $\mathrm{Tor}_i^A(M, N) \simeq \mathrm{Tor}_i^A(N, M)$.

Proof. In order to compute $\mathrm{Tor}_i^A(M, N)$, we first have a free resolution of N, say it's $\mathscr{R}_1$.

$$\cdots \to A^{\oplus n_2} \to A^{\oplus n_1} \to A^{\oplus n_0} \to N \to 0$$

Similarly, we have a free resolution of M for the computation of $\mathrm{Tor}_i^A(N, M)$, say it's $\mathscr{R}_2$.

$$\cdots \to A^{\oplus m_2} \to A^{\oplus m_1} \to A^{\oplus m_0} \to M \to 0$$

Then we can produce the following third quadrant double complex:

$$\begin{tikzcd}
\cdots \arrow[r] & A^{\oplus n_1} \otimes_A A^{\oplus m_0} \arrow[r] & A^{n_0} \otimes_A A^{\oplus m_0} \\
\cdots \arrow[u] & \cdots & A^{n_0} \otimes_A A^{\oplus m_1} \arrow[u] \\
A^{n_i} \otimes_A A^{\oplus m_j} \arrow[u] \arrow[r] & \cdots \arrow[r] & \cdots \arrow[u]
\end{tikzcd}$$

Here the differential is defined to be the differential tensor with the identity. You can easily check this is indeed a double complex. We first compute the spectral sequence in the rightward orientation: since tensoring free objects is exact, we know that the first page is:

$$\begin{tikzcd}
\cdots \\
N \otimes A^{\oplus m_0} \arrow[u] \\
N \otimes A^{\oplus m_1} \arrow[u] \\
N \otimes A^{\oplus m_2} \arrow[u] \\
\cdots \arrow[u]
\end{tikzcd}$$

But notice that this is the complex we use to compute $\mathrm{Tor}(N, M)$, then we know that the second page looks like:

$$\begin{tikzcd}
0 & & \\
0 & 0 & \\
0 & \mathrm{Tor}_0^A(N, M) \arrow[uul] & \\
 & \mathrm{Tor}_1^A(N, M) \arrow[uul] & \\
 & \mathrm{Tor}_2^A(N, M) \arrow[uul] & 0 \arrow[uul] \\
 & \cdots & 0 \arrow[uul] \\
 & & 0 \arrow[uul]
\end{tikzcd}$$

Therefore the spectral sequence stabilizes at the 2-th page. Then we know that using the filtration of the cohomology of the double complex, we conclude that the cohomology of the double complex is $\mathrm{Tor}_i^A(N, M)$ on various degree. For example, the cohomology of degree $-i$ is $\mathrm{Tor}_i^A(N, M)$. Then you compute the spectral sequence on the upward direction. We find the first page is the complex we use to compute $\mathrm{Tor}_i^A(M, N)$, and it's again stabilizes in the second degree, then we conclude again that the cohomology of the double complex at degree $-i$ is $\mathrm{Tor}_i^A(M, N)$, therefore we get the desired isomorphism. $\square$

Using similar argument, we can prove that the two definitions we gave in the previous section for Ext functor actually agree:

Proposition 4.2. We have an isomorphism between the two definitions of $\mathrm{Ext}^i_A(M,N)$ given in last section.

Proof. Recall that one definition we given is that in order to compute $\mathrm{Ext}^i_A(M,N)$, we think of it as the right derived functor of the left exact functor $\mathrm{Hom}(M,-)$, therefore we first will need to find an injective resolution of N:

$$0 \to N \to I^0 \to I^1 \to \cdots$$

We can also define it to be the left derived functor of the right exact contravariant functor $\mathrm{Hom}(-,N)$, therefore we will need a free resolution of M:

$$\cdots \to A^{\oplus n_1} \to A^{\oplus n_0} \to M \to 0$$

Then we will have a first quadrant double complex:

$$\begin{tikzcd}
 & \cdots & \cdots & \cdots & \\
0 \arrow[r] & \mathrm{Hom}(A^{\oplus n_2}, I_0) \arrow[r] \arrow[u] & \mathrm{Hom}(A^{\oplus n_2}, I_1) \arrow[r] \arrow[u] & \mathrm{Hom}(A^{\oplus n_2}, I_1) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \mathrm{Hom}(A^{\oplus n_1}, I_0) \arrow[r] \arrow[u] & \mathrm{Hom}(A^{\oplus n_1}, I_1) \arrow[r] \arrow[u] & \mathrm{Hom}(A^{\oplus n_1}, I_1) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \mathrm{Hom}(A^{\oplus n_0}, I_0) \arrow[r] \arrow[u] & \mathrm{Hom}(A^{\oplus n_0}, I_1) \arrow[r] \arrow[u] & \mathrm{Hom}(A^{\oplus n_0}, I_1) \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Since we know that $A^{\oplus n_i}$ are projective, then $\mathrm{Hom}(A^{\oplus n_i},-)$ is exact, therefore we know that if we compute the spectral sequence in the rightward diection, the first page looks like:

$$\begin{tikzcd}
\cdots & \cdots & \cdots \\
\mathrm{Hom}(A^{\oplus n_2}, N) \arrow[u] & 0 \arrow[u] & 0 \arrow[u] \\
\mathrm{Hom}(A^{\oplus n_1}, N) \arrow[u] & 0 \arrow[u] & 0 \arrow[u] \\
\mathrm{Hom}(A^{\oplus n_0}, N) \arrow[u] & 0 \arrow[u] & 0 \arrow[u] \\
0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

Notice that this is the complex that computes the section definition of $\mathrm{Ext}^i_A(M,N)$.

We compute the spectral sequence in the upward direction, I will let you see that you just get the complex that computes the first definition. Both spectral sequences stabilizes on the 2-th page, therefore we are done. ☐

Remark 4.3. Not like the Tor functor, the Ext functor is in general not symmetric in the two variables. The main reason behind this is that free modules is not injective. So if you try to mimic the argument that proves the symmetry for Tor, we will get the double complex $\mathrm{Hom}(A^{\oplus n_i}, A^{\oplus m_j})$, but since $A^{\oplus m_j}$ is not injective, we can not conclude that the computation in one direction stabilizes in the second page, therefore our argument falls apart.

If you want a counter example. Take the ring to be $\mathbb{Z}$ and let's compute $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n,\mathbb{Z})$, notice that we have a projective resolution of $\mathbb{Z}/n$:

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/n \to 0$$

We know that after we apply $\mathrm{Hom}(-, \mathbb{Z})$, we get:

$$0 \to 0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to 0 \to \cdots$$

Then you get $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}) = \mathbb{Z}/n$.

If we compute $\mathrm{Ext}^1_A(\mathbb{Z}, \mathbb{Z}/n)$, notice that since $\mathbb{Z}$ is projective, we know that $\mathrm{Ext}^1_A(\mathbb{Z}, \mathbb{Z}/n) = 0$.

Suppose that $F : \mathcal{A} \to \mathcal{B}$ is a right exact functor and $\mathcal{A}$ has enough projective. Then we know that we have left derived functors $L_i F$. We will say that an object $A \in \mathcal{A}$ is F-**acyclic** if $L_i F = 0$ for all $i > 0$. We will soon show that to compute the derived functor $L_i B$ for an object $B \in \mathcal{A}$, instead of taking a projective resolution of B, we can use an F-acyclic resolution and apply F and then take cohomology. But to show this, we need to build some preparations:

Suppose that we have a LES in an abelian category with enough projectives:

$$0 \to E_2 \to E_1 \to E_0 \to 0$$

I claim that we can build a double complex of projectives:

$$\begin{tikzcd}
 & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \\
\cdots \arrow[r] & P_{2,1} \arrow[r] \arrow[d] & P_{1,1} \arrow[r] \arrow[d] & P_{0,1} \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & P_{2,0} \arrow[r] \arrow[d] & P_{1,0} \arrow[r] \arrow[d] & P_{0,0} \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & E_2 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d] & E_0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with $P_{i,j}$ projective objects and the rows and columns all exact.

To see why we can build such a complex, suppose that we have already built part of the complex:

$$\begin{tikzcd}
? \arrow[r, dashed] \arrow[d, dashed] & P_{m-1,n} \arrow[r] \arrow[d] & P_{m-2,n} \arrow[d] \\
P_{m,n-1} \arrow[r] \arrow[d] & P_{m-1,n-1} \arrow[r] \arrow[d] & P_{m-2,n-1} \\
P_{m,n-2} \arrow[r] & P_{m-1,n-2} &
\end{tikzcd}$$

We want to ask how can we build $P_{m,n}$. Let's first assume that for each $b \in P_{m-1,n-1}$ whose image in both $P_{m-1,n-2}$ and $P_{m-2,n-1}$ is zero, there is a $c \in P_{m,n-1}$ whose image in $P_{m-1,n-1}$ is b and whose image $P_{m,n-2}$ is 0. Symmetrically, we also assume that there is a $c' \in P_{m-1,n}$ whose image in $P_{m-1,n-1}$ is b and whose image in $P_{m-2,n}$ is 0.

Let's consider the kernel of the following obvious map (we take the difference at the $P_{m-1,n-1}$ component):

$$K = \ker(P_{m,n-1} \oplus P_{m-1,n} \to P_{m-2,n} \oplus P_{m-1,n-1} \oplus P_{m,n-2})$$

Notice that if K surjects onto both the $\ker(P_{m-1,n} \to P_{m-2,n})$ and $\ker(P_{m,n-1} \to P_{m,n-2})$ (via the obvious projection), then we know that we can consider the following diagram:

$$\begin{tikzcd}
P_{m,n} = P \arrow[dr, two heads] & & & \\
& K \arrow[r, dashed] \arrow[d, dashed] & P_{m-1,n} \arrow[r] \arrow[d] & P_{m-2,n} \arrow[d] \\
& P_{m,n-1} \arrow[r] \arrow[d] & P_{m-1,n-1} \arrow[r] \arrow[d] & P_{m-2,n-1} \\
& P_{m,n-2} \arrow[r] & P_{m-1,n-2} &
\end{tikzcd}$$

where $P \to K$ is a surjective map from a projective morphism. This will ensure the commutativity, exactness and P is projective. Notice that commutativity will be ensure by the fact that we take the difference at $P_{m-1,n-1}$ component.

Now let's check that K indeed surjects to the two kernel. Suppose that $a \in \ker(P_{m-1,n} \to P_{m-2,n})$ and let b be the image in $P_{m-1,n-1}$. We want to find $a' \in P_{m,n-1}$ such that it maps to b in $P_{m-1,n-1}$ and maps to 0 in $P_{m,n-2}$. Notice that the image of b in $P_{m-2,n-1}$ and the image of b in $P_{m-1,n-2}$ are 0. Then by assumption, there will be some $c' \in P_{m,n-1}$ such that it maps to $b \in P_{m-1,n-1}$ and maps to 0 in $P_{m,n-2}$. Then notice that $(-c', a)$ is the desired element. Using a symmetric argument we are done with another surjection.

Then I claim that we can use this procedure to build the double complex in the following order:

$$\begin{tikzcd}
& & & 6 \arrow[r] \arrow[d] & 0 \\
& & 5 \arrow[r] \arrow[d] & 3 \arrow[r] \arrow[d] & 0 \\
& 4 \arrow[r] \arrow[d] & 2 \arrow[r] \arrow[d] & 1 \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & E_2 \arrow[r] & E_1 \arrow[r] & E_0 \arrow[r] & 0
\end{tikzcd}$$

where we use the diagonal order to build the entire double complex.

So we need to make sure this process can be carried out further since we want to use induction. We know that using the following order, we can assume that after we build $P_{m,n}$, we have the following diagram:

$$\begin{tikzcd}
& ? \arrow[r] \arrow[d] & P_{m-2,n+1} \arrow[r] \arrow[d] & P_{m-3,n+1} \arrow[d] \\
P_{m,n} \arrow[r] \arrow[d] & P_{m-1,n} \arrow[r] \arrow[d] & P_{m-2,n} \arrow[r] \arrow[d] & P_{m-3,n} \\
P_{m,n-1} \arrow[r] & P_{m-1,n-1} \arrow[r] & P_{m-2,n-1} &
\end{tikzcd}$$

Here we first want to build ?, therefore we have to check a similar condition for $P_{m-2,n}$, which is in the same diagonal of $P_{m-1,n-1}$. Then let's assume by induction assumption that it satisfies the condition for all the objects on the same diagonal line, which we will check later. Therefore we can build the ? and further all the objects on the same diagonal. Notice that for the induction to work, the starting condition is that in the following diagram:

$$\begin{tikzcd}
1 & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
E_0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 &
\end{tikzcd}$$

Then the assumption is trivially satisfied. T

Therefore out job is now to check the condition for the assumption on the next diagonal line. For example the diagram is:

$$\begin{tikzcd}
 & 1 \arrow[r] \arrow[d] & 0 \arrow[d] \\
E_1 \arrow[r] \arrow[d] & E_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 &
\end{tikzcd}$$

That is we need to check for each $x \in E_0$, there is some $y \in E_1$ that maps to x and there is some $z \in 1$ such that it maps to $x \in E_0$, but this is again trivial since $E_1 \to E_0$ is surjective and $1 \to E_0$ is surjective. What we notice that if you consider one step further:

$$\begin{tikzcd}
 & 2 \arrow[r] \arrow[d] & 1 \arrow[d] \\
E_2 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d] & E_0 \\
0 \arrow[r] & 0 &
\end{tikzcd}$$

You find that the condition is trivially satisfied since the complex of E^i is exact. Therefore we find that to build the diagonal of 1, we need to check the diagonal that consists of all 0's. Then To build the diagonal of $2, 3$, we need to check the diagonal of $E_0, 0$, which is again satisfied, then to build the diagonal of $4, 5, 6$ we need to check the diagonal of $E_1, 1$. But the condition at E_1 is trivially satisfied, therefore we only need to check it at 1. Then we keep going, this implies that after we build $P_{m,n}$, we need to check the condition $P_{m-1,n}$. I will let you figure this out as this is indeed a bit confusing the first time you see it.

