Fourier–Mukai Transforms Notes

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This is Chapter 4 of the book, where we present several results due to Bondal and Orlov which tells the story that the complexity of the derived category of coherent sheaves of a smooth projective variety is pretty much determined by it's geometry, in the sense that if it has ample (anti)-canonical bundles, the derived category completely describes the variety itself which in some sense tells you that the derived category of varieties with (anti)-canonical bundles contains more information.

However, the other half of the story of the complexity of the derived category is told by studying the autoequivalence group of the category, which shows that the derived category of varieties with (anti)-canonical bundles has simpler autoequivalences. All of these will be made more precise in the develoment.

Before we do any arguments that is too special, we start with a general property that applies to any smooth projective varieties:

Proposition 0.1. Let X and Y be smooth projective varieties over a field k, if there is an exact equivalence $D^b(X) \to D^b(Y)$, then $dimX = dimY$. Moreover, the order of ω_X is the same as the order of ω_Y where the order of ω_X is defined to be the minimal nonnegative integer n such that $\omega_X^n \simeq \mathcal{O}_X$. If there is no such integer, we define the order to be infinite and the above statement also applies to that case.

Proof. Since we are working with the derived category of smooth projective varieties, both derived category comes with natural Serre functors S_X and S_Y, we know by a previous lemma that, since F is an exact equivalence, it commute with both Serre functors.

Let $x \in X$ be a closed point, then we can consider the sheaf $k(x)$, which is coherent, moreover we have the following sequence of isomorphisms:

$$k(x) \simeq k(x) \otimes \omega_X \simeq S_X(k(x))[-dimX]$$

Therefore, we have:

$$F(k(x)) \simeq F(S_X(k(x))[-dimX])$$

Using the fact that F is exact, thus commutes with shifts, we get:

$$F(k(x)) \simeq FS_X(k(x))[-dimX] \overset{F\circ S_X \simeq S_Y \circ F}{\simeq} S_Y F(k(x))[-dimX] = F(k(x)) \otimes \omega_Y[dimY - dimX]$$

Now, since $k(x)$ is a nontrivial object in $D^b(X)$, from the fact that F is an equivalence, $F(k(x))$ is not trivial, then suppose that i is the maximal such that $\mathcal{H}^i(Fk(x))$ is not trivial, then we know that:

$$0 \neq \mathcal{H}^i(Fk(x)) = \mathcal{H}^i(Fk(x) \otimes \omega_Y[dimY - dimX]) \simeq \mathcal{H}^{i+dimY-dimX}(Fk(x)) \otimes \omega_Y$$

The last isomorphism comes from the fact that tensoring with a line bundle is exact. Therefore we know that $0 \neq \mathcal{H}^{i+dimY-dimX}(Fk(x))$, thus $dimY \leqslant dimX$.

Similarly, by choose a minimal i such that $\mathcal{H}^i(F(k(x))) \neq 0$, we can conclude that $dimY \geqslant dimX$, thus we are done with the first assertion that $dimX = dimY$.

To see the second assertion, let's first consider the case where one of ω_X and ω_Y has finite order. We will denote $dimX = dimY = n$. As you can see below, the proof is completely symmetric, thus we will just assume that $\omega_X^k = \mathcal{O}_X$, thus we know that $S_X^k[-kn] = id$, then you have:

$$F^{-1} \circ S_Y^k[-kn] \circ F = S_X[-kn] = id$$

but this implies that $S_Y^k[-kn] = id$, thus we must have $\omega^k = \mathcal{O}_X$, this is because that in the derived category, morphism between two complexes concentrated at degree 0 is the same as the morphism between two objects that lies in the 0-th slot. ∎

1 Ample (anti)-Canonical Bundle

Before we actually show that how the derived category determines the variety X, let's first study how to use the derived category to characterize geometric structures, such as points or line bundles. As you will see, they will be thought of as objects intrinsically lying in the category:

Definition 1.1. Let $\mathcal{D}$ be a k-linear triangulated category with a Serre functor S, an object P is called point like of codimension d if:

1. $SP \simeq P[d]$.
2. $Hom(P, P[i]) = 0$ for $i < 0$, and
3. $k(P) := Hom(P, P)$ is a field.

An object only satisfying condition 3 above is called simple, since as usual we assume all Hom sets are finite dimensional, we know that $k(P)$ is automatically a finite field extension of k. So if k is algebraically closed, it's just k.

Example 1.2. As the name suggests, a point-like object is likely to have some kind of connections with a point.

