Fourier–Mukai Transforms Notes

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In this chapter, we introduce the central notion of a Fourier-Mukai transform between derived categories.

As you will see, functors of Fourier-Mukai form behaves well in a lot of aspects. For example, they are exact, they admit left and right adjoints and can be composed. Even better, we will state without proof one of Orlov's celebrated result that says that any equivalence between derived categories of smooth projective varieties is of geometric origin, i.e. is of Fourier-Mukai type.

After this introduction, we will try to use Homological methods to study Fourier-Mukai transforms such as how they behave under gradings, Hodge structure and Mukai-mapping.

1 Orlov's Result

Let X, Y be smooth projective varieties and we will denote the two projections by:

$$p : X \times Y \to X \qquad q : X \times Y \to Y$$

Definition 1.1. Let $\mathcal{P} \in D^b(X \times Y)$, the induced Fourier-Mukai transform is the functor:

$$\Phi_{\mathcal{P}} : D^b(X) \to D^b(Y), \qquad \mathcal{E}^\bullet \mapsto p_*(q^*\mathcal{E}^\bullet \otimes \mathcal{P})$$

as before, we still use $p_*, q^*, \otimes$ to denote the derived version of that functor.

Remark 1.2. Recall that q is always a flat morphism, thus the pullback is exact, thus q^* here is just the usual pullback just we are applying that to each of the terms. Also, if $\mathcal{P}$ is a complex of locally free sheaves, the tensor product is also exact as what we discussed before, then in this case $\otimes\mathcal{P}^\bullet$ is just the usual tensor product then take the total complex.

Remark 1.3. In the literature, $\Phi_{\mathcal{P}}$ in the above definition is sometimes only called an integral functor and only called Fourier-Mukai when it's an equivalence.

If $\Phi_{\mathcal{P}}$ is indeed an equivalence, we will call X, Y Fourier-Mukai partners.

The reason why this is called Fourier-Mukai transform is it indeed has analogy with the classical Fourier-Transform, which will be mentioned in a later chapter discussing Abelian varieties.

Since the kernel $\mathcal{P}$ can also be used to define an exact functor $D^b(Y) \to D^b(X)$ in a similar fashion, one sometimes writes $\Phi_{\mathcal{P}}^{X\to Y}$ to indicate we are using the one that goes from $D^b(X)$ to $D^b(Y)$.

Remark 1.4. Notice that $\Phi_{\mathcal{P}}$ is the composition of three exact functors, thus is exact itself.

Example 1.5. In this example, we will see that some of the equivalences we already saw before are in fact Fourier-Mukai:

1. Consider the identity:
$$id : D^b(X) \to D^b(X)$$
We claim that it's naturally isomorphic to $\Phi_{\mathcal{O}_\Delta}$ where Δ is the diagonal $\Delta \subset X \times X$. Indeed, if we use $\iota : X \simeq \Delta \hookrightarrow X \times X$ to denote the diagonal embedding one has:
$$\Phi_{\mathcal{O}_\Delta}(\mathcal{E}^\bullet) = p_*(q^*\mathcal{E}^\bullet \otimes \mathcal{O}_\Delta) = p_*(q^*\mathcal{E}^\bullet \otimes \iota_*\mathcal{O}_X)$$

Then using the fact that derived tensor product is associative and the projection formula, we have:

$$p_*(q^*\mathcal{E}^\bullet \otimes \iota_*\mathcal{O}_X) = p_*(\iota_*\mathcal{O}_X \otimes q^*\mathcal{E}^\bullet) = p_*(i_*(\mathcal{O}_X \otimes \iota^* q^*\mathcal{E}^\bullet))$$

Then use the fact that $q \circ \iota = id = p \circ \iota$, we know this is just $\mathcal{E}^\bullet \otimes \mathcal{O}_X = \mathcal{E}^\bullet$.

2. Let $f : X \to Y$ be a morphism of projective smooth varieties, then we claim:

$$f_* \simeq \Phi_{\mathcal{O}_{\Gamma_f}} : D^b(X) \to D^b(Y)$$

where Γ_f is the graph of f. The reason of this is basically the same as the previous claim except this time we are considering $\iota : X \simeq \Gamma_f \hookrightarrow X \times Y$ and $q \circ \iota = id_X$ while $p \circ \iota = f$.

As a special instance, consider the structure morphism $f : X \to Speck$, apply the above discussion, we know that:

$$R\Gamma \simeq \Phi_{\Gamma_f} : D^b(X) \to D^b(Speck) = D^b(Vec_f(k))$$

Here the graph is just $X \subset X \times k$.

Also, we can also use Γ_f to define the Fourier-Mukai tranform in the opposite direction which is just the derived f^* as you can check.

3. Let L be a line bundle on X, then $\mathcal{E}^\bullet \mapsto \mathcal{E}^\bullet \otimes L$ is an exact equivalence $D^b(X) \to D^b(X)$, you can that this is isomorphic to the Fourier-Mukai transform with kernel $\iota_*(L)$, where $\iota : X \to \Delta \to X \times X$ is still the diagonal embedding. The check of this is similar to the first example and omitted here.

4. The shift functor can be thought of as $\Phi_{\mathcal{O}_\Delta[1]}$. The check is again stright-forward.

5. We still use $\iota : X \hookrightarrow X \times X$ as the diagonal embedding. Then we claim that:

$$S^k[-nk] = \Phi_{\iota_*\omega_X^k}$$

where S is the Serre functor and $n = dimX$.

6. Suppose that $\mathcal{P}$ is a coherent sheaf on $X \times Y$, which is flat over X, then we claim that $\Phi_P(k(x)) \simeq \mathcal{P}_x$ where x is a closed point in X with $k(x) = k$ and $\mathcal{P}_x$ is defined to be $\mathcal{P}|_{x\times Y}$.

To check this, notice that P is flat over X thus the derived tensor with P is just the ordinary tensor. Then notice that q is flat then the derived version is itself, thus:

$$q^*k(x) = i_*\mathscr{O}_{x\times Y}$$

where $i : x \times Y \to X \times Y$ is the natural morphism. Then the rest of the check is simple.

References