Fourier–Mukai Transforms Notes

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This notes series is my personal reading notes of the book Fourier–Mukai transforms in algebraic geometry by D. Huybrechts before my summer research 2025. Before we begin, a statement here is that we won't bother to much to distinguish Mor and Hom.

1 Triangulated Categories

1.1 Additive Categories

In this section we basically will review the results and notions we need in order to use the language of triangulated categroies, thus it won't be a complete introduction.

Definition 1.1. A functor $F : \mathscr{A} \to \mathscr{B}$ is called an equivalence of categroies if there exists a functor $F^{-1} : \mathscr{B} \to \mathscr{A}$ such that $F^{-1} \circ F$ is isomorphic to $id_{\mathscr{A}}$ and $F \circ F^{-1}$ is isomorphic to $id_{\mathscr{B}}$. One calls F^{-1} an **inverse** of F or sometimes a **quasi-inverse** of F.

Two categories are equivalent if there is an equivalence of categories $F : \mathscr{A} \to \mathscr{B}$.

You should be able to show that an equivalence is always fully-faithful: you can either prove that $Hom(A, B) \to Hom(FA, FB)$ is full and faithful separately or you can use that that $Hom(A, B) \to Hom(FA, FB) \to Hom(F^{-1}FA, F^{-1}FB)$ and $Hom(C, D) \to Hom(F^{-1}C, F^{-1}D) \to Hom(FF^{-1}C, FF^{-1}D)$ are all isomorphisms (why?). But a fully faithful functor may not be an equivalence, the following proposition gives you a necessary and sufficient criterion to determine when a fully faithful functor is an equivalence:

Proposition 1.2. Let $F : \mathscr{A} \to \mathscr{B}$ be a fully faithful functor, it's an equivalence of category iff every object $B \in \mathscr{B}$ is isomorphic to $F(A)$ for some A.

Proof. We need to construct an inverse of F in order to show that F is an equivalence.

For $B \in \mathscr{B}$, by our assumption, we can choose an $A_B \in \mathscr{A}$ such that B is isomorphic to $F(A_B)$ via the isomorphism $\varphi_B : F(A_B) \to B$. But if $B = F(A)$ for some A we won't bother to find another A' such that $B \cong F(A')$, we will just choose $A_B = A$.

Then we define $F^{-1} : \mathscr{B} \to \mathscr{A}$ to be the functor that maps B to A_B and for B_1, B_2, we define $Hom(B_1, B_2) \to Hom(F^{-1}B_1, F^{-1}B_2)$ to be the composition of two bijections(you want it to be an equivalence thus it must be fully faithful):

$$Hom(B_1, B_2) \to Hom(F(A_{B_1}), F(A_{B_2}))$$

where $f \mapsto \varphi_{B_2}^{-1} \circ f \circ \varphi_{B_1}$, and the inverse of the map:

$$Hom(A_{B_1}, A_{B_2}) \to Hom(F(A_{B_1}), F(A_{B_2}))$$

since F is assumed to be a fully faithful functor.

One question is: is this definition of F^{-1} really a functor? It seems like we are only defining it really randomly. But it turns out that this is a canonical definition which makes F^{-1} a functor. To check this, we only have to make sure that this definition respects composition. First, the conjugate like map which map f to $\varphi_{B_2}^{-1} \circ f \circ \varphi_{B_1}$ obviously respects composition thus we only have to show the second map in the composition also respects composition, but this is just the fact that F is a fully faithful functor (makes sure you understand why fully faithful functor is enough).

Finally, we need to verify F^{-1} is indeed the inverse of F. To check this, consider the following two diagrams:

$$\begin{tikzcd}
B_1 \arrow[r] \arrow[d, "\varphi_{B_1}"'] & B_2 \arrow[d, "\varphi_{B_2}"] \\
FF^{-1}B_1 = F(A_{B_1}) \arrow[r] & FF^{-1}B_2 = F(A_{B_2})
\end{tikzcd}$$

and

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, dashed, "id"] & A_2 \arrow[d, dashed, "id"] \\
F^{-1}F(A_1) = A_1 \arrow[r] & F^{-1}F(A_2) = A_2
\end{tikzcd}$$

you can easily check this is indeed commutative, hence proving the proposition. □

One immediate corollary of this proposition is:

Corollary 1.3. Any fully faithful functor $F : \mathscr{A} \to \mathscr{B}$ defines an equivalence between $\mathscr{A}$ and the full subcategory of $\mathscr{B}$ of all objects of $\mathscr{B}$ such that it's isomorphic to $F(A)$ for some objects $A \in \mathscr{A}$

Proof. The proof is a straight-forward application of the above proposition. □

Now we also review Yoneda's lemma shortly since we will use it extensively later, you can find more discussion of Yoneda's lemma in the AG notes:

We will use $Fun(\mathscr{A})$ to denote the category of all contravariant functors from $\mathscr{A}$, i.e. all contravariant functors from the category of $\mathscr{A}$ to the category of sets, then we have a natural functor:

$$\mathscr{A} \to Sets \qquad A \mapsto Hom(-, A)$$

you should be aware how we map a morphism $A_1 \to A_2$ via this functor, and again more details can be found in the AG notes:

Proposition 1.4. This functor defined above is an equivalence of categories between $\mathscr{A}$ and the full subcategory of representable functors F, i.e. those F such that they are isomorphic to $Hom(-, A)$ for some $A \in \mathscr{A}$.

Proof. The key of proving this proposition is to show that the above functor is fully faithful, i.e. there is a bijection between morphisms $A_1 \to A_2$ and natural transformations between $Hom(-, A_1)$ and $Hom(-, A_2)$. This is proved in the AG notes and we won't bother to prove it here since it's not a very complicated result.

Once we have this, notice that the above corollary quickly implies that this functor is an equivalence of categories between $\mathscr{A}$ and the full subcategroy of representable functors. □

In this notes, we will rarely work with arbitrary categories, those categories we care about will at least be additive:

Definition 1.5. A category $\mathscr{A}$ is an additive category if for any pair of objects in $\mathscr{A}$, the set $Hom(A_1, A_2)$ are endowed with the structure of an abelian group (we don't see the empty set as an abelian group so we are implicitly assuming that for any pair of objects, there is at least one morphism between them) such that the following three conditions hold:

1. The composition operations:

$$Hom(A_1, A_2) \times Hom(A_2, A_3) \to Hom(A_1, A_3) \qquad (f, g) \mapsto g \circ f$$

is bilinear, i.e. $(g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f$ and $g \circ (f_1 + f_2) = g \circ f_1 + g \circ f_2$.