Indeed, suppose that x is a closed point on a smooth projective variety X, then we claim that $k(x)[m]$ for any integer is a point like object of codimension $dimX$.

Condition 1 and 2 above is easy to check, to check condition 3, we need to verify $Hom_{D^b(X)}(k(x), k(x))$ is a field. We claim that it's indeed a field, and the field should be $k(x)$. To see this, notice that:

$$Hom_{D^b(X)}(k(x), k(x)) \simeq Hom_{\mathcal{O}_X}(k(x), k(x)) \simeq Hom_{\mathcal{O}_{X,x}}(k(x), k(x)) = Hom_{k(x)}(k(x), k(x))$$

the last equation holds since we have $k(x) = \mathcal{O}_{X,x}/m_{X,x}$.

Proposition 1.3. Suppose that X is a smooth projective variety, then any point like object in $D^b(X)$ is of codimension $d = dimX$.

Proof. Recall that if we work with a smooth projective variety, the Serre functor is $- \otimes \omega_X[dimX]$, then we know that we have an isomorphism:

$$P \otimes \omega_X[dimX] \simeq P[d]$$

Notice that this isomorphism induces isomorphism of cohomology, thus we have:

$$\mathcal{H}^{n+dimX}(P) \otimes \omega_X \simeq \mathcal{H}^{n+d}(P)$$

since $P \in D^b(X)$, it's $A^\bullet$, let's assume that i is the maximal number such that $\mathcal{H}^i(P) \neq 0$, by this you can prove that $dimX \leqslant d$ by taking stalks of the above isomorphism, also similarly you can choose i to be the minimal number such that $\mathcal{H}^i$ is not 0, then we can deduce that $d \leqslant dimX$, thus we are done. $\square$

Definition 1.4. A sheaf $\mathcal{F}$ on X is simple, if $Hom(\mathscr{F}, \mathscr{F})$ is a field.

Proposition 1.5. If X is a smooth projective variety with $\omega_X \simeq \mathcal{O}_X$, then any simple sheaf defines a point-like object.

Proof. Condition 2 and 3 of a point like object is easy to prove, thus we only have to check the first condition, which is also easy since $\omega_X \simeq \mathcal{O}_X$, thus the Serre functor is just $[dimX]$, we are done. $\square$

The following lemma turns out to be useful later:

Lemma 1.6. Suppose that $\mathcal{F}^\bullet$ is a simple object in $D^b(X)$ with zero-dimensional support. Then if $Hom(\mathcal{F}^\bullet, \mathcal{F}^\bullet[i]) = 0$ for $i < 0$, then:

$$\mathcal{F}^\bullet \simeq k(x)[m]$$

for some closed point x and some integer m.

Proof. The first thing we will prove is that $\mathcal{F}^\bullet$ has support concentrated at a closed point. To see this, notice that by assumption, the support is 0-dimensional (as the union of all the support of cohomology), thus a finite collection of points. Then if it's not just a single point, as what we proved in the previous chapter, we can always decompose it into:

$$\mathcal{F}^\bullet \simeq \mathcal{F}_1^\bullet \oplus \mathcal{F}_2^\bullet$$

where none of $\mathcal{F}_i$ is 0. Here we don't need the extra conclusion that the support of $\mathcal{F}_i^\bullet$ are disjoint. However, if this is the case, we know that:

$$Hom(\mathcal{F}^\bullet, \mathcal{F}^\bullet) \simeq Hom(\mathcal{F}_1^\bullet \oplus \mathcal{F}_2^\bullet, \mathcal{F}_1^\bullet \oplus \mathcal{F}_2^\bullet)$$

But this is a contradiction, since the projection is not invertible, thus it's never a field.

Now, since we know that the support of $\mathcal{F}^\bullet$ is just one closed point, by definition, all of it's cohomology has support at the same closed point, say x. Then we will call m_0 the maximal number such that $\mathcal{H}^{m_0}(\mathcal{F}^\bullet)$ is not 0 and m_1 the minimal number such that $\mathcal{H}^{m_1}(\mathcal{F}^\bullet)$ is not 0. We claim that $m_0 = m_1$.

To prove this, notice that $\mathcal{H}_{m_i}$ are both sheaves concentrated at the point m, then we claim that there is a nontrivial morphism between these two guys. To be more specific, suppose that M is a finite module over a local Noetherian ring (A, m), there is a surjection $M \to A/m$ and a injection $A/m \hookrightarrow M$.