Therefore, suppose that $b' \in P_{m-1,n}$ that maps to $(0, 0)$ in $P_{m-1,n-1} \oplus P_{m-2,n}$, there is an element of $P_{m,n}$ that maps to $(0, b')$ in $P_{m,n-1} \oplus P_{m-1,n}$. Take $Q = \ker(P_{m-1,n} \to P_{m-1,n-1} \oplus P_{m-2,n})$ and $Q'' = \ker(P_{m-1,n} \to P_{m,n-2} \oplus P_{m-1,n-1})$. We know that we have surjections $P' \to Q'$ and $P'' \to Q''$. Then we replace $P_{m,n}$ we described above by $P_{m,n} \oplus P' \oplus P''$ with the obvious map, then this still makes the diagram commute and exact with $P_{m,n} \oplus P' \oplus p''$ still being projective as a direct sum of projective objects. Moreover since this is a direct sum, it doesn't change the conditions that is originally satisfied by $P_{m,n} \to P_{m,n-1}$. Then we also have to make sure that $P_{m-1,n+1} \to P_{m-1,n}$ satisfies the conditions, we again use similar method to renovate $P_{m-1,n+1}$, we are done.

Notice that we used the fact that the complex $E_\bullet$ is exact in the way that the first mapping condition we check in each diagonal is automatically satisfied, therefore, to check the condition at E_i, we don't have to modify E_{i+1}.

Then we can apply the above construction to:

Proposition 4.4. Suppose that F is a right exact functor from $\mathcal{A}$ to $\mathcal{B}$ and $\mathcal{A}$ has enough projectives.

Then we know that we can compute $L_i F B$ using an F-acyclic resolution, i.e. a resolution:

$$\cdots \to A_2 \to A_1 \to A_0 \to B \to 0$$

where A_i are F-acyclic.

Proof. Since $\cdots \to A_2 \to A_1 \to A_0 \to B \to 0$ is exact. Then we know that by the above procedure, we have a third quadrant double complex:

$$\begin{tikzcd}
\cdots \arrow[r] & A_2 \arrow[r] & A_1 \arrow[r] & A_0 \arrow[r] & B \arrow[r] & 0 \\
\cdots \arrow[r] & P_{2,0} \arrow[r] \arrow[u] & P_{1,0} \arrow[r] \arrow[u] & P_{0,0} \arrow[r] \arrow[u] & P_0 \arrow[r] \arrow[u] & 0 \\
\cdots \arrow[r] & P_{2,1} \arrow[r] \arrow[u] & P_{1,1} \arrow[r] \arrow[u] & P_{0,1} \arrow[r] \arrow[u] & P_1 \arrow[r] \arrow[u] & 0 \\
& \cdots \arrow[u] & \cdots \arrow[u] & \cdots \arrow[u] & \cdots \arrow[u] &
\end{tikzcd}$$

We know that by truncating the first row and the first column, we get a new double complex and we can apply F:

$$\begin{tikzcd}
& 0 & 0 & 0 & \\
\cdots \arrow[r] & FP_{2,0} \arrow[r] \arrow[u] & FP_{1,0} \arrow[r] \arrow[u] & FP_{0,0} \arrow[r] \arrow[u] & 0 \\
\cdots \arrow[r] & FP_{2,1} \arrow[r] \arrow[u] & FP_{1,1} \arrow[r] \arrow[u] & FP_{0,1} \arrow[r] \arrow[u] & 0 \\
& \cdots \arrow[u] & \cdots \arrow[u] & \cdots \arrow[u] &
\end{tikzcd}$$

Let's compute the spectral sequence using the rightward orientation, we get that the first page is:

$$\begin{tikzcd}
& 0 & 0 & 0 \\
\cdots & L_2P_0 = 0 \arrow[u] & L_1P_0 = 0 \arrow[u] & L_0P_0 \arrow[u] \\
\cdots & L_2P_1 = 0 \arrow[u] & L_1P_1 = 0 \arrow[u] & L_0P_1 \arrow[u] \\
& \cdots \arrow[u] & \cdots \arrow[u] & \cdots \arrow[u]
\end{tikzcd}$$

Therefore we know that the spectral sequence stabilizes at 2-th page and the homology of the double complex is $P_0, \cdots, P_n, \cdots$.

Then we compute the spectral sequence on the upward direction, we know 1-st page is:

$$\begin{tikzcd}
\cdots & L_0A_2 \arrow[r] & L_0A_1 \arrow[r] & L_0A_0 \arrow[r] & 0 \\
\cdots & L_1A_2 = 0 \arrow[r] & L_1A_1 = 0 \arrow[r] & L_1A_0 = 0 \arrow[r] & 0 \\
& \cdots & \cdots & \cdots &
\end{tikzcd}$$

So again, the spectral sequence stabilizes again in the 2-page. Notice that $L_0A_i = FA_i$, then we know that the second page of this spectral sequence is just the homology of the complex:

$$\cdots \to FA_2 \to FA_1 \to FA_0 \to 0 \to \cdots$$

Therefore the homology of the double complex is the homology of the above complex.

Similarly, we know that the homology of the double complex can also be computed in the rightward orientation, therefore is the homology of:

$$\cdots \to FP_2 \to FP_1 \to FP_0 \to 0$$

The homology is the definition of the derived functors. Therefore we are done. $\square$

Remark 4.5. After you finish the above proposition, you probably will wonder is there a similar result for left exact functors. The answer is there is, and as you might have guessed, to prove the result, we need a dual result of the above double complex of projectives objects, i.e. we need to construct a double complex for a LES consisting of injective objects and is exact in both the rows and columns.

Suppose that:

$$0 \to I^0 \to I^1 \to \cdots$$

is a LES in $\mathcal{A}$ and $\mathcal{A}$ is with enough injectives, then I claim that there is a double complex:

$$\begin{tikzcd}
 & \cdots & \cdots & \cdots & \\
0 \arrow[r] & I^{0,1} \arrow[r] \arrow[u] & I^{1,1} \arrow[r] \arrow[u] & I^{2,1} \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & I^{0,0} \arrow[r] \arrow[u] & I^{1,0} \arrow[r] \arrow[u] & I^{2,0} \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & I^{0} \arrow[r] \arrow[u] & I^{1} \arrow[r] \arrow[u] & I^{2} \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

such that each row and column is an injective resolution.

To see how to construct it, we know that $I^{0,0}$, $I^{1,0}$ and the map between then is easy to find using lifting property of injectives. Then you can observe that the situation is the dual of the above projective case, where we have diagrams:

$$\begin{tikzcd}
I^{m-2,n} \arrow[r] & I^{m-1,n} & ? \\
I^{m-2,n-1} \arrow[r] \arrow[u] & I^{m-1,n-1} \arrow[r] \arrow[u] & I^{m,n-1} \\
 & I^{m-1,n-2} \arrow[r] \arrow[u] & I^{m,n-2} \arrow[u]
\end{tikzcd}$$

Notice that if we have a map:

$$I^{m-2,n} \bigoplus I^{m-1,n-1} \bigoplus I^{m,n-2} \to I^{m-1,n} \bigoplus I^{m,n-1}$$

Then if we can show that the cokernels of: $I^{m-2,n} \to I^{m-1,n}$, and $I^{m,n-2} \to I^{m,n-1}$ injectives into the cokernel of the above map, then we are done as we can just pick a injective resolution of the cokernel of $I^{m-2,n} \bigoplus I^{m-1,n-1} \bigoplus I^{m,n-2} \to I^{m-1,n} \bigoplus I^{m,n-1}$. We have canonical maps between the two kernels and here we don't need extra assumptions like in the projective case to get the desired result. Then you just build this double complex inductively along the diagonal. Then you can prove our assertion of computing spectral sequences using acyclic objects using similar spectral sequence argument.

Suppose that $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ are two left exact additive functors between abelian categories with $\mathcal{A}, \mathcal{B}$ having enough injectives. Then we have the derived functors R^iF, R^iG and $R^i(G \circ F)$. A reasonable question is to ask how they are related:

Theorem 4.6. (Grothendieck Composition of Functors Spectral Sequence) Using the above notation and assume that F sends injectives objects to G-acyclic elements of $\mathcal{B}$. Then for each $X \in \mathcal{A}$, there is a spectral sequence with:

$$_{\to}E_2^{p,q} = R^qG(R^pF) \Rightarrow R^{p+q}(G \circ F)(X)$$

We will see in the next section that the Leray spectral sequence is just a special case of the above spectral sequence:

To prove this theorem, recall that if we want to compute the derived functor of F applied to X, we choose an injective resolution of X:

$$0 \to X \to I^0 \to I^1 \to \cdots$$

To compute the derived functor $R^pF(X)$, we apply to the injective resolution $I^\bullet$:

$$0 \to F(I^0) \to F(I^1) \to F(I^2) \to \cdots$$

Notice that by our assumption, $F(I^p)$ will be G-acyclic. Then we pick an injective resolution of each $F(I^0)$. Recall that if we have the following diagram of two injectve resolutions:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots \\
0 \arrow[r] & B \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & \cdots
\end{tikzcd}$$

Then we can extend $A \to B$ to a complex map. First, notice that J_0 is injective and $A \to I^0$ is injective, then we get a factorization $I^0 \to J^0$. We call this f^0. Then suppose that we have build f^n and we want to build f^{n+1}, the diagram is the following:

$$\begin{tikzcd}
I^{n-1} \arrow[r] \arrow[d, dashed, "f^{n-1}"] & I^n \arrow[r] \arrow[d, dashed, "f^n"] & I^{n+1} \arrow[r] \arrow[d, dashed, "?"] & \cdots \\
J^{n-1} \arrow[r] & J^n \arrow[r] & J^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

We know that we have a monomorphism $I^n/\mathrm{Im}(I^{n-1} \to I^n) \to I^{n+1}$:

$$\begin{tikzcd}
I^n \arrow[rr] \arrow[dr] \arrow[dd, "f^n"] & & I^{n+1} \arrow[dd, dashed] \\
& I^n/\mathrm{Im} \arrow[ur, hook] \arrow[dr, dashed] & \\
J^n \arrow[rr] & & J^{n+1}
\end{tikzcd}$$

According diagram, if $(J^n \to J^{n+1}) \circ f^n$ factors through $I^n/\mathrm{Im}(I^{n-1} \to I^n)$, we are done. That is if $x \in I^{n-1}$, then we know that if we first map it to I^n, then by f^n then by $(J^n \to J^{n+1})$, it's the same as first by f^{n-1} then $J^{n-1} \to J^n \to J^{n+1}$, therefore is 0.