2. There exists an object 0 in $\mathscr{A}$, called the zero object, such that the group $Hom(0, 0)$ is the trivial group with only one element.

3. For any two objects $A_1, A_2 \in \mathscr{A}$, there is an object $B \in \mathscr{A}$ together with morphisms $j_i : A_i \to B$ and $p_i : B \to A_i$ where $i = 1, 2$ which makes B the direct sum and direct product of A_1, A_2 respectively. We also assume $p_i \circ j_i = id$ $p_2 \circ j_1 = p_1 \circ j_2 = 0$ and $j_1 \circ p_1 + j_2 \circ p_2 = id$.

To remember this definition, you should only remember we are assuming additional condition for Hom sets, we require a special object 0, and we need the direct sum and direct product condition.

Proposition 1.6. For any object $A \in \mathscr{A}$, an additive category, there exists unique morphisms $0 \to A$ and $A \to 0$.

Proof. Let's first consider $Hom(0, A)$, suppose that $f, g \in Hom(0, A)$, we can consider $id \circ f$ and $id \circ g$, notice that in $Hom(0, 0)$ the identity element is equal to the zero element in $Hom(0, 0)$, thus $id \circ f = id \circ g = f = g = 0$ in $Hom(0, A)$ since we assumed $Hom(0, 0) \times Hom(0, A) \to Hom(0, A)$. (Make sure you understand why!)

To study $Hom(A, 0)$, we follow the same argument and the details are left to you. □

If you work with additive categories, we usually don't study arbitrary functors, but instead we will work with **additive functors**, which are functors F such that the induced map $Hom(A_1, A_2) \to Hom(FA_1, FA_2)$ are group homomorphisms.

What we discussed above for arbitrary functors all have their counterparts in additive categories. For example, and additive functor which is also an equivalence of categories is called and **additive equivalence** of categories, and in particular it's inverse F^{-1} is also an additive functor (you should make sure you understand why it's inverse is also additive!) Here we can modify Yoneda's lemma as follows: we will use $Fun(\mathscr{A})$ now to denote the category of **additive functors** from $\mathscr{A}$ to the category of **abelian groups**, not just plain sets. In the case where $\mathscr{A}$ is additive, we can easily show that $Fun(\mathscr{A})$ is also additive, and the functor we defined in Yoneda's lemma above is now additive. Then everything we discussed in Yoneda's lemma remains true, the bijection between morphisms from A_1 to A_2 and natrual transformations between $Hom(-, A_1)$ and $Hom(-, A_2)$ are now a bijection as a group homomorphism.

We will go even one step further, since in the end we will eventually be interested in categories of varieties over a base field k, thus we will usually deal with the following special type of additive categories:

Definition 1.7. A k-linear category is an additive category $\mathscr{A}$ such that the group $Hom(A, B)$ are k-vector spaces over the same base field k and all the compositions are k-bilinear.

Additive functors between two k-linear categories over a common base field k will be assumed to be k-linear, i.e. the induced map $Hom(A, B) \to Hom(FA, FB)$ is k-linear.

Again, what we discussed above for additive categories applies here and we also have the corresponding Yoneda's lemma where we will use $Fun(\mathscr{A})$ to denote the functors from a k-linear category to $Vec(k)$ the category of all k-vector spaces.

Except for requiring the additive structure to be k-linear, we can impose conditions of another flavor to additive categories and obtain the so called abelian category:

Definition 1.8. An additive category $\mathscr{A}$ is called abelian if the following additional condition holds:

Every morphism $f \in Hom(A, B)$ admits a kernel and cokernel and the natural map $Coim(f) \to Im(f)$ is an isomorphism.

Recall that the notion of kernel and cokernel makes sense in any category with an zero object in terms of certain universal properties that involves the notion of 0-morphism. In the case where we work with additive categories, the 0-morphism from A to B is just the 0-element in $Hom(A, B)$ with the abelian group structure. Though we won't use it, you will find that this 0-morphism is equal to the composition of $A \to 0$ and $0 \to B$ which are all unique. Therefore, even if we don't work with additive categories, as long as this category has an 0-object (defined as an object that is both terminal and initial, not what we defined above), we can define the 0-morphism from A to B to be the composition $A \to 0 \to B$, since the morphism from A to 0 and from 0 to B are unique, this makes sense.

Also, the image is defined to be the kernel of the cokernel, and the coimage is defined to be the cokernel of the kernel, thus we have the following natural diagram:

$$
\begin{tikzcd}
Kerf \arrow[r] & A \arrow[r, "f"] \arrow[d] \arrow[dr, dashed] & B \arrow[r] & Cokerf \\
& Coim(f) \arrow[r, dashed] \arrow[ur, dashed] & Im(f) \arrow[u] &
\end{tikzcd}
$$

and we require the induced natural map from $Coim(f)$ to $Im(f)$ to be an isomorphism.

Sometimes we have equivalent definition of abelian categories, where we still require the exsitence of the kernel and cokernel, but we don't require the above isomorphism, instead we require every monomorphism to be the kernel of it's cokernel and every epimorphism to be the cokernel of it's kernel. The two different definitions are indeed equivalent if you are aware that in an additive category, the kernel of an monomorphism is the 0-object and the cokernel of an epimorphism is the 0 object(try to prove it if you haven'r done this before since it's not hard). Of course there are a few facts about monomorphisms and epimorphisms in additive categories you need to know, but they are all not that hard to prove. On the other hand, if we assume that monomorphism is the kernel of it's cokernel and epimorphism is the cokernel of it's kernel, we also can retrieve the isomorphism $Coim(f) \cong Im(f)$. The stragety we use here is to prove it's both monic and epic, thus an isomorphism this can be proved by the fact that if the equalizer of the morphisms is epic, it's necessarily an isomorphism. Then since a monic morphism is the kernel of it's cokernel, then it's the equalizer of the cokernel and the 0-morphism.

We won't bother too much here for these kind of technical details, the reason we want to introduce abelian categories is we want to discuss **exact sequences**. In an abelian category, a sequence $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$ is called exact if $Ker(f_2) = Im(f_1)$, or more precisely the composition $f_2 \circ f_1$ is the 0 morphism and the natural monomorphism (it's a monomorphism since the morphism from kernel and image to A_2 are all monomorphisms) from $Im(f_1)$ to $Ker(f_2)$ is an isomorphism.

