Fourier–Mukai Transforms Notes

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The second chapter is called Derived categories, which is divided into three parts, corresponding to the three sections of this notes: derived category of an abelian category, derived functors and spectral sequences that appears when you try to compose two derived functors.

1 Derived Categories of Coherent Sheaves

The third part of the notes is about the basics of derived category of coherent sheaves. We will basically apply what we have developed in the previous two chapters to study a scheme or a smooth projective variety.

1.1 Basic Structures

Definition 1.1. Let X be a scheme. It's derived category $D^b(X)$ is by definition the bounded derived category of the abelian category $\mathbf{Coh}(X)$, i.e.:

$$D^b(X) := D^b(Coh(X))$$

Definition 1.2. Two schemes X, Y defined over a field k are called derived equivalent or simply D-equivalent if there is a k-linear exact equivalence $D^b(X) \simeq D^b(Y)$.

Similarly, one can introduce categories like $D(X), D^+(X)$ or $D^-(X)$, but as you will see later, the hero of the story is going to be the bounded derived category. Unfortunately, the underlying abelian category $Coh(X)$ usually doesn't contain non-trivial coherent injective objects, thus it's unavoidable to consider a larger abelian category, which is $Qcoh(X)$, the abelian category of quasi-coherent sheaves of X, with it's derived category $D^*(Qcoh(X))$, or sometimes, we even have to consider the category of $\mathscr{O}_X$-modules $Sh_{\mathscr{O}_X}(X)$.

Whenever the scheme is defined over a field k, the derived category is tacitly considered as a k-linear category.

As for the matter of notation, in order to avoid any possible confusion between the sheaf cohomology $H^i(X, \mathscr{F})$ and the cohomology of a complex of sheaves $H^i(\mathscr{F}^\bullet)$, we will use $\mathcal{H}^i(\mathscr{F}^\bullet)$ for the later.

We will also quote a lot of basic results about sheaves and other stuff without proof, you can basically find them in almost any standard book of algebraic geometry, so we won't bother to explain them here.

The first important we aim to prove is that the derived category of a noetherian scheme is indecomposable iff it's connected. In order to prove it, we need the following results:

Proposition 1.3. On a noetherian scheme X, any quasi-coherent sheaf $\mathscr{F}$ admites a resolution by quasi-coherent sheaves $\mathscr{I}^i$ which are injective as $\mathscr{O}_X$-modules.

Proof. *** omitted. □

Remark 1.4. Due to the above proposition, for example, if we are studying smooth projective varieties, we can quote it.

Recall that in the previous chapter we prove that if $\mathscr{A} \subset \mathscr{B}$ is a thick subcategory and any object of $\mathscr{A}$ can be embedded into an injective object of $\mathscr{B}$, then there is an equivalence via the inclusion functor $D^*(\mathscr{A}) \hookrightarrow D^*(\mathscr{B})$:

$$D^+(\mathscr{A}) \simeq D^+_{\mathscr{A}}(\mathscr{B})$$

or:

$$D^b(\mathscr{A}) \simeq D^b_{\mathscr{A}}(\mathscr{B})$$

Recall that the quasi-coherent sheaves form a thick subcategory of the category of $\mathscr{O}_X$-modules, thus $D^b(Qcoh(X))$ can either be viewed as the bounded derived category of $Qcoh(X)$ or as a full triangulated subcategory of $D^b(Sh_{\mathscr{O}_X}(X))$ whose cohomology are quasi-coherent sheaves(this also works for bounded below derived categories). This is the following corollary:

Corollary 1.5. For any Noetherian scheme X, there are natural equivalences:

$$D^*(Qcoh(X)) \simeq D^*_{Qcoh(X)}(\mathscr{O}_X(X))$$

with $* = b, +$.

The passage from quasi-coherent to coherent world is a bit trickier:

Proposition 1.6. Let X be a Noetherian scheme, then the natural functor:

$$D^b(X) \to D^b(Qcoh(X))$$

defines an equivalence between the derived category of X and the full triangulated subcategory of $D^b(Qcoh(X))$ which consists of bounded complexes of quasi-coherent sheaves with coherent cohomology.

In order to prove it, we need the following lemma:

Lemma 1.7. If $\mathscr{G} \to \mathscr{F}$ is a surjective $\mathscr{O}_X$-module morphism from a quasi-coherent sheaf $\mathscr{G}$ to a coherent sheaf $\mathscr{F}$ on a Noetherian scheme X, then there exists a coherent subsheaf $\mathscr{G}' \subset \mathscr{G}$ such that the composition:

$$\mathscr{G}' \subset \mathscr{G} \to \mathscr{F}$$

is still surjective.

