Fourier–Mukai Transforms Notes

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June. 2025

The second chapter is called derived categories, which is divided into three parts, corresponding to the three sections of this notes: derived category of an abelian category, derived functors and spectral sequences that appears when you try to compose two derived functors.

1 Derived Categories

1.1 Derived category of an abelian category.

Remark 1.1. Before we start our treatment, we would like to first explain a little bit of how in a given abelian category, an object is always studied in terms of resolutions.

Let's give an example first. Recall that a coherent sheaf $\mathscr{F}$ on a scheme X is locally given by finitely many generators and finitely many relations. In other words, at least locally there exists an exact sequence

$$\mathscr{O}_X^{\oplus m_1} \to \mathscr{O}_X^{\oplus m_2} \to \mathscr{F} \to 0$$

Moreover, if X is a smooth projective variety, any coherent sheaf $\mathscr{F}$ can be resolved by locally free objects of length $n = dim(X)$, i.e. there exists an exact sequence of the form:

$$0 \to \mathscr{E}_n \to \cdots \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{D} \to 0$$

where all $\mathscr{E}_i$'s are locally free sheaves. Therefore, in order to study arbitrary coherent sheaves on X, we can switch to study locally free sheaves and complexes of them.

More generally, when we study an abelian category $\mathscr{A}$, it's important to allow complexes of objects in $\mathscr{A}$, which leads to the notion of category of complexes:

Let's recall some basic notions in the category of complexes before we give the definition. The category of complexes of an abelian category $\mathscr{A}$ is denoted by $Kom(\mathscr{A})$ where Kom stands for the German word for complex. The objects of $Kom(\mathscr{A})$ and morphisms are the usual ones as you expect.

Definition 1.2. The category of complexes $Kom(\mathscr{A})$ of an abelian category $\mathscr{A}$ is the category whose objects are complexes $A^\bullet$ in $\mathscr{A}$ whose morphisms are morphisms of complexes.

One important feature of the category of complexes of an abelian category $\mathscr{A}$ is that $Kom(\mathscr{A})$ is again an abelian category, the proof is simple, recall the three conditions for an additive category to be an abelian category. First, $Kom(\mathscr{A})$ is obviously an additive category with zero objects the complex $0^\bullet$. Also, given a morphism of complexes, the kernel, cokernel are just the complex consisting of kernel and cokernels. The last condition is also easy to check since we know $\mathscr{A}$ is already an abelian category.

Also notice that the functor that maps an object $A \in \mathscr{A}$ to the complex $A^\bullet$ with $A^0 = 0$ and all the other terms 0 identifies $\mathscr{A}$ as a full subcategory of $Kom(\mathscr{A})$.

The category $Kom(\mathscr{A})$ has two more features, shifts and cohomology. We will start with shifts:

Definition 1.3. Let $A^\bullet \in Kom(\mathscr{A})$ be a given complex. Then $A^\bullet[1]$ is the complex with $(A^\bullet[1])^i = A^{i+1}$ and differential $d^i_{A[1]} = -d^{i+1}_A$.

The shift $f[1]$ of a morphism of complexes $f : A^\bullet \to B^\bullet$ is the complex morphism $A^\bullet[1] \to B^\bullet[1]$ is given by $f[1]^i = f^{i+1}$.

To have a concrete intuition about the shift functor, you can see the diagram below:

$$\begin{array}{ccccc}
\cdots & -1 & 0 & 1 & \cdots \\
\cdots & A^{-1} \xrightarrow{d^{-1}} & A^{0} \xrightarrow{d^{0}} & A^{1} & \cdots \\
\cdots & A^{0} \xrightarrow{-d^{0}} & A^{1} \xrightarrow{-d^{1}} & A & \cdots \\
\cdots & -1 & 0 & 1 & \cdots
\end{array}$$

where you can think of the vertical lines are absolute marks indicating the lable of the slots, the first row is $A^\bullet$ and you can see that in this row, everything stays at their designated place. The second row is $A^\bullet[1]$ so you can see that the 0-th slot, A^1 is there and the situation of other slots are similar.

The following fact is easily verified:

Corollary 1.4. The shift functor $T : Kom(\mathscr{A}) \to Kom(\mathscr{A})$ $A^\bullet \mapsto A^\bullet[1]$ and $f \mapsto f[1]$ is an equivalence of categories.

Proof. Indeed, this functor has a real inverse T^{-1} where $A^\bullet \mapsto A^\bullet[-1]$ and $(A^\bullet[-1])^i = A^{i-1}$ with $d^i_{A^\bullet[-1]} = -d^{i-1}_{A^\bullet}$. Similar to T, if $f : A^\bullet \to B^\bullet$ is a morphism of complexes, then $f[-1]^i = f^{i-1}$. $\square$

Notice that, however, $Kom(\mathscr{A})$ with the shift functor, doesn't define a triangulated category. Indeed, we need to specify the set of distinguished triangles and how to complete morphisms into distinguished triangles. Unfortunately, the usual canonical choices all fails:

Proposition 1.5. A short exact sequence $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ which can be viewed as a triangle with trivial $C^\bullet \to A^\bullet[1]$, do not in general satisfy the conditions imposed on distinguished triangles in a triangulated category.

Proof. This is an example of why it's not so easy to define the set of distinguished triangles in $Kom(\mathscr{A})$.

This set of distinguished triangles is not even closed under rotation since:

$$0 \to B^\bullet \to C^\bullet \xrightarrow{0} A^\bullet[1] \xrightarrow{0} B^\bullet[1]$$

is not exact. $\square$

We also have the notion of cohomology which is the usual quotient of kernel by image, and notice that any complex morphism $f : A^\bullet \to B^\bullet$ induces natural homomorphisms:

$$H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$$

If the all the cohomology of $A^\bullet$ is 0, we will say this complex is acyclic.

One easy exercise is that:

Proposition 1.6. Suppose that $F : \mathscr{A} \to \mathscr{B}$ is an additive functor between two abelian categories then F is exact iff the image of $F(A^\bullet)$ of any acyclic complex is also acyclic.

Proof. If F maps acyclic objects to acyclic objects then we can take the complex to be a SES then we are done.

Conversely, suppose that F is exact, then given any LES:

$$\cdots \longrightarrow A^{-2} \xrightarrow{d^{-2}} A^{-1} \xrightarrow{d^{-1}} A^{0} \xrightarrow{d^{0}} A^{1} \xrightarrow{d^{1}} A^{1} \longrightarrow \cdots$$

we can break that into a series of SES, as indicated in the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & A^{-2} \arrow[r, "d^{-2}"] & A^{-1} \arrow[r, "d^{-1}"] & A^{0} \arrow[r, "d^{0}"] \arrow[dr] & A^{1} \arrow[r, "d^{1}"] & A^{1} \arrow[r] & \cdots \\
0 \arrow[r] & Imd^{-2} = Kerd^{-1} \arrow[u, hook] \arrow[urr, hook] & & & Imd^{0} = Kerd^{1} \arrow[u, hook] \arrow[r] & 0
\end{tikzcd}$$

then apply F, since we assumed that F is exact, then it commutes with images and kernels, we are done. $\square$

Another point we would like to mention is that a SES in $Kom(\mathscr{A})$ gives a LES in $\mathscr{A}$ with cohomologies being the terms. We won't go deep into the details here but the LES goes like:

$$\cdots \to H^i(\mathscr{A}) \to H^i(\mathscr{B}^\bullet) \to H^i(\mathscr{C}^\bullet) \xrightarrow{\delta} H^{i+1}(\mathscr{A}^\bullet) \to \cdots$$

If you are working with modules over a ring, the connecting morphism δ is easy to construct explicitly.

The induced map for cohomology $H^i(f)$ is used to define quaisi-isomorphisms which is central in the passage to derived categories.

Definition 1.7. A morphism of complexes $f : A^\bullet \to B^\bullet$ is a quasi-isomorphism (or qis ,for short) if for all $i \in \mathbb{Z}$, the induced map $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$ is an isomorphism.

Usually, a qis is not a real isomorphism between two complexes. Such an example is easy to find if you look at the category of abelian groups. However, in the following development, you will see that the central idea of derived category is to make quasi-isomorphisms real isomorphisms. As an early explanation, as what we discussed above, for a coherent sheaf $\mathscr{F}$ over a projective variety X, we have a resolution of locally free sheaves of length n $\mathscr{E}^\bullet \to \mathscr{F} \to 0$. Notice that this gives a quasi-isomorphism $(\mathscr{E}^n \to \cdots \to \mathscr{E}^0) \to \mathscr{F}$:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & \mathscr{E}^1 \arrow[r] \arrow[d] & \mathscr{E}^0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \mathscr{F} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

In order to study $\mathscr{F}$ using such complexes, we don't want it to be that different from $\mathscr{F}$, which gives a reason of why we want quasi-isomorphisms to be isomorphisms.

Without wasting time, let's introduce the following existence theorem of derived categories, whose proof is given by the subsequent discussions and this will be the main goal of this section:

Theorem 1.8. Let $\mathscr{A}$ be an abelian category and let $\mathrm{Kom}(\mathscr{A})$ be it's categories of complexes, then there exists a category $\mathrm{D}(\mathscr{A})$, the derived category of $\mathscr{A}$ and a functor:

$$Q : Kom(\mathscr{A}) \to D(\mathscr{A})$$

such that the following two conditions hold:

1. If $f : A^\bullet \to B^\bullet$ is a quasi-isomorphism in $Kom(\mathscr{A})$, then $Q(f)$ is an isomorphism in $D(\mathscr{A})$.

2. Any functor $F : Kom(\mathscr{A}) \to \mathscr{D}$ satisfying condition 1 uniquely factors over Q, i.e. there exists a unique functor (up to isomorphism) $G : D(\mathscr{A}) \to \mathscr{D}$ with $F \simeq G \circ Q$:

$$\begin{tikzcd}
Kom(\mathscr{A}) \arrow[rr, "Q"] \arrow[dr, "F"'] & & D(\mathscr{A}) \arrow[dl, dashed, "G"] \\
& \mathscr{D} &
\end{tikzcd}$$

This theorem is purely a existence result which is not proved. In the subsequence discussion, we will provide a construction which helps us understand which objects become isomorphic under Q and more complicated, what are the morphisms in $D(\mathscr{A})$. This discussion will help us to work with the derived category in the future. Also, along with our discussion, we will be able to see the following facts:

Corollary 1.9. 1. Under this functor Q, the objects of the two categories $Kom(\mathscr{A})$ and $D(\mathscr{A})$ are identified.

2. The cohomology objects $H^i(A^\bullet)$ of an object $A^\bullet \in D(\mathscr{A})$ is an well defined object in $\mathscr{A}$. We have seen this in $Kom(\mathscr{A})$, but the result for $D(\mathscr{A})$ should wait for the proof below.

3. Viewing any object in $\mathscr{A}$ as a complex concentrated in degree 0 yields an equivalence between $\mathscr{A}$ and the full subcategory of $D(\mathscr{A})$ that consists of all complexes $A^\bullet$ with $H^i(A^\bullet) = 0$ for $i \neq 0$.

Contrary to $Kom(\mathscr{A})$, $D(\mathscr{A})$ is in general not abelian, but it is always triangulated, as you will see later. The shift functor in $Kom(\mathscr{A})$ does descend to $D(\mathscr{A})$ and a natural class of distinguished triangles can be found, which will also be explained shortly.

Now, let's move to the construction of the derived category. First, we need to introduce the so called homotopy category of complexes, which is going to be an intermediate step for us to pass to $D(\mathscr{A})$, as indicated in the following diagram:

$$\begin{tikzcd} Kom(\mathscr{A}) \arrow[rr, "Q"] \arrow[dr] & & D(\mathscr{A}) \\ & K(\mathscr{A}) \arrow[ur] & \end{tikzcd}$$

The reason for this is that we want quais-isomorphisms to be isomorphisms in the derived category, so if we suppose $C^\bullet \to A^\bullet$ is a quasi-isomorphism, for any morphism $C^\bullet \to B^\bullet$, like what the following diagram shows:

$$\begin{tikzcd} & C \arrow[dl, "qis"'] \arrow[dr] & \\ A \arrow[rr, dashed] & & B \end{tikzcd}$$

Notice that this roof like stuff can have different tips, i.e. a morphism from A to B in $D(\mathscr{D})$, if everything works out, can be construct via an intermediate object, not necessarily C. Therefore, we would like to define the morphisms in $D(\mathscr{A})$ to be such roofs. But some realistic questions are: how are we able to say that two roofs are the same and how to compose such roofs. To answer this question, we will need the homotopy category of complexes:

Definition 1.10. Two morphism of complexes:

$$f, g : A^\bullet \to B^\bullet$$

are called homotopically equivalent (written as $f \sim g$) if there exists a sequence of morphisms $h^i : A^i \to B^{i-1}$ such that the following holds:

$$f^i - g^i = h^{i+1} \circ d^i_{A^\bullet} + d^{i-1}_{B^\bullet} \circ h^i$$

as shown in the diagram below:

$$\begin{tikzcd} \cdots \arrow[r] & A^{i-1} \arrow[r] \arrow[d] \arrow[dl] & A^i \arrow[r] \arrow[d] \arrow[dl] & A^{i+1} \arrow[r] \arrow[d] \arrow[dl] & \cdots \arrow[dl] \\ \cdots \arrow[r] & B^{i-1} \arrow[r] & B^i \arrow[r] & B^{i+1} \arrow[r] & \cdots \end{tikzcd}$$

The following results are helpful for us to define the homotopy category:

Proposition 1.11.
1. Homotopy equivalence between morphisms $A^\bullet \to B^\bullet$ of complexes is an equivalence relation.
2. Homotopically trivial morphisms form an 'ideal' of the morphisms of $Kom(\mathscr{A})$, i.e. if f is a homotopically trivial morphism, then any composition involving f will be homotopically trivial.
3. If $f \sim g$, then $H^i(f) = H^i(g)$ for all i.
4. If $f : A^\bullet \to B^\bullet$ and $g : B^\bullet \to A^\bullet$ are given such that $f \circ g \sim id$ and $g \circ f \sim id$, then f, g are quasi-isomorphisms and more precisely $H^i(f)^{-1} = H^i(g)$.

Proof. We won't write out every detail of proof.

For 1, the key part is to prove that this relation is transitive, which is the result of we are working with an additive category.