Notice that if $f_2 \circ f_1 = 0$, the natural morphism $Im(f_1) \to Ker(f_2)$ is always a monomorphism as what we mentioned above, thus if the composition is 0, we always have the image is a subobject of the kernel. Now, giving an additive functor, by definition, it maps 0-morphism to 0-morphism, thus after you apply an additive functor to $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$ the composition is still 0, we are interested in the case when if $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$ is exact, after you apply F, it's still exact, which is the definition below:

Definition 1.9. An additive functor F is left(right) exact if any short exact sequence:

$$0 \to A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \to 0$$

is mapped to a sequence:

$$0 \to FA_1 \xrightarrow{Ff_1} FA_2 \xrightarrow{Ff_2} FA_3 \to 0$$

that is exact except for possibly in FA_3(respectively in FA_1), and F is said to be exact if it's both left and right exact.

One classic exercise of exact sequence is:

Proposition 1.10. *F is left exact if and only if for any exact sequence $0 \to A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$, after you apply F, it's still exact. (Here we don't require $A_2 \to A_3$ to be surjective!)*

Proof. Notice that from $0 \to A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$ we can produce a new exact sequence by replacing A_3 by $Im(f_2)$ and we obtain:

$$0 \to A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} Im(f_2) \to 0$$

then you you can apply F and then delete $F(Im(f_2))$.

The other direction is trivial since if $0 \to A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \to 0$ is exact then $0 \to A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$ is still exact, then you apply F and then add on $F(A_3)$ to the sequence. $\square$

Now we can head to the discussion of adjointness where Yoneda's lemma is going to be used a lot:

Definition 1.11. Let $F : \mathscr{A} \to \mathscr{B}$ be a functor between arbitrary categories, then we will say that another functor $H : \mathscr{B} \to \mathscr{A}$ is right adjoint to F (we write it as $F \dashv H$) if there exist isomorphisms:

$$Hom(FA, B) \simeq Hom(A, HB)$$

for any pair of object $A \in \mathscr{A}$ and $B \in \mathscr{B}$, which is functorial in both A and B. To be more precise, if you have a morphism $f : A_1 \to A_2$ then the following diagram commutes:

$$\begin{tikzcd} Hom(FA_2, B) \arrow[r, "\circ Ff"] \arrow[d, "iso"] & Hom(FA_1, B) \arrow[d, "iso"] \\ Hom(A_2, HB) \arrow[r, "\circ f"'] & Hom(A_1, HB) \end{tikzcd}$$

and similarly if you have a morphism $g : B_1 \to B_2$, the following diagram should commute:

$$\begin{tikzcd} Hom(FA, B_1) \arrow[r, "g\circ"] \arrow[d, "iso"] & Hom(FA, B_2) \arrow[d, "iso"] \\ Hom(A, HB_1) \arrow[r, "Hg\circ"'] & Hom(A, HB_2) \end{tikzcd}$$

Similarly, we will say that G is left adjoint to F, written as $G \dashv F$ if there exists isomorphisms $Hom(B, FA) = Hom(GB, A)$ for any pair of objects $A \in \mathscr{A}$ and $B \in \mathscr{B}$ and these isomorphisms should also be functorial in both A and B.

Clearly, F is left adjoint to H iff H is right adjoint to F. Here are some important remarks that is worth mentioning:

Remark 1.12.

1. The first remark partially explains why we want the notion of adjoint functors: in some sense, if $F \dashv H$, even though F and V are not necessarily inverses of each other, there is a natural functor morphism or in other words natural transformation obtained from this adjoint relation. Suppose that $F \dashv H$, then we can consider $id_{FA} \in Hom(FA, FA)$, which is mapped to a map in $Hom(A, HFA)$, then if you consider the following diagram:

$$\begin{tikzcd} Hom(A_1, A_1) \arrow[r, "g\circ"] \arrow[d] & Hom(A_1, A_2) \arrow[d] & Hom(A_2, A_2) \arrow[l, "\circ g"'] \arrow[d] \\ Hom(FA_1, FA_1) \arrow[r, "Fg\circ"] \arrow[d, "iso"] & Hom(FA_1, FA_2) \arrow[d, "iso"] & Hom(FA_2, FA_2) \arrow[l, "\circ Fg"'] \arrow[d] \\ Hom(A_1, HFA_1) \arrow[r, "HFg\circ"] & Hom(A_1, HFA_2) & Hom(A_2, HFA_2) \arrow[l, "\circ g"'] \end{tikzcd} \qquad \begin{tikzcd} A_1 \arrow[r, "g"'] \arrow[d] & A_2 \arrow[d] \\ HFA_1 \arrow[r, "HFg"'] & HFA_2 \end{tikzcd}$$

   you can check that the diagram above implies that the right bottom square is commutative by tracing where id_{A_1} and id_{A_2} go. Therefore, this adjointness relation gives us a natural transformation between $id_{\mathscr{A}}$ and HF.

   Following the similar argument, you can show that this adjointness relation also gives a natural transformation between $id_{\mathscr{B}}$ and FH. Temporarily, we will use h and g to denote the natural transformations $id_{\mathscr{A}} \to HF$ and $FH \to id_{\mathscr{B}}$ respectively.

   It's natural to wonder when this natural transformation is a natural isomorphism, we will answer this question below.

2. Another point we would like to add here is that if F has a two sided inverse F^{-1}, i.e. F is an equivalence, then $F^{-1} \dashv F \dashv F^{-1}$, i.e. the inverse is both left and right adjoint to F. I will let you to find why this is true. However, when F only have a one sided inverse, we are not able to conclude that the inverse is an adjoint.

3. Using Yoneda's lemma, we can prove that if a left or right adjoint exists at all, it's unique up to a unique isomorphism, more explicitly, for two right adjoint functors H and H' of F, you can define a natural isomorphism between H and H' via Yoneda's lemma.