Then we get a double complex:

$$\begin{tikzcd}
& \cdots & \cdots & \cdots & \\
0 \arrow[r] & I^{0,1} \arrow[r] \arrow[u] & I^{1,1} \arrow[r] \arrow[u] & I^{2,1} \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & I^{0,0} \arrow[r] \arrow[u] & I^{1,0} \arrow[r] \arrow[u] & I^{2,0} \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & F(I^0) \arrow[r] \arrow[u] & F(I^1) \arrow[r] \arrow[u] & F(I^2) \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

We know that the columns are exact, but we don't know if the rows are exact. Then we apply G, and get:

$$\begin{tikzcd}
 & \cdots & \cdots & \cdots & \\
0 \arrow[r] & GI^{0,1} \arrow[r] \arrow[u] & GI^{1,1} \arrow[r] \arrow[u] & GI^{2,1} \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & GI^{0,0} \arrow[r] \arrow[u] & GI^{1,0} \arrow[r] \arrow[u] & GI^{2,0} \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Let's compute the spectral sequence in the upward orientation, we know that the first page is:

$$\begin{tikzcd}
0 \arrow[r] & \cdots \arrow[r] & \cdots \arrow[r] & \cdots \arrow[r] & 0 \\
0 \arrow[r] & R^1G(F(I_0)) = 0 \arrow[r] & R^1G(F(I_1)) = 0 \arrow[r] & R^1G(F(I_2)) = 0 \arrow[r] & \cdots \\
0 \arrow[r] & R^0G(F(I_0)) \arrow[r] & R^0G(F(I_1)) \arrow[r] & R^0G(F(I_2)) \arrow[r] & \cdots
\end{tikzcd}$$

Here the higher derived functors of $F(I_i)$ vanish by our assumption that F maps injective objects to G-acyclic objects. Then we know that this spectral sequence stabilizes in the 2-page and the objects in the second page is the derived functor of $G \circ F$ applied to X, i.e. we have:

$$\begin{tikzcd}
0 & R^0(G \circ F)(X) \arrow[dr] & R^0(G \circ F)(X) \arrow[dr] & R^1(G \circ F)(X) \arrow[dr] & \cdots \\
 & & 0 & 0 & 0
\end{tikzcd}$$

Therefore we know that the 2-page $E_2^{p,q} = 0$ if $q \neq 0$ and $E_2^{p,q} = R^p(G \circ F)(X)$ if $q = 0$. This is just the limit that the allegedly existing spectral sequence abuts to.

However, to fully prove the theorem, only the upward orientation is not enough. We need to make the double complex more pleasant to work with. So let's build the double complex more carefully.

Suppose that we have a complex $C^\bullet$ in $\mathcal{B}$:

$$0 \to C^0 \to C^1 \to \cdots \to C^{p-1} \to C^p \to C^{p+1} \to \cdots$$

We will denote by $Z^p(C^\bullet)$ the kernel of the p-th differential of the complex $C^\bullet$. We denote $B^{p+1}(C^\bullet)$ the image of the p-th differential of $C^\bullet$. As usual, $H^p(C^\bullet)$ is the cohomology of the complex $C^\bullet$.

Then we will construct an injective resolution of $C^\bullet$:

$$\begin{tikzcd}
 & \cdots & \cdots & \cdots & \\
0 \arrow[r] & I^{0,1} \arrow[r] \arrow[u] & I^{1,1} \arrow[r] \arrow[u] & I^{2,1} \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & I^{0,0} \arrow[r] \arrow[u] & I^{1,0} \arrow[r] \arrow[u] & I^{2,0} \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & C^0 \arrow[r] \arrow[u] & C^1 \arrow[r] \arrow[u] & C^2 \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

For each fixed p, we will have three complexes:

$$
\begin{array}{ccc}
\cdots & \cdots & \cdots \\
\uparrow & \uparrow & \uparrow \\
Z^p(I^{\bullet,1}) & B^p(I^{\bullet,1}) & H^p(I^{\bullet,1}) \\
\uparrow & \uparrow & \uparrow \\
Z^p(I^{\bullet,0}) & B^p(I^{\bullet,0}) & H^p(I^{\bullet,0}) \\
\uparrow & \uparrow & \uparrow \\
Z^p(C^\bullet) & B^p(C^\bullet) & H^p(C^\bullet) \\
\uparrow & \uparrow & \uparrow \\
0 & 0 & 0
\end{array}
$$

We will construct the injective resolution of $C^\bullet$ such that all the above three are injective resolutions.

Now let's choose injective resolutions $Z^{p,\bullet}$ and $H^{p,\bullet}$ of $B^p(C^\bullet)$ and $H^p(C^\bullet)$ such that we have the following diagram:

$$
\begin{array}{ccccccccc}
 & & \cdots & & & & \cdots & & \\
 & & \uparrow & & & & \uparrow & & \\
 & & B^{p,1} & & & & H^{p,1} & & \\
 & & \uparrow & & & & \uparrow & & \\
 & & B^{p,0} & & & & H^{p,0} & & \\
 & & \uparrow & & & & \uparrow & & \\
0 & \longrightarrow & B^p(C^\bullet) & \longrightarrow & Z^p(C^\bullet) & \longrightarrow & H^p(C^\bullet) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

I claim that we can mimic the horseshoe construction for the projective resolution and fill in the middle vertical column and get an injective resolution of $Z^p(C^p)$ such that we have a SES of complexes:

$$
\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & B^{p,1} & \longrightarrow & Z^{p,1} & \longrightarrow & H^{p,1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & B^{p,0} & \longrightarrow & Z^{p,0} & \longrightarrow & H^{p,0} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & B^p(C^\bullet) & \longrightarrow & Z^p(C^\bullet) & \longrightarrow & H^p(C^\bullet) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

To prove this, notice that if I_1, I_2 are both injective, then $I^1 \oplus I_2$ is also injective. Then we mimic the proof for the projective horseshoe construction and take the middle column to be the direct sum with the obvious morphism from $B^{p,\bullet}$ to $Z^{p,\bullet}$ and from $Z^{p,\bullet}$ to $H^{p,\bullet}$.

We again need to figure out the morphism within this complex to make sure that it's an injecitve resolution. Notice that using the LES of cohomology applied to a SES of complexes, we only need the middle column to be a complex and it's automatically exact. For example we know that $Z^{p,0} = B^{p,0} \oplus H^{p,0}$, we need to define a morphism $Z^p(C^\bullet) \to B^{p,0} \oplus H^{p,0}$ such that the bottom two squares commute. Notice that $B^p(C^\bullet) \to Z^p(C^\bullet)$ is injective and $B^{p,0}$ within $B^{p,0} \oplus H^{p,0}$ is injective, then the lifting property gives $f^0 : Z^p(C^\bullet)$ to $B^{p,0}$, then to $B^{p,0} \oplus H^{p,0}$. Then if we set $g^0 = Z^p(C^\bullet) \to H^p(C^\bullet) \to H^{p,0}$ we can define the map $Z^p(C^\bullet) \to Z^{p,0}$ to be $f^0 + g^0$.

Then going forward, we know that we have an injecion $B^{p,0} \to Z^{p,0}$, which gives a map $Z^{p,0} \to B^{p,1}$, then we define $Z^{p,0} \to Z^{p,1}$ to be this map plus $Z^{p,0} \to H^{p,0} \to H^{p,1}$. Then we can always keep going, we are done.

Following the same argument, we also have a injective resolution:

$$\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Z^{p,1} & \longrightarrow & I^{p,1} & \longrightarrow & B^{p+1,1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Z^{p,0} & \longrightarrow & I^{p,0} & \longrightarrow & B^{p+1,0} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Z^p(C^\bullet) & \longrightarrow & C^p & \longrightarrow & B^{p+1}(C^\bullet) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Using all these above ingredients, we can build the so called **Cartan-Eilenberg resolution**. We define this resolution $I^{\bullet,\bullet}$ to be $I^{p,q}$ is the above resolution we build and the map $I^{p,q} \to I^{p+1,q}$ is defined to be:

$$I^{p,q} \to B^{p+1,q} \to Z^{p+1,q} \to I^{p+1,q}$$

Then if we first fix p, q and compute what is $Z^p(I^{\bullet,q})$, it's going to be the kernel of the above map, which is just $Z^{p,q}$. Similarly, $B^p(I^{\bullet,q})$ is just the image of:

$$I^{p-1,q} \to B^{p,q} \to Z^{p,q} \to I^{p,q}$$

which is just $B^{p,q}$. Finally, the cohomology $H^p(I^{\bullet,q})$ is again just $H^{p,q}$. Therefore we know that with this injective resolution $I^{\bullet,\bullet}$, the following three complexes are all injective resolutions:

$$\begin{array}{ccc}
\cdots & \cdots & \cdots \\
\uparrow & \uparrow & \uparrow \\
Z^p(I^{\bullet,1}) & B^p(I^{\bullet,1}) & H^p(I^{\bullet,1}) \\
\uparrow & \uparrow & \uparrow \\
Z^p(I^{\bullet,0}) & B^p(I^{\bullet,0}) & H^p(I^{\bullet,0}) \\
\uparrow & \uparrow & \uparrow \\
Z^p(C^\bullet) & B^p(C^\bullet) & H^p(C^\bullet) \\
\uparrow & \uparrow & \uparrow \\
0 & 0 & 0
\end{array}$$

Such a resolution is called a **Cartan-Eilenberg resolution**, i.e. an injective resolution that makes the above three complexes all injective resolutions.

Using such a resolution, we can prove the Grothendieck composition of functors Spectral sequence:

Proof. Recall that before we have this Cartan-Eilenberg resolution, we already computed a upward spectral sequence.

We first have an injective resolution $I^\bullet$ of X, then we apply F to get $FI^\bullet$, not we take a Cartan-Eilenberg resolution of $FI^\bullet$ and apply G. We already know that the cohomology of the double complex is $R^p(G \circ F)(X)$.

Now we compute the rightward spectral sequence. What is important about the Cartan-Eilenberg resolution is that we have SES:

$$0 \to B^p(I^{\bullet,q}) \to Z^p(I^{\bullet,q}) \to H^p(I^{\bullet,q}) \to 0$$

$$0 \to Z^p(I^{\bullet,q}) \to I^{p,q} \to B^{p+1}(I^{\bullet,q}) \to 0$$

Moreover every term in the two SES are injective, therefore we know that both sequences actually split. Therefore after apply G, these two sequences still remain exact.

Then consider the exact sequence:

$$0 \to Z^p(I^{\bullet,q}) \to I^{p,q} \to I^{p+1,q}$$

Since G is left exact, we know that $GZ^p(I^{\bullet,q})$ is going to be the kernel of $G(I^{p,q}) \to GI^{p+1,q}$.

Then consider the following exact sequences:

$$0 \to GZ^p(I^{\bullet,q}) \to GI^{p,q} \to GB^{p+1}(I^{\bullet,q}) \to 0$$

$$0 \to GB^p(I^{\bullet,q}) \to GZ^p(I^{\bullet,q}) \to GH^p(I^{\bullet,q}) \to 0$$

The first exact sequence tells us that $GB^{p+1}(I^{\bullet,q})$ is the image of $G(I^{p,q}) \to G(I^{p+1,q})$, therefore we know that the second diagram tells us that the cohomology of $G(I^{\bullet,q})$ at degree p is isomorphic to $GH^p(I^{\bullet,q})$. Therefore when you apply G to this double complex, G just behaves like an exact functor.

Then we compute the spectral in the rightward orientation. The 1-th page is with objects $E_1^{p,q} = H^p(GI^{\bullet,q}) \simeq GH^p(I^{\bullet,q})$. Moreover, under this identification $H^p(GI^{\bullet,q}) \to H^p(GI^{\bullet,q+1})$ is just:

$$G(H^p(I^{\bullet,q}) \to H^p(I^{\bullet,q+1}))$$

Then notice that by our construction $H^p(I^{\bullet,q})$ is an injective resolution of $(R^pF)(X)$, therefore the cohomology, i.e. the 2-th page is with $E_2^{p,q} = R^qG(R^pF(X))$. Then since this is a first quadrant double complex, we have the desired assertion that:

$$E_2^{p,q} = R^qG(R^pF(X)) \Rightarrow H^{p+q}(GI^{\bullet,\bullet}) = R^{p+q}(G \circ F)(X)$$

$\square$

Remark 4.7. If you have any question of identification general abelian categories after you apply an exact functor, consider the following diagram:

$$\begin{tikzcd}
\ker(\mathrm{coker} f) \arrow[rr] \arrow[dr] & & \ker g \arrow[dl] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
F\ker(\mathrm{coker} f) \arrow[rr] \arrow[dr, hook] & & F\ker g \arrow[r] \arrow[dl, hook'] & FH \\
FA \arrow[r, "Ff"] & FB \arrow[r, "Fg"] & FC & \simeq \\
\ker\mathrm{coker}(Ff) \arrow[rr] \arrow[ur, hook] \arrow[uu, dashed] & & \ker(Fg) \arrow[r] \arrow[ul, hook'] \arrow[uu, dashed] & H(F) \arrow[uu, dashed]
\end{tikzcd}$$

The right hand side is what happens after you apply F to the left hand side. The monomorphism $\ker(\mathrm{coker} f) \to \ker g$ is the unique map that maps the triangle commutes.

Then since F is exact, the two dotted vertical isomorphisms in the right hand side diagram are the only morphisms that make the left and right triangle commute, then I will let you figure out the identification your self.

$$\begin{tikzcd}
 & F\ker(\mathrm{coker} f) \arrow[d] & F\mathrm{coker} f \\
FA \arrow[r, "Ff"] & FB \arrow[ur] \arrow[dr] & \simeq \\
 & \ker(\mathrm{coker} Ff) \arrow[u] \arrow[uu, dashed, bend left] & \mathrm{coker} Ff \arrow[uu, dashed]
\end{tikzcd}$$

Remark 4.8. The above result is for left exact functors, where we have to use the Cartan-Eilenberg resolution to prove what we need. In fact, similar result holds for right exact functors as well, it's just we need to produce a projective resolution sort of like this Cartan Eilenberg resolution. This is not hard to produce as in the Cartan-Eilenberg resolution, we only used the horseshoe construction, which indeed holds for projective objects as well. We omit the proof here.