For 2, if f is homotopically trivial, then $f^i = h^{i+1} \circ d^i_{A^\bullet} + d^{i-1}_{B^\bullet} \circ h^i$, then if we post compose g, then we get $g^i \circ f^i = g^i \circ h^{i+1} \circ d^i_{A^\bullet} + g^i \circ d^{i-1}_{B^\bullet} \circ h^i$. One important feature of g is that $g^i \circ d^{i-1}_{B^\bullet} = d^{i-1}_{C^\bullet} \circ g^{i-1}$, thus we know that: $g^i \circ f^i = g^i \circ h^{i+1} \circ d^i_{A^\bullet} + d^{i-1}_{C^\bullet} \circ g^{i-1} \circ h^i$ thus if we define the new $h'^i = g^{i-1} \circ h^i$, we are done. Similarly, when we precomposing, things will work out similarly.

For 3, we only have to show that if $f \sim 0$, then $H^i(f) = 0$ for all i. Again, we won;t write out every detail, but the basic thinking here is that since $H^i(f)$ comes out of the kernel, then after d^i everything goes to 0, also since the target of $H^i(f)$ is something quotient the image of d^{i-1}, then after d^{i-1}, everything thing again goes to 0.

For 4, we only have to know that H is a functor then $H^i(f \circ g) = H^i(f) \circ H^i(g)$. $\square$

Via the above results, we can now define:

Definition 1.12. The homotopy category of complexes $K(\mathscr{A})$ is the category whose objects are the objects of $Kom(\mathscr{A})$, i.e. $Ob(K(\mathscr{A})) = Ob(Kom(\mathscr{A}))$ and whose morphisms $Hom_{K(\mathscr{A})}(A^\bullet, B^\bullet) = Hom_{Kom(\mathscr{A})}(A^\bullet, B^\bullet)/\sim$, i.e. the morphisms in $Kom(\mathscr{A})$ modding out the relation of being homotopically equivalent.

Remark 1.13. In this definition, we still need to verify that the composition of morphisms makes sense, in the sense that if $f_1 \sim f_2$ and $g_1 \sim g_2$ then $g_1 \circ f_1 \sim g_2 \circ f_2$. This comes from 2 in the proposition above and $g_1 \circ f_1 \sim g_1 \circ f_2 \sim g_1 \circ f_2$.

Remark 1.14. Notice that the definition of $Kom(\mathscr{A})$ and $K(\mathscr{A})$ also makes sense for additive categories, not necessarily only for abelian categories. Only defining $Kom(\mathscr{A})$ and $K(\mathscr{A})$ and show that they are well defined doesn't involve any use of kernel cokernel or anothing that's special to abelian category. We will use this remark later.

Now comes the precise definition of derived category. The first step is to describe the objects in $D(\mathscr{A})$, which is simple since it will have the same object as $Kom(\mathscr{A})$ and $K(\mathscr{A})$:

$$Ob(Kom(\mathscr{A})) = Ob(K(\mathscr{A})) = Ob(D(\mathscr{A}))$$

Next, the set of morphisms between $A^\bullet$ and $B^\bullet$ is viewed as the set of equivalent classes of diagrams of the form, where $C^\bullet \to A^\bullet$ is a quasi-isomorphism:

$$\begin{tikzcd}
& C^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\
A^\bullet & & B^\bullet
\end{tikzcd}$$

Two of such diagrams are equivalent if they are dominated **in the homotopy category, not necessarily in the complex category** $K(\mathscr{A})$ by a third one of the same sort, i.e. there exists a commutative diagram in $K(\mathscr{A})$ of the form:

$$\begin{tikzcd}
& & C^\bullet \arrow[dl, "qis"'] \arrow[dr] & & \\
& C_1^\bullet \arrow[dl, "qis"'] \arrow[drrr] & & C_2^\bullet \arrow[dlll, "qis"] \arrow[dr] & \\
A^\bullet & & & & B
\end{tikzcd}$$

Notice that we know that composition of quasi-isomorphisms are quasi-isomorphisms thus in particular $C^\bullet \to C_2^\bullet \to A^\bullet$ is also a quasi-isomorphism. The reason of why the commutativity is required in the homotopy category (i.e. up to homotopy and they don't have to be exactly the same morphism) will be explained later. To put it in simple words, if we impose the stronger condition where we don't want to do thing up to homotopy but require exact the same morphism, the composition of such roofs can not be well defined.

In this way, we have gather the basic building blocks of $D(\mathscr{D})$, i.e. the objects and morphisms. However, we still need to make sense of how these morphisms compose. Before we begin the construction, we can imagine that if we want two roofs to compose, and the thing you get is another roof, if you have ever played lego or stuff of that sort, a natural choice is to use another roof and put that one on top of the two that you want to compose, which is the following construction:

If two morphisms are given:

$$\begin{tikzcd}[column sep=small] & C_1^\bullet \arrow[dl, "qis"'] \arrow[dr] & & & & C_2^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet & & B^\bullet & & C^\bullet \end{tikzcd}$$

we would like the composition of such two diagrams to be a diagram of the following form:

$$\begin{tikzcd}[column sep=small] & & C_0^\bullet \arrow[dl, "qis"'] \arrow[dr] & & \\ & C_1^\bullet \arrow[dl, "qis"'] \arrow[dr] & & C_2^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet & & C^\bullet \end{tikzcd}$$

However, one problem is that does this diagram exist and since we want the composition to be well defined, if we have multiple choices of the third roof, do we get equivalent bigger roofs in the end. To answer these questions, we need to introduce the notion of mapping cones, which is also useful to put the triangulated structure on $D(\mathscr{A})$ later:

Definition 1.15. Let $f : A^\bullet \to B^\bullet$ be a complex morphism, then it's mapping cone, denoted by $C(f)$ is defined to be a complex with:

$$C(f)^i = A^{i+1} \oplus B^i$$

the differential is defined to be:

$$d^i_{C(f)} = \begin{pmatrix} -d^{i+1}_{A^\bullet} & 0 \\ f^{i+1} & d^i_{B^\bullet} \end{pmatrix}$$

the first row stands for the the morphism coming out of A and the second row stands for the morphisms coming out of B. If we use a diagram to show this:

$$\begin{tikzcd} & A^i \arrow[r, "-d"] \arrow[dr, "f^i"] & A^{i+1} \arrow[r, "-d"] \arrow[dr, "f^{i+1}"] & A^{i+2} \\ B^{i-1} \arrow[r, "d"] & B^i \arrow[r, "d"] & B^{i+1} & \end{tikzcd}$$

The first thing you can check is that this indeed defines a complex, which can be verified using the universal property. (How to check a morphism form a direct sum to another direct sum is 0, it's zero iff the morphsim from each component to each component is 0). Moreover, onece we had this mapping cone, there exists two natural complex morphisms:

$$\tau : B^\bullet \to C(f) \qquad \pi : C(f) \to A^\bullet[1]$$

given by the natural inclusion and natural projection. The following are a few stuff you need to know about the mapping cones:

Proposition 1.16. The following sequence is a SES:

$$0 \to B^\bullet \to C(f) \to A^\bullet[1] \to 0$$

Proof. This is just a very explicit observation. ☐

Remark 1.17. In particular, using the above exact sequence, we have a LES in cohomology:

$$H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C(f)) \to H^{i+1}(A^\bullet) \to \cdots$$

Proposition 1.18. The following composition is homotopic to the trivial morphism:

$$A^\bullet \to B^\bullet \to C(f)$$

Proof. To show this, consider the following diagram, and we claim that the inclusion maps will suffice:

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r, "d"] \arrow[d, "f^{i-1}"'] & A^i \arrow[r, "d"] \arrow[d, "f^i"] \arrow[dl, "i"'] & A^{i+1} \arrow[r] \arrow[d, "f^{i+1}"'] \arrow[dl, "i"'] & \cdots \\
\cdots \arrow[r] & A^i \oplus B^{i-1} \arrow[r] & A^{i+1} \oplus B^i \arrow[r] & A^{i+2} \oplus B^i \arrow[r] & \cdots
\end{tikzcd}$$

I will let you verify this. ☐

Moreover, any diagram of the following form can be completed via a morphism $C(f_1) \to C(f_2)$(the first square is assumed to be commutative):

$$\begin{tikzcd}
A_1^\bullet \arrow[r, "f_1"] \arrow[d] & B_1^\bullet \arrow[r] \arrow[d] & C(f_1) \arrow[r] \arrow[d, dashed] & A_1^\bullet[1] \arrow[d] \\
A_2^\bullet \arrow[r, "f_2"] & B_2^\bullet \arrow[r] & C(f_2) \arrow[r] & A_2^\bullet[1]
\end{tikzcd}$$

Not surprisingly, the morphism connecting $C(f_1)$ and $C(f_2)$ is constructed using the data in the first two vertical arrows, I will let you check the details. Of course, the above result should reminds you of **TR3** in a triangulated category. Similarly, the following result should be viewed as **TR2**, that tells you that you can rotate the distinguished triangle and still get something distinguished, which will also be crucial for defining the composition of roofs:

Proposition 1.19. Let $f : A^\bullet \to B^\bullet$ be a morphism of complexes and let $C(f)$ be it's mapping cone that comes with the two natural morphisms $\tau : B^\bullet \to C(f)$ and $\pi : C(f) \to A^\bullet[1]$. Then there exists a complex morphism $g : A^\bullet[1] \to C(\tau)$ which is an isomorphism in $K(\mathscr{A})$, notice that not in $Kom(\mathscr{A})$, such that the following diagram is commutative in $K(\mathscr{A})$:

$$\begin{tikzcd}
B^\bullet \arrow[r, "\tau"] \arrow[d, "="'] & C(f) \arrow[r, "\pi"] \arrow[d, "="'] & A^\bullet[1] \arrow[r, "-f"] \arrow[d, dashed, "g"] & B^\bullet[1] \arrow[d, "="] \\
B^\bullet \arrow[r, "\tau"'] & C(f) \arrow[r, "\tau_\tau"'] & C(\tau) \arrow[r, "\pi_\tau"'] & B^\bullet[1]
\end{tikzcd}$$

Proof. Before we prove it, you shoule be able to see that this plus **TR1**, shows that **TR3** holds.

To find this g, we need to for every i, find a $g^i : A^\bullet[1]^i \to C(\tau)^i$, which is a morphism from A^{i+1} to $B^{i+1} \oplus C(f)^i = B^{i+1} \oplus A^{i+1} \oplus B^i$. A natural choice is to consider $(-f^{i+1}, id, 0)$ where the negative sign is to compensate the negative sign in $A^\bullet[1] \to B^\bullet[1]$.

Let's first check that it's indeed a complex morphism. It should be easier via the following diagram, though indeed very tedious since the bottom horizontal morphism is complicated:

$$\begin{tikzcd}
A^{i+1} \arrow[r, "-d"] \arrow[d, "g"'] & A^{i+2} \arrow[d, "g"] \\
B^{i+1} \oplus A^{i+1} \oplus B^i \arrow[r] & B^{i+2} \oplus A^{i+2} \oplus B^{i+1}
\end{tikzcd}$$

After you check this, we need to show that g is an isomorphism in $K(\mathscr{A})$, which means that we need to find a morphism it's inverse. Now, if we define g^{-1} by projecting $B^{i+1} \oplus A^{i+1} \oplus B^i$ to it's middle factor, we can check that $g^{-1} \circ g$ is the identity even if in $Kom(\mathscr{A})$. But for $g \circ g^{-1}$, we need to show that it's homotopic to the identity. I will let you check that the map:

$$h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & id \end{pmatrix}$$

should suffice for the homotopy.

You can also easily check that the square in the most right indeed commutes even in $Kom(\mathscr{A})$, thus certainly in $K(\mathscr{A})$.

However, the square in the middle doesn't commute in $Kom(\mathscr{A})$, but we need to pass to the homotopy category. One thing that is clear is that $g^{-1} \circ \tau_\tau = \pi$ in $Kom(\mathscr{A})$, which you can easily check, therefore you know that in the homotopy category $K(\mathscr{A})$ you have:

$$g^{-1} \circ \tau_\tau = \pi$$

Now you compose g in $K(\mathscr{A})$ to get the desired result. $\square$

After our brief discussion of mapping cones, we can show how to use it to compose two roofs in the derived category. In order to do this, consider the following proposition:

Proposition 1.20. Let $f : A^\bullet \to B^\bullet$ be a quasi-isomorphism and $g : C^\bullet \to B^\bullet$ be an arbitrary morphism, then there exists a commutative diagram in $K(\mathscr{A})$:

$$\begin{tikzcd}
C_0^\bullet \arrow[r, "qis"'] \arrow[d] & C^\bullet \arrow[d, dashed, "g"] \\
A^\bullet \arrow[r, "f=qis"'] & B^\bullet
\end{tikzcd}$$

Proof. Before we prove it, please notice that if we don't require the top morphism $C_0^\bullet \to C^\bullet$ be a quasi-isomorphism, the existence of such a diagram even in $Kom(\mathscr{A})$ is trivial. The difficulty comes in when we need the top horizontal morphism to be a quasi-isomorphism.

What we have to start with is just the following diagram:

$$\begin{tikzcd}
 & C^\bullet \arrow[d, dashed, "g"] \\
A^\bullet \arrow[r, "f=qis"'] & B^\bullet
\end{tikzcd}$$

we hope to find a way to add on an object on the left top to fulfill the requirement. The trick is to consider the following diagram:

$$\begin{tikzcd}
 & C^\bullet \arrow[r, "\tau\circ g"] \arrow[d, dashed, "g"] & C(f) \arrow[r] \arrow[d, dashed, "="] & C(\tau \circ g) \\
A^\bullet \arrow[r, "f"'] & B^\bullet \arrow[r, "\tau"'] & C(f) \arrow[r, "\pi"'] & A^\bullet[1]
\end{tikzcd}$$

The important step is that by the above proposition, the sequence $B^\bullet \to C(f) \to A^\bullet$ is isomorphic to $B^\bullet \to C(f) \to C(\tau)$, thus we have a diagram of the following shape:

$$\begin{tikzcd}
 & C^\bullet \arrow[r, "\tau\circ g"] \arrow[d, dashed, "g"] & C(f) \arrow[r] \arrow[d, dashed, "="] & C(\tau \circ g) \arrow[d, dashed] \\
 & B^\bullet \arrow[r, "\tau"'] \arrow[d, "="'] & C(f) \arrow[r, "\tau_\tau"'] \arrow[d, "="'] & C(\tau) \\
A^\bullet \arrow[r, "f"'] & B^\bullet \arrow[r, "\tau"'] & C(f) \arrow[r, "\pi"'] & A^\bullet[1] \arrow[u, dashed, "g"]
\end{tikzcd}$$

where the morphism $C(\tau \circ g) \to C(\tau)$ is the natural morphism from $C^{i+1} \bigoplus C(f)^i \to B^{i+1} \bigoplus C(f)^i$ which is defined by (g, id). I will leave some of the details to you to consider why the above discussion implies that we have the following commutative diagram in $K(\mathscr{A})$:

$$\begin{tikzcd}
C(\tau \circ g)[-1] \arrow[r] \arrow[d, dashed] & C^\bullet \arrow[r] \arrow[d, "="'] & C(f) \arrow[r] \arrow[d, "="'] & C(\tau \circ g) \arrow[d, dashed] \\
A^\bullet \arrow[r, "f"'] & B^\bullet \arrow[r, "\tau"'] & C(f) \arrow[r, "\pi"'] & A^\bullet[1]
\end{tikzcd}$$

Now the question is that why $C(\tau \circ g)[-1] \to C^\bullet$ is a quasi-isomorphism. The main reason is that since f is a quasi-isomorphism then all the cohomology of $C(f)$ vanish, I will let you to check this, via the LES induced by $0 \to B^\bullet \to C(f) \to A^\bullet[1] \to 0$.