   To be more precise about how to do it: since both H, H' are right adjoint to F, we know that for any $B \in \mathscr{B}$, we have functorial isomorphisms:

$$Hom(HB, HB) \cong Hom(FHB, B) \cong Hom(HB, H'B)$$

   which is in the first slot HB, thus we know from Yoneda's lemma that since this is a natural isomorphism between $Hom(-, HB)$ and $Hom(-, H'B)$, it correspond to an isomorphism $H'B \simeq HB$ for each B, we only need to show that these isomorphisms assemble to a functor isomorphism. This is the content of the following diagram:

$$\begin{tikzcd} Hom(HB_1, HB_1) \arrow[r] \arrow[d, "Hg\circ"'] & Hom(HB_1, H'B_1) \arrow[d, "H'g\circ"] \\ Hom(HB_1, HB_2) \arrow[r] & Hom(HB_1, H'B_2) \\ Hom(HB_2, HB_2) \arrow[r] \arrow[u, "\circ Hg"] & Hom(HB_2, H'B_2) \arrow[u, "\circ H'g"'] \end{tikzcd} \qquad \begin{tikzcd} HB_1 \arrow[r, "Hg"] \arrow[d] & HB_2 \arrow[d] \\ H'B_1 \arrow[r, "H'g"'] & H'B_2 \end{tikzcd}$$

   the conclusion of the Yoneda's lemma is important, but what's more important is the argument we used to proved it where we know that that isomorphism between two representing objects is given by the image of the identity.

4. We worked with arbitrary functors in the above arguments, but if we want to restrict ourselves to additive categories and additive functors, the isomorphism in the definition of adjointness should also be isomorphism as abelian groups, not just arbitrary bijections. Similarly, if we want to study k-linear categories, we will want the isomorphism to be k-linear. A priori, one can't exclude the pathological case where the adjoint of an additive functor is not additive, or the adjointness of two additive functor is given by non group homomorphism bijections, but what we will see below will avoid those unnatural cases.

To enhance our understanding of adjointness, we provide a few propositions:

Proposition 1.13. Suppose that $F \dashv G$, then the adjunction isomorphism $Hom(FA, B) \simeq Hom(A, HB)$ is given by:

$$f \mapsto (A \xrightarrow{h_A} HFA \xrightarrow{Hf} HB)$$

Proof. Consider the following diagram:

$$\begin{tikzcd} Hom(FA, B) \arrow[d] & Hom(FA, FA) \arrow[l, "f"'] \arrow[d] & Hom(A, A) \arrow[l] \\ Hom(A, HB) & Hom(A, HFA) \arrow[l, "Hf"] & \end{tikzcd}$$

again, by tracing where does the identity in $Hom(A, A)$ go, you will prove the desired property. ☐

Remark 1.14. The above proposition shows the map $Hom(FA, B) \to Hom(A, HB)$, it's natural to ask what is the inverse, i.e., what is the map $Hom(A, HB) \to Hom(FA, B)$. We can use the same method as the proposition above, which is the diagram below:

$$\begin{tikzcd} Hom(B, B) \arrow[r] & Hom(HB, HB) \arrow[r, "\circ f"] \arrow[d] & Hom(A, HB) \arrow[d] \\ & Hom(FHB, B) \arrow[r, "\circ Ff"] & Hom(FA, B) \end{tikzcd}$$

and you get the conclusion: $Hom(A, HB) \to Hom(FA, B)$ is $f \mapsto (g_B \circ Ff)$.

Another property we will use quite often is:

Proposition 1.15. Suppose that $\mathscr{A}, \mathscr{B}$ are abelian categories, and $F \dashv H$ where both are additive functors, then F is right exact, H is necessarily left exact.

Proof. Notice that we don't have to assume any thing except for adjointness for F, H, and we can show that one is right exact and another is left exact.

One easy way to prove this is to use the following fact: in any abelian category, left adjoint commutes with colimits and right adjoint commutes with limits, thus since kernel is a limit, H commutes with kernel, we claim that if an additive functor preserves kernel, then it's left exact. To show this, we need a few preliminary results. The first is that if $f : A \to B$ is a monomorphism, then $A \simeq Im(f)$. This is not hard to prove by hands, as we can use the image factorization:

$$\begin{tikzcd} & Imf \arrow[d, hook] & \\ A \arrow[ur] \arrow[r, hook, "f"'] & B \arrow[r] & Cokerf \end{tikzcd}$$

we already know that $A \to Im(f)$ is epic, if we can prove that it's monic, then it's an isomorphism(again from the equlizer argument we proved above). But notice that $A \to B$ is monic $Im(f) \to B$ is monic, then $A \to Im(f)$ must be monic.

Then if H preserves kernel, it preserves monomorphisms, then if you have an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C$, after you apply H, you obtain:

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

$$0 \longrightarrow HA \xrightarrow{Hf} HB \xrightarrow{Hg} HC$$

and HA is still the kernel of $HB \xrightarrow{Hg} HC$. Therefore, we only have to prove $Ker(Hg) \simeq Im(Hf)$, but you know that the image of f is just A, thus we have:

$$\begin{tikzcd} & A \arrow[r, "iso"] & Kerg \arrow[d, hook] & \\ 0 \arrow[r] & A \arrow[u, "id"] \arrow[r, "f"] & B \arrow[r, "g"] & C \end{tikzcd}$$

then you apply H to obtain:

$$\begin{tikzcd} & HA \arrow[r, "iso"] & HKerg \arrow[d, hook] & \\ 0 \arrow[r] & HA \arrow[u, "id"] \arrow[r, "Hf"] & HB \arrow[r, "Hg"] & HC \end{tikzcd}$$

since Hf is still monomorphism, then it's image is still itself, also $Hkerg = KerHg$, thus we are done.

Similarly, if we can prove left adjoint functors preserves colimits, since cokernels are colimits, they preserve cokernels, and similar arguments will show that left adjoint functors are right exact.

Finally to show that if $F \dashv H$ then F preserve colimits and H preserves limits, we only have to check the universal property. Let's work with H first, suppose that $A = limA_i$ is a limit in $\mathscr{A}$ of a diagram indexed by $\mathscr{I}$, then we can consider the corresponding diagram indexed by $\mathscr{I}$ involving GA and GA_i, we have to prove that GA with these new arrows is now still a limit. We check the universal property, suppose that W has arrows into all GA_i which commutes all the maps in between GA_i indexed by $\mathscr{I}$, we want to show that there is a unique morphism $W \to GA$ extending $W \to GA_i$. But we can restate this via adjointness, we claim this above universal property is equivalent to the following: suppose that we have morphisms $FW \to A_i$ such that they commute with all the arrows indexed by $\mathscr{I}$, we wish there is a unique morphism $FW \to A$ extending $FW \to A_i$.

To see why we have such translations, see that usual diagram we draw for adjointness:

$$\begin{tikzcd} Hom(W, GA_i) \arrow[r] \arrow[d] & Hom(W, GA_j) \arrow[d] \\ Hom(FW, A_i) \arrow[r] & Hom(FW, A_j) \end{tikzcd}$$

commuting with all the arrows between GA_i's indexed by $\mathscr{I}$ can be translated to the fact that the arrow from W to GA_i composed with the arrow from GA_i to GA_j is the arrow from W to GA_j, then the other details are left to you.