Now, notice that we have two more morphisms we can use:

$$\mathcal{F}^\bullet[m_0] \to \mathcal{H}^{m_0} \qquad \mathcal{H}^{m_1} \to \mathcal{F}^\bullet[m_1]$$

which induces identity after you apply the cohomology functor. Thus the composition:

$$\mathcal{F}^\bullet[m_0] \to \mathcal{H}^{m_0} \to \mathcal{H}^{m_1} \to \mathcal{F}^\bullet[m_1]$$

is not trivial and you can check that by applying the cohomology. But since we assume that $Hom(\mathcal{F}^\bullet, \mathcal{F}^\bullet[i])$ is 0 for $i < 0$, we know that here $m_1 \geqslant m_0$, plus the fact that $m_0 \geqslant m_1$, we get $m_0 = m_1$, thus $\mathcal{F}^\bullet \simeq \mathcal{F}[m]$ for some m and some coherent sheaf $\mathcal{F}$.

Now we claim that this implies that $\mathcal{F} = k(x)$. To see this, we need to show that if $\mathcal{F}$ is not $k(x)$, then there is a nontrivial morphism in $Hom(\mathcal{F}, \mathcal{F})$ such that it's not invertible, which contradicts the assumption that $\mathcal{F}$ is simple. To construct such a morphism, we still use the composition:

$$\mathcal{F} \to k(x) \hookrightarrow \mathcal{F}$$

where the first one is surjective and second one is injective. Obviously this morphism is not trivial, but if $\mathcal{F}$ is not $k(x)$, the image of the second morphism is strictly smaller than $\mathcal{F}$, thus not invertible. □

The following result is the first example where you see why the condition that the (anti)-canonical bundle is ample is so important:

Proposition 1.7. Let X be a smooth projective variety such that ω_X or ω_X^* is ample, then the point like objects in $D^b(X)$ are the objects that are isomorphic to $k(x)[m]$, where x is a closed point and $m \in \mathbb{Z}$.

Proof. We have already proved that $k(x)[m]$ is point like. For the converse:

Suppose that $P \in D^b(X)$ is point like, then we have proved that it's of codimension $dimX = n$, thus we know that $P \otimes \omega_X[n] = P[n]$, if we denote $\mathcal{H}^i$ by the cohomology of P, we know that $\mathcal{H}^i \otimes \omega \simeq \mathcal{H}^i$.

This implies that $\mathcal{H}^i$ is supported in something of dimension 0 since we know that the Hilbert polynomial $P_{\mathcal{F}}(k) = \chi(\mathcal{F} \otimes \omega_X^k)$ of any coherent sheaf $\mathcal{F}$ is of degree $dimsupp(\mathcal{F})$, then since the Hilbert polynomial of $\mathcal{H}^i$ is the same as the Hilbert polynomial of $\mathcal{F} \otimes \omega_X$, we know that the dimension of the support of $\mathcal{F}$ must be 0, since $P_{\mathcal{F}}(k) = P_{\mathcal{F}}(k+1)$ for all k. □

Remark 1.8. In the above proof, the assumption that ω_X or ω_X^* is ample is necessary. For example, if you have an abelian variety that is smooth projective, the canonical bundle is the structure sheaf, then since $\mathcal{O}_X$ is simple, condition 1 and 2 are easily check since $\omega_X \simeq \mathcal{O}_X$, thus $\mathcal{O}_X$ is a point like object, which is not as desired.

Just as how we characterize point as objects in the derived category, you can also characterize line bundles:

Definition 1.9. Let $\mathcal{D}$ be a k-linear triangulated category with a Serre functor S. An object $L \in \mathcal{D}$ is called invertible if for any point-like object $P \in \mathcal{D}$, there exists an integer depending on both P and L, $n_P \in \mathbb{Z}$, such that:

$$Hom(L, P[i]) = \begin{cases} k(P) & \text{if } i = n_p, \\ 0 & \text{otherwise} \end{cases}$$

Proposition 1.10. Let X be a smooth projective variety, any invertible object in $D^b(X)$ is of the form $L[m]$ with L a line bundle on X and $m \in \mathbb{Z}$. Conversely, if we assume ω_X or ω_X^* is ample, then for any line bundle L and any integer m, the object $L[m] \in D^b(X)$ is invertible.

Proof. This is going to be a rather long argument.

First, suppose that L is an invertible object, we will use m to denote the maximal number such that $\mathcal{H}^m$ is not zero. Now, choose a closed point $x_0 \in supp\mathcal{H}^m$, which gives you a nontrivial morphism:

$$\mathcal{H}^m \to k(x_0)$$

Again, notice that we have a natural morphism $f : L \to \mathcal{H}^m[-m]$ such that the m-th cohomology functor induces identity. Then if you consider the Hom set $Hom(L, k(x_0)[-m])$, I claim that this is the same as $Hom(\mathcal{H}^m, k(x_0))$, which is not 0.