Knowing this, notice that we have another SES: $0 \to C(f) \to C(\tau \circ g) \to C^\bullet[1] \to 0$, this also induces a LES in cohomology, then using this LES and the fact that all the cohomology of $C(f)$ vanish to conclude the result. $\square$

The proposition above makes the composition of roofs immediate:

Corollary 1.21. The composition of roofs proposed in the earlier discussion makes sense.

Proof. Recall the diagram we drew there in $K(\mathscr{A})$:

$$
\begin{tikzcd}
 & & C_0^\bullet \arrow[dl, dashed, "qis"'] \arrow[dr, dashed] & & \\
 & C_1\bullet \arrow[dl, "qis"'] \arrow[dr] & & C_2^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\
A^\bullet & & B^\bullet & & C^\bullet
\end{tikzcd}
$$

We wish to find a $C_0^\bullet$ and morphisms in $K(\mathscr{A})$ as indicated in the daigram such that everything is commuatative. But this comes as we apply the above proposition to:

$$
\begin{tikzcd}
 & C_2^\bullet \arrow[d] \\
C_2^\bullet \arrow[r, "qis"'] & B^\bullet
\end{tikzcd}
$$

Finally we need to show that the equivalence class of such roofs appearing above is unique, i.e. if there are two such roofs:

$$
\begin{tikzcd}
 & & D^\bullet \arrow[dl, dashed, "qis"'] \arrow[dr, dashed] & & \\
 & C_1\bullet \arrow[dl, "qis"'] \arrow[dr] & & C_2^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\
A^\bullet & & B^\bullet & & C^\bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & E^\bullet \arrow[dl, dashed, "qis"'] \arrow[dr, dashed] & & \\
 & C_1\bullet \arrow[dl, "qis"'] \arrow[dr] & & C_2^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\
A^\bullet & & B^\bullet & & C^\bullet
\end{tikzcd}
$$

they are again dominated by something else. To show this, consider the following diagram:

$$
\begin{tikzcd}
 & & D^\bullet \arrow[dl, dashed, "qis"'] \arrow[drr, dashed] & F^\bullet \arrow[l] \arrow[dl] & \\
 & C_1\bullet \arrow[dl, "qis"'] \arrow[dr] & E^\bullet \arrow[l, dashed, "qis"] \arrow[rr, dashed] & & C_2^\bullet \arrow[dll, "qis"] \arrow[dr] & \\
A^\bullet & & B^\bullet & & & C^\bullet
\end{tikzcd}
$$

where we can apply the above proposition to:

$$
\begin{tikzcd}
 & E^\bullet \arrow[d, "qis"] \\
D^\bullet \arrow[r, "qis"'] & C_1^\bullet
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
F^\bullet \arrow[r, "qis"] \arrow[d, "qis"'] & E^\bullet \arrow[d, "qis"] \\
D^\bullet \arrow[r, "qis"'] & C_1^\bullet
\end{tikzcd}
$$

However, notice that this isn't enough to prove the commutativity of the diagram since the commutativity of $F^\bullet \to D^\bullet \to C_2^\bullet$ and $F^\bullet \to E^\bullet \to C_2^\bullet$ is not clear at this point. To prove this is really commutative, we need the following proposition: $\square$

Proposition 1.22. Suppose that $f : A^\bullet \to B^\bullet$ is a morphism of complex such that there is a quasi-isomorphism $s : B^\bullet \to C^\bullet$ such that $sf = 0$, then there is a quasi-isomorphism $t : D^\bullet \to A^\bullet$ such that $ft = 0$.

Proof. This is due to Gelfand, which is the most important step in proving that in $K(\mathscr{A})$, not in $Kom(\mathscr{A})$, the class of quasi-isomorphisms is localizing, which is the notion that we will briefly introduce later in a remark. Please note in the following proof that why the construction only works in $K(\mathscr{A})$ but not in $Kom(\mathscr{A})$.

Now, let $f : K^\bullet \to L^\bullet$ in $K(\mathscr{A})$ be a morphism, if there is a quasi-isomorphism $s : L^\bullet \to L'^\bullet$ with $sf = 0$. Since we are working in $K(\mathscr{A})$, there is a homotopy between sf and 0, say h with $h^i : K^i \to L'^{i-1}$. Then let's consider the following diagram:

$$\begin{tikzcd}
C(s)[-1] \arrow[r, "\pi_s[-1]"] \arrow[d, "="'] & L^\bullet \arrow[r, "s"] & L'^\bullet \\
C(s)[-1] & K^\bullet \arrow[l, "g"] \arrow[u, "f"'] & C(g)[-1] \arrow[l, "\pi_g[-1]"']
\end{tikzcd}$$

where g is defined to be $g^i : K^i \to L^i \oplus L'^{i-1}$ is just $(f^i, -h^i)$. Then we can verify that g is a morphism of complexes and the commutativity of the square, which is left for you to do. We also add on the $\pi_g[-1] : C(g)[-1] \to K^\bullet$ at the bottom right. Notice that the composition of the bottom row is 0 also in $K(\mathscr{A})$, not necessarily in $Kom(\mathscr{A})$ since we are quoting Prop 1.19 here and the dotted g in that proposition is only invertible in $K(\mathscr{A})$.

Finally, we define $t = \pi_g[-1]$, then it's easy to see that $ft = 0$, once we can show that t is a quasi-isomorphism, we are done. To show this, we know that since s is a quasi-isomorphism, then $C(s)$ is acyclic, i.e. all cohomology vanishes, then consider the LES of cohomology induced from the SES:

$$0 \to C(s)[-1] \to C(g) \to K[1] \to 0$$

we are done from the fact that all cohomology of $C(s)[-1]$ vanishes. $\square$

Due to the above proposition, we can continue the proof of the composition is well-defined:

Proof. I will leave it for you to check that even though we don't know if $F^\bullet \to D^\bullet \to C_2^\bullet$ and $F^\bullet \to E^\bullet \to C_2^\bullet$ is commutative or not, but we know that after we post-compose with $C_2^\bullet$, they are equal. Then from the above proposition, we can find a quasi-isomorphism $k : G^\bullet \to F^\bullet$ such that if precompose this with $F^\bullet \to D^\bullet \to C_2^\bullet$ and $F^\bullet \to E^\bullet \to C_2^\bullet$, they are equal, we are done. $\square$

Remark 1.23. Notice that the construction of the derived category depends a lot on the fact that Prop 1.20 holds. If you didn't know the proof we gave before, you would probably think $C_0^\bullet$ can be defined to be the fibered product (this is definitely a reasonable guess since this $C_0^\bullet$ is also called the homotopy pull-back, which is related to the pull-back, often being the fibered product), but below is an example which tells you that this doesn't work.

Let's just take $\mathscr{A}$ to be the category of abelian groups and we will take $B^\bullet$ to be a surjection $d : B^0 \to B^1$ and we will take $A^\bullet$ to be the kernel of d, concentrated at degree 0 and $C^\bullet = B^1[-1]$, then if you consider the fibered product, which is defined to the the fibered product on each degree with obvious differentials. I will left it for you to check that this won't give you a quasi-isomorphism from $C_0^\bullet$ to $C^\bullet$.

Remark 1.24. A common mistake people tend to have when they first deal with $K(\mathscr{A})$ is that they tend to think that if f is a quasi-isomorphism, then it's indeed invertible in $K(\mathscr{A})$, which is not true! It is just because this is not true, we start to consider the derived category where we can indeed invert them.

For an example, just think of the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[u, "0"] & \mathbb{Z} \arrow[r] \arrow[u, dashed, "p"] & 0
\end{tikzcd}$$

this gives a quasi-isomorphism. However, you should notice that there is no non-zero morphism going the other way around, thus this morphism in the diagram, even though being a quasi-isomorphism, won't have a right inverse forever.

Proposition 1.25. A complex $A^\bullet$is isomorphic to 0 in $D(\mathscr{A})$ if and only if $H^i(A^\bullet) = 0$ for all i.

Proof. First, I will let you show that for any $A^\bullet$ in $D(\mathscr{A})$, the following two roofs all can represent the identity morphism $id_{A^\bullet}$(which is in particular):

$$\begin{tikzcd}
 & A^\bullet \arrow[dl, "id"'] \arrow[dr, "id"] & \\
A^\bullet & & A^\bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
 & B^\bullet \arrow[dl, "f=qis"'] \arrow[dr, "f=qis"] & \\
A^\bullet & & A^\bullet
\end{tikzcd}$$

then using this, I will let you show that if we have the following roof, then $A^\bullet$ is isomorphic to $B^\bullet$:

$$\begin{tikzcd}
 & C^\bullet \arrow[dl, "f=qis"'] \arrow[dr, "g=qis"] & \\
A^\bullet & & B^\bullet
\end{tikzcd}$$

Now let's go to this specific problem, first if $H^i(A) = 0$, then the zero morphism from A to 0 is a quasi-isomorphism, then we have the following roof $0 \leftarrow A^\bullet \rightarrow A^\bullet$ is an isomorphism.

Conversely, suppose that $A^\bullet$ is isomorphic to 0 via two roofs:

$$\begin{tikzcd}
 & C^\bullet \arrow[dl, "f=qis"'] \arrow[dr] & \\
A^\bullet & & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
 & D^\bullet \arrow[dl, "0=qis"'] \arrow[dr] & \\
0 & & A^\bullet
\end{tikzcd}$$

then we know that the composition should be in the equivalence class of $id_{A^\bullet}$. Then using this, I will let you see we have the following diagram:

$$\begin{tikzcd}
 & & & F^\bullet \arrow[dl, "h=qis"'] \arrow[d] & \\
 & & E^\bullet \arrow[dl, "g=qis"'] \arrow[dr] & A^\bullet \arrow[ddlll] \arrow[ddr] & \\
 & C^\bullet \arrow[dl, "f=qis"'] \arrow[dr] & & D^\bullet \arrow[dl, "0=qis"'] \arrow[dr] & \\
A^\bullet & & 0 & & A^\bullet
\end{tikzcd}$$

Then we claim that after we apply H^i, $F^\bullet \rightarrow A^\bullet$ gives you an isomorphism, which is 0, which proves that $H^i(A^\bullet) = 0$. To see that it's indeed an isomorphism, we use the triangle:

$$\begin{tikzcd}
 & F^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\
A^\bullet & & A^\bullet \arrow[ll, "id"]
\end{tikzcd}$$

to see it's the 0-morphism, we use the triangle:

$$\begin{tikzcd}
 & F^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\
E^\bullet \arrow[rr] & & A^\bullet
\end{tikzcd}$$

if we can prove that after we apply H^i the morphism $E^\bullet \rightarrow E^\bullet$ gives you 0, we are done. To see that it's 0, notice that $E \rightarrow D \rightarrow 0$ is 0 while $D \rightarrow 0$ is a quasi-isomorphism, we are done. $\square$

Now, suppose that $f : A^\bullet \to B^\bullet$ is a morphism in $K(\mathscr{A})$, by abuse of notation we will use the roof $A^\bullet \leftarrow A^\bullet \xrightarrow{f} B^\bullet$ to denote the morphism in $D(\mathscr{A})$, then we have the following proposition:

Proposition 1.26. Suppose that $f : A^\bullet \to B^\bullet$ is a morphism in $K(\mathscr{A})$, then by the above convention, f is zero in $D(\mathscr{A})$ iff there exists a qis $g : C^\bullet \to A^\bullet$ such that fg is homotopic to 0.

Proof. Notice that the zero morphism from $A^\bullet$ to $B^\bullet$ can be represented by:

$$\begin{tikzcd} & A^\bullet \arrow[dl, "id"'] \arrow[dr, "0"] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

then we know that if f is zero in $D(\mathscr{A})$, we will have the following diagram:

$$\begin{tikzcd} C^\bullet \arrow[d, "qis"'] \arrow[dr] & & \\ A^\bullet \arrow[d, "id"'] \arrow[drr, "f"] & A^\bullet \arrow[dl, "id"'] \arrow[dr, "0"] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

everything is in $K(\mathscr{A})$, thus we are done.

Conversely, suppose that we have such a g, then you can easily construct the above diagram easily, we are done. $\square$

We have mentioned before that $D(\mathscr{A})$ is not an abelian category, but it is indeed an additive category, now we begin to see what is the additive structure in $D(\mathscr{A})$. We first need to figure out what is the abelian group structure of $Hom_{D(\mathscr{A})}(A^\bullet, B^\bullet)$. One of the biggest trouble is that for $\varphi, \varphi' \in Hom_{D(\mathscr{A})}(A^\bullet, B^\bullet)$, the roof representing them probably have a different top, but this can be solved:

Proposition 1.27. Suppose that we have $\varphi, \varphi' \in Hom_{D(\mathscr{A})}(A^\bullet, B^\bullet)$, represented by the following roofs:

$$\begin{tikzcd} & C_1^\bullet \arrow[dl, "s"'] \arrow[dr, "f"] & & & C_2^\bullet \arrow[dl, "t"'] \arrow[dr, "g"] & \\ A^\bullet & & B^\bullet & A^\bullet & & B^\bullet \end{tikzcd}$$

then there exists an object C such that the roofs above can be represented by:

$$\begin{tikzcd} & C^\bullet \arrow[dl, "q"'] \arrow[dr, "f'"] & & & C^\bullet \arrow[dl, "q"'] \arrow[dr, "g'"] & \\ A^\bullet & & B^\bullet & A^\bullet & & B^\bullet \end{tikzcd}$$

respectively.