5 Derived functor cohomology of $\mathcal{O}$-modules

In this section, we want to apply the machinery we built in the previous discussion and define the derived functor cohomology of quasicoherent $\mathcal{O}_X$ modules on a scheme X. However, in order to make our argument easier, we will work with the abelian category of the $\mathcal{O}_X$-modules, not just quasicoherent $\mathcal{O}_X$-modules. The reason for this will be given in a later remark. For now, let's just focus on $\mathcal{O}_X$-modules.

Theorem 5.1. Suppose that $(X, \mathcal{O}_X)$ is a ringed space, not necessarily a scheme, then the abelian category of $\mathcal{O}_X$-modules have enough injective.

Remark 5.2. Recall that if X is a topological space, we use $\underline{\mathbb{Z}}$ to denote the constant sheaf of $\mathbb{Z}$ on X. This is a ringed space and if we take $\mathcal{O}_X$ to be $\underline{\mathbb{Z}}$, then we know that an $\mathcal{O}_X$-module on X is just a sheaf of abelian groups. A more precise statement is that we have an isomorphism of categories between the category of the sheaf of abelian groups on X with the category of $\mathcal{O}_X$-modules.

Recall that an $\mathcal{O}_X$-module structure of $\mathcal{F}$ where $\mathcal{F}$ is a sheaf of abelian groups on X is just a morphism:

$$\mathcal{O}_X \times \mathcal{F} \to \mathcal{F}$$

such that it's $\mathbb{Z}$-bilinear such that it's compatible with multiplication in $\mathcal{O}_X$. Conversely, if $\mathcal{F}$ is a sheaf of abelian groups, we have a morphism:

$$\mathbb{Z} \times \mathcal{F} \to \mathcal{F}$$

where $\mathbb{Z}$ is the presheaf of constant $\mathbb{Z}$, that is $\mathbb{Z}$-linear.

Then suppose that we have such a $\mathcal{O}_X$-module structure on $\mathcal{F}$, then we consider:

$$\mathbb{Z} \times \mathcal{F} \to \mathcal{O}_X \times \mathcal{F} \to \mathcal{F}$$

Then clearly make $\mathcal{F}$ a sheaf of abelian groups.

Then conversely, suppose that $\mathcal{F}$ is a sheaf of abelian groups, i.e. we have a map:

$$\mathbb{Z} \times \mathcal{F} \to \mathcal{F}$$

We know that we have a canonical diagram:

$$\begin{tikzcd} \mathbb{Z} \times \mathcal{F} \arrow[rr] \arrow[dr] & & \mathcal{F} \\ & \mathcal{O}_X \times \mathcal{F} \arrow[ur, dashed] & \end{tikzcd}$$

I claim that $\mathcal{O}_X \times \mathcal{F}$ is the sheafification of $\mathbb{Z} \times \mathcal{F}$, thus we have the unique extension. In the way of prove this assertion, you will see that this map is indeed bilinear. To show that it's compatible with the multiplication, you check locally using the concrete description of $\mathcal{O}_X$ as locally constant stalks, and I will let you do it.

Note that if $\mathcal{M}$ is a presheaf of abelian groups and $\mathcal{N}$ is a sheaf of abelian groups, then we know that the canonical map:

$$\mathcal{M} \times \mathcal{N} \to \mathcal{M}^a \times \mathcal{N}$$

is the shefification map. To see why, notice that a map $\mathcal{M} \times \mathcal{N} \to \mathcal{F}$ is equivalent a map:

$$\mathcal{M} \to \mathcal{H}om(\mathcal{N}, \mathcal{F})$$

Notice that since $\mathcal{N}, \mathcal{F}$ is a sheaf then $\mathcal{H}om(\mathcal{N}, \mathcal{F})$ is also a sheaf. Then by the universal property, this gives a unique morphism:

$$\mathcal{M}^a \to \mathcal{H}om(\mathcal{N}, \mathcal{F})$$

which is a map $\mathcal{M}^a \times \mathcal{N} \to \mathcal{F}$. Therefore there is only one sheaf morphism $\mathcal{M}^a \times \mathcal{N} \to \mathcal{F}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{M} \times \mathcal{N} \arrow[r] \arrow[d] & \mathcal{F} \\ \mathcal{M}^a \times \mathcal{N} \arrow[ur] & \end{tikzcd}$$

Therefore $\mathcal{M}^a \times \mathcal{F}$ will be the sheafification of $\mathcal{M} \times \mathcal{N}$.

One consequence of the above remark is that we can define the derived functor cohomology of sheaves of the abelian group on X. If the topological space X satisfies some property, then this derived functor cohomology will agree with the classical singular cohomology. But we won't use it later.

Proof. Then let's begin to prove this theorem. We will do it in a series of steps. $\square$

First, suppose that $\mathcal{F}$ is an $\mathcal{O}_X$-module, we will show an injective morphism $\mathcal{F} \hookrightarrow \mathcal{Q}$ where $\mathcal{Q}$ is an injective $\mathcal{O}_X$-module. Recall that for each $p \in X$, we know that $\mathcal{F}_p$ is an $\mathcal{O}_{X,p}$-module. Recall that he category of rings have enough injectives, therefore we can choose an injective map of $\mathcal{O}_{X,p}$-modules:

$$\mathcal{F}_p \hookrightarrow Q_p$$

where Q_p is an injective $\mathcal{O}_{X,p}$-module. Then we can consider the sky-scraper sheaf $\mathcal{Q}_p = i_{p,*}Q_p$ on X where i_p is the map that includes p to X. Notice that clearly $\mathcal{Q}_p$ has an $\mathcal{O}_X$-module structure. I claim that $\mathcal{O}_p$ is injective as an $\mathcal{O}_X$-module.

We could do this using one categorical abstract nonsense we prove later. But let's directly verify it. Recall that an $\mathcal{O}_X$-module morphism from any $\mathcal{O}_X$-module $\mathcal{F}$ is just an $\mathcal{O}_{X,p}$-module map $\mathcal{F}_p \to Q_p$, then consider the following diagram:

$$\begin{tikzcd} & \mathcal{Q}_p \\ \mathcal{F} & \mathcal{G} \arrow[l, hook'] \arrow[u] \end{tikzcd}$$

Notice that this gives a morphism $\mathcal{G}_p$ to Q_p and an injective map $\mathcal{G}_p$ to $\mathcal{F}_p$, then we can use the injectivity of Q_p.

Remark 5.3. Notice that this doesn't mean that the stalk of $\mathcal{Q}_p$ is only nonzero at p. For example if $(X, \mathcal{O}_X)$ is a scheme, then we can consider the affine case. Suppose that p is a prime ideal in $SpecA$ and q is a prime ideal containing p, then notice that if $D(x)$ contains q, then we know that $x \notin q$, then since $p \subset q$, $x \notin q$, therefore $D(x)$ contains p. Therefore every open containing $p \subset q$ contains p. Then the stalk at q is also going to be Q_p.

Note that if $\mathcal{I}_i$ are all injective $\mathcal{O}_X$-modules, then the product $\prod_i \mathcal{I}_i$ is also injective. We want to extend the following diagram:

$$\begin{tikzcd} & \prod \mathcal{I}_i \\ \mathcal{F} & \mathcal{G} \arrow[l, hook'] \arrow[u] \end{tikzcd}$$

Recall that for each single I_i, we can extend this map, say it's f_i. Then we define the extension $\mathcal{F} \to \prod_i I_i$ on each U to be $\prod_i f_i(U)$. You can check that this indeed is commutative.

Therefore we know that $\prod_{p \in X} \mathcal{Q}_p$ is an injective module on X and we know that we have a canonical map:

$$\mathcal{F} \to \mathcal{Q}' = \prod_{p \in X} \mathcal{Q}_p$$

which is defined by the canonical map $\mathcal{F} \to \mathcal{Q}_p$ for each p, i.e. a map $\mathcal{F}_p \to Q_p$. Say this map is f_i, then we define define $f : \mathcal{F} \to \prod_p \mathcal{Q}_p$ on U to be $\prod_i f_i(u)$. This will be an injective map. To check this, we check on each stalk at $q \in X$. The stalk of $\mathcal{Q}'$ at q is still a product of $\mathcal{Q}_{p'}$ where p' is point in X such that p' is contained in any open that contains q. (Again, if $(X, \mathcal{O})X)$ is a scheme, then affine locally speaking these p' are primes contained in q) Then the morphism is just:

$$\mathcal{F}_q \to \prod_{p'} Q_{p'}$$

It's determined by:

$$\mathcal{F}_q \hookrightarrow \prod_{p'} \mathcal{F}_p \hookrightarrow \prod_{p'} Q_p$$

Here clearly the second map being the product of injective maps is injective. The first map is also obviously injective since the product has a factor $\mathcal{F}_q$. This finishes the proof of the theorem.

Definition 5.4. If $(X, \mathcal{O}_X)$ is a ringed space, we define:

$$H^i(X, \mathcal{F}) = R^i\Gamma(X, \mathcal{F})$$

Furthermore, suppose that $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a map of ringed space, then the pushforward π_* functor is left exact from the category $\mathrm{Mod}_{\mathcal{O}_X}$ to the category $\mathrm{Mod}_{\mathcal{O}_Y}$. Then we define the **higher pushforwards** (or derived **pushforwards**) to be:

$$R^i\pi_* : \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_Y}$$

Recall that back when we discussed the Cech cohomology, we defined these notions in the context of working with quasicoherent sheaves on a quasicompact and separated scheme. We will later prove that the above definitions agree with our earlier definitions given previously. Therefore we can think of this derived functor cohomology as a generalization of the Cech cohomology. In practice, we will still mostly use Cech cohomology to do computation, but for example, the following result is still very important:

Theorem 5.5. Leray Spectral Sequences Let $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a map of ringed spaces, then for any $\mathcal{O}_X$-module $\mathcal{F}$, there is a spectral sequence with E_2 term given by:

$$E_2^{p,q} = H^q(Y, R^p\pi_*\mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F})$$

Notice that once we prove that the pushforward functor π_* maps injective objects to Γ-acyclic functors then this spectral sequence is just a direct consequence of the Grothendieck composition of functors spectral sequence.

Definition 5.6. We make an intermediate definition that is use to show that pushforward of injective $\mathcal{O}_X$-modules is Γ-acyclic. This definition is important in itself.

A $\mathcal{F}$ of say, abelian groups on a topological space is **flasque** (sometimes also called **flabby**) if all restriction maps are surjective. That is for any $U \subset V$, $\mathrm{Res}_{U\subset V} : \mathcal{F}(V) \to \mathcal{F}(U)$ is surjective.

Proposition 5.7. Suppose that $(X, \mathcal{O}_X)$ is a ringed space, then if:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is an exact sequence of $\mathcal{O}_X$-modules with $\mathcal{F}'$ being flasque, then we know that for every U the induced map:

$$0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$$

is exact.

Proof. We only need to show that the right end is exact, i.e. $\mathcal{F}(U) \to \mathcal{F}''(U)$ is surjective. Say $x \in \mathcal{F}''(U)$, we consider any $p \in U$ and the image x_p of x in any $\mathcal{F}''_p$, notice that we have an exact sequence of stalks:

$$0 \to \mathcal{F}'_p \to \mathcal{F}_p \to \mathcal{F}''_p \to 0$$

Then there will be $y_p \in \mathcal{F}_p$ whose image in $\mathcal{F}''_p$ is x_p.

Notice that we can assume that there is a small open U_p around p such that $y_p \in \mathcal{F}(U_p)$ and $x_p \in \mathcal{F}''(U_p)$. Then now we have a cover of U by U_p with $y_p \in \mathcal{F}(U_p)$ whose image is x_p. We would like these y_p to glue to an element in $\mathcal{F}(U)$ then we are done. However, we only know that on the intersection $y_p - y_q$ is mappped to 0.

Note that using the exactness of:

$$0 \to \mathcal{F}'(U_p \cap U_q) \to \mathcal{F}(U_p \cap U_q) \to \mathcal{F}''(U_p \cap U_q)$$

We know that $y_p - y_q$ comes from a unique $z_{p,q}$ in $\mathcal{F}'(U_p \cap U_q)$.