To see this, notice that $Hom(\mathcal{H}^m, k(x_0)) = Hom(\mathcal{H}^m[-m], k(x_0)[-m])$ we have a bijection between $Hom(\mathcal{H}^m[-m], k(x_0)[-m])$ and $Hom(L, k(x_0)[-m])$ by precomposing $L \to \mathcal{H}^m[-m]$. To see this is indeed a bijection, let's show that it's surjective first. Suppose that you have a $\varphi \in Hom(L, k(x_0)[-m])$, after you apply $\mathcal{H}^m$, you get a morphism $\psi : \mathcal{H}^m \to k(x_0)$, I claim that $\psi[-m] \circ f = \varphi$. In order to prove this, you need to recall how exactly did we constructed such a morphism $f : L \to \mathcal{H}^m[-m]$ in the derived category, the hint is to use the following diagram:

$$\begin{tikzcd}
 & \mathcal{F} \arrow[dl] \arrow[dr, "id"] & & \\
\mathcal{E}^\bullet \arrow[d] \arrow[drrr] & & \mathcal{F} \arrow[r] \arrow[dll] & \mathcal{H}^m \arrow[d] \\
L^\bullet & & & k(x_0)[-m]
\end{tikzcd}$$

where $\mathcal{F}$ is the intermediate stuff we constructed to have a morphism from L to $\mathcal{H}^m$.

Injectivity is a little bit simpler, you just have to prove that if some morphism $\mathcal{H}^m \to k(x_0)$ is mapped to 0, then it's 0 itself. All you have to do is to apply cohomology. Therefore, by the definition of $n_{k(x_0)}$, we know that $n_{k(x_0)} = -m$.

Or if you find the above argument not elegant, you can also use spectral sequences to make this argument simpler. With the above notation, consider the following spectral sequence:

$$E_2^{p,q} = Hom(\mathcal{H}^{-q}(L), k(x_0)[p]) \Rightarrow_p Hom(L, k(x_0)[p+q])$$

Let $p = 0, q = -m$, you get:

$$E_2^{0,-m} = Hom(\mathcal{H}^m(L), k(x_0)) \Rightarrow_p Hom(L, k(x_0)[-m])$$

Now, notice that by definition of the spectral sequence anything $E_2^{p,q}$ with $p < 0$ is 0 since it's the negative Ext group. Also, by our choice of m, for $q < -m$, $E_2^{p,q}$ is also 0, this implies that $E_2^{0,-m} \simeq E_\infty^{0,-m}$, which is not 0, thus $Hom(L, k(x_0)[-m])$ is not 0, which again proves $n_{k(x_0)} = -m$.

Now, we can use the same spectral sequence and essentailly the same method to prove that $E_2^{1,-m} \simeq E_\infty^{1,-m}$, thus it abuts to $Hom(L, k(x_0)[1 + n_{k(x_0)}]) = 0$, thus $E_2^{1,-m} = Ext^1(\mathcal{H}^m, k(x_0)) = 0$.

For later use, let's state a result from commutative algebra without proof. Suppose that M is a finitely generated module over a Noetherian local ring (A, m), if $Ext^1(M, A/m) = 0$ then M is free.

Now, let's consider the local to global spectral sequence:

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(\mathcal{H}^m, k(x_0))) \Rightarrow_p Ext^{p+q}(\mathcal{H}^m, k(x_0))$$

We claim that $E_2^{2,0} = 0$, this is because if you write it out, you have:

$$H^2(X, \mathcal{E}xt^0(\mathcal{H}^m, k(x_0)))$$

Notice that $\mathcal{E}xt^0(\mathcal{H}^m, k(x_0))_x \simeq Ext^0_{\mathcal{O}_{X,x}}(\mathcal{H}^m_x, k(x_0)_x)$, which is 0 except for when $x = x_0$, thus the second degree cohomology of a dimension 0 coherent sheaf is 0. This proves $E_2^{2,0} = 0$, then if you consider the following morphisms:

$$E_2^{-2,2} \to E_2^{0,1} \to E^{2,0}$$

you find $E_2^{-2,2} = 0$ since for $p < 0$, everthing is 0, we also just proved that $E_2^{2,0} = 0$, thus $E_2^{0,1} \simeq E_2^{0,1}$, but as you can see, when $r \geqslant 3$, if you consider the following morphisms:

$$E_3^{-3,3} \to E_3^{0,1} \to E_3^{3,-1}$$

it's easy to see that since $E_2^{-3,3} = 0 = E_2^{3,-1}$, $E_3^{-3,3} = 0 = E_3^{3,-1}$, thus $E_3^{0,1} \simeq E_4^{0,1}$, similar picture holds later, thus you know that $E_2^{0,1} = E_\infty^{0,1}$, thus you can deduce from the fact $Ext^1(\mathcal{H}^m, k(x_0)) = 0$ that $\mathcal{E}xt^1(\mathcal{H}^m, k(x_0))$.