Proof. First, consider the following diagram:

$$\begin{tikzcd} C^\bullet \arrow[r, dashed, "j"] \arrow[d, dashed, "k"] & C_2^\bullet \arrow[d, "t"] \\ C_1^\bullet \arrow[r, "s"] & A^\bullet \end{tikzcd}$$

this comes from a previous proposition and since s, t, j are quasi-isomorphisms, k is also a quasi-isomorphism, then we have the following diagram:

$$\begin{tikzcd} & & C \arrow[dl] \arrow[dr] & & \\ & C_2^\bullet \arrow[dl] \arrow[dr] & & C_1^\bullet \arrow[dl] \arrow[dr] & \\ B^\bullet & & A^\bullet & & B^\bullet \end{tikzcd}$$

now if we can prove that the roofs on the left and right are the original morphisms we are done.

This comes from the fact that if you have a roof and on the top of the roof, you post-compose a quasi-isomorphism, you get the same morphism. I will let you to prove it, which is not hard. □

After we prove this, we define that $\varphi + \varphi'$ is just:

$$\begin{tikzcd} & C^\bullet \arrow[dl, "q"'] \arrow[dr, "f'+g'"] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

This is indeed a well-defined notion since if you two ways to represent φ, φ':

$$\begin{tikzcd} & C^\bullet \arrow[dl, "t"'] \arrow[dr, "f"] & & & C^\bullet \arrow[dl, "t"'] \arrow[dr, "g"] & \\ A^\bullet & & B^\bullet & A^\bullet & & B^\bullet \end{tikzcd} \quad \Bigg| \quad \begin{tikzcd} & C'^\bullet \arrow[dl, "s"'] \arrow[dr, "g"] & & & C'^\bullet \arrow[dl, "s"'] \arrow[dr, "h"] & \\ A^\bullet & & B^\bullet & A^\bullet & & B^\bullet \end{tikzcd}$$

I will let you show that we have we have the diagram below, which shows the well-definedness:

$$\begin{tikzcd} & D^\bullet \arrow[dl, "qis"'] \arrow[dr] & & E \arrow[ll, "qis"'] \arrow[rr, "qis"] & & D'^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ C^\bullet \arrow[d, "t"'] \arrow[drr, "s"] & & C'^\bullet \arrow[d, "h"] \arrow[dll, "f"] & & C^\bullet \arrow[d, "t"'] \arrow[drr, "s"'] & & C'^\bullet \arrow[d, "k"] \arrow[dll, "g"'] \\ A^\bullet & & B^\bullet & & A^\bullet & & B^\bullet \end{tikzcd}$$

We still have to show that this definition satisfies the axioms of an additive category. For example, we need to check that composition is bilinear, which I will let to you (a suggestion when you try to check it: start from adding the composed stuff up, don't start with the added morphism $f + g$ and then compose). You should also find the zero object, which is 0 and check that it indeed satisfies $Hom_{D(\mathscr{A})}(0, 0) = 0$. Finally, you need to make sure that direct sum of any two objects exist, which is still $A \oplus B$. To check it's really the direct sum, we need to find the four canonical morphisms, which are:

$$\begin{tikzcd} & A \arrow[dl, "id"] \arrow[dr, "i"] & \\ A & & A \oplus B \end{tikzcd} \quad \begin{tikzcd} & B \arrow[dl, "id"'] \arrow[dr, "j"] & \\ B & & A \oplus B \end{tikzcd} \quad \begin{tikzcd} & A \oplus B \arrow[dl, "id"'] \arrow[dr, "p"] & \\ A \oplus B & & A \end{tikzcd} \quad \begin{tikzcd} & A \oplus B \arrow[dl, "id"'] \arrow[dr, "q"] & \\ A \oplus B & & B \end{tikzcd}$$

and you need to check a few composition conditions, which are all easy to do. To conclude our discussion above:

Proposition 1.28. For any abelian category $\mathscr{A}$, it's derived category is additive.

Proof. See the discussion above. □

Remark 1.29. We mentioned localizing sets when we were proving the composition in $D(\mathscr{A})$ is well-defined. This actually comes from what is known as the localization of a category, and what we have proved above shows that the collection of quasi-isomorphisms indeed localizes in the category $K(\mathscr{A})$, but not in $Kom(\mathscr{A})$.

Definition 1.30. A triangle in $K(\mathscr{A})$ or respectively in $D(\mathscr{A})$:

$$A_1^\bullet \to A_2^\bullet \to A_3^\bullet \to A_1[1]$$

is distinguished if it's isomorphic in $K(\mathscr{A})$ or respectively in $D(\mathscr{A})$ to a triangle of the form:

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\tau} C(f) \xrightarrow{\pi} A^\bullet[1]$$

with f a complex morphism. If we work in $D(\mathscr{A})$, then the morphisms in the second row is those identified with the morphism in $K(\mathscr{A})$.

Proposition 1.31. Distinguished triangles defined above in $K(\mathscr{A})$ or $D(\mathscr{A})$ and the shift functor that maps $A^\bullet$ to $A^\bullet$ makes the homotopy category or the derived category a triangulated category.

Moreover the natural functor:

$$Q_{\mathscr{A}} : K(\mathscr{A}) \to D(\mathscr{A})$$

is an exact functor of triangulated categories.

Proof. It's pretty easy to see that $Q_{\mathscr{A}}$ is an exact functor once we have shown that $K(\mathscr{A})$ and $D(\mathscr{A})$ are indeed triangulated.

For the proof of they are triangulated, please see Gelfand's book since the real proof is not that important for our development later. □

As an exercise of practicing the definitions:

Proposition 1.32. Let $\mathscr{A} = Vec_f(k)$, the category of finite dimensional vector spaces over a fixed field k. Then $D(\mathscr{A})$ is equivalent to $\prod_{i\in\mathbb{Z}} \mathscr{A}$. More precisely, there exist an equivalence of categories

Proof. The key in proving this proposition is to notice that since each vector space is free, thus each each SES sequence splits in $\mathscr{A}$. Also, the main ingredient in the proof is that quasi-isomorphisms become isomorphism in $D(\mathscr{A})$.

The first thing we can prove is that for any acyclic complex with objects in $\mathscr{A}$ there is a homotopy between the zero morphism and the identity morphism. To be precise, suppose that $K^\bullet$ is an acyclic complex, then there is a homotopy $h : K^i \to K^{i-1}$ such that:

$$d \circ h + h \circ d = id$$

To see this, consider the following sequence:

$$\cdots \longrightarrow K^{i-1} \xrightarrow{d^{i-1}} K^i \xrightarrow{d^i} K^{i+1} \longrightarrow \cdots$$

then I will leave it to you to show that since each SES split, by dividing this LES into SES, if we denote $Kerd^i$ by Z^i, we will have the above sequence is isomorphic in both $Kom(\mathscr{A})$ and $K(\mathscr{A})$ to:

$$\cdots \longrightarrow Z^{i-1} \oplus Z^i \longrightarrow Z^i \oplus Z^{i+1} \longrightarrow Z^{i+1} \oplus Z^{i+2} \longrightarrow \cdots$$

with projections as the differentials. Then it's not hard for us to show that the identity is homotopic to the zero map.

Now, let's deal with the case of an arbitrary complex:

$$\cdots \longrightarrow A^{i-1} \longrightarrow A^i \longrightarrow A^{i+1} \longrightarrow \cdots$$

we will still denote Z^i to be the kernel of d^i and B^i to be the image of d^i. Then first consider the surjection $A^i \to B^i$, which has the kernel Z^i, then you know that $A^i \simeq Z^i \oplus B^i$, and the complex now is isomorphic in $Kom(\mathscr{A})$ to:

$$\cdots \longrightarrow Z^{i-1} \oplus B^{i-1} \longrightarrow Z^i \oplus B^i \longrightarrow Z^{i+1} \oplus B^{i+1} \longrightarrow \cdots$$

with differentials being $f : B^i \to Z^{i+1}$. Furthermore, we can decompose Z^i, consider the surjection $Z^i \to H^i$, which has the kernel B^{i-1}, again, from the fact that SES splits, the above complex is again isomorphic in $Kom(\mathscr{A})$ to:

$$\cdots \longrightarrow H^{i-1} \oplus B^{i-2} \oplus B^{i-1} \longrightarrow H^i \oplus B^{i-1} \oplus B^i \longrightarrow H^{i+1} \oplus B^i \oplus B^{i+1} \longrightarrow \cdots$$

I will let you figure out the connecting morphisms but it's actually quite as expected.

Now I will let you check that, if you write the above complex as $\oplus_{i\in\mathbb{Z}} H^i(A^\bullet)[-i] \oplus K^\bullet$ $K^\bullet$, then since $Q_{\mathscr{A}}$ is additive hence commutes with direct sums, then after you apply $Q_{\mathscr{A}}$, the acyclic complex $K^\bullet$ vanishes, thus $Q_{\mathscr{A}}(A^\bullet) \simeq \oplus_{i\in\mathbb{Z}} H^i(A^\bullet)[-i]$.

Finally let's use this to define an equivalence:

$$H^*: D(\mathscr{A}) \to \prod_i \mathscr{A} \qquad Kom: \prod_i \mathscr{A} \to D(\mathscr{A})$$

where H^* is defined by $A^\bullet \mapsto (H^i(A^\bullet))$ and Kom is defined to be $(V_i) \mapsto \oplus V_i[-i]$ with trivial differentials. As for morphisms, how Kom will map morphism is clear and natural. I will let you figure out how to map a roof like morphism in $D(\mathscr{A})$, and also it's well defined. You should also show that this is indeed a functor by showing that it respects composition, and you can also show that both functors are additive.

Now, to show this is indeed an equivalence pair. First, notice that $H^* \circ Kom = Id$ for trivial reasons. To study $Kom \circ H^*$ we know that $A^\bullet$ is mapped to $\oplus_i H^i(A^\bullet)[-i]$ with 0 differentials, we have already show that it's isomorphic to $A^\bullet$ in $D(\mathscr{A})$, as what we have shown. You can also easily show this is a functorial isomorphism, we are done. □

If you think about the above proof, we are only using that each SES in $Vec_f(k)$ split and not any other property of finite dimensional vector spaces, thus the above proof actually gives:

Corollary 1.33. If $\mathscr{A}$ is a semi-simple abelian category, i.e. every SES splits, then $D(\mathscr{A})$ is equivalent to $\prod_i \mathscr{A}$.

Proof. The above proof for vector spaces works. □

It's a good place here to see how cohomology functor play in derived categories: Suppose that $\varphi : A^\bullet \to B^\bullet$ is a morphism in $D(\mathscr{A})$, represented by the roof $A^\bullet \xleftarrow{t} C^\bullet \xrightarrow{f} B^\bullet$, what is $H^*(\varphi)$ in this case? Well, we know that since t and f are all just ordinary morphisms in $K(\mathscr{A})$, after you apply H^*, they give you well defined morphisms, moreover, $H^*(t)$ will be isomorphisms for all indices, therefore, a natural thinking is to define $H^*(\varphi)$ as the composition $H^*(f) \circ H^*(t)^{-1}$. It turns out to be the right defintion and we just need to check that it's well-defined and respects composition of morphisms in $D(\mathscr{A})$, which I will let you do.

The next proposition shows how SES in the original abelian category and the derived category are related:

Proposition 1.34. Suppose that $0 \to A \xrightarrow{f} B \to C \to 0$ is an exact sequence in an abelian category $\mathscr{A}$, then under the full-embedding $\mathscr{A} \to K(\mathscr{A})$ or $\mathscr{A} \to D(\mathscr{A})$, this becomes a distinguished triangle $A \to B \to C \xrightarrow{\delta} A[1]$ in $K(\mathscr{A})$ or respectively in $D(\mathscr{A})$ where $\delta : C \to A[1]$ is given as the inverse (in $K(\mathscr{A})$ and $D(\mathscr{A})$ respectively) of the quasi-isomorphism $C(f) \to C$ and the natural morphism $C(f) \to A[1]$.

Conversely, if $A \to B \to C \to A[1]$ is distinguished in $K(\mathscr{A})$ or $D(\mathscr{A})$, then $0 \to A \to B \to C \to 0$ is indeed exact.

Proof. Well, let's first check from the below diagram that $C(f) \to C$ is indeed a quasi-isomorphism:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d] & B \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \bullet \\
0 \arrow[r] & 0 \arrow[r] & C \arrow[r] & 0 \arrow[r] & \bullet
\end{tikzcd}$$

Now, I will again let you check that if we still assume that the category is semi-simple then $C(f) \to C$ is indeed invertible in $K(\mathscr{A})$ (Hint: again consider the split of $B \to C$).

Well, now let's show that this is indeed a distinguished triangle. First, in $K(\mathscr{A})$, we still assume that everything splits, then the following diagram suffices(similarly in $D(\mathscr{A})$):

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "="'] & B \arrow[r] \arrow[d, "="'] & C(f) \arrow[rr, "\pi"] \arrow[d, "qis"'] & & A[1] \\
A \arrow[r, dashed, "f"] & B \arrow[r] & C \arrow[r] & C(f) \arrow[r, "\pi"] & A[1]
\end{tikzcd}$$

Now conversely, suppose that we have a distinguished triangle $A \to B \to C \to A[1]$ (in $D(\mathscr{A})$ or $K(\mathscr{A})$), to get the desired exact sequence, we quote the result of the following proposition, which allows us to produce a LES via the distinguished triangle:

$$\cdots \longrightarrow H^0(A) \longrightarrow H^0(B) \longrightarrow H^0(C) \longrightarrow H^1(A) \longrightarrow \cdots$$

notice that A, B, C are complexes that are concentrated in degree 0, thus $H^0(A) = A$, similarly for B, C and all the other cohomology vanishes, which gives us the desired SES;

$$0 \to A \to B \to C \to 0$$

□

Proposition 1.35. Suppose that $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ is a distinguished triangle in $K(\mathscr{A})$ or $D(\mathscr{A})$, then it naturally induces a LES:

$$\cdots \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to \cdots$$

Proof. By definition, being a distinguished triangle means that the following diagram is commutative:

$$\begin{tikzcd} A^\bullet \arrow[r, "g"] \arrow[d] & B^\bullet \arrow[r] \arrow[d] & C^\bullet \arrow[r] \arrow[d] & A^\bullet[1] \arrow[d] \\ X^\bullet \arrow[r, "f"] & Y^\bullet \arrow[r, "\tau"] & C(f) \arrow[r, "\pi"] & X^\bullet[1] \end{tikzcd}$$

where all vertical arrows are isomorphisms.