Proof. This statment is clearly true for modules since if you have a module morphism $M \to N$ which is surjective and N is finitely generated, then notice that pick the submodule of M that is generated by the preimage of the generators of N could work. Then we know that the statement in the lemma at least holds true locally. Then the lemma boils down to the well-known result for Noetherian schemes:

Let $U \subset X$ be an open subscheme of a Noetherian scheme ane let $\mathscr{F} \subset \mathscr{G}|_U$ be a coherent subscheme of the the restriction of a quasi-coherent sheaf $\mathscr{G}$ on X, then there is a coherent subsheaf $\mathscr{F}' \subset \mathscr{G}$ such that $\mathscr{F}'|_U = \mathscr{F}$. Then use the fact that finite sums of coherent sheaves are still coherent (not direct sums! since we still want the result to be a subscheme of $\mathscr{G}$). □

Then we can prove the above proposition:

Proof. Let $\mathscr{G}^\bullet$ be a bounded complex of quasi-coherent sheaves:

$$\cdots \to 0 \to \mathscr{G}^n \to \cdots \to G^m \to 0 \to \cdots$$

with coherent complex cohomology $\mathcal{H}^i$. Since the complex is bounded, we may assume that for $i > j$, all $\mathscr{G}^i$ is coherent (in particular, the zero sheaf in coherent), then you can apply the above proposition to the following surjections:

$$d^j : \mathscr{G}^j \to Im(d^j) \subset \mathscr{G}^{j+1} \qquad Ker(d^j) \to \mathcal{H}^j$$

notice that since we are working with Noetherian schemes, $Im(d^j)$ as a subsheaf of a coherent sheaf $\mathscr{G}^{j+1}$, is coherent, and by assumption $\mathcal{H}^j$ is also coherent.

Then we will let $\mathscr{G}_1^j$ and $\mathscr{G}_2^j$ be coherent subsheaves of $\mathscr{G}^j$ and $Ker(d^j)$ respectively such that the composition is still surjective, then we replace $\mathscr{G}^j$ by the submodule generated by the sum of $\mathscr{G}_1^j$ and $\mathscr{G}_2^j$, which give us the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathscr{G}^{j-1} \arrow[r] & \mathscr{G}^j \arrow[r, "d^j"] & \mathscr{G}^{j+1} \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathscr{G}^{j-1'} \arrow[r] \arrow[u, hook] & \mathscr{G}_1^j + \mathscr{G}_2^j \arrow[r] \arrow[u, hook] & \mathscr{G}^{j+1} \arrow[r] \arrow[u, "="] & \cdots
\end{tikzcd}$$

where $\mathscr{G}^{j-1'}$ is the preimage of $\mathscr{G}^{j-1} \to \mathscr{G}_1^j + \mathscr{G}_2^j$, which is still quasi-coherent. We claim that in this diagram, the morphism $\mathscr{G}_1^j + \mathscr{G}_2^j \to \mathscr{G}^j$ gives you an isomorphism after you apply cohomology, since the morphism is just inclusion, we only have to prove that both of them have the same cohomology. Since the kernel of $\mathscr{G}_1^j + \mathscr{G}_2^j \to \mathscr{G}^{j+1}$ is still the kernel of $\mathscr{G}^j \to \mathscr{G}^{j-1}$, we only have to care about the image, which I will left for you to do.

Then notice that the natural functor $D^b(\mathscr{B}) \to D^b(QCoh(X))$ is obviously fully-faithful, then we just proved that it's essential to the full-triangulated subcategory $D^b_{Coh(X)}(Qcoh(X))$ □

Remark 1.8. This remark is mainly concerned with when the hom set $Hom_{D^b(X)}(\mathscr{E}, \mathscr{F})$ is a finite dimensional vector space over k, which makes our discussion of the Serre functor later much easier:

1. Suppose that we work with a Noetherian scheme X, then we already know that $Qcoh(X)$ has enough injective, thus it makes sense to define $RHom^\bullet(\mathscr{E}^\bullet, \mathscr{F}^\bullet)$ for all $\mathscr{E}^\bullet \in D(Coh(X))$ and $\mathscr{F}^\bullet$ in $D^+(Coh(X))$, thus in particular, this works for $\mathscr{E}, \mathscr{F}$ in $D^b(X)$. Then we know that the i-th cohomology of this complex:
$$Ext^i(\mathscr{E}^\bullet, \mathscr{F}^\bullet) = Hom_{D^b(X)}(\mathscr{E}, \mathscr{F}^\bullet[i])$$
Therefore, if we can prove that $Ext^i(\mathscr{E}^\bullet, \mathscr{F}^\bullet)$ is finite dimensional over k, we are done.