Then consider the set of pairs (V, t) with $V \subset U, t \in \mathcal{F}(V)$ such that the image of t in $\mathcal{F}''(V)$ is $x|_V$. This is not empty since we know that (U_p, y_p) is. We partially order this set by:

$$(V,t) \leqslant (U,s), \text{iff } V \subset U \text{ and } s|_V = t$$

Notice that given any well-order subset $\{(V_i, t_i)\}$, we can take the union and there will be a new element in $\mathcal{F}(\cup_i V_i)$ such that it restricts to t_i on each V_i. Then by Zorn's lemma there is a maximal such pair (V, t). I claim that this V is U. If not, consider $x \in U - V$. Then we know that there is the pair (U_p, y_p). Then we know that in the intersection $y_p - t$ is mapped to 0, therefore it comes from a unique z in $\mathcal{F}'(V \cap U_p)$. Since $\mathcal{F}'$ is flasque, we know that we can pick z_p in U_p such that it restrict to z on $U_p \cap V$. Then consider the pair $(U_p, y_p - z)$, then in the intersection $y_p - z$ agrees with t, therefore we know that they glue, this contradicts the maximality. ☐

Remark 5.8. We will see prove that flasque $\mathcal{O}_X$-modules are Γ-acyclic. Therefore the above result should not be a surprise as we know that in the LES, the degree 1 cohomology of $\mathcal{F}'$ vanishes.

Proposition 5.9. Using the above notation, if $\mathcal{F}'$ is flasque, then $\mathcal{F}$ is flasque iff $\mathcal{F}''$ is flasque.

Proof. Consider $U \subset V$, then we have:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}'(V) \arrow[r] \arrow[d, dashed, "\alpha"] & \mathcal{F}(V) \arrow[r] \arrow[d, dashed, "\beta"] & \mathcal{F}''(V) \arrow[r] \arrow[d, dashed, "\gamma"] & 0 \\
0 \arrow[r] & \mathcal{F}'(U) \arrow[r] & \mathcal{F}(U) \arrow[r] & \mathcal{F}''(U) \arrow[r] & 0
\end{tikzcd}$$

Here we use α, β, γ to denote the various restriction maps. Then we use the snake lemma to conclude that $\operatorname{coker}\beta = 0$ iff $\operatorname{coker}\gamma = 0$. ☐

Proposition 5.10. Suppose that $\pi : X \to Y$ is a continuous map of topological spaces, and $\mathcal{F}$ is a flasque sheaf of sets on X. Show that $\pi_*\mathcal{F}$ is a flasque sheaf on Y. If π is a morphism of ringed spaces, then similar results hold.

Proof. By definition of the pushforward. Omitted. ☐

Now we introduce a new functor, that is an occasional left adjoint to the inverse image functor. Recall that we always have a right adjoint to the inverse image functor, which is the pushforward, no matter what sort of morphisms we are using. However we will see that only when π is an open embedding of ringed spaces will π^{-1}, the inverse image functor have a left adjoint, called the **extension by zero** functor.

Suppose that $(Y, \mathcal{O}_Y)$ is a ringed space, and $i : U \hookrightarrow Y$ is an open embedding of ringed spaces, we will define the **extension of i by zero** $i_! : \mathrm{Mod}_{\mathcal{O}_U} \to \mathrm{Mod}_{\mathcal{O}_X}$ as follows.

Suppose that $\mathcal{F}$ is an $\mathcal{O}_U$-module. Then we will define for each $W \subset Y$:

$$(i_!^{pre}\mathcal{F})(W) = \begin{cases} \mathcal{F}(W) & \text{if } W \subset U, \\ 0 & \text{otherwise.} \end{cases}$$

It's easy to check that this is a presheaf of $\mathcal{O}_Y$-modules, then we sheafify it to get $i_!\mathcal{F}$ which will be come an $\mathcal{O}_Y$-module. To see why, if $\mathcal{F}^{pre}$ is a presheaf of $\mathcal{O}_X$-module, then we have the following diagram:

$$\begin{tikzcd}
\mathcal{F}^{pre} \times \mathcal{O}_Y \arrow[r] \arrow[d] & \mathcal{F}^{pre} \arrow[d] \\
\mathcal{F} \times \mathcal{O}_Y \arrow[r, dashed] & \mathcal{F}
\end{tikzcd}$$

where the dotted map is induced since $\mathcal{F} \times \mathcal{O}_Y$ is the sheafification of $\mathcal{F}^{pre} \times \mathcal{O}_Y$. Again, this $\mathcal{F} \to \mathcal{O}_Y$ is bilinear and you check that this compatible with multiplications in $\mathcal{O}_Y$ locally using concrete descriptions of $\mathcal{F}$ as locally constant stalks.

I will let you this extension by zero indeed defines a functor.

Remark 5.11. In fact the presheaf $i_!^{pre}\mathcal{F}$ always satisfies the identity axiom. To see this, since if W is not contained in U, everything is 0. Then we only need to worry about those W that is contained in U. But $\mathcal{F}$ is itself a sheaf, so the identity axiom is always satisfied.

To see why it may not satisfy the gluing axiom. Let's consider the sheaf of continuous functions on the interval $(-1, 1)$, we can extend it by zero to $\mathbb{R}$. Notice that we have a open cover of $\mathbb{R}$ by $(-\infty, -1/2)(1/2, \infty)$ and $(-3/4, 3/4)$. Then consider the function on $(-3/4, 3/4)$ such that it's not 0 on $(-1/4, 1/4)$ bu that is 0 on $(-3/4, -1/2)$ and on $(1/2, 3/4)$. We pick 0 function on $(-\infty, -1/2)$ and 0 function on $(1/2, \infty)$. Then we know that they agree on intersections, but since there is only 0 function on $\mathbb{R}$, obvious it doesn't restrict to the function on $(-3/4, 3/4)$ we chose.

This example may give you a feeling of why this is called extension by 0.

Proposition 5.12. There is a natural identification of the stalk of $i_!\mathcal{F}$ at q with $\mathcal{F}_q$ if $q \in U$, otherwise the stalk is just 0.

Proof. Notice that there is a natural identification between the stalk of $i_!\mathcal{F}$ at q with the stalk of $i_!^{pre}\mathcal{F}$ at q. It's clear that if $q \in U$, then the stalk of the presheaf if the same as that of $\mathcal{F}$. It's also clear that if q is not in U, then all opens around U gives you 0, therefore we are done. $\square$

One immediate result is that $i_!$ is an exact functor since exactness can be checked on stalks.

Proposition 5.13. Suppose that $\mathcal{G}$ is an $\mathcal{O}_Y$-module and $i : U \hookrightarrow Y$ is an open embedding. Then there is a natrual injective morphism:

$$i_!i^{-1}\mathcal{G} \to \mathcal{G}$$

Proof. Since U is just an open subset of Y, then in this case i^{-1} is just the restriction map, then we are just trying to give a morphism:

$$i_!\mathcal{G}|_U \to \mathcal{G}$$

It's easy how we can define a natural morphism:

$$i_!^{pre}\mathcal{G}|_U \to \mathcal{G}$$

Then sheafification gives you the desired morphism. Notice that the maps induced on stalks is either the identity or 0 to something, therefore all injective. $\square$

Remark 5.14. Actually something more interesting can be given here. Suppose that Z is a closed subset of Y and U is the complement of Z, then we have the embedding of topological spaces $i : U \hookrightarrow Y$ and the embedding of the closed subset $j : Z \hookrightarrow Y$. Then I claim that for any sheaf $\mathcal{G}$ of $\mathcal{O}_Y$-modules, we also have a canonical map of $\mathcal{O}_Y$-modules:

$$\mathcal{G} \to j_*j^{-1}\mathcal{G}$$

We give Z a ringed space structure by define $\mathcal{O}_Z = j^{-1}\mathcal{O}_Y$. Then we know that $j^{-1}\mathcal{G}$ will be an $\mathcal{O}_Z$-module and the map $j : Z \hookrightarrow Y$ can actually be promoted to a morphism of ringed spaces(I will let you figure out what is it). We also know that (j^{-1}, j_*) is an adjoint pair between $\mathcal{O}_Z$-modules to $\mathcal{O}_Y$-modules, then the morphism:

$$\mathcal{G} \to j_*j^{-1}\mathcal{G}$$

is the one defined by $\mathrm{Id} : j^{-1}\mathcal{G} \to j^{-1}\mathcal{G}$. I claim this is surjective, which can be checked on stalks. Recall that j embed the closed subset Z to Y, therefore the stalk of the pushforward by j can be easily computed. Then I will let you check the morphism is either the identity or a morphism to 0, all surjective. The following diagram may help you do it:

$$\begin{tikzcd}
& j_*(j^{-1}\mathcal{G})^{pre} \arrow[d] \\
\mathcal{G} \arrow[ur] \arrow[r] & j_*j^{-1}\mathcal{G}
\end{tikzcd}$$

You can also check that we have a SES of $\mathcal{O}_Y$-modules:

$$0 \to i_*i^{-1}\mathcal{G} \to \mathcal{G} \to j_*j^{-1}\mathcal{G} \to 0$$

We now show that $(i_!, i^{-1})$ is indeed an adjoint pair:

Proposition 5.15. There is a functorial isomorphism:

$$\mathrm{Hom}_{\mathcal{O}_Y}(i_!\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}, \mathcal{G}|_U)$$

Proof. Notice that using the fact that sheafification is left adjoint to forgetful functor, we have functorial isomorphisms:

$$\mathrm{Hom}_{\mathcal{O}_Y}(i_!\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathcal{O}_Y}(i_!\mathcal{F}^{pre}, \mathcal{G})$$

It's easy to define a natural morphism:

$$\mathrm{Hom}_{\mathcal{O}_Y}(i_!\mathcal{F}^{pre}, \mathcal{G}) \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}, \mathcal{G}|_U)$$

by just restricting to U. This is clearly an isomorphism. You can check the functoriality yourself. $\square$

Remark 5.16. In particular, we have an isomorphism:

$$\mathrm{Hom}_{\mathcal{O}_Y}(i_!\mathcal{O}_U, \mathcal{G}) \to \mathrm{Hom}(\mathcal{O}_U, \mathcal{G}|_U) \simeq \mathcal{G}(U)$$

Then we can use this extension by zero as a tool to prove several facts about flasque sheaves:

Proposition 5.17. If $(X, \mathcal{O}_X)$ is a ringed space and $\mathcal{Q}$ is an injective $\mathcal{O}_X$-module, then $\mathcal{Q}$ is flasque.

Proof. Suppose that $U \subset V \subset X$, then notice that we have an injective $\mathcal{O}_X$-module morphism:

$$0 \to (i_U)_!\mathcal{O}_U \to (i_V)_!\mathcal{O}_U$$

Since we know that $\mathcal{Q}$ is injective, then the functor $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{Q})$ is exact, thus we know that if we apply this functor to the above exact sequence, we have a surjective map:

$$\mathrm{Hom}_{\mathcal{O}_X}((i_V)_!\mathcal{O}_U, \mathcal{Q}) \to \mathrm{Hom}_{\mathcal{O}_X}((i_U)_!\mathcal{O}_U, \mathcal{Q})$$

Using the previous remark, I will let you check that the above map can be identified with:

$$\mathcal{Q}(V) \to \mathcal{Q}(U)$$

Therefore is surjective. $\square$

Then let's prove that flasque modules are Γ-acyclic:

Proposition 5.18. Recall that we have proved that the abelian category of $\mathcal{O}_X$-modules have enough injectives and define $\mathrm{H}^i(X, -)$ as the right derived functor of $\Gamma(X, -)$. Then I claim that if $\mathcal{F}$ is a flasque sheaf, then $\mathrm{H}^i(X, \mathcal{F}) = 0$ for all $i > 0$.

Proof. Notice that if $\mathcal{F}$ is flasque, we can first embed it to an injective $\mathcal{O}_X$-module, say $\mathcal{Q}$, then we know that we have a SES:

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0$$

Recall that $\mathcal{I}$ is flasque, therefore $\mathcal{Q}$ is flasque. Then consider the LES of the above SES:

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{I}) \to \Gamma(X, \mathcal{Q}) \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{I}) = 0$$

since $\mathcal{F}$ is flasque, we know that $H^1(X, \mathcal{F})$ must be 0. We prove that the degree 1 cohomology of all flasque sheaves vanish. Then since $\mathcal{Q}$ is also a flasuqe sheaf, we know that it's degree 1 cohomology also vanishes, then we continue the LES:

$$H^1(X, \mathcal{Q}) = 0 \to H^2(X, \mathcal{F}) \to H^2(X, \mathcal{I}) = 0$$

Then we prove that $H^2(X, \mathcal{F}) = 0$ as well. Then we can use induction. $\square$

Remark 5.19. Notice that if $\pi : X \to Y$ is a morphism of ringed spaces and $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, therefore we know that it's flasque. Then $\pi_*\mathcal{I}$ is flasque. Therefore pushforward takes an injective object to a $\Gamma(Y, -)$-acyclic object, therefore we are done with the proof of Leray spectral sequence.