But after the translation, this is just the universal property of limits. □

Proposition 1.16. Again, we still assume that $F \dashv H$, then we already showed that we have two induced natural transformations:$g : FH \to id_{\mathscr{B}}$ and $h : id_{\mathscr{A}} \to HF$, then the following composition is the identity transformation:

$$H \xrightarrow{h_{H()}} (H \circ F) \circ H \longrightarrow H \circ (F \circ H) \xrightarrow{H(g)} H$$

Proof. This is basically an application of the previous proposition, and you should easily guess this when the statement involves composition. To see this, suppose that B is an object in $\mathscr{B}$, the first map in the composition is just h_{HB}, according to the previous proposition, since the second map is just H of the image of the identity under the adjoint map $Hom(HA, HA) \to Hom(FHA, A)$, thus this composition is just the adjoinction morphism: $\varphi : Hom(FHA, A) \to Hom(HA, HA)$ and we are only tracing where $\varphi^{-1}(id_{HA})$ goes, and you know that it's just $\varphi \circ \varphi^{-1}(id_{HA}) = id_{HA}$, we are done. □

Remark 1.17. We can also see what happens when we still $F \dashv H$, but we start with F where we have the following composition:

$$F \stackrel{F(h)}{\to} F \circ (H \circ F) \to (F \circ H) \circ F \stackrel{g_{F()}}{\to} F$$

we claim that this is still the identity, and the reason is similar to the proposition above. We know that $h : A \to HFA$ is the image of the identity id_{FA} under the inverse of the adjunction map $\varphi : Hom(A, HFA) \to Hom(FA, FA)$, then the composition above is the adjunction map apply to $\varphi^{-1}(id_{FA})$, then we are done.

The following lemma is useful in determining when the natural transformations g, h we described earlier are natural isomorphisms:

Lemma 1.18. Suppose that $G \dashv F$, then the following diagram commutes:

$$\begin{tikzcd} & Hom(A, B) \arrow[dl, "\circ g_A"'] \arrow[dr, "F"] & \\ Hom(GFA, B) \arrow[rr, leftrightarrow, "iso"] & & Hom(FA, FB) \end{tikzcd}$$

similarly if $F \dashv G$, the following diagram commutes:

$$\begin{tikzcd} & Hom(A, B) \arrow[dl] \arrow[dr, "h_B \circ"'] & \\ Hom(FA, FB) \arrow[rr, leftrightarrow, "iso"] & & Hom(A, GFB) \end{tikzcd}$$

the two isomorphisms are given by the adjoint morphisms.

Proof. **** later* □

Given the result of the previous lemma, we are able to prove a useful result about fully faithful functors:

Proposition 1.19. If $F : \mathscr{A} \to \mathscr{B}$ is a fully faithful functor such that it admits a left adjoint $G \dashv F$, then the natural transformation: $g : G \circ F \to id_{\mathscr{A}}$ is a natural isomorphism.

Similarly, if F admits a right adjoint $F \dashv H$, then the natural transformation $id_{\mathscr{A}} \to H \circ F$ is also a natural isomorphism.

Proof. Let's prove the first assertion since the second one is similar. First note that F is fully faithful, then for any A, B, we get a bijection:

$$Hom(A, B) \to Hom(FA, FB)$$

composing with the isomorphism $Hom(FA, FB) \to Hom(GHFA, B)$, we obtain that for any objects A, B, $Hom(A, B) \to Hom(GFA, B)$ is an isomorphism, then since this morphism is also obtained by precomposing g_A, Yoneda's lemma shows that g_A must be a isomorphism, and this holds for every A. $\square$

Following a similar argument, we can also show the following:

Proposition 1.20. Let $F : \mathscr{A} \to \mathscr{B}$ be a functor such that it admits a left a left adjoint $G \dashv F$, then if the natural transformation $G \circ F \to id_{\mathscr{A}}$ is a natural isomorphism then F is fully faithful.

Similarly, if $F \dashv H$ and the natural transformation $id_{\mathscr{A}} \to H \circ F$ is a natural isomorphism, then F is fully faithful.

Proof. We still only prove the first assertion. When the natural transformations are natural isomorphisms, g_A is going to be an isomorphism for every A, thus again by Yoneda's lemma, precomposing these morphisms give you isomorphisms between hom sets, we are done. $\square$

Since adjoint functors are extremely useful, you probably want to ask when do such functors exist, one special case where they do exist is for equivalences:

Proposition 1.21. Let F be an equivalence of categories then F admits a left adjoint and a right adjoint. More precisely, if F' is an inverse of F, then $F \dashv F' \dashv F$.

Proof. We only prove $F \dashv F'$ since they other direction is proved just by switching the role of F and F'.

The key points here are F' is also an equivalence and in particular fully faithful and $F'F \cong id_{\mathscr{A}}$. Using these two, we get the following sequence of isomorphisms: $Hom(FA, B) \to Hom(F'FA, F'B) \cong Hom(A, F'B)$. Notice that the first isomorphism is functorial in both A and B since we are just applying F' which is a functor. More precisely, the following diagram commutes:

$$\begin{tikzcd}
Hom(FA_1, B) \arrow[r] \arrow[d] & Hom(F'FA_1, F'B) \arrow[d] \\
Hom(FA_2, B) \arrow[r] & Hom(F'FA_2, F'B)
\end{tikzcd} \qquad \begin{tikzcd}
Hom(FA, B_1) \arrow[r] \arrow[d] & Hom(F'FA, F'B_1) \arrow[d] \\
Hom(FA, B_2) \arrow[r] & Hom(F'FA, F'B_2)
\end{tikzcd}$$

similarly, the second isomorphism is also functorial in both A and B. It's functorial in A because we have a functor isomorphism $F'F \cong id$, it's functorial in B for trivial reasons. Thus the composition is functorial in both A and B, then we are done. $\square$

Remark 1.22. We conclude by a useful remarl:

The above discussion shows a way we will follow in the future to prove certain functros are equivalences: You will first have a candidate F which you want to be an equivalence. Usually F comes with a left adjoint $G \dashv F$ or right adjoint $F \dashv H$, then by our previous result, if the natural transformation $G \circ F \to id$ is a natural isomorphism, F is fully faithful, which usually turns out to be the most important part of proving F is an equivalence. Finally, you show that any object in the target category is isomorphism to the image of some object in the source catgeory.