Then if you take the stalk at x_0, the aforementioned result from commutative algebra tells you that $\mathcal{H}^m$ is free at x_0. This in particular shows that for each closed point in $supp\mathcal{H}^m$, there is an open near that point such that $\mathcal{H}^m$ doesn't vanish, thus the support of $\mathcal{H}^m$ is both open and closed, thus since X is irreducible, $supp\mathcal{H}^m = X$.

Therefore, for any closed point x, there is a nontrivial morphism $\mathcal{H}^m \to k(x)$, thus you can prove that:

$$Hom(L, k(x)[-m]) \neq 0$$

therefore we have the following equality(proved using the same argument as above):

$$k(x) = Hom(L, k(x)[-m]) = Hom(\mathcal{H}^m, k(x))$$

This implies that $\mathcal{H}^m_x / m_{X,x}\mathcal{H}^m_x$ is a one dimensional $k(x)$-vector space, thus the fiber dimension of $\mathcal{H}^m$ is constant and is equal to 1, we are done.

What's left to prove right now is $\mathcal{H}^i = 0$ for $i < 0$. We need a few preparations before we do this. First, recall that $\mathcal{H}^m$ is a line bundle, an in particular, locally free, thus we have the following isomorphism:

$$E_2^{p,-m} = Ext^p(\mathcal{H}^m, k(x)) \simeq H^p(X, \mathcal{H}^{m*} \otimes k(x))$$

where the second isomorphism comes from the fact that for any vector bundle $\mathcal{E}$ and a arbitrary $\mathcal{O}_X$-module $\mathcal{F}$, we have:

$$Ext^p(\mathcal{E}, \mathcal{F}) \simeq H^n(X, \mathcal{E}^* \otimes \mathcal{F})$$

But notice that $\mathcal{H}^{m*}$ is a skyscraper sheaf, thus all higher cohomology vanishes. This implies that $E_2^{p,-m}$ is zero except when $p = 0$. Now consider $i_0 = m - 1$, then we claim that $E_2^{0,i_0} = E_\infty^{0,i_0}$, also notice that the abutment is $Hom(L, k(x)[-i_0]) = 0$, we know that $E_2^{0,i_0} = 0$, thus $Hom(\mathcal{H}^{i_0}, k(x)) = 0$ for every x, thus the fiber dimension of $\mathcal{H}^{i_0}$ is 0 for each $x \in X$, thus $\mathcal{H}^{i_0} = 0$.

Then you can continue by induction. □

After our description of the geometric objects, we are able to prove the fundamental result:

Proposition 1.11. Let X, Y be smooth projective varieties and assume that the (anti)-canonical bundle of X is ample, if there is an exact equivalence $D^b(X) \simeq D^b(Y)$, then X and Y are isomorphic, in particular, the (anti)-canonical bundle of Y is also ample.

Proof. Before we dive deep into the technicalities, let's first sketch the idea of the proof, which is surprisingly simple.

We will first assume something and prove that we can reduce the question to this setting later. Let's first assume that under this exact equivalence $F : D^b(X) \to D^b(Y)$, the structure sheaf $\mathcal{O}_X$ is mapped to $\mathcal{O}_Y$.

Then we have proved that we always have $dimX = dimY = n$ and since F always commutes with Serre functors and shifts, we know that:

$$F(\omega_X^k) = F(S_X^k(\mathcal{O}_X))[-kn] \simeq S_Y^k(F(\mathcal{O}_X))[-kn] \simeq S_Y^k(\mathcal{O}_Y)[-nk] = \omega_Y^k$$

Then using the fact that F is fully faithful, we can conclude that:

$$H^0(X, \omega_X^k) = Hom(\mathcal{O}_X, \omega_X^k) \simeq Hom(F\mathcal{O}_X, F\omega_X^k) = Hom(\mathcal{O}_Y, \omega_Y^k) = H^0(Y, \omega_Y^k)$$

for all k.