Now if Z is just a one point topological space, we know that taking the global section functor in Y is naturally identified with the pushforward of

$$\rho : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$$

where $\mathcal{O}_Z = \pi_*\mathcal{O}_Y$. Then we will naturally ask, if $\rho : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ is an arbitrary morphism of ringed spaces and we consider the composition of:

$$(X, \mathcal{O}_X) \xrightarrow{\pi} (Y, \mathcal{O}_Y) \xrightarrow{\rho} (Z, \mathcal{O}_Z)$$

Do we still have similar spectral sequence $E_2^{p,q} = R^q\rho_*(R^p\pi_*\mathcal{F}) \Rightarrow R^{p+q}((\rho \circ \pi)_*)\mathcal{F}$. The answer this spectral sequence indeed exists and we only need to prove that the pushforward functor maps injective objects to ρ_*-acyclic objects. Therefore we want to prove that flasque object is ρ_*-acyclic.

We follow the same argument as above: first if $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, we prove that if we have SES of $\mathcal{O}_X$-modules:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

with $\mathcal{F}'$ flasque, then after we pushforward by π_*, the sequence remains exact:

$$0 \to \pi_*\mathcal{F}' \to \pi_*\mathcal{F} \to \pi_*\mathcal{F}'' \to 0$$

Then recall that injective objects are flasque and is π_*-acyclic, then the same argument as above shows that flasque objects are π_*-acyclic.

Remark 5.20. In this remark we give an example of a ringed space where we don't have enough projective objects.

Consider the projective line $X = \mathbb{P}^1_k$ with the Zariski topology. In fact we only need this space to be a space that is not **Alexandrov**. We say that a space is Alexandrov if arbitray intersections of open subsets is open. Most spaces we are familiar is not Alexandrov. One important feature of a Alexandrov space is that for every point of the space, it has a smallest open neigborhood. Therefore is a space is not Alexandrov, there is a point who has no smallest neighborhood. Notice that $Speck[x]$ is not Alexandrov since for each $D(f)$ where f is irreducible, for any polynomial g with degree at least 1, we know that $D(fg)$ strictly lies in $D(f)$. (i.e. we proved that closed points of $Speck[1]$ has no smallest neighborhood.)

Then we give X a ring space structure by putting $\mathcal{O}_X = \underline{\mathbb{Z}}$. I claim that there is no projective P objects with a surjective map:

$$\psi : P \to \mathcal{O}_X$$

Suppose that there is such a map, then if we fix a closed point $q \in X$, we will show that the induced map on stalks is the 0 map, therefore contradicting the surjectivity.

For $U \subset X$ open, we will use $\underline{\mathbb{Z}}_U$ to denote the extension by zero sheaf of the constant sheaf with values in $\mathbb{Z}$ in U. Let V be an open neighborhood of q, let W be a strictly smaller neighborhood of q.

Notice that we have a surjective morphism:

$$\underline{\mathbb{Z}}_{X-q} \oplus \underline{\mathbb{Z}}_W \to \underline{\mathbb{Z}}$$

By projectivity of P, we have a factorization:

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d, two heads] \\ \underline{\mathbb{Z}}_{X-q} \oplus \underline{\mathbb{Z}}_W \arrow[r, two heads] & \underline{\mathbb{Z}} \end{tikzcd}$$

Then we know that $P(V) \to \underline{\mathbb{Z}}(V)$ factors though:

$$P(V) \to \underline{\mathbb{Z}}_{X-q}(V) \oplus \underline{\mathbb{Z}}_W(V)$$

Now notice that we can take V, W are connected and $X - q$ is also connected. Then I claim that $\underline{\mathbb{Z}}_{X-q}(V) \oplus \underline{\mathbb{Z}}_W(V) = 0$ recall that by definition the elements are locally constant stalks while the stalk at q of $\underline{\mathbb{Z}}_{X-q}$ is 0 and the stalk of $\underline{\mathbb{Z}}_W$ at p outside W is 0.

Then suppose that $(x_p)_{p\in V}$ is such a system of locally constant stalks. We consider the subset of V where x_p is 0. Notice that this is open by definition. Then we consider the subset where x_p is not 0, then we can assume that x_p comes from something nonzero in $\mathbb{Z}$, therefore in a small open where the stalk is constant, it's nonzero. Therefore we know that the place where $x_p = 0$ is open and closed, therefore is a union of connected components, but we know that V is connected, therefore it's V. Similarly we can argue that $\underline{\mathbb{Z}}_W(V) = 0$ since V is open.

Therefore $\psi : P(V) \to \underline{\mathbb{Z}}(V) = 0$, then I claim that this implies the induced map on the stalk $P_q \to \mathbb{Z}$ is 0. This is because this applies to every neighborhood V of q. This is a contradiction.

6 Cech cohomology and derived functor cohomology agree

Theorem 6.1. Suppose that X is a quasi-compact and separated scheme and $\mathcal{F}$ is a quasicoherent sheaf. Then the Cech cohomology of $\mathcal{F}$ agrees with the derived functor cohomology of $\mathcal{F}$.

Before we prove this, we must say that this theorem is not precise enough. For example, we not only want to know there is such an isomorphism, we also want this isomorphism to be functorial in $\mathcal{F}$ and it should also respect the connecting homomorphisms in the induced LES. Moreover, we also would like to have similar results for the higher pushforward. But we will leave these to the end of this section.

The way we prove this theorem is again to use a spectral sequence argument by computing the sequence in two orientations. We will need to cohomology vanishing ingridients:

1. If $(X, \mathcal{O}_X)$ is a ringed space and $\mathcal{Q}$ is an injective $\mathcal{O}_X$-module. For any finite open cover $X = \cup U_i$, the $\mathcal{Q}$ has no higher Cech cohomology.

2. If X is an affine scheme and $\mathcal{F}$ is a quasicoherent sheaf on X, then $R^i\Gamma(\mathcal{F}) = 0$ for $i > 0$.

The first condition above tells you that the building blocks of derived functor cohomology have no higher Cech cohomology. The second condition above tells you that the building blocks of Cech cohomology have no higher derived functor cohomology.

Proposition 6.2. Suppose that $(X, \mathcal{O}_X)$ is a ringed space and $\mathcal{Q}$ is an injective $\mathcal{O}_X$-module. Let $i : U \hookrightarrow X$ be an open embedding, then $\mathcal{Q}|_U$ is an injective $\mathcal{O}_U$-module.

Proof. We prove that i^{-1} is the right adjoint of the extension by zero functor, which you can easily prove to be exact. Then the adjoint property implies that i^{-1} preserves injective objects, therefore if $\mathcal{Q}$ is injective then so is $\mathcal{Q}|_U$.

You can either prove this abstractly by manipulating the Hom sets. Or you will find that the abstract argument translates to the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A} \arrow[r] \arrow[d] & \mathcal{B} \arrow[dl, dashed, "?"] \\ & \mathcal{Q}|_U & \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & i_!\mathcal{A} \arrow[r] \arrow[d] & i_!\mathcal{B} \arrow[dl, dashed, "?"] \\ & \mathcal{Q} & \end{tikzcd}$$

□

Now let's first assume condition 1 and 2 above and prove the theorem. We will then go back and prove condition 1 and 2:

Proof. We will only do one thing that is reasonable. First to compute the derived functor cohomology of $\mathcal{F}$, we choose an injective resolution:

$$0 \to \mathcal{F} \to \mathcal{Q}^0 \to \mathcal{Q}^1 \to \cdots$$

We also need to include the information of the Cech cohomology cover and make these a double complex. Finally we toss it into the spectral sequence machine.

Let's choose an affine open cover $X = \cup_i U_i$ and consider the double complex:

$$\begin{tikzcd}
 & \cdots & \cdots & \cdots & \\
0 \arrow[r] & \oplus_i \mathcal{Q}_2(U_i) \arrow[r] \arrow[u] & \oplus_{i,j} \mathcal{Q}_2(U_{ij}) \arrow[r] \arrow[u] & \oplus_{i,j,k} \mathcal{Q}_2(U_{ijk}) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \oplus_i \mathcal{Q}_1(U_i) \arrow[r] \arrow[u] & \oplus_{i,j} \mathcal{Q}_1(U_{ij}) \arrow[r] \arrow[u] & \oplus_{i,j,k} \mathcal{Q}_1(U_{ijk}) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \oplus_i \mathcal{Q}_0(U_i) \arrow[r] \arrow[u] & \oplus_{i,j} \mathcal{Q}_0(U_{ij}) \arrow[r] \arrow[u] & \oplus_{i,j,k} \mathcal{Q}_0(U_{ijk}) \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Recall that the differential in the Cech complex is defined to be either 0 or just the restriction map with a sign modification. Then it's to check this double complex is indeed a double complex.

We take this as the E_0 page in the spectral sequence. We consider the rightward orientation. We know that we are just computing the Cech cohomology of these injective objects. Since we assumed that the higher Cech cohomology vanishes for injective objects, then the first page looks like:

$$\begin{tikzcd}
\cdots & \cdots & \cdots \\
0 \arrow[u] & \mathcal{Q}_2(X) \arrow[u] & 0 \arrow[u] \\
0 \arrow[u] & \mathcal{Q}_1(X) \arrow[u] & 0 \arrow[u] \\
0 \arrow[u] & \mathcal{Q}_0(X) \arrow[u] & 0 \arrow[u] \\
0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

Then we continue to compute the second page which stabilizes and is:

$$\begin{tikzcd}
\cdots & \cdots & \cdots \\
0 & R^2\Gamma(X, \mathcal{F}) & 0 \\
0 & R^1\Gamma(X, \mathcal{F}) \arrow[uul] & 0 \arrow[uul] \\
0 & \Gamma(X, \mathcal{F}) \arrow[uul] & 0 \arrow[uul] \\
0 & 0 \arrow[uul] & 0 \arrow[uul]
\end{tikzcd}$$

This tells you that the cohomology of the total complex of the double complex are $R^i\Gamma(X, \mathcal{F})$.

Then we start over with the E_0 page and do the computation in the upward orientation. Recall that we know that the restriction of an injective module on an open subset remains injective, and we know that taking the cohomology commutes with direct sums, therefore we know that we are justing computing the derived functor cohomology of $\mathcal{F}|_{U_i}$, $\mathcal{F}|_{U_{ij}}$ and so on. By our assumption, we know that the higher cohomology vanishes, therefore the 1-st page is:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
0 \arrow[r] & \oplus_i \Gamma(U_i, \mathcal{F}) \arrow[r] & \oplus_{i,j} \Gamma(U_{ij}, \mathcal{F}) \arrow[r] & \oplus_{i,j,k} \Gamma(U_{ijk}, \mathcal{F}) \arrow[r] & \cdots \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Then we know that this spectral sequence stabilizes at 2-page and on the second page we have the Cech cohomology of $\mathcal{F}$, therefore we are done. $\square$

Now we need to show condition 1 and 2 indeed hold. First we show that injectives have no Cech cohomology:

Proposition 6.3. Suppose that $X = \cup_j U_j$ where U_j is a finite cover of X by open sets and $\mathcal{F}$ is a flasque sheaf on X. Then the Cech cohomology of $\mathcal{F}$ with respect to this cover vanish in positive degree.

Proof. Let's assume that these index j starts from 0 and ends in n. We will do induction on the number n, first if $n = 1$, then we know that the Cech complex is:

$$0 \to \mathcal{F}(U_0) \oplus \mathcal{F}(U_1) \to \mathcal{F}(U_1 \cap U_0) \to 0$$

since $\mathcal{F}$ is flasque we know that the map $\mathcal{F}(U_0) \oplus \mathcal{F}(U_1) \to \mathcal{F}(U_1 \cap U_0)$ is surjective, therefore the cohomology of degree 1 is 0. Then we suppose that the higher Cech cohomology of $\mathcal{F}$ vanishes for a cover with n opens in it. Then we consider a new cover $U_0, \cdots, U_n$, i.e. a cover with $n+1$ element.

Recall that we have a SES of complexes:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \prod_{|I|=1, 0 \in I} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \prod_{|I|=2, 0 \in I} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] \arrow[d] & \prod_{|I|=1} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \prod_{|I|=2,} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \prod_{|I|=1, 0 \notin I} \mathcal{F}(U_I) \arrow[r] & \prod_{|I|=2, 0 \notin I} \mathcal{F}(U_I) \arrow[r] & \cdots
\end{tikzcd}$$

We can consider the first row as the Cech complex with respect to the cover of U_0 by $U_0 \cap U_j$ for $j \neq 0$, but we append $\mathcal{F}(U_0)$ in the beginning. The second row is the Cech complex with respect to the cover of X by U_j. The last row is the Cech complex of $\mathcal{F}|_{X_i}$ where X_i is the union of U_j for $j \neq 0$. This tells you that the first row is computing Cech cohomology using a cover of n opens, the last row is also computing the Cech cohomology using a cover of n-elements. (though the sheaf and the space the sheaf is on are different.) Let's denote the cover $U_0, \cdots, U_n$ by $\mathfrak{U}$ and the cover $U_1, \cdots, U_n$ by $\mathfrak{U}'$. Finally we denote the cover of U_0 by $U_0 \cap U_1, \cdots, U_0 \cap U_n$ by $\mathfrak{V}$. Recall that the restriction of $\mathcal{F}$ to U_0 and $X' = U_1 \cup \cdots \cup U_n$ are still flasque.