Notice that in the bottom row, $0 \to Y^\bullet \to C(f) \to X^\bullet[1] \to 0$ is exact (we can say this in $K(\mathscr{A})$ not in $D(\mathscr{A})$, but even if we work in $D(\mathscr{A})$, notice that we are identifying this sequence via $Q_{\mathscr{A}}$ and in such situations, the induced morphism after we apply cohomology is the same no matter we work in $K(\mathscr{A})$ or $D(\mathscr{A})$), thus induces a LES:

$$\cdots \longrightarrow H^{i-1}(A[1]) \longrightarrow H^i(B^\bullet) \longrightarrow H^i(C^\bullet) \longrightarrow H^i(A^\bullet[1]) \longrightarrow \cdots$$

also notice that via the commutativity of the two squares on the right, we can obtain the following commutative diagram:

$$\begin{tikzcd} \cdots \arrow[r] & H^{i-1}(A[1]) \arrow[r, dashed] \arrow[d] & H^i(B^\bullet) \arrow[r] \arrow[d] & H^i(C^\bullet) \arrow[r] \arrow[d] & H^i(A[1]) \arrow[d] & \cdots \\ \cdots \arrow[r] & H^{i-1}(X[1]) \arrow[r] & H^i(Y^\bullet) \arrow[r] & H^i(C(f)) \arrow[r] & H^i(X^\bullet[1]) \arrow[r] & \cdots \end{tikzcd}$$

where the above sequence is also exact by choosing the dotted arrow to be the composition, then via identifying $H^{i-1}(A^\bullet[1])$ as $H^i(A^\bullet)$, it's a well-known result in homological algebra that: $H^i(X^\bullet) = H^{i-1}(X^\bullet[1]) \to H^i(Y^\bullet)$ is $H^i(f)$, then I will let you to show that the dotted morphism is actually $H^i(g)$, thus we are done. This argument works in both $D(\mathscr{A})$ and $K(\mathscr{A})$. □

In the above discussion, all the complexes are unbounded, but most of the times, it's more convenient to work with bounded complexes, which are the topic we will talk about right now:

Definition 1.36. Let $Kom^*(\mathscr{A})$, with $* = +, -, b$, be the category of complexes $A^\bullet$ with $A^i = 0$ for $i \ll 0$, $i \gg 0$, respectively $|i| \gg 0$.

Now, notice that we can still considering dividing out homotopy equivalence and obtain $K^*(\mathscr{A})$, then the above construction of derived category still works. For example, if we work with $K^+(\mathscr{A})$, when you try to compose roofs, you need to make sure the top of roof is still a bounded below complex. Say the roof is $A^\bullet \leftarrow C^\bullet \to B^\bullet$, since $A^\bullet$ is bounded below, you have a diagram like the one below, a quasi-isomorphism:

$$\begin{tikzcd} \cdots \arrow[r] & C^{I-2} \arrow[r] \arrow[d] & C^{I-1} \arrow[r] \arrow[d] & C^i \arrow[r] \arrow[d] & C^{I+1} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & A^i \arrow[r] & A^{I+1} \arrow[r] & \cdots \end{tikzcd}$$

I claim that there is another quasi-isomorphism $C'^{\bullet} \to C^{\bullet}$ such that $C'^{\bullet}$ is bounded below. The construction is illustrated as in the diagram below:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "0"] & ker(d^{I-1}) \arrow[r] \arrow[d, "0"] & C^{I-1} \arrow[r, "d^{I-1}"] \arrow[d, "="] & C^i \arrow[r] \arrow[d, "="] & C^{I+1} \arrow[r] \arrow[d, "="] & \cdots \\
\cdots \arrow[r] & C^{I-2} \arrow[r] \arrow[d] & C^{I-1} \arrow[r] \arrow[d] & C^i \arrow[r] \arrow[d] & C^{I+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & A^i \arrow[r] & A^{I+1} \arrow[r] & \cdots
\end{tikzcd}$$

Then via this, you know that the roof $A^{\bullet} \leftarrow C^{\bullet} \to B^{\bullet}$ is equal to the one that has the roof $C'^{\bullet}$. Similarly, the existence bounded above and bounded category can be justified.

You also have a natural functor $D^*(\mathscr{A}) \to D(\mathscr{A})$ by just forgetting the boundedness condition, the below proposition shows that this functor defines an equivalence with a full-triangulated subcategory of $D(\mathscr{A})$:

Proposition 1.37. The natural functor $D^*(\mathscr{A}) \to D(\mathscr{A})$ defines equivalences of $D^*(\mathscr{A})$ with full triangulated subcategories of all complexes $A^{\bullet}$ in $D(\mathscr{A})$ with $H^i(A^{\bullet}) = 0$ for $i \ll 0, \gg 0$ or $|i| \gg 0$.

Proof. I will let you check that this is indeed a full triangulated subcategory using the techniques we built in the previous chapter and the LES we just built in the previous proposition.

The idea of proving that this is really an equivalence is that you first know that this functor $D^*(\mathscr{A}) \to D(\mathscr{A})$ is fully faithful for trivial reasons. Then you only have to show that every object in the full category is isomorphic to the image.

Now suppose that, for example $A^{\bullet}$ is a cohomologically bounded below complex, which means that it's cohomology vanishes for $i \ll 0$. We would like to find a quasi-isomorphism, no matter from it or to it, such that it's quasi-isomorphic to a real bounded complex. We already saw a way to do in the discussion above about $D^*(\mathscr{A})$ is well-defined. However, that is not the ideal result, since we assumed there $H^i(C^{\bullet}) = 0$ for $i < i_0$ but the bounded below complex which is quasi-isomorphic to $C^{\bullet}$ may have a nonzero object at $i_0 - 1$ and $i_0 - 2$. However, that is the sacrifice we have to make if we want a quasi-isomorphism to $C^{\bullet}$. It we can allow a quasi-isomorphism from $C^{\bullet}$, you can consider the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i_0-1} \arrow[r] \arrow[d] & A^{i_0} \arrow[r] \arrow[d] & A^{i_0+1} \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & Coker(d^{i_0-1}) \arrow[r] & A^{i_0+1} \arrow[r] & \cdots
\end{tikzcd}$$

similarly, if you want bounded above, you can consider the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i_0-1} \arrow[r] \arrow[d] & Kerd^{i_0} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & A^{i_0-1} \arrow[r] & A^{i_0} \arrow[r] & A^{i_0+1} \arrow[r] & \cdots
\end{tikzcd}$$

Then in $D(\mathscr{A})$, these quasi-isomorphisms become isomorphisms. □

The same argument shows one of the corollaries we didn't prove:

Corollary 1.38. $\mathscr{A}$ is canonically equivalent to the full subcategory of all objects $A^{\bullet} \in D(\mathscr{A})$ such that $H^i(A^{\bullet}) = 0$ for $i \neq 0$.

Proof. Apply the above proposition to $i_0 = 0$. □

Now we prove a few properties of bounded complexes following similar line of thinking as above:

Proposition 1.39. Let $A^{\bullet}$ be a complex with $m := max\{i | H^i(A^{\bullet}) \neq 0\} < \infty$, then there exists a morphism in the derived category $D(\mathscr{A})$:

$$\varphi : A^{\bullet} \to H^m(A^{\bullet})[-m]$$

such that $H^m(\varphi) : H^m(A^\bullet) \to H^m(A^\bullet)$ is the identity.

Similarly, if $m := min\{i | H^i(A^\bullet) \neq 0\} > -\infty$, then there is a morphism $\psi : H^m(A^\bullet)[-m] \to A^\bullet$ in the derived category with $H^m(\psi) = id$.

Proof. In the proof of this proposition, you can see why this only holds in the derived category, not in the homotopy category.

Consider the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & A^{m-1} \arrow[r] & A^m \arrow[r] & A^{m+1} \arrow[r] & \cdots \\
\cdots \arrow[r] & A^{m-1} \arrow[r] \arrow[u, "id"] \arrow[d, dotted, "0"] & Ker \arrow[r] \arrow[u, hook] \arrow[d, "p"] & 0 \arrow[r] \arrow[u, dotted, "0"] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & H^m(A^\bullet) \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Then, if we assume that all the cohomology beyond m is 0, then the proposition above tells that the vertical arrows on top assemble to a quasi-isomorphism, thus this gives a morphism φ in $D(\mathscr{A})$ and you can easily see that apply H^m to this you get id.

On the other hand, if we assume that all the cohomology below A^m vanishes, then the vertical arrows at the bottom give you a quasi-isomorphism which further gives a morphism ψ from $H^m(A^\bullet)[-m]$ to $A^\bullet$ and $H^m(\psi)$ is the identity is clear. $\square$

Proposition 1.40. Suppose that $H^i(A^\bullet) = 0$ for all $i < m$, then there exists a distinguished triangle:

$$H^m(A^\bullet)[-m] \to A^\bullet \xrightarrow{\varphi} B^\bullet \to H^m(A^\bullet)[1-m]$$

in $D(\mathscr{A})$ with $H^i(B^\bullet) = 0$ for $i \leqslant m$ and φ inducing isomorphisms $H^i(A^\bullet) \simeq H^i(B^\bullet)$ for $i > m$.

Proof. As we know, the most important morphism in a distinguished triangle is the first arrow, since we know that more or less, the second arrow is just the morphism into the mapping cone. Therefore, what could be the first arrow, why not try the one we just had and define B to be the mapping cone. Therefore, we have the following diagram:

$$H^m(A^\bullet)[-m] \xrightarrow{\psi} A \longrightarrow C(\psi) = B \longrightarrow H^m(A^\bullet)[1-m]$$

Then we need to verify this indeed satisfy the desired properties. For example, to check $H^i(B^\bullet) = 0$ for $i \leqslant m$, we need to consider the following LES given by this distinguished triangle, I will let you verify that this indeed suffice. You can also verify that this suffice to prove $H^i(A^\bullet) \simeq H^i(B^\bullet)$. $\square$

Corollary 1.41. If now the complex $A^\bullet$ satisfies $H^i(A^\bullet) = 0$ for all $i > m$, then there is a distinguished triangle:

$$A \to H^m(A^\bullet)[-m] \to B^\bullet \to A[1]$$

such that $H^i(B^\bullet) = 0$ for $i \geqslant m$ and we have $H^i(B^\bullet) \simeq H^{i+1}(A^\bullet)$ for all $i < m$.

Proof. Consider $\varphi : H^m(A^\bullet)[-m] \to A^\bullet$ and taking the mapping cone. In the end, take the induced LES. $\square$

Now, we begin to introduce a very special class of complexes for which morphisms in the derived category is the same as the morphisms in the homotopy category, which is really good since you have notice that not only the morphisms in derived categories are complicated, compose them is even more cumbersome. We begin by injective or projective resolutions:

Definition 1.42. An abelian category $\mathscr{A}$ contains enough injective (respectively enough projectives), if for any object A there exits an injective morphism $A \to I$ with I an injective object (respectively a surjective morphism $P \to A$ with P projective).

An injective resolution of A is an exact sequence:

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$$

with all I^i's injective and you can dually define a projective resolution.

One useful view when we think of injective and projective resolution is worth-mentioning here. For example, an injective resolution can be viewed as a quasi-isomorphism from A to a complex $I^\bullet$ such that $I^i = 0$ for $i = 0$ and all I^i's are injective objects for $i \geqslant 0$. Make sure you understand what's going on here and the similar view can also be applied to projective objects.

Proposition 1.43. Suppose that $\mathscr{A}$ is an abelian category with enough injectives, then for any $A^\bullet \in K^+(\mathscr{A})$, bounded below complex, there exists a complex $I^\bullet \in K^+(\mathscr{A})$ with $I^i \in \mathscr{A}$ injective objects and a quasi-isomorphism $A^\bullet \to I^\bullet$.

In particular, an object A admits an injective resolution.

Proof. As $A^\bullet$ is bounded below, we will proceed by induction as follows:

Let's spell out the strategy first and then do the construction. Suppose that we have constructed a morphism:

$$f_i : A^\bullet \to (\cdots \to I^{i-1} \to I^i \to 0 \to \cdots)$$

such that $H^j(f_i)$ is an isomorphism for $j < i$ and is injective for $j = i$, then we can produce $f_{i+1} : A^\bullet \to (\cdots \to I^{i-1} \to I^i \to I^{i+1} \to 0 \to \cdots)$ such that f^j_{i+1} coincides with f^j_i for $j \leqslant i$, and now we have $H^i(f_{i+1})$ is an isomorphism while $H^{i+1}(f_{i+1})$ is injective, then we keep going.

For the first step, we can assume that $A^\bullet$ is just $0 \to A^0 \to A^1 \to A^2 \to \cdots$, then by assumption we can find $A^0 \to I^0$, injective, which gives us a morphism of complexes:

$$f_0 : A^\bullet \to (0 \to I^0 \to 0 \to \cdots)$$

, you can see that for $i < 0$, it induces an isomorphism and for $i = 0$, it induces an injective morphism.

For the induction step, let's consider the object $(I^0 \oplus A^1)/A^0$, then choose an injective morphism $(I^0 \oplus A^1)/A^0 \to I^1$, then you have:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A^0 \arrow[r] \arrow[d, hook] & A^1 \arrow[r] \arrow[d] & A^2 \\
0 \arrow[r] & I^0 \arrow[r] & I^0 \oplus A^1/A^0 \arrow[r, hook] & I^1
\end{tikzcd}$$

Now we can check the desired cohomological properties easily.

This general ideal works for the definition of $I^{i+1} = I^i \oplus A^{i+1}/A^0$, but I will leave the more technical here and you can check Gelfand's book for a full proof. ☐

The corollary of which is:

Corollary 1.44. Suppose that $\mathscr{A}$ is an abelian category with enough injectives, any $A^\bullet \in D(\mathscr{A})$ with $H^i(A^\bullet) = 0$ for $i \ll 0$ is isomorphic as an object in the derived category to a complex $I^\bullet$ of injective objects and $I^i = 0$ for $i \ll 0$.

Proof. We have already proved that if $A^\bullet$ is cohomologically bounded below, then in the derived category, it's isomorphic to a bounded below object, thus we can assume that $A^\bullet$ is bounded below, then we can apply the above proposition. ☐

We can also spell out the dual statement for a category with enough projectives, we will only state them without proof since they are similar to the injective case:

Proposition 1.45.

1. Suppose that $\mathscr{A}$ is an abelian category with enough projectives, then for any $A^\bullet \in K^-(\mathscr{A})$, there is a complex $P^\bullet$ with each P^i projective objects and a quasi-isomorphism $P^\bullet \to A^\bullet$.
2. Suppose that $\mathscr{A}$ is an abelian category with enough projectives, then any cohomologically bounded above object is isomorphic to a complex $P^\bullet$ of projective objects such that $P^i = 0$ for $i \gg 0$.