2. To prove the above assertion, we will use spectral sequences **** later*

Thus, in the sequel, we will just assume that $Hom(\mathscr{E}^\bullet, \mathscr{F}^\bullet)$ is finite dimensional for each $\mathscr{E}^\bullet, \mathscr{F}^\bullet$ in $D^b(X)$.

Definition 1.9. The support of a complex $\mathscr{F}^\bullet \in D^b(X)$ is the union of the supports of all it's cohomology sheaves, i.e. the closed subset:
$$supp(\mathscr{F}^\bullet) := \bigcup supp(\mathcal{H}^i(\mathscr{F}^\bullet))$$

Lemma 1.10. Suppose that $\mathscr{F}^\bullet \in D^b(X)$ and $supp(\mathscr{F}^\bullet) = Z_1 \cup Z_2$ for two disjoint closed subsets Z_1, Z_2, then $\mathscr{F}^\bullet \simeq \mathscr{F}_1^\bullet \oplus \mathscr{F}_2^\bullet$ with $supp(\mathscr{F}_i^\bullet) \subset Z_i$ for $i = 1, 2$.

Proof. Notice that when the length of the complex is just 0, which means that there is only one coherent sheaf in the complex, then it's a well-know result in algebraic geometry. The general time will be proved by induction.

Let $\mathscr{F}^\bullet$ be a complex of at least 1, suppose that m is minimal with $0 \neq \mathcal{H}^m(\mathscr{F}^\bullet) := \mathcal{H}$. Notice that this $\mathcal{H}$ can be decomposed to a direct sum $\mathcal{H} \sim \mathcal{H}_1 \oplus \mathcal{H}_2$ with $supp(\mathcal{H}_i) \subset Z_i$ since the support of $\mathcal{H}$ is still a disjoint union of clossed subsets contained in Z_i respectively.

Then recall that we have a distinguished triangle:
$$\mathcal{H}[-m] \to \mathscr{F}^\bullet \to \mathscr{G}^\bullet \to \mathcal{H}[1-m]$$
and the first arrow induces identity on the m-th cohomology.

The LES induced by the above LES also tells us that $\mathcal{H}^q(\mathscr{G}^\bullet) = \mathcal{H}^q(\mathscr{F}^\bullet)$ for $q > m$ and $\mathcal{H}^q(\mathscr{G}^\bullet) = 0$ for $q \leqslant m$. Then we know that in the derived category, $\mathscr{G}$ is isomorphic to a complex that is really bounded by m, then we reduced the length of the complex by 1, thus the induction hypothesis comes in and allows us to decompose $\mathscr{G}^\bullet$ into $\mathscr{G}_1^\bullet \oplus \mathscr{G}_2^\bullet$ with support contained in Z_i respectively.

Then we claim that $Hom(\mathscr{G}_1^\bullet, \mathcal{H}_2[1-m]) = Hom(\mathscr{G}_2^\bullet, \mathcal{H}_1[1-m]) = 0$. We only prove it for the first one since the proof of the second one is similar. To do this, we need to use the spectral sequence we introduced in the spection Introduction to Spectral Sequences:
$$E_2^{p,q} = Hom_{D^b(X)}(\mathcal{H}^{-q}(\mathcal{G}_1^\bullet), \mathcal{H}_2[p]) \Rightarrow_p Hom(\mathcal{G}_1^\bullet, \mathcal{H}_2[p+q])$$
If we can prove the abutment for every p, q is 0, then $1-m$ is just a special case. To do this, we would like to prove that $E_2^{p,q}$ is 0. Therefore, we have simplified this question from proving the morphism between to complexes is 0 to the morphism between two objects is 0 in the derived category.

Notice that we have the identification:
$$Ext^p(\mathcal{H}^{-q}(\mathcal{G}_1^\bullet), \mathcal{H}_2) = Hom_{D^b(X)}(\mathcal{H}^{-q}(\mathcal{G}_1^\bullet), \mathcal{H}_2[p])$$

Thus, we now only have to prove that every Ext group of $\mathcal{H}^{-q}(\mathcal{G}_1^\bullet), \mathcal{H}_2$ is 0. Recall that we know that $\mathcal{H}^{-q}(\mathcal{G}_1^\bullet), \mathcal{H}_2$ are of disjoint supports, thus if we can reduce the problem to the stalk of these two sheaves, it's going to be easier, thus we can use the local to global spectral sequence which we also introduced in that section:

$$E^{p,q} = H^p(X, \mathcal{E}xt^q(\mathcal{H}^{-q}(\mathcal{G}_1^\bullet), \mathcal{H}_2)) \Rightarrow_p Ext^{p+q}(\mathcal{H}^{-q}(\mathcal{G}_1^\bullet), \mathcal{H}_2)$$