Then we consider the LES induced by taking cohomology:

$$0 \to H^1(X, \mathcal{F})_{\mathfrak{U}} \to H^1(X', \mathcal{F}|_{X'})_{\mathfrak{U}'} \to H^1(U_0, \mathcal{F})_{\mathfrak{V}} \to H^2(X, \mathcal{F})_{\mathfrak{U}} \to H^2(X', \mathcal{F}|_{X'})_{\mathfrak{U}'} \to H^2(U_0, \mathcal{F})_{\mathfrak{V}} \to \cdots$$

Since we know that the Cech cohomology with respect to $\mathfrak{U}'$ and $\mathfrak{V}$ all vanish, the LES shows the assertion. $\square$

Since we have proved that injective $\mathcal{O}_X$-modules are flasque, we are done. Even though this is all the algebro-geometric application of the cohomology of flasque sheaves we will use, the following example shows how general this machinery is:

Example 6.4. Suppose that X is a finite set and it's a finite union of some finite subsets $U_i, 1 \leqslant i \leqslant n$. We define U_I as usual as the intersection of $U_i, i \in I$. Then recall that the famous inclusion-exclusion formula:

$$|X| = \sum |U_i| - \sum_{|I|=2} |U_I| + \sum_{|I|=3} |U_I| - \cdots$$

We can actually prove this using sheaf theory. If we endow X with discrete topology, then let's first show that the constant sheaf $\underline{\mathbb{Q}}$ is flasque. (we don't have to use $\mathbb{Q}$, we can use for example $\mathbb{R}, \mathbb{C}$ instead.)

To see this is flasque, we know that the section of the constant sheaf $\underline{\mathbb{Q}}$ in U_I is $\mathbb{Q}^{\oplus(i \in I)}$, if $U_J \subset I$ we know that the restriction map is:

$$\mathbb{Q}^{\oplus(i \in I)} \to \mathbb{Q}^{\oplus(j \in J)}$$

where if $i \in I$ is not in J, we define the map by project the i-th component to 0. Otherwise we define the map to be the identity $\mathbb{Q} \to \mathbb{Q}$. Then it's easy to see that this is indeed a flasque sheaf. Then we know that we can compute the Cech complex with respect to this cover. Recall that if we have a complex of k-vector spaces:

$$0 \to A^0 \to A^1 \to \cdots \to A^n \to 0 \to \cdots$$

We have:

$$\sum_{i=0}^{n} (-1)^i \dim A^i = \sum_i (-1)^i h^i(A^\bullet)$$

We know that each of the Cech complex is a complex of $\mathbb{Q}$-vector spaces. Then we can apply the above formula. We know that the alternating sum of the dimension of the cohomology is just the dimension of $\Gamma(X, \underline{\mathbb{Q}})$ since the complex is exact after degree 1. We also know the dimension of $\Gamma(X, \underline{\mathbb{Q}})$ is just the cardinality of the set. We are done.

Then we prove that quasicoherent sheaves on an affine scheme doesn't have any derived functor cohomology. Let X be an affine scheme and $\mathcal{F}$ a quasicoherent sheaf on X, we want to prove that $R^i\Gamma(X, \mathcal{F}) = 0$ for all $0 < i \leqslant k$. We will use induction, suppose that $k = 1$ and $\alpha \in R^1\Gamma(X, \mathcal{F})$, we want to show that it's 0. First we choose an injective resolution of $\mathcal{F}$ by $\mathcal{O}_X$-modules (not necessarily quasicoherent):

$$0 \to \mathcal{F} \to \mathcal{Q}_0 \to \mathcal{Q}_1 \to \cdots$$

By definition of how to compute the derived functor cohomolog, we apply $\Gamma(X, -)$ and take cohomology. Then we know that α has a representative α' in $\Gamma(X, \mathcal{Q}_1)$. Since we know that the injective resolution is exact, we know , by working on stalk-level, where have an exact sequence, that for any $p \in X$ by restricting α' to an open neighborhood of p, we can assume that α' comes from a section in $\mathcal{Q}_0$. Let's call this neighborhood V_p, we can also assume that each V_p is affine. Since X is affine, it's quasicompact, therefore we can choose finitely many such V_p to cover X. Rename these as U_i, then we have an affine open cover of X. We will consider the Cech complex of X with respect to this open cover. Then we have a double complex:

$$\begin{tikzcd}
 & \cdots & \cdots & \cdots & \\
0 \arrow[r] & \oplus_i \mathcal{Q}_2(U_i) \arrow[r] \arrow[u] & \oplus_{i,j} \mathcal{Q}_2(U_{ij}) \arrow[r] \arrow[u] & \oplus_{i,j,k} \mathcal{Q}_2(U_{ijk}) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \oplus_i \mathcal{Q}_1(U_i) \arrow[r] \arrow[u] & \oplus_{i,j} \mathcal{Q}_1(U_{ij}) \arrow[r] \arrow[u] & \oplus_{i,j,k} \mathcal{Q}_1(U_{ijk}) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \oplus_i \mathcal{Q}_0(U_i) \arrow[r] \arrow[u] & \oplus_{i,j} \mathcal{Q}_0(U_{ij}) \arrow[r] \arrow[u] & \oplus_{i,j,k} \mathcal{Q}_0(U_{ijk}) \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Again, we compute the spectral sequence in two orientations. First, let's compute it in the rightward direction. As what we computed before, we know that the 1-page is the Cech cohomology of $\mathcal{Q}_i$. By condition 1 we proved, we know that injective objects have no higher Cech cohomology. Therefore we know that the 1-page is again just one vertical comlumn, thus the spectral sequence stabilizes on the 2-th page with the derived functor cohomology of $\mathcal{F}$ appearing on the 0-th column.

Then we compute the upward direction, we know that the 1-st page this time is:

$$\begin{array}{l}
0 \longrightarrow \oplus_i R^2\Gamma(U_i, \mathcal{F}|_{U_i}) \longrightarrow \oplus_{ij} R^2\Gamma(U_{ij}, \mathcal{F}|_{U_{ij}}) \longrightarrow \oplus_{ijk} R^2\Gamma(U_{ijk}, \mathcal{F}|_{U_{ijk}}) \longrightarrow \cdots \\
0 \longrightarrow \oplus_i R^1\Gamma(U_i, \mathcal{F}|_{U_i}) \longrightarrow \oplus_{ij} R^1\Gamma(U_{ij}, \mathcal{F}|_{U_{ij}}) \longrightarrow \oplus_{ijk} R^1\Gamma(U_{ijk}, \mathcal{F}|_{U_{ijk}}) \longrightarrow \cdots \\
0 \longrightarrow \oplus_i \mathcal{F}(U_i) \longrightarrow \oplus_{i,j} \mathcal{F}(U_{ij}) \longrightarrow \oplus_{i,j,k} \mathcal{F}(U_{ijk}) \longrightarrow \cdots
\end{array}$$

the bottom row is just the Cech complex of $\mathcal{F}$. Then let's compute the 2-page, notice that since $\mathcal{F}$ is quasicoherent on X, which is affine, it has no higher Cech cohomology, therefore we know that the term at $(0,1)$-slot is 0. Then we need to figure out what is the kernel of:

$$\oplus_i R^1\Gamma(U_i, \mathcal{F}|_{U_i}) \to \oplus_{ij} R^1\Gamma(U_{ij}, \mathcal{F}|_{U_{ij}})$$

which is $E_2^{0,1}$, which is also the derived functor cohomology as $E_2^{1,0} = 0$. (This is because we know that $\mathcal{Q}_i|_{U_i}$ is still injective, thus can still be used to compute derived functor cohomology.) Moreover after the second page all maps to $E_\bullet^{0,1}$ and from $E_\bullet^{0,1}$ have 0 source and 0 image, therefore we know that $E_\infty^{0,1} = E_2^{0,1}$. Therefore we know that the degree 1 cohomology of the double complex is computed by $E_2^{0,1}$.

Nows, recall that $\alpha \in R^1(X, \mathcal{F})$ has a representative $\alpha' \in \Gamma(X, \mathcal{Q}_1)$. We can first consider it's restriction in $\Gamma(U_i, Q_1)$, recall that the image in each $\Gamma(U_i, Q_1)$ comes from something in $\Gamma(U_i, Q_0)$ by how we choose U_i!

Therefore we know that since $\oplus_i R^1\Gamma(U_i, \mathcal{F}|_{U_i})$ is computed as the homology of:

$$\oplus_i \mathcal{Q}_0(U_i) \to \oplus_i \mathcal{Q}_1(U_i) \to \oplus_i \mathcal{Q}_2(U_i)$$

we know that the image of α' in the homology is 0. Now if we can show that the stuff we get this way is still α, then we proved α is 0.

Recall that we have the following diagram:

$$\begin{tikzcd}
\mathcal{Q}_2(X) \arrow[r, "\mathrm{Res}"] & \oplus_i \mathcal{Q}_2(U_i) \\
\mathcal{Q}_1(X) \arrow[u, "d_{1X}"] \arrow[r, "\mathrm{Res}"] & \oplus_i \mathcal{Q}_1(U_i) \arrow[u, "d_{1U_i}"'] \\
\mathcal{Q}_0(X) \arrow[u] \arrow[r, "\mathrm{Res}"] & \oplus_i \mathcal{Q}_0(U_i) \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\alpha \in R^1\Gamma(X, \mathcal{F}) \arrow[r, "\alpha \mapsto (\alpha_i)"'] & \oplus_i R^1\Gamma(U_i, \mathcal{F}|_{U_i}) \\
\alpha' \in \ker d_{1,X} \arrow[u] \arrow[r, "\mathrm{Res}"] & \oplus_i \ker d_{1U_i} \arrow[u]
\end{tikzcd}$$

We know that the left diagram commutes which implies that the right diagram commutes. In this diagram we know that $R^1\Gamma(X, \mathcal{F}) \to \oplus_i R^1\Gamma(U_i, \mathcal{F}|_{U_i})$ is induced by restriction from X to U_i.

In the above discussion we proved that α is mapped to (α_i) which is 0. But we don't know if $R^1\Gamma(X, \mathcal{F}) \to \oplus_i R^1\Gamma(U_i, \mathcal{F}|_{U_i})$ is injective or not. What we have from the spectral sequence machine is an isomorphism of the kernel of $\oplus_i R^1\Gamma(U_i, \mathcal{F}|_{U_i}) \to \oplus_i R^1\Gamma(U_i, \mathcal{F}|_{U_i})$ with the cohomology of the total complex, which is $R^1\Gamma(X, \mathcal{F})$ as shown in the following diagram:

$$\ker \longrightarrow \oplus_i R^1\Gamma(U_i, \mathcal{F}|_{U_i}) \longrightarrow \oplus_{ij} R^1\Gamma(U_{ij}, \mathcal{F}|_{U_{ij}})$$

But what we really want is an isomorphism like the following:

$$\begin{tikzcd}
\ker \arrow[dr] & & \\
& \oplus_i R^1\Gamma(U_i, \mathcal{F}|_{U_i}) \arrow[r] & \oplus_{ij} R^1\Gamma(U_{ij}, \mathcal{F}|_{U_{ij}}) \\
R^1\Gamma(X, \mathcal{F}) \arrow[uu, dashed, "\simeq"] \arrow[ur, "\alpha \mapsto (\alpha_i)"'] & &
\end{tikzcd}$$

To establish such an isomorphism, recall that in the upward orientation, we have a SES:

$$0 \to E_2^{1,0} = 0 \hookrightarrow H^1(T^\bullet) \to E_2^{0,2} \to 0$$

where we use $T^\bullet$ to indicate the total complex. We will need more information on what is the map $H^1(T^\bullet) \to E_2^{0,2}$. And we will do this in the section of the formalism of spectral sequences. If this is the first time you read the spectral sequence, you should most definitely skip this part as the above discussion is convincing enough on a first reading. **** to be completed later*

Then after we finish with the degree 1 case, we can do induction. Suppose that we have proved that quasicoherent sheaves on affine scheme have no derived functor cohomology of degree $1 \leqslant i \leqslant k$ and we want to prove for $k+1$. You find that exactly the same argument above works as we know that in the first page, if you consider the $k+1$-th antidiagonal, the first row is again the Cech complex of $\mathcal{F}$, therefore on the 2-page it vanishes as quasicoherent sheaves on an affine scheme have no higher Cech cohomology. Then going up in the antidiagonal, we know that by assumption, all the terms in the 1-st pages vanishes being higher derived functor cohomology. Therefore, the only term of interest is $\oplus_i R^{k+1}\Gamma(U_i, \mathcal{F}|_{U_i})$, then you can use the same argument as above.

We tie up some loose ends before we head to the formalism of spectral sequences:

First, we prove that the theorem also holds for higher pushforward $R^i\pi_*$:

Theorem 6.5. Suppose that $\pi : X \to Y$ is a quasicompact separated morphism of scheme and $\mathcal{F}$ is a quasicoherent sheaf on X, then there is an isomorphism between the derived functor higher pushforward and the Cech cohomology version higher pushforward we defined before.