Next small topic we will mention briefly is Serre functor:

Definition 1.23. Let $\mathscr{A}$ be a k-linear category, a Serre functor is a k-linear equivalence $S : \mathscr{A} \to \mathscr{A}$ such that for any two objects $A, B \in \mathscr{A}$, there exists an isomorphism of k-vector spaces:

$$\eta_{A,B} : Hom(A, B) \to Hom(B, SA)^*$$

that is functorial in both A and B. For notational simplicity, we will write the induced pairing as:

$$Hom(B, SA) \times Hom(A, B) \to k \qquad (f, g) \mapsto < f|g >$$

Remark 1.24. In the original definition, we require the commutativity of the following diagram:

$$\begin{tikzcd} Hom(A,B) \arrow[r, "\eta_{A,B}"] \arrow[d, "S"'] & Hom(B,SA)^* \\ Hom(SA,SB) \arrow[r, "\eta_{SA,SB}"'] & Hom(SB,S^2A)^* \arrow[u, "S^*"'] \end{tikzcd}$$

here this $S^* : Hom(SB, S^2A)^* \to Hom(B, SA)^*$ is just the dual map of the map $S : Hom(B, SA) \to Hom(SB, S^2A)$.

However, this turns out to be automatically true, following directly from the definition above. To see why: **** later, especially the commutativity of the second triangle*

The following result shows that if a Serre functor exists it's unique up to isomorphism, and more generally we have the following result:

Proposition 1.25. Let $\mathscr{A}$ and $\mathscr{B}$ be k-linear categories over a field k with all finite dimensional Hom sets. If $\mathscr{A}$ and $\mathscr{B}$ are endowed with Serre functors S_A and S_B respectively then for any k-linear equivalence:

$$F : \mathscr{A} \to \mathscr{B}$$

it commutes with Serre duality, i.e. we have an isomorphism:

$$F \circ S_A \simeq S_B \circ F$$

Proof. Notice that after we prove this proposition, since the identity is a k-linear equivalence between $\mathscr{A}$ and itself, then we know that any two Serre functor of $\mathscr{A}$ will be isomorphic.

To prove this proposition, we still use Yoneda's lemma. Notice that an equivalence is fully fatithful, then for any two objects $A, B \in \mathscr{A}$, we have the following k-linear isomorphisms:

$$Hom(A, SB) \simeq Hom(FA, FSB) \qquad Hom(B, A) \cong Hom(FB, FA)$$

together with:

$$Hom(A, SB) \simeq Hom(B, A)^* \qquad Hom(FB, FA) \simeq Hom(FA, SFB)^*$$

combine these isomorphsms, we get $Hom(FA, FSB) \simeq Hom(FA, SFB)$ for every A, B. Notice that every isomorphism is functorial in both slots, thus the final isomorphism we got is also functorial in both A, B. Notice that F is an equivalence, thus we know that for every object B in $\mathscr{B}$, there is some object A in $\mathscr{A}$ such that $B \simeq FA$, thus the upshot is that for any two objects B_1, B_2 in $\mathscr{B}$, we have:

$$Hom(B_1, FSB_2) \simeq Hom(FA, FSB_2) \simeq Hom(FA, SFB_2) \simeq Hom(B_1, SFB_2)$$

We claim that $Hom(B_1, FSB_2) \simeq Hom(B_1, SFB_2)$ is functorial in both B_1 and B_2, then applying Yoneda's lemma we can conclude $FS \simeq SF$. To show that the isomorphism is functorial, we only need to notice the following fact:

suppose that we have $C_1, C_2 \in \mathscr{B}$ and $f : C_2 \to C_1$, if $C_1 \simeq FA_1$ and $C_2 \simeq FA_2$ via the equivalence F, consider the following diagram:

$$\begin{tikzcd} C_1 & FA_1 \arrow[l, "iso"'] \\ C_2 \arrow[u, "f"] & FA_2 \arrow[l, "iso"] \arrow[u] \end{tikzcd}$$

you can always find a morphism $FA_2 \to FA_1$ such that the diagram commutes, thus we can use this morphism to obtain the following diagram:

$$\begin{tikzcd} Hom(C_1, FSB) \arrow[r, leftrightarrow] \arrow[d] & Hom(FA_1, FSB) \arrow[r, leftrightarrow] \arrow[d] & Hom(FA_1, SFB) \arrow[r, leftrightarrow] \arrow[d] & Hom(C_1, SFB) \arrow[d] \\ Hom(C_2, FSB) \arrow[r, leftrightarrow] & Hom(FA_2, FSB) \arrow[r, leftrightarrow] & Hom(FA_2, SFB) \arrow[r, leftrightarrow] & Hom(C_2, SFB) \end{tikzcd}$$

thus we have $Hom(B_1, FSB_2) \simeq Hom(B_1, SFB_2)$ is functorial in the first variable. The reason why it's also functorial in the second variable is left to you. $\square$

Before we move to another topic, we conclude this early discussion of Serre functor by how the existence of Serre functors help us to construct adjoint functors:

Remark 1.26. Let $F : \mathscr{A} \to \mathscr{B}$ be a functor between k-linear categories and $\mathscr{A}, \mathscr{B}$ are endowed with Serre functors S_A, S_B respectively, if $G \dashv F$, we can consturct a right adjoint of F, and more precisely, it's going to be:

$$F \dashv S_A \circ G \circ S_B^{-1}$$

as before, we still assume all hom sets are finite dimensional and S_B^{-1} is an inverse of S_B. To show this, we consider the following sequence of functorial isomorphisms:

$$\begin{aligned} Hom(A_1, S_A \circ G \circ S_B^{-1} A_2) &\simeq Hom(G \circ S_B^{-1} A_2, A_1)^* \\ &\simeq Hom(S_B^{-1} A_2, FA_1) \\ &\simeq Hom(FA_1, S_B S_B^{-1} A_2) \\ &\simeq Hom(FA_1, A_2) \end{aligned}$$

Notice that each of the isomorphisms are functorial in both A_1 and A_2, which gives us the adjointness.