The following two lemma are crucial for our main goal of discussing resolutions:

Lemma 1.46. Suppose that $A^\bullet \to B^\bullet$ is a quasi-isomorphism between two complex $A^\bullet, B^\bullet \in K^+(\mathscr{A})$, then for any complex $I^\bullet$ of injective objects I^i with $I^i = 0$ for $i \ll 0$, the induced morphism $Hom_{K(\mathscr{A})}(B^\bullet, I^\bullet) \to Hom_{K(\mathscr{A})}(B^\bullet, I^\bullet)$ is bijective. I.e, in this case, any morphism from $A^\bullet$ to $I^\bullet$ can be obtained by precomposing a morphism from $B^\bullet \to I^\bullet$ with $A^\bullet \to B^\bullet$.

Proof. Let's first complete $A^\bullet \to B^\bullet$ to a distinguished triangle $A^\bullet \to B^\bullet \to C^\bullet \to B^\bullet[1]$ in the homotopy category $K^+(\mathscr{A})$. Notice that since $B^\bullet \to A^\bullet$ is a quasi-isomorphism, you can prove that $C^\bullet$ is an acyclic object via the LES induced by the distinguished triangle, then we reduce the question to proving that for any acyclic object $C^\bullet$, $Hom(C^\bullet, I^\bullet) = 0$. This requires us to construct a homotopy for any complex morphism $g : C^\bullet \to I^\bullet$ and the zero morphism.

This is a standard construction and the idea goes by the following: It's inductively constructed, say that h^j has already been given for $j \leqslant i$, then to construct h^{i+1}, consider the morphism $g_i - d_I^{i-1} \circ h^i$. Notice that since the complex $C^\bullet$ is acyclic, it factors through $C^i/Im(d_C^{i-1})$, also, the differential $C_i \to C^{i+1}$, factors through the quotient by the kernel, which is the same as the image, thus we have the following diagram:

$$\begin{tikzcd}
C^I/Im(d_C^{I-1}) \arrow[r] \arrow[d, hook] & I^i \\
C^{I+1} \arrow[ur, dashed] &
\end{tikzcd}$$

then since I^i is an injective object we have an induced morphism from C^{i+1} to I^i, and you can check that this morphism precompose with d plus $d^{i-1} \circ h^i$ is g^i. ☐

Lemma 1.47. Let $A^\bullet, I^\bullet \in Kom^+(\mathscr{A})$ such that all I^i's are injective, then:

$$Hom_{K(\mathscr{A})}(A^\bullet, I^\bullet) = Hom_{D(\mathscr{A})}(A^\bullet, I^\bullet)$$

i.e. the morphism in the homotopy category is the same as in the derived category.

Proof. Clearly, there is a natural map $Hom_{K(\mathscr{A})}(A^\bullet, I^\bullet) \to Hom_{D(\mathscr{A})}(A^\bullet, I^\bullet)$, at this point, we don't know any thing about this morphism, but our goal is to show that for any morphism in $\varphi \in Hom_{D(\mathscr{A})}(A^\bullet, I^\bullet)$, there is a unique morphism in $Hom_{K(\mathscr{A})}(A^\bullet, I^\bullet)$ such that under the natural morphism above, the image is φ.

Suppose that φ is the following roof:

$$\begin{tikzcd}
& B^\bullet \arrow[dl, "t=qis"'] \arrow[dr] & \\
A^\bullet & & I^\bullet
\end{tikzcd}$$

then if there is a morphism $g : A^\bullet \to B^\bullet$ such that the above diagram commutes in $K(\mathscr{A})$, then we know that the image of g is φ, which will prove the natural map is surjective. But this is the proposition above.

Is this natural map injective? The answer is yes, say the image of g is φ, then if we have the image of another g' is also φ, then I will let you check that we have the following diagram:

$$\begin{tikzcd}
& & C^\bullet \arrow[dl, "s=qis"'] \arrow[d, "s=qis"] \\
& B^\bullet \arrow[dl, "t=qis"'] \arrow[dr] & A^\bullet \arrow[dll, "id"'] \arrow[d, "g'"] \\
A^\bullet \arrow[rr, dashed, "g"'] & & I^\bullet
\end{tikzcd}$$

which proves that $g' = g$ in $K(\mathscr{A})$.

There is a subtlety in this proof: since we are quoting the proceeding lemma, this object on the top of the roof should also be bounded below, but the quasi-isomorphism only makes it cohomologically bounded below. But we have already seen how to convert a cohomologically bounded below object to a bounded below object in $D(\mathscr{A})$. ☐

In the following proposition, we consider the full additive subcategory $\mathscr{I} \subset \mathscr{A}$, which consists of all injective objects. You can check that this is still an addtive category by showing that direct sum of two injective objects is still injective. Then since we can define for each additive category $K^*(-)$, we can define $K^*(\mathscr{I})$, which is still triangulated.

Also, we have a natural inclusion functor $K^*(\mathscr{I}) \to K^*(\mathscr{A})$, which can be composed by $Q_{\mathscr{A}} : K^*(\mathscr{A}) \to D(\mathscr{A})$, the result of which is called $\iota : K(\mathscr{I}) \to D(\mathscr{A})$. Both functors are exact, thus the composition is still exact, and in the following proposition, we will show that this functor is an equivalence:

Proposition 1.48. Suppose that $\mathscr{A}$ is an abelian category with enough injectives, then the natural functor $\iota : K(\mathscr{I}) \to D(\mathscr{A})$ is an equivalence.

Proof. We will prove that ι is a fully-faithful functor which is essentially surjective.

Even without the hypothesis that $\mathscr{A}$ has enough injectives, ι is always fully-faithful. Since morphisms in $K^+(\mathscr{A})$ are complex morphisms between injective objects, then the two lemma above shows that this is indeed a bijection for $Hom_{K(\mathscr{A})}(I^\bullet, J^\bullet) \to Hom_{D(\mathscr{A})}(I^\bullet, J^\bullet)$ with $I^\bullet, J^\bullet$ complexes of injective objects which is bounded below.

Now we need to show that it's essentially surjective. This means that for any bounded below complex $A^\bullet$, there is a quasi-isomorphism from $A^\bullet$ to a bounded below complex consisting of injective objects, but this is also a previous proposition. $\square$

Eventually after we finished all the basic discussions of derived categories, we will be interested in $D^b(Coh(X))$, the derived category of coherent sheaves of a scheme. In order to study this, the following inclusion of abelian categories is useful:

$$Coh(X) \subset Qcoh(X) \subset Sh(X)$$

In many cases, even we only want to study coherent sheaves on X, we need to go out of the category of coherent sheaves, to study quasi-coherent sheaves or even sheaves. To study this case, we introduce the following defintion:

Definition 1.49. A thick subcategory $\mathscr{A}$ of an abelian category $\mathscr{B}$ is a full abelian subcategory such that any extension in $\mathscr{B}$ of objects in $\mathscr{A}$ is again in $\mathscr{A}$.

Proposition 1.50. Let $\mathscr{A} \subset \mathscr{B}$ be a thick subcategory and suppose that any $A \in \mathscr{A}$ can be embedded in an object $A' \in \mathscr{A}$ which is injective as an object of $\mathscr{B}$. Then the natural functor:

$$D_+(\mathscr{A}) \to D_+(\mathscr{B})$$

induces an equivalence $D^+(\mathscr{A}) \to D^+_{\mathscr{A}}(\mathscr{B})$ of $D^+()$ and the full triangulated subcategory $D^+_{\mathscr{A}}(\mathscr{B}) \subset D_+(\mathscr{B})$ of complexes with cohomology in $\mathscr{A}$.

Analogously, one has $D^b(\mathscr{A}) \simeq D^b_{\mathscr{A}}(\mathscr{B})$.

Proof. I will first let you check that being a thick subcategory indeed ensures that $D^*_{\mathscr{A}}(\mathscr{B})$ is a full-triangulated subcategory of $D^*(\mathscr{B})$. (Hint, use the LES of a distinguished triangle and the fact that $\mathscr{A}$ itself is abelian thus kernels and cokernels are in $\mathscr{A}$).

We will leave the assertion that this is fully faithful and essentially surjective to the interested readers to check Gelfand's book. $\square$

1.2 Derived Functors

After we have developed the basic settings of derived categories, we head to the discussion of derived functors, which is essential in a lot of the geometric stuff we will discuss later.

Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor between abelian categories, we have seen that the image of an acyclic complex is not in general acyclic. Remember that we know that acyclic objects are quasi-isomorphic to 0, thus if we want to naively extend F to a functor $F : D(\mathscr{A}) \to D(\mathscr{B})$ by $F(A) = F(A)$, a 0 object may be mapped to a non-zero object, thus F can't be additive, thus this is not ideal. To resolve this problem, we need to introduce the derived functors to extend F to the derived category. But before we do it, we can verify the following lemma:

Lemma 1.51. Let $F : K^*(\mathscr{A}) \to K^*(\mathscr{B})$ be an exact functor of triangulated categories, then F naturally induces a commutative diagram:

$$\begin{tikzcd} K^*(\mathscr{A}) \arrow[r] \arrow[d] & K^*(\mathscr{B}) \arrow[d] \\ D^*(\mathscr{A}) \arrow[r, dashed] & D^*(\mathscr{B}) \end{tikzcd}$$

if one of the following conditions is true.

1. Under F, a quasi-isomorphism is mapped to a quasi-isomorphism.
2. The image of an acyclic complex is still acyclic.

Proof. Suppose that the first condition holds, then the universal property of derived category induces the dotted arrow and you can easily guess what is the explicit construction.

Now suppose that the second conditions holds, we claim that it implies the first condition if we assume that F is exact. Now suppose that $A^\bullet \to B^\bullet$ is a quasi-isomorphism, then the mapping cone is acyclic, consider the corresponding distinguished triangle:

$$A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet = Cone \longrightarrow A^\bullet[1]$$

but after you apply F, it's still distinguished and $F(C^\bullet)$ is still acyclic, thus $F(A^\bullet) \to F(B^\bullet)$ is still a quasi-isomorphism.

Therefore, in particular, if F comes from a functor G between $\mathscr{A}$ and $\mathscr{B}$, if G is exact, then it maps acyclic objects to acyclic objects, thus extends to a functor between derived categories. ☐

We now begin to discover what we will do if F is not exact, which gives you the construction of the derived functor and explains why the original functor fails to be exact:

To begin with, we start with an additive functor between two categories $F : \mathscr{A} \to \mathscr{B}$, in order to construct the derived category, we have to always assume some exactness, even though F fails to be exact. Therefore, if F is left exact, we will construct the right derived functor:

$$RF : D^+(\mathscr{A}) \to D^+(\mathscr{B})$$

and if F is right exact, we can construct the left derived functor:

$$LF : D^-(\mathscr{A}) \to D^-(\mathscr{B})$$

Both constructions are similar to each other, thus we will only discuss RF.

To start with, we again assume that $\mathscr{A}$ has enough injectives, then by the previous discussion, we have an equivalence:

$$K^+(\mathscr{I}_{\mathscr{A}}) \overset{\iota}{\hookrightarrow} K^+(\mathscr{A}) \overset{Q_{\mathscr{A}}}{\to} D^+(\mathscr{A})$$

Remember that our goal is to produce the dotted arrow from $D^+(\mathscr{A}) \to D^*(\mathscr{B})$, but what we have right now is the following diagram:

$$\begin{tikzcd} K^+(\mathscr{I}_{\mathscr{A}}) \arrow[r, hook] \arrow[dr, "\iota"'] & K^+(\mathscr{A}) \arrow[r, "K(F)"] \arrow[d, "Q_{\mathscr{A}}"] & K^+(\mathscr{B}) \arrow[d, "Q_{\mathscr{B}}"] \\ & D^+(\mathscr{A}) & D^+(\mathscr{B}) \end{tikzcd}$$

since ι is an equivalence, it will have a quasi-inverse, which we will call ι^{-1}, thus, if we follow ι^{-1}, then $K(F)$ (you can guess it's definition), finally $Q_{\mathscr{B}}$, we get a functor from $D^+(\mathscr{A})$ to $D^+(\mathscr{B})$, this will be the right derived functor of F:

Definition 1.52. The right derived functor of F is the functor:

$$RF := Q_{\mathscr{B}} \circ K(F) \circ \iota^{-1}$$

Remark 1.53. I have to admit that due to the reason that we don't want to go too deep into the details of the construction, you could feel confused if you really paid attention to the writing.

The above definition seems totally fine even if F is not left exact. This is indeed true and this definition totally makes sense even if F is not left exact. The reason we want this to be left exact is that in this case, $K(F)$ will map

Below are some of the main properties of the derived functors:

Proposition 1.54. 1. There exist a natural morphism of functors:

$$Q_{\mathscr{B}} \circ K(F) \to RF \circ Q_{\mathscr{A}}$$

2. The right derived functor $RF : D^+(\mathscr{A}) \to D^+(\mathscr{B})$ is an exact functor of triangulated categories.

3. Suppose that $G : D^+(\mathscr{A}) \to D^+(\mathscr{A})$ is an exact functor, then any functor morphism $Q_{\mathscr{B}} \circ K(F) \to G \circ Q_{\mathscr{A}}$ factors through a unique functor morphism $RF \to G$:

$$\begin{tikzcd} Q_{\mathscr{B}} \circ K(F) \arrow[r] \arrow[dr] & RF \circ Q_{\mathscr{A}} \arrow[d, dashed] \\ & G \circ Q_{\mathscr{A}} \end{tikzcd}$$

Proof. Let's first consider the first assertion. For a complex $A^\bullet \in D^+(\mathscr{A})$, we will denote $I^\bullet$ by $\iota^{-1}(A^\bullet)$, then the natural isomorphism between id and $\iota \circ \iota$ gives you a functorial isomorphism $A \to I^\bullet$ in $D^+(\mathscr{A})$, which is given by a roof $A^\bullet \leftarrow C^\bullet \to I^\bullet$, then since $I^\bullet$ is injective, then we know that there is a unique morphism in $K^+(\mathscr{A})$ $A^\bullet \to I^\bullet$ such that the following diagram commutes:

$$\begin{tikzcd} & C^\bullet \arrow[dl] \arrow[dr] & \\ A^\bullet \arrow[rr] & & I^\bullet \end{tikzcd}$$

and we have proved that this $A^\bullet \to I^\bullet$ is independent of the choice of the representing object.

I will let you to prove that this $A^\bullet \to I^\bullet$ is functorial in $A^\bullet$, then you apply $K(F)$, which gives you a functorial morphism from $KF(A^\bullet) \to KF(I^\bullet) = RF(A^\bullet)$.

For the second assertion, we only have to prove that $\iota^{-1} : D^+(\mathscr{A}) \to K^+(\mathscr{I}_{\mathscr{A}})$ is exact, but this follows from that ι is obviously exact, and as the inverse of an exact equivalence between triangulated categories, we have proved in the first notes that ι^{-1} is exact.