If you form the direct sum $\bigoplus_k H^0(X, \omega_X^k)$, you find out that this is a ring by the identification $H^0(X, \omega_X^k) = Hom(\mathcal{O}_X, \omega_X^k)$, the for $s_i \in Hom(\mathcal{O}_X, \omega_X^{k_i})$, we can define the product to be:

$$s_1 \cdot s_2 = S_X^{k_1}(s_2)[-k_1 n] \circ s_1$$

and you can indeed check that the right hand side lies in $Hom(\mathcal{O}_X, \omega_X^{k_1+k_2})$.

Thus, under this definition of product, the induced bijection:

$$\bigoplus_k H^0(X, \omega_X^k) \simeq \bigoplus_k H^0(X, \omega_Y^k)$$

is actually a ring isomorphism. Then it's a well-known result that if you have ample line bundle L over X, a smooth projective variety, then $X = Proj(\bigoplus_k H^0(X, L^k))$, thus if we can also assume the canonical or anti-canonical bundle is ample, we will have the following isomorphism:

$$X \simeq Proj(H^0(X, \omega_X^k)) \simeq Proj(\bigoplus_k H^0(X, \omega_Y^k)) \simeq Y$$

which gives you the desired isomorphism, thus we only have to show that we can reduce the problem to the case where $F(\mathcal{O}_X) = \mathcal{O}_Y$ and the canonical bundle or anti-canonical bundle of Y is ample.

Let's first deal with the assertion that we can assume $F(\mathcal{O}_X)$ is $\mathcal{O}_Y$. Recall that since the (anti)canonical bundle of X is ample, each line bundle defines an invertible object, thus in particular, $\mathcal{O}_X$ defines a invertible object, thus we know that in Y, even if we don't assume the canonical bundle is ample, we can still know that $F(\mathcal{O}_X)$, as an equivalence of derived categories F maps invertible objects to invertible object, $F(\mathcal{O}_X)$ is still invertible, thus is of the form $M[m]$ for some line bundle M and some integer m, recall that tensoring with a line bundle is an exact equivalence, so is shifting, thus by tensoring with M^* and shift back $[-m]$, we can assume that $F(\mathcal{O}_X) = \mathcal{O}_Y$.

Then let's deal with the ampleness of ω_Y or ω_Y^*. To do this, the first step is to prove that the point like objects in $D^b(Y)$ are all of the form $k(x)[m]$ for some closed point x and integer m, without any positivity assumption on ω_Y, just based on the equivalence.

Suppose that there is a $P \in D^b(Y)$ that is not of the form $k(y)[m]$, since we know that F is an equivalence, there is some closed point x_P, such that $P \simeq F(k(x_P)[m_P])$. Also, recall that for each closed point $y \in Y$, there is a closed $x_y \in X$ and an integer m_y, such that $F(k(x_y)[m_y]) \simeq k(y)$. Since we assumed that P is not of the form $k(y)[m]$, we know that x_P is not any of the x_y. Knowing this, we have the following sequence of isomorphisms:

$$Hom(P, k(y)[m]) = Hom(F(k(x_P))[m_P], F(k(x_y))[m_y + m]) = Hom(k(x_p), k(x_y)[m_y + m - m_P]) = 0$$

where the last equation comes from the fact that x_P is not any of x_y, thus the two sheaves are of disjoint support, thus the morphisms are all trivial. This equation holds for every y and every m, thus recall that we proved $k(y)$ spans the derived category, thus $P = 0$, a contradiction. In particular, we know for every $x \in X$, there is a $y \in Y$ such that $F(k(x)) = k(y)$, with no shifts. This is because that:

$$Hom(\mathcal{O}_X, k(x)) \simeq Hom(\mathcal{O}_Y, k(y)[m]) \neq 0$$

thus we know that m can only be 0.

Finally, it remains to prove that ω_Y^k is (very)-ample for each k if ω_X^k is (very)-ample. Let's prove this with a very geometric way by showing that some power ω_Y^k separates points and tangent vectors. This will use an extra assumption that k is algebraically closed, but there is actually a less geometric way to prove this without assuming k is algebraically closed.

We need to continue using the assumption that for any $y \in Y$, there is a unique $x \in X$ such that $F(k(x)) = k(y)$, then we know that by definition of separating points ω_Y^k separates points iff for any $y_1, y_2 \in Y$, the morphism on global sections $H^0(r_{y_1,y_2}) : H^0(Y, \omega_Y^k) \to H^0(k(y_1) \oplus k(y_2))$ induced by the restriction morphism:

$$r_{y_1,y_2} : \omega_Y^k \to \omega_Y^k(y_1) \oplus \omega_Y^k(y_2) \simeq k(y_1) \oplus k(y_2)$$

is surjective.