Recall that for the local $\mathcal{E}xt$ group, stalkification functor behaves well in the sense that we always have:

$$\mathcal{E}xt^p(\mathcal{E}, \mathcal{F})_x \simeq Ext^p_{\mathcal{O}_{X,x}}(\mathcal{E}_x, \mathcal{F}_x)$$

Then since $\mathcal{H}^{-q}(\mathcal{G}_1^\bullet), \mathcal{H}_2$ are of disjoint support, then for any $x \in X$, one of their stalk is going to be 0, thus each $E_2^{p,q} = 0$, then we are done.

Then let's choose $\mathscr{F}_j^\bullet$ such that the following sequences are distinguished triangles:

$$\mathscr{F}_j^\bullet \longrightarrow \mathscr{G}_j^\bullet \longrightarrow \mathcal{H}_j[1-m] \longrightarrow \mathscr{F}_j^\bullet[1]$$

where the morphism $\mathscr{G}_j^\bullet \to \mathcal{H}_j[1-m]$ comes from :

$$\begin{tikzcd}
\mathscr{G}^\bullet \arrow[r] \arrow[d, "iso"] & \mathcal{H}[1-m] \arrow[d, "iso"'] \\
\mathscr{G}^\bullet \oplus \mathscr{G}^\bullet \arrow[r, dashed] & \mathcal{H}_1[1-m] \oplus \mathcal{H}_2[1-m]
\end{tikzcd}$$

We claim that this suffice to show that $\mathscr{F}^\bullet \simeq \mathscr{F}_1^\bullet \oplus \mathscr{F}_2^\bullet$ and the support of each $\mathscr{F}_i^\bullet$ is contained in Z_i.

To see the assertion about the support is true, use the inudced LES, which I will left to you. To see that $\mathscr{F}^\bullet \simeq \mathscr{F}_1^\bullet \oplus \mathscr{F}_2^\bullet$, consider the following diagram:

$$\begin{tikzcd}
\mathcal{H}[-m] \arrow[r] & \mathscr{F}^\bullet \arrow[r] & \mathscr{G}^\bullet \arrow[r] \arrow[d, "iso"] & \mathcal{H}[1-m] \arrow[d, "iso"'] \\
\mathcal{H}_1[-m] \oplus \mathcal{H}_2[-m] \arrow[r] & \mathscr{F}^\bullet \oplus \mathscr{F}^\bullet \arrow[r] & \mathscr{G}^\bullet \oplus \mathscr{G}^\bullet \arrow[r] & \mathcal{H}_1[1-m] \oplus \mathcal{H}_2[1-m]
\end{tikzcd}$$

where the bottom is the direct sum of distinguished triangles thus still distinguished. Then the fact that $Hom(\mathscr{G}_1^\bullet, \mathcal{H}_2[1-m]) = Hom(\mathscr{G}_2^\bullet, \mathcal{H}_1[1-m]) = 0$ tells you that the last square is indeed commutative, then we can complete this diagram, such that it's a diagram between two distinguished triangles, and two arrows in the morphism are isomorphisms, thus the last one is also an isomorphism, we are done. ☐

Now we can prove our promise earlier about the structure of the derived category, which need the above lemma:

Proposition 1.11. Let X be a Noetherian scheme and let $D^b(X)$ be the bounded derived category of coherent sheaves, then $D^b(X)$ is indecomposable iff X is connected.

Proof. We already explained in chapter 1 what is meant by decomposing a triangulated category.

If X is not connected, we will prove that $D^b(X)$ is decomposable. Suppose that $X = X_1 \coprod X_2$, which are two open and closed subsets of X. Consider X_1 and X_2 first as open subschemes of X, we claim that $D^b(X)$ is decomposed into $D^b(X_1)$ and $D^b(X_2)$.

Notice that by the above lemma, for any $\mathscr{F}^\bullet \in D^b(X)$, we can decompose $\mathscr{F}^\bullet \simeq \mathscr{F}_1^\bullet \oplus \mathscr{F}_2^\bullet$ with the support in X_1 and X_2 respectively. We claim that $\mathscr{F}_i^\bullet$ is in $D^b(X_i)$ for $i = 1, 2$. This is because that the cohomology of the complex has support in X_i, then can be viewed as a coherent sheaf on X_1 and X_2 respectively, then since the subcategory $Coh(X_i)$ of $Coh(X)$ are thick, then we know that $D^b(X_i)$ is equivalent to $D^b_{Coh(X_i)}(X)$.