For the last assertion, I will again redirect you to Gelfand's book. This is the universal property of RF, and these properties uniquely determines RF up to isomorphism. ☐

Definition 1.55. Let $RF : D^+(\mathscr{A}) \to D^+(\mathscr{B})$ be the right derived functor of a left exact functor $F : \mathscr{A} \to \mathscr{B}$, then for any complex $A^\bullet \in D^+(\mathscr{A})$, one defines:

$$R^iF(A^\bullet) = H^i(RF(A^\bullet)) \in \mathscr{B}$$

Remark 1.56. Recall that we have a natural functor $\mathscr{A}^\bullet \to D^+(\mathscr{A})$, then pre-compose R^iF with this functor yields a natural additive functor $R^iF : \mathscr{A} \to \mathscr{B}$, called the higher derived functors of F.

First, what happens if $i < 0$, well, by definition, $R^iF(A) = H^i(RF(A))$, recall how $RF(A)$ is defined, we are trying to find a complex of injectives that is isomorphic to $A^\bullet$, thus it's just the injective resolution of A, then you take the cohomology of this complex after we apply F, thus for $i < 0$, you get 0. What happens if $i = 0$, I claim that $R^0F \simeq A$, I will let you to check this but the idea is still the same, you find an injective resolution, apply F and take cohomology, but remember that F is left exact.

Corollary 1.57. Under the above assumption where we defined the higher derived functors, for any SES in $\mathscr{A}$:

$$0 \to A \to B \to C \to 0$$

there is a induced LES:

$$0 \to FA \to FB \to FC \to R1FA \to \cdots$$
$$\cdots \to R^iFB \to R^iFC \to R^{i+1}FA \to \cdots$$

Proof. Remember that this SES can be viewed as a SES in $D^+(\mathscr{A})$, which gives rise to a distinguished triangle as what we proved before:

$$A \to B \to C \to A[1]$$

Then since RF is exact, then we can apply RF to get:

$$RFA \to RFB \to RFC \to RFA[1]$$

then take the induced LES to get the desired result. □

The above discussion is the usual way one defines derived functors, however, in a lot of geometric cases, some of the assumptions above is not available, thus the following remark helps us weaken the assumptions we had above:

Remark 1.58. Let's first state some of the possible situations we may find ourselves in: Sometimes, the functor we are interested in is only given between the homotopy category of complexes, they are not induced by a functor between the original abelian categories. It's also possible that we may have to work with categories where it's problematic or even impossible to find enough injectives.

There is a general statement that can solve the two problems at the same time, which we will give here right away:

Suppose that $\mathscr{A}$ and $\mathscr{B}$ are abelian categories and:

$$F : K^+(\mathscr{A}) \to K(\mathscr{B})$$

is an exact functor between triangulated categories. Then the right derived functor $RF : D^+(\mathscr{A}) \to D(\mathscr{B})$ (well, this is not the same derived functor as in the proposition above, but they share the same name), which satisfies the conditions 1 to 3 in the above proposition about basic property of derived functors, exists whenever there is a triangulated subcategory $\mathcal{K}_F \subset K^+(\mathscr{A})$ which is adapted to F, which means that the following conditions hold:

1. If $A^\bullet \in \mathcal{K}_F$ is acyclic, then $F(A^\bullet)$ is acyclic.
2. Any $A^\bullet \in K^+(\mathscr{A})$ is quasi-isomorphic to a complex in $\mathcal{K}_F$.

We will need this general facts later for defining $RHom(A^\bullet, B^\bullet)$ and the left derived functor for tensor products.

Beside this most general case, we have another way to construct derived functors when F is indeed defined on the original abelian categories but instead, this time finding enough injectives is hard.

In this case, one will define the notion of adapted on the abelian categories: A class of objects $\mathscr{I}_F$ in $\mathscr{A}$ is called F-adapted if the following two conditions hold true:

1. If $A^\bullet \in K^+(\mathscr{A})$ is acyclic with objects in $\mathscr{I}_F$, then $F(A^\bullet)$ is still acyclic.
2. An object can be embedded into an object in $\mathscr{I}_F$.

With these conditions, we can consider localization of $K^+(\mathscr{I}_F)$ by quasi-isomorphisms, just like what we do for $K^+(\mathscr{I})$, the complex of injective objects, then we can similarly define derived functors.

As an example, if $\mathscr{A}$ has enough injectives, then the class of injective objects is a F-adapted class for any left exact functor F. In the following exercise, we will show that you can enlarge this class by adding on F-acyclic objects and get a larger F-adapted class.

Proposition 1.59. Let $\mathscr{I}_F$ be an F-adapted class, e.g. the class of all injective objects in an abelian category with enough injectives and F is left exact, then enlarging $\mathscr{I}_F$ to be all F-acyclic ones yields again an F-adapted class.

Proof. *** *later* □

Proposition 1.60. Let $\mathscr{I}_F$ be an adapted class for a left exact functor $F : \mathscr{A} \to \mathscr{B}$, then $R^iF(A) = 0$ for all $A \in \mathscr{I}_F$ and $i \neq 0$.

Proof. *** *later* ∎

Proposition 1.61. Suppose that we know the right derived functor of F exist, and $A^\bullet$ is a complex of F-acyclic objects then $RF(A^\bullet) \simeq KF(A^\bullet)$.

Proof. *** *later* ∎

In the next notes, we will study in detail a lot of functors of geonmetric meaning, so here the only abstract setting we would consider right now is the covariant functor $Hom(A, -) : \mathscr{A} \to Ab$. We already know that $Hom(A, -)$ is left exact and if $\mathscr{A}$ contains enough injectives, we can define $Ext^i_{\mathscr{A}}(A, -) := H^i \circ RHom(A, -)$. It turns out that these Ext groups can be described by certain homomorphism groups inside the derived category:

Proposition 1.62. Suppose that $A, B \in \mathscr{A}$ are objects of an abelian category containing enough injectives, then there are natural isomorphisms:

$$Ext^i_{\mathscr{A}}(A, B) \simeq Hom_{D(\mathscr{D})}(A, B[i])$$

where A, B on the right hand side is consider to be complexes concentrated in degree 0.

Proof. Suppose that we have an injective resolution of B:

$$B \to I^0 \to I^1 \to \cdots$$

then by construction of the derived functor, $RHom(A, B)$ is going to be, as an object of $D^+(Ab)$, isomorphic to the complex $Hom(A, I^i)$, thus Ext^i is just the cohomology of this complex.

First, what is the kernel of $Hom_{\mathscr{A}}(A, I^i) \to Hom_{\mathscr{A}}(A, I^{i+1})$. Notice that this map is still take $f \in Hom_{\mathscr{A}}(A, I^i)$ and then compose it with $I^i \to I^{i+1}$, thus f is in the kernel iff f defines a complex morphism $A \to I^\bullet[i]$, thus the kernel is $Hom_{Kom(\mathscr{A})}(A, I^\bullet[i])$.

Then what is the image of $Hom_{\mathscr{A}}(A, I^{i-1}) \to Hom_{\mathscr{A}}(A, I^i)$, I will let you check that f is in the image of this map iff f is homotopically trivial, thus, we quotient out this to get the i-th cohomology is just $Hom_{K(\mathscr{A})}(A, I^\bullet[i])$.

Since $I^\bullet$ is a complex of injectives, thus we have proved that in this case:

$$Hom_{K(\mathscr{A})}(A, I^\bullet[i]) = Hom_{D(\mathscr{A})}(A, I^\bullet[i])$$

Then, notice that $B \simeq I^\bullet$ in the derived category, then we substitute $I^\bullet$ with $B[i]$, to obtain:

$$Hom_{K(\mathscr{A})}(A, I^\bullet[i]) = Hom_{D(\mathscr{A})}(A, B[i])$$

∎

Remark 1.63. Actually the above proposition can be taken further: To do this, let's consider a new functor, again we work with an abelain category $\mathscr{A}$ with enough injectives, then suppose that $A^\bullet \in Kom(\mathscr{A})$, then there is a covariant functor:

$$Hom^\bullet(A^\bullet, -) : K^+(\mathscr{A}) \to K(Ab)$$

which associate to each $B^\bullet \in K^+(\mathscr{A})$ it's **inner hom**: $Hom^\bullet(A^\bullet, B^\bullet)$. So what is it?

By definition, the inner hom $Hom^\bullet(A^\bullet, B^\bullet)$ should be a complex with each term being:

$$Hom^i(A^\bullet, B^\bullet) = \bigoplus_k Hom(A^k, B^{k+i})$$

Then what is the differential, it should be component-wise:

$$d(f) = d_B \circ f - (-1)^i f \circ d_A$$

then you can easily check that this is a complex and $Hom^\bullet(A^\bullet, \varphi)$ for $\varphi : B_1^\bullet \to B_2^\bullet$ should be obvious.

It's worth-noting that this functor is an exact functor between triangulated categories, which we won't prove here. Moreover, you can prove that the full triangulated subcategory of complex of injectives is adapted to this functor, thus we can define:

$$RHom^\bullet : D^+(\mathscr{A}) \to D(Ab)$$

and we will set $Ext^i(A^\bullet, B^\bullet) = H^i(RHom^\bullet(A^\bullet, B^\bullet))$.

Finally, the proof of the above proposition can be adapted to a proof of:

$$Ext^i(A^\bullet, B^\bullet) = Hom_{D(\mathscr{A})}(A^\bullet, B^\bullet[i])$$

From the above discussion, you can see that $Ext^i(A^\bullet, B^\bullet)$ for a fixed $B^\bullet$ only depends on the objects that is isomorphic to $A^\bullet$ in the derived category.

There are also discussions of how to think about $A^\bullet$ as the variable and compare the difference of two different constructions, but we won't discuss them here.

Remark 1.64. Via the isomorphisms discussed above, we can define the following map:

$$Ext^i(A^\bullet, B^\bullet) \times Ext^j(B^\bullet, C^\bullet) \to Ext^{i+j}(A^\bullet, C^\bullet)$$

How this map works, notice that $Ext^i(A^\bullet, B^\bullet) \simeq Hom_{D(\mathscr{A})}(A^\bullet, B^\bullet[i])$ and $Ext^j(B^\bullet, C^\bullet) \simeq Hom_{D(\mathscr{A})}(B^\bullet, C^\bullet[j]) = Hom_{D(\mathscr{A})}(B^\bullet[i], C^\bullet[i+j])$, then we have a composition and get something that lies in $Hom_{D(\mathscr{A})}(A^\bullet, C^\bullet[i+j]) \simeq Ext^{i+j}(A^\bullet, C^\bullet)$.

Proposition 1.65. Let $F_1 : \mathscr{A} \to \mathscr{B}$ and $F_2 : \mathscr{B} \to \mathscr{C}$ be left exact functors of abelian categories. Assume that there exist adapted classes $\mathscr{I}_{F_1} \subset \mathscr{A}$ and $\mathscr{I}_{F_2} \subset \mathscr{B}$ such that $F(\mathscr{I}_{F_1}) \subset \mathscr{I}_{F_2}$. Then the derived functors $RF_1, RF_2, R(F_2 \circ F_1)$ all exist and there is a natural isomorphism:

$$R(F_2 \circ F_1) \simeq RF_2 \circ RF_1$$

Proof. This basically is just the universal property of derived functors, since we didn't really prove it in the notes, we will skip this proof.

However, in the following remark, we will briefly talk about how to use this proposition. □

Remark 1.66.

1. Suppose that $\mathscr{A}$ and $\mathscr{B}$ are abelian categories both with enough injectives, then if we know that F is a functor that maps injective objects to injective objects, the assumptions in the above propositions holds.

   However, it's usually hard to verify this directly. Therefore, a common way of doing this is to enlarge the class of F-adapted objects. For example, we will always enlarge $\mathscr{I}_{\mathscr{B}}$, the class of injective objects to the class of F_2-acyclic objects, which gives you a new class of F_2-adapted objects. Then we only have to prove that in this case F maps injective objects in $\mathscr{A}$ to F_2-acyclic objects.

2. The result also holds true for the derived functor of exact functors between the homotopy category, but in this case, one has to assume the existence of adapted triangulated subcategories $\mathcal{K}_{F_1}, \mathscr{K}_{F_2}$ and $F_1(\mathcal{K}_{F_1}) \subset \mathscr{K}_{F_2}$.

1.3 Spectral Sequences

Before we end the abstract discussion of category theory and head to real geometric objects, we would like to briefly introduce the usage of spectral sequences for later use, especially when we want to compose derived functors later. We will try to still keep away from technical details and instead to provide the amount necessary for applications in the later notes.

Let's go straight into the definition of spectral sequences: The main data of a spectral sequence in an abelian category $\mathscr{A}$ is a collection of objects:

$$(E_r^{p,q}, E^n), p, q, r, n \in \mathbb{Z}, r \geqslant 1$$

and morphisms

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

subject to the following conditions:

1. For all p, q, r, we have $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$, which thus yields a complex $E_r^{p+\bullet \cdot, q-\bullet \cdot r+\bullet}$ where $\bullet$ stands for the label $i \in \mathbb{Z}$. The meanings of these labels will be more clear if you can see an visualization of the data of a spectral sequence, but here you can image that if you for the moment don't worry about the E^n's, $E_r^{p,q}$, with the three labels, represent a three dimensional grid, with coordinates p, q, r where p, q are the x, y coordinate and r represent the vertical z coordinate.

2. There are isomorphisms $E_{r+1}^{p,q} \simeq H^0(E_r^{p+\bullet \cdot, q-\bullet \cdot r+\bullet})$, which is part of the data of the spectral sequence.

3. For any p, q, there exists an r_0 such that $d_r^{p,q} = d_r^{p-r,q+r-1} = 0$ for all $r \geqslant r_0$. In particular, the isomorphism in 2 tells you that $E_r^{p,q} \simeq E_{r_0}^{p,q}$ for all $r \geqslant r_0$, we will call this $E_{r_0}^{p,q}$ $E_\infty^{p,q}$.

4. There is a decreasing filtration
$$F^{p+1}E^n \subset F^pE^n \subset \cdots \subset E^n$$
such that $\bigcap F^pE^n = 0$ and $\bigcup F^pE^n = E^n$ and isomorphisms:
$$E_\infty^{p,q} \simeq F^pE^{p,q}/F^{p+1}E^{p,q}$$
thus in some sense, the objects $E_r^{p,q}$, as r goes to infinity, is converging to a subquotient of a certain filtration of E^n.

Remark 1.67. Though the definition of a spectral sequence looks complicated and terrifying, in most of the applications, one in general doesn't go beyond $r = 2$. Typical arguments goes likes this: for some reason, we know in the first place that the differentials beyond $r = 2$ level vanish, hence E^n admits a filtration of which the subquotients are isomorphic to $E_2^{p,n-p}$. For example, if everything are vector spaces, i.e. all SES splits, then this yields a non-canonical isomorphism $E^n = \bigoplus E_2^{p,n-p}$.