1.2 Triangulated categories and exact functors

We are finally able to introduce the main character of our show right now, i.e. triangulated categories. Via this notion, we have a different definition of exactness which generalizes the usual notion of exactness in abelian categories. To begin with, we will directly give the definition of triangulated category, it puts an additional structure on an additive category:

Definition 1.27. Let $\mathscr{D}$ be an additive category, the structure of a triangulated category on $\mathscr{D}$ is given by an additive equivalence $T : \mathscr{D} \to \mathscr{D}$ called the shift functor and a set of distinguished triangles (also called exact triangles) that satisfies the following axioms(TR1-TR4) illustrated below:

$$A \to B \to C \to T(A)$$

it's called a triangle since when we are talking about a distinguished triangle, we are thinking about the following diagram:

$$\begin{tikzcd} A \arrow[rr] & & B \arrow[dl] \\ & C \arrow[ul] & \end{tikzcd}$$

where the arrow from C to A is view as an arrow from C to TA. For the notation, we will use $A[1]$ to denote TA for any object $A \in \mathscr{D}$ and $f[1]$ to denote $T(f) \in Hom(A[1], B[1])$ for any morphism $f : A \to B$. Similarly, one writes $A[n] = T^n(A)$ and $f[n] = T^n(f)$ for any $n \in \mathbb{Z}$, when n is negative, we are thinking about apply an inverse of T $-n$ times. Therefore, a triangle is also denoted by $A \to B \to C \to A[1]$.

We also have morphisms between triangles, which are just a commutative diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, dashed, "f"] & B \arrow[r] \arrow[d, dashed, "g"] & C \arrow[r] \arrow[d, dashed, "h"] & A[1] \arrow[d, dashed, "f[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$

notice that the arrow between $A[1]$ and $A'[1]$ is not arbitrary but $f[1]$ where f is the arrow between A and A'. It's an isomorphism if f, g, h are all isomorphisms, thus we can talk about isomorphic triangles.

Now we are ready to give the axioms **TR1** to **TR4**. You might be a little bit confused with why we need these axioms, to learn more about where did all these axioms originate, you can see the notes by Caldararu or keep reading this notes where we will explain more in later parts.

TR1:

1. Any triangle of the form:
$$A \xrightarrow{id} A \to 0 \to A[1]$$
is distinguished.

2. Any triangle isomorphism to a distinguished triangle is distinguished.

3. Any morphism $f : A \to B$ can be completed to a distinguished triangle:
$$A \xrightarrow{f} B \to C \to A[1]$$

TR2:The triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ is distinguished iff:
$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$$
is distinguished triangle.

TR3:Suppose that there exists a commutative diagram of distinguished triangles with vertical arrows f and g:
$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] & A[1] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$
it can be completed to a morphism of triangles by a not necessarily unique morphism $h : C \to C'$
$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, dashed, "h"] & A[1] \arrow[d, "f[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$

TR4: This is called the octahedron axiom, and it's very complicated to state and to draw the diagram. Since we never explicitly use it in the book, we will not include it here. If you are interested, just google or see other literature to find the precise formulation. One thing that is worth mentioning is that without this TR4, or octahedron axiom, the category we are dealing with is called the pre-triangulated category.

It's also worth-noting that in **TR3**, we only require that we can complete the morphism of triangles, but the completion may not be unique. You can also see the notes by Caldararu to learn more on this.

Remark 1.28. Except for the special **TR3** we mentioned above, **TR1** and **TR2** are naturally arising axioms since if you think about it, they are describing what are the distinguished triangles and given a distinguished triangle, what other new triangles can you produce from the old ones.

First, there are always enough distinguished triangles since every morphism can be completed to a distinguished triangle. Second, distinguished triangles are closed under shifts and isomorphisms. Finally, the most simple triangles are distinguished. There is another point we would like to mention about **TR2**. It seems that we are only allowed to shift forward and applying the inverse T^{-1} is not as good as applying T. This is true since by definition, any distinguished triangle must start by A and end by $A[1] = T(A)$. Therefore, stuff like $C[-1] \to A \to B \to C$, unless we have $C[-1][1] = C$, is not a triangle by definition, thus we can not expect in general that $A \to B \to C \to A[1]$ is distinguished iff $C[-1] \to A \to B \to C$ is distinguished. However, there are certain $\mathscr{D}$ and T such that $T^{-1}T = TT^{-1} = id$, such triangulated structure and twisting functors are extremely good, we will have some examples later.

Also, for **TR3**, in the description above, it seems a little bit strange to require we can only complete morphisms of triangles at the position of C. If we have something like:
$$\begin{tikzcd} A \arrow[r] & B \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, "g"] & A[1] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$

can we also complete it to a morphism of triangles. To answer this question, consider:

$$\begin{tikzcd}
B \arrow[r] \arrow[d, dashed, "f"] & C \arrow[r] \arrow[d, dashed, "g"] & A[1] \arrow[r] \arrow[d, dashed, "h"] & B[1] \arrow[d, dashed, "f[1]"] \\
B' \arrow[r] & C' \arrow[r] & A'[1] \arrow[r] & B'[1]
\end{tikzcd}$$

notice that apply T^{-1} to the square on the right, you get a new commutative square since you have $T^{-1}T \cong id$:

$$\begin{tikzcd}
A \arrow[r, leftrightarrow] \arrow[d, dashed] & A[1][-1] \arrow[r] \arrow[d, dashed, "h[-1]"] & B[1][-1] \arrow[d, dashed, "f[1][-1]"] & B \arrow[l, leftrightarrow, "iso"'] \arrow[d, dashed, "f"] \\
A' \arrow[r, leftrightarrow] & A'[1][-1] \arrow[r] & B'[1][-1] & B' \arrow[l, leftrightarrow, "iso"']
\end{tikzcd}$$

notice that T is fully faithful, thus h is $j[1]$ for some $j \in Hom(A, A')$, thus the above diagraqm is:

$$\begin{tikzcd}
A \arrow[r, leftrightarrow] \arrow[d, dashed, "j"] & A[1][-1] \arrow[r] \arrow[d, dashed, "{h[-1]=j[1][-1]}"] & B[1][-1] \arrow[d, dashed, "f[1][-1]"] & B \arrow[l, leftrightarrow, "iso"'] \arrow[d, dashed, "f"] \\
A' \arrow[r, leftrightarrow] & A'[1][-1] \arrow[r] & B'[1][-1] & B' \arrow[l, leftrightarrow, "iso"']
\end{tikzcd}$$

then we completed the morphism of triangles. Similarly, if the arrow between B and B' is missing, we can also complete it.

If you recall that, we mentioned before that distinguished triangles are also called exact triangles, and in an abelian category if $0 \to A \to B \to C \to 0$ is exact, we know that the composition $A \to C$ should be the 0-morphism, here in a triangulated category, we would also like something similar, thus if $A \to B \to C \to A[1]$ is distinguished, we want $A \to C$ to be the 0-morphism, this is not given in the axioms, but it's an easy consequence of the axioms, as shown in the following proposition:

Proposition 1.29. If $A \to B \to C \to A[1]$ is a distinguished triangle, then the composition $A \to C$ is 0.