Recall the definition of being decomposable: first, since X_1 and X_2 are not empty, then there is obviously objects that is not isomorphic to 0. Then I will let you to figure out why for any $\mathscr{F}^\bullet \in D^b(X)$, $\mathscr{F}$ is isomorphic to $\mathscr{F}_1^\bullet \oplus \mathscr{F}_2^\bullet$ with $\mathscr{F}_i^\bullet \in D^b(X_i)$ satisfies condition 2 in the definition. (Hint:direct sums of distinguished triangles are still distinguished). Finally, since the support are disjoint, then the only possible morphism between them are 0's.

Now conversely, we will suppose that X is connected, but $D^b(X)$ is not decomposable, which gives you a contradiction. Well, suppose that $D^b(X)$ is decomposed by D_1 and D_2, then according to Grothendieck's philosophy, without any other information, one of the two canonical line bundles available to us is the structure sheaf $\mathscr{O}_X$, thus we can decompose:

$$\mathscr{O}_X \simeq \mathscr{F}_1^\bullet \oplus \mathscr{F}_2^\bullet$$

with $\mathscr{F}_i^\bullet$ in D_i. Also, since the cohomology of $\mathscr{F}_1^\bullet \oplus \mathscr{F}_2^\bullet$ is concentrated in degree 0, we can assume that $\mathscr{F}_1^\bullet$ and $\mathscr{F}_2^\bullet$ are just coherent sheaves, not complexes. Then we can just assume that they are ideal sheaves $\mathscr{I}_i$ of $\mathscr{O}_X$, thus corresponding to closed subschemes X_i of X. Notice that $\mathscr{O}_X = \mathscr{I}_1 + \mathscr{I}_2 \subset \mathscr{I}_{X_1 \cap X_2}$, thus $X_1 \cap X_2 = \varnothing$, and $\mathscr{I}_{X_1 \cup X_2} \subset \mathscr{I}_{X_1} \cap \mathscr{I}_{X_2} = 0$, thus $X_1 \cup X_2 = X$, thus since X is connected, we know that either X_1 or X_2 is empty. Thus we know that $\mathscr{O}_X$ is either in D_1 or D_2, since you will find that they are symmetric, let's assume that X is in D_1.

Recall that any Noetherian scheme has a closed point, then notice that the decomposition of $k(x)$ with respect to D_1 and D_2 must be trivial, or otherwise it's the direct sum of two coherent sheaves, which contradicts to it's being a closed point. Then, $k(x)$ is either in D_1 or D_2, but notice that we have a nontrivial sheaf morphism $\mathscr{O}_X \to k(x)$, thus $k(x)$ is in D_1, thus for any closed point, it's corresponding object in the derived category is in D_1.

Remember that since we assumed the decomposition, then there has to be something nontrivial in D_2, say $\mathscr{F}^\bullet$, then we will construct a nontrivial morphism from $\mathscr{F}^\bullet$ to $k(x)[m]$ for some m, which is a contradiction. To do this, suppose that m is minimal with $\mathcal{H}^m := \mathcal{H}^m(\mathcal{F}^\bullet) \neq 0$, i.e. m is the minimal number such that the cohomology doesn't vanish. Then since $\mathcal{H}^m$ doesn't vanish, there is a closed point in the support, thus in particular, there is a surjection $\mathcal{H}^m \to k(x)$, which we will use to construct a nontrivial morphism. To do this, notice that we have a quasi-isomorphism:

$$(\cdots \to \mathscr{F}^{m-1} \to Kerd^m \to 0 \to \cdots) \to \mathscr{F}^\bullet$$

then notice that we can compose it's inverse in $D^b(X)$ with:

$$(\cdots \to \mathscr{F}^{m-1} \to Kerd^m \to 0 \to \cdots) \to\to H^m[-m] \to k(x)[-m]$$

the result is still nontrivial since we know that after we take cohomology, the first morphism just gives you the identity at m-th slot, and the surjection is not trivial.