Or sometimes, one only knows the vanishing of $d_r^{p,q}$ and $d_r^{p-r,q+r-1}$ for some specifically fixed p, q and all r, in this case, for example if you know that $E_2^{p,q}$ is not trivial, then you can conclude that E^{p+q} is not trivial if you examine the filtration.

Remark 1.68. In the above definition we require that the spectral sequence starts from $r = 1$, but in practice, we may as well just define the spectral sequence to be starting from $r = m$, the amount of information that's useful for us is the same.

For us, you will see that for most of the applications, r will start from 2.

Usually, to simplify the notation, if we have given $E_r^{p,q}$ for a certain layer r, we will condense the other informations and only write $E_r^{p,q} \Rightarrow E^{p+q}$.

The standard source of studying spectral sequence comes from nice filtrations of complexes and such nice filtrations occur naturally on the total complex of a double complex concentrated in, e.g. the first quadrant, thus we will briefly sketch this story:

Definition 1.69. A double complex $K^{\bullet,\bullet}$ consists of objects $K^{i,j}$ for $i, j \in \mathbb{Z}$ and morphisms $d_I^{ij} : K^{i,j} \to K^{i+1,j}$ and $d_{II}^{ij} : K^{i,j} \to K^{i,j+1}$ satisfying:
$$d_I^2 = d_{II}^2 = d_I d_{II} + d_{II} d_I = 0$$
The total complex $K^\bullet := tot(K^{\bullet,\bullet})$ of the double complex is the complex $K^n = \oplus_{i+j=n} K^{i,j}$ with $d = d_I + d_{II}$, and you should check that this is indeed a complex.

Example 1.70. Recall the complex $Hom^\bullet(A^\bullet, B^\bullet)$ we defined before with $Hom^i(A^\bullet, B^\bullet) = \oplus Hom(A^k, B^i + k)$, we can realize this complex as the total complex of the doubloe complex $K^{i,j} = Hom(A^{-i}, B^j)$ endowed with the differentials $d_I = (-1)^{j-i+1} d_A$ and $d_{II} = d_B$. This is a pain to check but there is nothing deep here.

On the total complex $K^\bullet$ of a double complex, there is a natural decreasing filtration (or to be more precise, there are two of such natural due to the symmetry of the situation):
$$F^lK^n := \oplus_{j \leqslant l} K^{n-j,j}$$

or

$$F^lK^n = \oplus_{j\leqslant l}K^{j,n-j}$$

with the first one corresponding to the direct towards the left top and the second one corresponding to the direct sum to the bottom right.

Notice that this filtration satisfies $d(F^lK^n) \subset F^lK^{n+1}$

A natural generalization of this conditions leads us to the notion of a filtered complex:

Definition 1.71. A filtered complex is a complex $K^\bullet$ together with a decreasing filtration $\cdots F^lK^n \subset F^{l-1}K^n \subset \cdots \subset K^n$ for each K^n such that $d^n(F^lK^n) \subset F^lK^{n+1}$, which means that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & K^{n-1} \arrow[r] & K^n \arrow[r] & K^{n+1} \arrow[r] & \cdots \\
\cdots \arrow[r] & F^lK^{n-1} \arrow[r] \arrow[u, hook] & F^lK^n \arrow[r] \arrow[u, hook] & F^lK^{n+1} \arrow[r] \arrow[u, hook] & \cdots \\
\cdots \arrow[r] & F^{l+1}K^{n-1} \arrow[r] \arrow[u, hook] & F^{l+1}K^n \arrow[r] \arrow[u, hook] & F^{l+1}K^{n+1} \arrow[r] \arrow[u, hook] & \cdots
\end{tikzcd}$$

Thus we see that any total complex of a double complex gives you a filtered complex. Moreover, we know that we have:

$$F^lK^n/F^{l+1}K^n = K^{n-l,l}$$

then if we denote this quotient by gl^lK^n, we know that this forms a natural complex $gl^lK^\bullet = K^{\bullet,l}[-l]$, which is up to a global sign change $(-1)^l$ since the twist changes the sign of the differential. Hence $H^k(gl^l(K^\bullet)) = H^{k-l}(K^{\bullet,l})$, again, this is still a complex with the morphism induced by d_{II}, thus we can take the cohomology again, which gives us $H^l_{II}H^{k-l}_I(K^{\bullet,\bullet})$, this is the object we are interested in in the following proposition:

Proposition 1.72. Suppose that $K^{\bullet,\bullet}$ is a double complex such that for any n, one has $K^{n-l,l} = 0$ for $|l| \gg 0$, then the natural filtration of the total complex of the double complex naturally induces a spectral sequence with:

$$E_2^{p,q} = H^p_{II}H^q_I(K^{\bullet,\bullet}) \Rightarrow H^{p+q}(K^\bullet)$$

Proof. The above finiteness condition of the double complex can be verified such that the above proposition works for any filtered complex, but here we won't care about that modified version.

The proof is not so hard, just very cumbersome, thus we will not try to prove it here. But you should keep in mind that you still need to specify what are the $E_r^{p,q}$'s for $r \neq 2$ and you have to give the filtration of $H^n(K^\bullet)$. ☐

The following proposition also gives us a very important result we won't prove:

Proposition 1.73. Let $F_1 : K^+(\mathscr{A}) \to K^+(\mathscr{B})$ and $F_2 : K^+(\mathscr{B}) \to K(\mathscr{C})$ be two exact functors between triangulated categories, then if $\mathscr{B}, \mathscr{A}$ contains enough injectives and that the image under F_1 of a complex $I^\bullet$ with I^i injective is contained in an F_2-adapted triangulated subcategory $\mathcal{K}_{F_2}$, then for any complex $A^\bullet \in D^+(\mathscr{A})$, there is a spectral sequence: $E_2^{p,q} = R^pF_2(R^qF_1(A^\bullet)) \Rightarrow E^{p+q} = R^{p+q}(F_2 \circ F_1)(A^\bullet)$.

Proof. The proof of this uses the special case mentioned in the remark below and something called the Cartan-Eilenberg resolution, thus we will omit it here. ☐

Remark 1.74. One special case of this proposition is worth-mentioning. Suppose that F_1 is the identity and F_2 is induced by a left exact functor F, then the above spectral sequence is just:

$$E_2^{p,q} = R^pF(H^q(A^\bullet)) \Rightarrow E^{p+q} = R^{p+q}F(A^\bullet)$$

The special case mentioned in the remark gives the following corollary:

Corollary 1.75. Suppose that $F : K^+(\mathscr{A}) \to K^+(\mathscr{B})$ is an exact functor which admits a right derived functor $RF : D^+(\mathscr{A}) \to K^+(\mathscr{B})$ and $\mathscr{A}$ has enough injectives. Then:

1. Suppose that $\mathscr{C} \subset \mathscr{B}$ is a thick subcategory with $R^iF(A) \in \mathscr{C}$ for every $A \in \mathscr{A}$ and that there exists $n \in \mathbb{Z}$ such that $R^iF(A) = 0$ for $i < n$ for all $A \in \mathscr{A}$, then RF actually takes values in $D +_{\mathscr{C}} (\mathscr{B})$.

2. If $RF(A) \in D^b(\mathscr{B})$ for every $A \in \mathscr{A}$, then $RF(A^\bullet) \in D^b(\mathscr{B})$ for any $A^\bullet \in D^b(\mathscr{A})$, i.e. RF induces:

$$D^b(\mathscr{A}) \to D^b(\mathscr{B})$$

Proof. **** omitted for now* □

We will use the following example in the next chapter:

Example 1.76. Let $A^\bullet, B^\bullet \in D(\mathscr{A})$ with $B^\bullet$ bounded below, and $\mathscr{A}$ has enough injectives, then there is a spectral sequence:

$$E_2^{p,q} = Hom_{D(\mathscr{A})}(A^\bullet, H^q(B^\bullet)[p]) \Rightarrow Hom_{D(\mathscr{A})}(A^\bullet, B^\bullet[p+q])$$

Here we are using the usual identification:

$$R^p Hom^\bullet(A^\bullet, H^q(B^\bullet)) = Ext^p(A^\bullet, H^q(B^\bullet)) \simeq Hom_{D(\mathscr{A})}(A^\bullet, H^q(B^\bullet)[p])$$

and the spectral sequence in the above remark.

1.4 Ample Sequences

We conclude this section of the notes by a discussion of ample sequences in abelian categories. You will find that any ample sequence turns out to be a spanning class in the associated derived category.

Definition 1.77. A sequence of objects $L_i \in \mathcal{A}$ with $i \in \mathbb{Z}$ in a k-linear category is called ample if for any object $A \in \mathcal{A}$, there exists an integer $i_0(A)$ such that for $i < i_0(A)$ the following conditions are satisfied:

1. The natural morphism $Hom(L_i, A) \otimes L_i \to A$ is surjective.
2. If $j \neq 0$, then $Hom(L_i, A[j]) = 0$ for all L_i.
3. $Hom(A, L_i) = 0$ for all L_i.

In order to define the tensor product $Hom(L_i, A) \otimes L_i \to A$ as an object in $\mathcal{A}$ properly, we either assume that all these $Hom(L_i, A)$ are finite dimensional, or assume that infinite direct sums exists in $\mathcal{A}$. We will just assume that every Hom set is finite dimensional, then we can just define this tensor product as the direct sum of L_i with itself, repeating $dim_k Hom(L_i, A)$ times.

Proposition 1.78. Let $L_i, i \in \mathbb{Z}$ be an ample sequence in a k-linear abelian category $\mathcal{A}$ of finite homological dimensional, then considered as objects in the derived category in $D^b(\mathcal{A})$, L_i's span $D^b(\mathcal{A})$.

Before we prove it, let's breifly recall what does it mean for the homological dimension of an abelian category to be finite. Suppose that $\mathcal{A}$ has enough injectives, then this just means that there is an integer i_0 such that for any $A, B \in \mathcal{A}$, $Ext^i(A, B) = 0$ for all $i \geqslant i_0$. If we don't assume $\mathcal{A}$ has enough injectives, we can just replace $Ext^i(A, B)$ with $Hom_{D(\mathcal{A})}(A, B[i])$.

Then if $\mathcal{A}$ has finite homological dimension, then we claim that for a fixed bounded complex $A^\bullet$, there exists $i_0(A^\bullet)$ such that $Hom(A^\bullet, B[i]) = 0$ for all $i > i_0(A^\bullet)$. A quick way to see this is via the following spectral sequence:

$$E_2^{p,q} = Hom(\mathcal{H}^{-q}(A^\bullet), B[p]) \Rightarrow_p Hom(A^\bullet, B[p+q])$$

Now, since $\mathcal{A}$ has finite homological dimension, we know that there is a l such that $E_2^{p,q}$ is trivial for all $p > l$, hence $Hom(A^\bullet, B[i]) = 0$ for $i \geqslant i_0(A^\bullet)$ where $i_0(A^\bullet)$ is defined to be:

$$i_0(A^\bullet) = l + max\{m : \mathcal{H}^{-m}(A^\bullet) \neq 0\}$$

This is because when $i > i_0$, we know that $E_2^{p,i-p}$ is 0 for all p, which follows from the fact that if $p > l$, then it's obviously trivial, if p is less than l, then $i - p$ is greater than $max\{m : \mathcal{H}^{-m}(A^\bullet) \neq 0\}$, we are done.

Proof. To prove that L_i's span the derived category. We need to prove two conditions, and if a Serre functor is available, we only have to prove the first one, while if there is no Serre functor, the proof is going to be more involved.

To prove condition 1 of the spanning class. We need to prove that if $A^\bullet \in D^b(\mathcal{A})$ such that $Hom(L_i, A^\bullet[j]) = 0$ for all i and j, then $A^\bullet$ is trivial. For the sake of contradition, we assume that $A^\bullet$ is not trivial and assume that $A^\bullet$ is isomorphic to:

$$\cdot \to 0 \to A^n \to A^{n+1} \to \cdots$$

with $\mathcal{H}^n$ not 0. Hence we have an injective morphism:

$$Hom(L_i, H^n(A^\bullet)) \to Hom(L_i, A^\bullet[n]) = 0$$

for all i induced by the morphism $H^n \hookrightarrow A^\bullet[n]$. I will let you check that this is ineed injective.

However, on the other hand, since we assumed L_i's form an ample sequence, we know that we have a surjection:

$$Hom(L_i, \mathcal{H}^n) \otimes L_i \to H^n$$

for $i < i_0(H^n)$, thus we get a contradiction since the cohomology is not 0.

Now as we mentioned, if we have a Serre functor, the second condition is equivalent to the first one, thus we are done. If we don't have a Serre functor, then the argument will need the assumption that homological dimension of $\mathcal{A}$ is finite.

Again, since $A^\bullet \in D^b(\mathcal{A})$, we can assume that $A^\bullet$ is of the form:

$$\cdot \to A^{n-1} \to A^n \to 0 \to \cdots$$

with $H^n \neq 0$. Then the ampleness of L_i gives you a surjection:

$$Hom(L_i, H^n) \otimes L_i \to H^n$$

we will call the kernel of this surjection B_1, which gives a SES:

$$0 \to B_1 \to Hom(L_i, H^n) \otimes L_i \to H^n \to 0$$

which induces a LES (here we consider the SES as a distinguished triangle in the derived category) that consists of the following morphisms:

$$Hom(A^\bullet, H^n) \to Hom(A^\bullet, B_1[1])$$

since the previous term is:

$$Hom(A^\bullet, Hom(L_i, H^n) \otimes L_i) \simeq Hom(L_i, H^n) \otimes Hom(A^\bullet, L_i) = 0$$

Therefore, this time you use B_1 and a surjection:

$$Hom(L_i, B_1) \otimes L_i \to B_1$$

for $i < i_0(B_1)$, then the same argument as above gives you an injective morphism:

$$Hom(A^\bullet, B_1[1]) \hookrightarrow Hom(A^\bullet, B_2[2])$$

Thus, the upshot is a recursively defined sequence of injective morphisms:

$$Hom(A^\bullet, H^n) \hookrightarrow Hom(A^\bullet, B_1[1]) \hookrightarrow Hom(A^\bullet, B_2[2])$$

since we assumed that H^n is not 0, we know that each term in this sequence is not 0, thus contradicting the finite homological dimension of $\mathcal{A}$. $\square$

Remark 1.79. If you go through the proof again, you will find that everything we used here is condition 1 in the definition of the ample sequence, you will find the later two conditions listed there useful in later chapters.

References