To do this, we notice that:

$$Hom(\omega_Y^k, k(y_1) \oplus k(y_2)) \simeq Hom(\omega_X^k, k(x_1) \oplus k(x_2))$$

where we use x_i to simplify the notation of x_{y_i} and we have a commutative diagram like below:

$$\begin{tikzcd}
H^0(Y, \omega_Y^k) \arrow[r, "H^0(r_{y_1,y_2})"] \arrow[d] & H^0(Y, k(y_1) \oplus k(y_2)) \arrow[d] \\
Hom(\mathcal{O}_Y, \omega_Y^k) \arrow[r, "r_{y_1,y_2}\circ"] \arrow[d] & Hom(\mathcal{O}_Y, k(y_1) \oplus k(y_2)) \arrow[d] \\
Hom(\mathcal{O}_X, \omega_X^k) \arrow[r, "r_{x_1,x_2}\circ"] \arrow[d] & Hom(\mathcal{O}_X, k(x_1) \oplus k(x_2)) \arrow[d] \\
H^0(X, \omega_X^k) \arrow[r, "H^0(r_{x_1,x_2})"'] & H^0(X, k(x_1) \oplus k(x_2))
\end{tikzcd}$$

the vertical morphisms are all isomorphisms and the bottom horizontal morphism is surjective since ω_X^k is very ample, thus the top is surjective, we are done.

We can prove ω_Y^k separates tangent vectors if ω_X^k does in a similar fashion. Recall the definition of separates tangent directions, we need to prove that for any closed point $y \in Y$, the subscheme supported at y with a tangent direction Z_y, we have the morphism $H^0(Y, \omega_Y^k) \to H^0(Y, \mathcal{O}_{Z_y})$, induced by the natural morphism $\omega_Y^k \to \mathcal{O}_{Z_y}$ is surjective.

Recall that, by the definition of Z_y, we have a SES:

$$0 \to k(y) \to \mathcal{O}_{Z_y} \to k(y) \to 0$$

this is a nontrivial class in $Ext^1(k(y), k(y))$, which is a nontrivial element in $Hom(k(y), k(y)[1])$, which is mapped to another nontrivial morphism in $Hom(k(x), k(x)[1])$, which again gives:

$$0 \to k(x) \to \mathcal{O}_{Z_x} \to k(x) \to 0$$

with $F(\mathcal{O}_{Z_x}) = \mathcal{O}_{Z_y}$ and we have:

$$F(\omega_X^k \to \mathcal{O}_{Z_x}) \simeq \omega_Y^k \to \mathcal{O}_{Z_y}$$

Then you again use the fact that ω_X^k separates tangent vectors and a commutative diagram as above to conclude the desired result. □

1.1 Derived functors in Algebraic Geometry

In this section, we will discuss all the derived functors in algebraic geometry that will be needed in the sequel. We will make sure all assumptions are carefully stated, but for certain details in the proof, we have to refer to the literature.

The first one we will be interested is the cohomology functor, let X be a Noetherian scheme over k, then we know the global section functor:

$$\Gamma : Qcoh(X) \to Vec(k)$$

is left exact. We already know that the category of quasicoherent sheaves on X has enough injectives, thus the right derived functor exists:

$$R\Gamma : D^+(Qcoh(X)) \to D^+(Vec(k))$$

The higher derived functors are denoted by $H^i(X, \mathcal{F}^\bullet) := R^i\Gamma(\mathcal{F}^\bullet)$

For a single sheaf, we just retrieve the usual sheaf cohomology $H^i(X, \mathcal{F})$, sometimes we will call $H^i(X, \mathcal{F}^\bullet)$ the hypercohomology groups and denote them by $\mathbb{H}^i(X, \mathcal{F}^\bullet)$.

Recall we proved before that if every exact sequence splits in a category, then the derived category has a simple description, thus since $R\Gamma$ lands in $D^+(Vec(k))$, we know that:

$$R\Gamma(\mathcal{F}^\bullet) \simeq \oplus_i H^i(X, \mathcal{F}^\bullet)$$

We will treat the following two theorems as common sense later:

Theorem 1.12. For any quasi-coherent sheaf $\mathcal{F}$ on a Noetherian scheme X, one has $H^i(X, \mathcal{F}) = 0$ for all $i > dimX$.