Therefore, we get the contradiction that there is a nontrivial morphism from D_1 to D_2. $\square$

Now let X be a smooth projective variety over a field k and let ω_X be it's canonical bundle, often, ω_X is also called the dualizing sheaf of X, which will be explained shortly, and it's related to the Serre functor on $D^b(X)$:

First, recall that tensor with a finite rank locally free sheaf $\mathscr{E}$ of X is an exact functor from $Coh(X)$ to itself, then since the canonical bundle is a line bundle, we know that it induces an exact functor, which descends to an exact functor of $D^*(X)$, which is denoted by $\mathscr{E} \otimes -$, also for any $i \in \mathbb{Z}$, we have an exact functor $[i] : D^*(X) \to D^*(X)$ which is shift by n, this gives the following definition:

Definition 1.12. Let X be a smooth projective variety of dimension n, then one defines the exact functor S_X as the composition:

$$D^*(X) \xrightarrow{\omega_X \otimes -} D^*(X) \xrightarrow{[n]} D^*(X)$$

with $* = -, +, b$.

In view of the following result S_X is called the Serre functor of X:

Theorem 1.13. Serre Duality Let X be a smooth variety over a field k, then:

$$S_X : D^b(X) \to D^b(X)$$

is a Serre functor of triangulated categories in the sense of the definition in Chapter 1.

Unpack the definition, we see that this just means that for any $\mathscr{E}^\bullet, \mathscr{F}^\bullet$ in $D^b(X)$, we have a functorial isomorphism:

$$\eta_{\mathscr{E}^\bullet,\mathscr{F}^\bullet} : Hom_{D^b(X)}(\mathscr{E}^\bullet, \mathscr{F}^\bullet) \to Hom_{D^b(X)}(\mathscr{F}^\bullet, \mathscr{E}^\bullet \otimes \omega_X[n])^*$$

Using the functorial isomorphisms:

$$Ext^i(\mathscr{E}^\bullet, \mathscr{F}^\bullet) = Hom_{D^b(X)}(\mathscr{E}^\bullet, \mathscr{F}^\bullet[i])$$

Serre duality is also written as:

$$Ext^i(\mathscr{E}^\bullet, \mathscr{F}^\bullet) \simeq Ext^{n-i}(\mathscr{F}^\bullet, \mathscr{E}^\bullet \otimes \omega_X)^*$$

Proof. **** omitted right now.* ☐

Serre duality as stated above yields a quick proof of:

Corollary 1.14. Suppose that $\mathscr{F}, \mathscr{G}$ are coherent sheaves on a smooth projective variety of dimension n, then:

$$Ext^i(\mathscr{F}, \mathscr{G}) = 0$$

for all $i > n$. In other words, we know that the homological dimension of $Coh(X)$ is n.

Proof. By viewing $\mathscr{F}$ and $\mathscr{G}$ as complexes which are concentrated at degree 0, we can apply the Serre duality above and get:

$$Ext^i(\mathscr{F}, \mathscr{F}) = Ext^{n-i}(\mathscr{G}, \mathscr{F} \otimes \omega_X)^*$$

Notice that when $i > n$, $n-i$ is negative, and we know that for a single object, Ext^i for $i < 0$ is always 0 (just recall how we defined this functor).

Also, we know that the homological dimension of $Coh(X)$ can't be smaller than n since $Ext^n(\mathscr{O}_X, \omega_X) \simeq Ext^{n-n}(\omega, \omega)^*$, which is not 0 since we at least have the identity map. ☐

Another easy application of Serre duality is to compute the derived category of a smooth projective curve:

Corollary 1.15. Let C be a smooth projective curve then any object in $D^b(C)$ is isomorphic to a direct sum $\oplus \mathscr{E}_i[i]$ where $\mathscr{E}_i$'s are coherent sheaves on C. More specifically, we can prove that $\mathscr{E}^\bullet \simeq \oplus_{i \leqslant i_0} \mathcal{H}^i(\mathscr{E}^\bullet)[-i]$ where i_0 is the minimal number such that $\mathcal{H}^i(\mathscr{E}^\bullet) \neq 0$.

Proof. Let's induct on the length of the complex. If the complex is of length 0, there is nothing to prove.

Then suppose that $\mathscr{E}^\bullet \in D^b(C)$ is of length at least 1, say length k, and since it's bounded, we can assume that $\mathcal{H}^i(\mathscr{E}^\bullet) = 0$ for $i < i_0$, then recall that we have a distinguished triangle:

$$H^{i_0}(\mathscr{E}^\bullet)[-i_0] \to \mathscr{E}^\bullet \to \mathscr{E}_1^\bullet \to H^{i_0}(\mathscr{E}^\bullet)[1-i_0]$$

Then using the induce LES you can see that $H^{i_0}(\mathscr{E}_1^\bullet) = 0$ thus you can assume that the length of $\mathscr{E}_1$ is $k-1$, then if the distinguished triangle splits, we can use the induction hypothesis for $\mathscr{E}_1^\bullet$ and conclude. Also, the LES also tells you that for $i > i_0$, we have $H^i(\mathscr{E}^\bullet) \simeq H^i(\mathscr{E}_1^\bullet)$.