Recall that the way we use Cech cohomology to define the higher pushforward is that for each affine open V on Y, we know that it's inverse image is quasicompact and separated over V, then we can define the Cech cohomology of the sheaf $\mathcal{F}$ restricted to the inverse image, and we define the sections of higher pushforward on V is this Cech cohomology group. The above discussion can be used to show that locally on every affine open of Y, the derived functor higher pushforward has isomorphic sections to the Cech cohomology higher pushforwards. Then we need to prove that this isomorphism is compatible with restricting to distinguished opens, if this holds then we can prove that they are isomorphic globally. In particular, the derived higher pushforward is also quasicoherent.

To prove this, recall that we again first choose an injective resolution of $\mathcal{F}$ using $\mathcal{O}_X$-modules:

$$0 \to \mathcal{F} \to \mathcal{Q}_0 \to \mathcal{Q}_1 \to \cdots$$

Then we apply the pushforward functor and take cohomology:

$$0 \to \pi_*\mathcal{F} \to \pi_*\mathcal{Q}_0 \to \pi_*\mathcal{Q}_1 \to \cdots$$

Then we know that this $R^i\pi_*\mathcal{F}$ will be the associated sheaf of the presheaf computed by on each open V, the section is given by the cohomology of:

$$\mathcal{Q}_{i-1}(\pi^{-1}V) \to \mathcal{Q}_i(\pi^{-1}V) \to \mathcal{Q}_{i+1}(\pi^{-1}V)$$

Now if V is an affine open, we choose a finite affine open cover U_i of $\pi^{-1}V$ and we will have a double complex that looks like the one we used to show that Cech cohomology agrees with derived functor cohomology:

$$\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \oplus_i \mathcal{Q}_2(U_i) & \longrightarrow & \oplus_{i,j} \mathcal{Q}_2(U_{ij}) & \longrightarrow & \oplus_{i,j,k} \mathcal{Q}_2(U_{ijk}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \oplus_i \mathcal{Q}_1(U_i) & \longrightarrow & \oplus_{i,j} \mathcal{Q}_1(U_{ij}) & \longrightarrow & \oplus_{i,j,k} \mathcal{Q}_1(U_{ijk}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \oplus_i \mathcal{Q}_0(U_i) & \longrightarrow & \oplus_{i,j} \mathcal{Q}_0(U_{ij}) & \longrightarrow & \oplus_{i,j,k} \mathcal{Q}_0(U_{ijk}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We know that using this double complex we get the following isomorphism, where we use $T_V^\bullet$ to indicate the total complex of the above double complex given by V:

$$\frac{\ker(\mathcal{Q}_i(\pi^{-1}V) \to \mathcal{Q}_{i+1}(\pi^{-1}V))}{\operatorname{Im}(\mathcal{Q}_{i-1}(\pi^{-1}V) \to \mathcal{Q}_i(\pi^{-1}V))} \simeq H^i(T_V^\bullet) \simeq H^i(\pi^{-1}V, \mathcal{F}|_{\pi^{-1}V})$$

Now if $W \subset V$ is a distinguished open, we know that we will have a finite affine open cover of $\pi^{-1}W$ by $W_i = U_i \cap \pi^{-1}W$, then we will have a morphism of double complex from the double complex above to the following double complex and the morphism is induced just by restriction:

$$\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \oplus_i \mathcal{Q}_2(W_i) & \longrightarrow & \oplus_{i,j} \mathcal{Q}_2(W_{ij}) & \longrightarrow & \oplus_{i,j,k} \mathcal{Q}_2(W_{ijk}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \oplus_i \mathcal{Q}_1(W_i) & \longrightarrow & \oplus_{i,j} \mathcal{Q}_1(W_{ij}) & \longrightarrow & \oplus_{i,j,k} \mathcal{Q}_1(W_{ijk}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \oplus_i \mathcal{Q}_0(W_i) & \longrightarrow & \oplus_{i,j} \mathcal{Q}_0(W_{ij}) & \longrightarrow & \oplus_{i,j,k} \mathcal{Q}_0(W_{ijk}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Recall that we had a remark in the introduction section of spectral sequence that spectral sequence is functorial in the first E_0 page, which means that the above morphism of double complexes will give us the following commutative diagram:

$$\begin{tikzcd}
\frac{\ker(\mathcal{Q}_i(\pi^{-1}V)\to\mathcal{Q}_{i+1}(\pi^{-1}V))}{\operatorname{Im}(\mathcal{Q}_{i-1}(\pi^{-1}V)\to\mathcal{Q}_i(\pi^{-1}V))} \arrow[r, leftrightarrow, "\simeq"] \arrow[d, "\text{res}"'] & H^i(T_V^\bullet) \arrow[r, leftrightarrow, "\simeq"] \arrow[d, "\text{res}"] & H^i(\pi^{-1}V, \mathcal{F}|_{\pi^{-1}V}) \arrow[d, "\text{res}"] \\
\frac{\ker(\mathcal{Q}_i(\pi^{-1}W)\to\mathcal{Q}_{i+1}(\pi^{-1}W))}{\operatorname{Im}(\mathcal{Q}_{i-1}(\pi^{-1}W)\to\mathcal{Q}_i(\pi^{-1}W))} \arrow[r, leftrightarrow, "\simeq"'] & H^i(T_W^\bullet) \arrow[r, leftrightarrow, "\simeq"'] & H^i(\pi^{-1}W, \mathcal{F}|_{\pi^{-1}W})
\end{tikzcd}$$

Notice that the right vertical morphism is the morphism we had for the Cech cohomology higher pushforwards, thus is isomorphic to the localization map. Therefore the left vertical morphism is also a localization map. This shows that the presheaf of the derived functor higher pushforward is already a sheaf on the distinguished basis which agrees with the data we used to define the quasicoherent Cech cohomology higher pushforward. Then after we do the sheafification, the two sheaves are isomorphic.

You will see why the spectral sequence is isomorphic in the first page in the detailed construction next section.

Remark 6.6. Using the same functorial argument, we can prove that the isomorphism we gave between:

$$H^i(X,\mathcal{F}) \to R^i\Gamma(X,\mathcal{F})$$

is also functorial in the sense that if we have a morphism $\mathcal{F} \to \mathcal{G}$ of quasicoherent sheaves on a quasicompact and separated scheme, then we have the following commutative diagram:

$$\begin{tikzcd} \mathrm{H}^i(X,\mathcal{F}) \arrow[r, leftrightarrow, "\simeq"] \arrow[d] & \mathrm{R}^i\Gamma(X,\mathcal{F}) \arrow[d] \\ \mathrm{H}^i(X,\mathcal{G}) \arrow[r, leftrightarrow, "\simeq"] & \mathrm{R}^i\Gamma(X,\mathcal{G}) \end{tikzcd}$$

where the vertical maps comes from the functoriality of $\mathrm{H}^i(X,-)$ and $\mathrm{R}^i\Gamma(X,-)$.

Finally, if we have a SES of quasicoherent sheaves:

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

We know that both the derived functor cohomology and the Cech cohomology gives LES, I claim that the ismorphism we constructed between then commutes with the connecting homomorphisms. This is also just an application of the functoriality of spectral sequences in the E_0-page. Recall that the connecting homomorphism can also be defined using spectral sequences and the morphisms on the second page, notice that we can choose injective resolutions for $\mathcal{F},\mathcal{H}$ and we know that by the horse show construction we have a SES of complexes:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{Q}_0 \arrow[r] \arrow[d] & \mathcal{Q}_1 \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & \mathcal{G} \arrow[r] \arrow[d] & \mathcal{R}_0 \arrow[r] \arrow[d] & \mathcal{R}_1 \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & \mathcal{H} \arrow[r] \arrow[d] & \mathcal{S}_0 \arrow[r] \arrow[d] & \mathcal{S}_1 \arrow[r] \arrow[d] & \cdots \\ & 0 & 0 & 0 & \end{tikzcd}$$

We know that since all $\mathcal{Q}_i,\mathcal{R}_i,\mathcal{S}_i$ are injective, we know that each vertical column is split exact, therefore we know that after we apply an additive functor, it remains exact. Therefore after we take the global section it is exact. We know that the connecting homomorphism is just given by considering the morphism in the second page of the spectral sequence of the SES of complexes after we take global sections.

Now choose a finite affine open cover of X, then we get morphisms between three double complexes that looks like:

$$\begin{tikzcd} & \cdots & \cdots & \cdots & \\ 0 \arrow[r] & \oplus_i \mathcal{Q}_2(U_i) \arrow[r] \arrow[u] & \oplus_{i,j} \mathcal{Q}_2(U_{ij}) \arrow[r] \arrow[u] & \oplus_{i,j,k} \mathcal{Q}_2(U_{ijk}) \arrow[r] \arrow[u] & \cdots \\ 0 \arrow[r] & \oplus_i \mathcal{Q}_1(U_i) \arrow[r] \arrow[u] & \oplus_{i,j} \mathcal{Q}_1(U_{ij}) \arrow[r] \arrow[u] & \oplus_{i,j,k} \mathcal{Q}_1(U_{ijk}) \arrow[r] \arrow[u] & \cdots \\ 0 \arrow[r] & \oplus_i \mathcal{Q}_0(U_i) \arrow[r] \arrow[u] & \oplus_{i,j} \mathcal{Q}_0(U_{ij}) \arrow[r] \arrow[u] & \oplus_{i,j,k} \mathcal{Q}_0(U_{ijk}) \arrow[r] \arrow[u] & \cdots \\ & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & \end{tikzcd}$$

With $\mathcal{Q}$ replaced by $\mathcal{R}, \mathcal{S}$. I claim that we can choose these U_i such that we get a SES of double complexes. Again, since we know that after we restrict injective objects to open, it remains injective, then we know that the map $\Gamma(U_I, \mathcal{R}_i) \to \Gamma(U_I, \mathcal{S}_i)$ remains surjective. Therefore we get a SES of double complexes, thus a SES of total complexes of these double complexes. You will later see in next section that in some special cases where we have a lot of vanishing terms, the connecting homomorphisms of the cohomology of the total complex commute with the connecting homomorphisms of the derived functors will be the same (as we will need more information on the filtration of the cohomology of the double complex), therefore we have:

$$\begin{tikzcd}
\mathrm{R}^{k+1}\Gamma(X,\mathcal{F}) \arrow[r, "\simeq"'] & H^{k+1}(T_\bullet) & H^{k+1}(X,\mathcal{F}) \arrow[l, "\simeq"'] \\
\mathrm{R}^{k}\Gamma(X,\mathcal{F}) \arrow[r, "\simeq"] \arrow[u, "\delta_R^k"'] & H^{k}(T_\bullet) \arrow[u, "\delta_T^k"] & H^{k}(X,\mathcal{F}) \arrow[l, "\simeq"'] \arrow[u, "\delta_C^k"']
\end{tikzcd}$$

The left and right squares commute, thus the rectangle commute.

Remark 6.7. Before we end this section, let's make a side remark on a minor question you might had questions on.

Recall that we defined the derived functor in the category of $\mathcal{O}_X$-modules, even when we work with quasicoherent sheaves. A natural question could be if we are only interested in the derived functor cohomology of quasicoherent sheaves, can we shrink the category to the abelian category of quasicoherent sheaves?

There are several reasons we don't use the category of quasicoherent sheaves. First, even if it's not hard to show that if we work with a scheme and the scheme is Noetherian, then QCoh_X has enough injective objects. However, there is no guarantee that the extension by zero functor still maps quasicoherent sheaves to quasicoherent sheaves. Therefore, we don't have enough tools to show that injective quasicoherent sheaves restrict to injective quasicoherent sheaves on a smaller open. (There is no guarantee that an injective object in the category of quasicoherent sheaves is still injective in the category of all $\mathcal{O}_X$-modules, which is much larger. For the same reason, it's hard to show that injective quasicoherent sheaves are flasqe.)

It's true that QCoh_X has enough injectives for any scheme X, but this is much harder to show. Therefore even you can define the derived functors, due to the lack of tools it's still hard to prove properties of injective quasicoherent sheaves that we need. Therefore, if we started with this, it will be much harder to show that the derived functor cohomology agrees with Cech cohomology in good situations.

In later sections, you will see that if we really need to compute any cohomology, without the separated assumption and Cech cohomology, it's really hard to compute anything using this derived functor definition. Rather, this derived functor cohomology definition is more of theoretic purposes which makes a lot of proof easier. If we want to do concrete computation, there is almost no other way but to retreat to the Cech cohomology case.

7 The formalism of spectral sequences

In this section, we will introduce how spectral sequences are formally defined and constructed, especially how the filtration of the cohomology of the total complex is done. To avoid distraction, we will just work in the category of A-modules, as situations are very similar in other abelian categories.

**** to be completed later*

References