Recall that we are studying $D^b(X)$, which is bounded, not just bounded below, thus we would like to have $R\Gamma$ have the property that when we apply $R\Gamma$ to a bounded complex, the result is also a bounded complex in the derived category, which is the following corollary:

Corollary 1.13. $R\Gamma$ defined above actually induces an exact functor:

$$R\Gamma : D^b(X) \to D^b(Vec(k))$$

for any Noetherian scheme X.

Proof. Recall another corollary we had after our introduction of Spectral sequences. We proved there is $R\Gamma(\mathcal{F})$ is bounded for every single sheaf $\mathcal{F}$, we can conclude what we want. ☐

Another theorem that is useful is due to Serre

Theorem 1.14. Let $\mathcal{F}$ be a coherent sheaf on a projective scheme X over a filed k, then all cohomology groups $H^i(X, \mathcal{F})$ are finite dimensional over k.

As usual we omit the proof here, but this theorem tells us that $\Gamma(X, -)$, if we work with projective schemes, actually gives you a left exact functor $Coh(X) \to Vec_f(k)$. However, as what we have discussed, the category of coherent sheaves doesn't admit enough injectives, thus the right derived functor can't be constructed directly.

However, if we consider the composition of exact functors:

$$D^b(X) \to D^b(Qcoh(X)) \to D^b(Vec(k))$$

where the first functor is just the natural one and the second functor is what we obtained just now. Again, notice that the cohomology of $R\Gamma(\mathcal{F}^\bullet)$ for $\mathcal{F}^\bullet \in D^b(X)$ are still finite dimension, we know that we can still view this functor as a functor to $D^b(Vec_f(k))$ via the following identification:

$$\begin{tikzcd} D^b(Coh(X)) \arrow[r] & D^b(Vec(k)) \\ D^b(X) \arrow[u, hook] \arrow[r, dashed, "\text{X is projective}"] & D^b(Vec_f(k)) \arrow[u, hook] \end{tikzcd}$$

Similarly, we can discuss this for the direct image functor, which is also left exact:

$$f_* : Qcoh(X) \to Qcoh(Y)$$

whose right derived functor is denoted by

$$R_*f : D^+(Qcoh(X)) \to D^+(Qcoh(Y))$$

The higher direct images $R^i f_*(\mathcal{F}^\bullet)$ is just the cohomology of the complex $Rf_*(\mathcal{F}^\bullet)$. This generalize the global section functor in the sense that when we work with schemes over $Speck$, consider the morphism $f : X \to Speck$, then f_* is just Γ.

In this point of view, we have the following generalization:

Theorem 1.15. For a quasi-coherent sheaf $\mathcal{F}$ on a Noetherian scheme X and a morphism $f : X \to Y$ between Noetherian schemes, the higher direct images of $\mathcal{F}$, $R^i f_* \mathcal{F} = 0$ for $i > dimX$.

Then again, we know Rf_* actually induces a functor:

$$D^b(Qcoh(X)) \to D^b(Qcoh(Y))$$

To stay in the coherent world, one has to use the following coherence criterion:

Theorem 1.16. If $f : X \to Y$ is a projective or proper morphism of Noetherian schemes, then the higher direct images $R^i f_*(\mathcal{F})$ of any coherent sheaf is still coherent.

Then using this theorem we can define for any proper morphism $f : X \to Y$ between Noetherian schemes a derived functor: $Rf_* : D^b(X) \to D^b(Y)$ as the composition $D^b(X) \hookrightarrow D^b(Qcoh(X)) \to D^b(Qcoh(Y))$, then we know that the image lies in the subcategory whose cohomology are coherent sheaves, thus equivalent to $D^b(Y)$ via the following diagram:

$$\begin{tikzcd}
D^b(Coh(X)) \arrow[r] & D^b(Qcoh(Y)) \\
D^b(X) \arrow[u, hook] \arrow[r, "\text{X is projective}"] & D^b(Y) \arrow[u, hook]
\end{tikzcd}$$

Sometimes, another f_*-adapted class is useful: recall that a sheaf on X is called flasque or flabby if for any open subset $U \subset X$, the restriction $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective. The following lemma is helpful for understanding flasque or flabby sheaves:

Lemma 1.17. Any injective $\mathcal{O}_X$-module is flabby and any flabby sheaf in X is f_*-adapted for any $f : X \to Y$, and $f_*\mathcal{F}$ is again flabby.

The next interesting example is the local homs, denoted by $\mathcal{H}om(-,-)$:

References