To show that it really splits, it suffices to show that $Hom(\mathscr{E}_1^\bullet, , \mathcal{H}^{i_0}(\mathscr{E}^\bullet)[1-i_0]) = 0$, then if we use the induction hypothesis and write $\mathscr{E}_1^\bullet \simeq \oplus_{i>i_0} \mathcal{H}^i(\mathscr{E}^\bullet[-i])$, then

$$Hom(\mathscr{E}_1^\bullet, \mathcal{H}^{i_0}(\mathscr{E}^\bullet)[1-i_0]) \simeq \oplus_{i>i_0} Ext^{1-i_0}(\mathcal{H}^i(\mathscr{E}^\bullet[-i]), \mathcal{H}^{i_0}(\mathscr{E}^\bullet))$$

Then simultaneously shifting both arguments give you $\oplus_{i>i_0} Ext^{1-i_0+i}(\mathcal{H}^i(\mathscr{E}^\bullet), \mathcal{H}^{i_0}(\mathscr{E}^\bullet)$, then since $i > i_0$ and the homological dimension of $Coh(C)$ is 1, we are done. ☐

Remark 1.16. If you think about the above proof, this actually works for every abelian category whose homological dimension is $\leqslant 1$.

1.2 Spanning classes in the Derived Category

Recall that the definition of a spanning class in a triangulated category $\mathcal{D}$: A collection of objects Ω is a spanning class of spans $\mathcal{D}$ if for any $D \in \mathcal{D}$ the following two conditions hold:

1. If $Hom(A, B[i]) = 0$ for all $A \in \Omega$ and all i, then $B = 0$.
2. If $Hom(B[i], A) = 0$ for all $A \in \Omega$ and i, then $B = 0$.

Also recall that the two above conditions are equivalent in the presence of a Serre functor.

In this section, we will introduce two most common spanning classes in $D^b(X)$. Of course, there are other spanning classes, but we will introduce them later in the following chapters.

Proposition 1.17. Let X be a smooth projective variety, then the objects of the form $k(x)$ with $x \in X$ a closed point span the derived category of $D^b(X)$.

Proof. In order to prove the assertion, by the definition of the spanning class, we need to prove that for any nontrivial $\mathcal{F}^\bullet \in D^b(X)$, there are closed points x_1, x_2 and integers i_1, i_2 such that:

$$Hom(\mathcal{F}^\bullet, k(x_1)[i_1]) \neq 0 \quad Hom(k(x_2), \mathcal{F}^\bullet[i_2]) \neq 0$$

Apply Serre duality to $Hom(k(x_2), \mathcal{F}^\bullet[i_2])$, we know it's isomorphic to:

$$Hom(\mathcal{F}^\bullet, k(x)[dim(X) - i_2])^*$$

Thus we only have to find x_1 and i_1. To do so, we will use the following spectral sequence:

$$E_2^{p,q} = Hom(\mathcal{H}^{-q}(\mathcal{F}^\bullet), k(x)[p]) \Rightarrow_p Hom(\mathcal{F}^\bullet, k(x)[p+q])$$

Now, since we assumed that $\mathcal{F}^\bullet$ is not trivial, then there is an integer m such that $\mathcal{H}^m$ is not trivial, we will take the maximum of such integers, call it m. Then consider $E_2^{0,-m} = Hom(\mathcal{H}^m(\mathcal{F}^\bullet), k(x))$. We first prove that there is a x such that this is not 0, which is simple since you can just choose a point in the support of $\mathcal{H}^m$. Then we claim that this suffice to prove that $Hom(\mathcal{F}^\bullet, k(x)[m])$ is not 0.

To see this, we claim that the morphisms out of and to $E_r^{0,-m}$ are all 0. This is because we know that we chose m to be the maximum of $\mathcal{H}^m$ not 0, thus we know that every $E_2^{p,n}$ with n smaller than $-m$ is going to be 0, thus we know that for any $r \geqslant 2$, morphisms going out of $E_r^{0,-m}$ has the target $E_r^{r,-m+1-r}$, which is 0 since $E_2^{r,-m+1-r} = 0$.

On the other hand, we know that morphisms with target $E_r^{0,m}$ is 0 since we know that all Ext^p groups with $p < 0$ between two coherent sheaves are 0. Thus, we know that $E_\infty^{0,m} \simeq E_2^{0,m}$, thus we know that there is a graded piece of $Hom(\mathcal{F}^\bullet, k(x)[-m])$ which is not 0, thus itself is not 0. ☐

Another choice for spanning classes is provided by powers of an ample line-bundle, you can recall the definition of an ample sequence in the preceding chapter, and here you can see the geometric realization of the notion is given by:

Proposition 1.18. Let X be a projective variety over a field, if L is an ample line bundle on X, then the powers L^i, $i \in \mathbb{Z}$ form an ample sequence in the abelian category $Coh(X)$.

Proof. Recall that by definition, a ample line L has the property that for any coherent sheaf $\mathcal{F}$ on X, there is an integer n_0 such that the sheaf $\mathcal{F} \otimes L^n$ is globally generated for all $n \geqslant n_0$, i.e. the following morphism is surjective:

$$H^0(X, \mathcal{F} \otimes L^n) \otimes \mathcal{O}_X \to \mathcal{F} \otimes L^n$$

Now tensoring with L^{-n}, you get another surjective morphism:

$$H^0(X, \mathcal{F} \otimes L^n) \otimes \mathcal{L}^{-n} \to \mathcal{F}$$

Then, if you recall that $H^0(X, \mathcal{F} \otimes L^n) = Hom(\mathcal{O}_X, \mathcal{F} \otimes L^n) \simeq Hom(L^{-n}, \mathcal{F})$, you proved the first condition in the definition.

To prove the second condition, we need the following fundamental result due to Serre. This says that for any coherent sheaf $\mathcal{F}$ over a projective scheme over a Noetherian ring, if L is an ample line bundle then there is a n_0, such that for all $i > 0$ and $n > n_0$, we have:

$$H^i(X, \mathcal{F} \otimes L^n) = 0$$

This proves condition two since we know that $Hom(L^n, \mathcal{F}[j]) = Hom(\mathcal{O}_X, \mathcal{F} \otimes L^{-n}) = Ext^j(\mathcal{O}_X, \mathcal{F} \otimes L^{-n}) = H^j(X, \mathcal{F} \otimes L^{-n})$, thus if n is less than $-n_0$, we are done.

To see the last condition, which is $Hom(\mathcal{F}, L^n) = 0$, we need to do a little bit of work. First, we know that for any $\mathcal{F}$ coherent sheaf, the set $Hom(\mathcal{F}, L^i)$ is finite dimensional, thus we can assume that L is ver ample. Otherwise, we know that a power of L is ample, say L^k, then we only have to prove the assertion for a finite number of sheaves $\mathcal{F} \otimes L^i$ where $i = 0, 1, \cdots, k-1$. I will let you figure out why this suffice.

By passing to a very ample line bundle, we can assume that for any closed point $x \in X$, there is a nonzero section $s_x \in H^0(X, L)$ with $0 = s_x(x) \in L(x)$ since L is the restriction of $\mathcal{O}(1)$. Let's assume that, for the sake of contradiction, we have a nontrivial $\varphi \in Hom(\mathcal{F}, L^i)$, then we know that there is a closed point x such that $\varphi(x)$ is not 0. Then if you consider the following SES:

$$0 \to L^{i-1} \xrightarrow{s_x} L^i \to L^i|_{Z(s_x)} \to 0$$

Then we know that if you apply $Hom(\mathscr{F}, -)$ to get the injective morphism $Hom(\mathscr{F}, L^{i-1}) \xrightarrow{\cdot s_x} Hom(\mathscr{F}, L^i)$, the morphism φ is not in the image, otherwise $\varphi(x)$ is 0.

Thus we dropped the dimension by 1 by passing to $Hom(\mathcal{F}, L^{i-1})$, thus since everything if finite dimensional, you repeat this process and get 0 after finitely many steps, thus we know that:

$$Hom(\mathscr{F}, L^i) = 0$$

for $i \ll 0$. □

Remark 1.19. If we assume instead that we work with a smooth projective variety, we can simpliy the proof of condition 3 a lot by invoking Serre duality. To be more specific, we can consider $Hom(\mathcal{F}, L^i) \simeq H^{dimX}(X, \mathcal{F} \otimes \omega_X \otimes L^{-i})$, which is 0 for $i \ll 0$ by the vanishing result of Serre we quoted in the proof above.

Corollary 1.20. Let X be a smooth projective variety and L is an ample line bundle on X, then the powers of L, L^i's, form a spanning class of $D^b(X)$.

Proof. Since we just proved that L^i's form an ample sequence, recall that in last chapter, we proved that the corresponding objects in the derived category forms a spanning class. □

References