Proof. Consider the following diagram and by **TR3**, you can complete this diagram via a the 0-morphism:

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r, "h"] & A[1] \\
A \arrow[r, dashed, "id"] \arrow[u, "id"] & A \arrow[r] \arrow[u, "f"] & 0 \arrow[r] \arrow[u, dashed, "0"] & A[1] \arrow[u, "id"]
\end{tikzcd}$$

thus you see that $A \to B \to C$ is the same as $A \to 0 \to C$, the zero morphism. ☐

The first result about triangles is the following:

Proposition 1.30. Let $A \to B \to C \to A[1]$ be a distinguished triangle in a triangulated category $\mathscr{D}$, then for any $A_0 \in \mathscr{D}$, the following two induced sequence are exact:

$$Hom(A_0, A) \to Hom(A_0, B) \to Hom(A_0, C)$$

$$Hom(C, A_0) \to Hom(B, A_0) \to Hom(A, A_0)$$

Proof. We will only prove the first one is exact since the proof of the second one is similar.

To prove the first sequence is exact, we need to show that if $f : A_0 \to B$ is a morphism such that after we post-compose $B \to C$, it becomes 0. Then f comes from some $g \in Hom(A_0, A)$ post-composed with $A \to B$. This is an easy application of **TR1** and **TR3** together with the remark we had before about how to complete the arrow even if it's not the third vertical arrow, then you can consider the following diagram:

$$\begin{tikzcd}
A_0 \arrow[r, "id"] & A_0 \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d, dashed, "0"] & A_0[1] \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & A[1]
\end{tikzcd}$$

apply **TR3**, we can lift f to a morphism from A_0 to A. ☐

Remark 1.31. Notice that due to the shift functor, we have $B \to C \to A[1] \to B[1]$ is distinguished, and using the preceding proposition, we have $Hom(A_0, B) \to Hom(A_0, C) \to Hom(A_0, A[1])$ is also exact, and you can keep going like this using the twisting functor.

Below are a few basic properties of distinguished triangles we need in the future and you can also use them to practice your understanding, after you finish them, you will find that this distinguished triangle is very similar to the notion of exactness:

Proposition 1.32. Suppose that $A \to B \to C \to C[1]$ is distinguished, then $A \to B$ is an isomorphism iff $C = 0$.

Proof. This is also very similar to the usual exactness you see in an abelian category, since given a SES $0 \to A \to B \to C \to 0$ $A \to B$ is an isomorphism iff $C = 0$.

Let's say first $A \to B$ is an isomorphism, we will show that $Hom(A_0, C) = 0$ for every A_0, which is equivalent to $C = 0$. This is because by definition, 0 in an addtive category is the element such that $Hom(0, 0) = 0$ and on the other hand, we proved that if $C = 0$, then $Hom(A_0, C) = 0$ for any A_0. We will use the exact sequence we obtained in the previous proposition:

$$Hom(A_0, A) \xrightarrow{iso} Hom(A_0, B) \longrightarrow Hom(A_0, C) \longrightarrow Hom(A_0, A[1]) \xrightarrow{iso} Hom(A_0, B[1])$$

we are done.

Conversely, if $C = 0$, we still use the above exact sequence and since this holds for every A_0, we conclude from Yoneda's lemma that $A \to B$ is an isomorphism. $\square$

Proposition 1.33. Consider a morphism of distinguished triangles with vertical arrows f, g, h:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & A[1] \arrow[d] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$

then if two of the three arrows are isomorphisms, then the the third one is also an isomorphism.

Proof. This is an application of the five lemma in an abelian category. First, notice that since T is an equivalence, to prove a morphism f is an isomorphism, we only need to prove that $T(f)$ is an isomorphism, this is because if $T(f)$ is an isomorphism, we can consider the following diagram and show f is also an isomorphism:

$$\begin{tikzcd} A[1][-1] \arrow[r, leftrightarrow, "{f[1][-1]}"] & B[1][-1] \\ A \arrow[u, leftrightarrow, "iso"] \arrow[r, "f"] & B \arrow[u, leftrightarrow, "iso"] \end{tikzcd}$$

then back to our question, suppose that we want to show f is an isomorphism when the other two are. Just consider the following diagram:

$$\begin{tikzcd} Hom(A_0, B) \arrow[r] \arrow[d, leftrightarrow] & Hom(A_0, C) \arrow[r] \arrow[d, leftrightarrow] & Hom(A_0, A[1]) \arrow[r] \arrow[d] & Hom(A_0, B[1]) \arrow[r] \arrow[d, leftrightarrow] & Hom(A_0, C[1]) \arrow[d, leftrightarrow] \\ Hom(A_0, B') \arrow[r] & Hom(A_0, C') \arrow[r] & Hom(A_0, A'[1]) \arrow[r] & Hom(A_0, B'[1]) \arrow[r] & Hom(A_0, C'[1]) \end{tikzcd}$$

then apply five lemma to conclude that $f[1]$ is an isomorphism. The proof for the other two is similar. $\square$

Proposition 1.34. Suppose that $A \to B \to C \to A[1]$ is distinguished with $f : A \to B$, $g : B \to C$ and $C \to A[1]$ is trivial, then the triangle splits, i.e $B \simeq A \oplus C$.

Proof. We still apply $Hom(-,)$ to the sequence and since $C \to A[1]$ is trivial, for any $A_0 \in \mathscr{D}$, we have the following exact sequence:

$$Hom(A_0, A) \longrightarrow Hom(A_0, B) \longrightarrow Hom(A_0, C) \longrightarrow 0$$

we also claim that the first morphism is injective, which gives us a SES:

$$0 \longrightarrow Hom(A_0, A) \longrightarrow Hom(A_0, B) \longrightarrow Hom(A_0, C) \longrightarrow 0$$

this comes from the fact that $Hom(A_0[1], A[1]) \to Hom(A_0[1], B[1])$ is injective, thus $Hom(A_0[1][-1], A[1][-1]) \to Hom(A_0[1][-1], B[1][-1])$ is injective(make sure you understand why), then since $T^{-1}T \simeq id$ we get the desired result.

Then we claim that in any additive category, the SES:

$$0 \longrightarrow Hom(A_0, A) \longrightarrow Hom(A_0, B) \longrightarrow Hom(A_0, C) \longrightarrow 0$$

is enough for us to say that $B \simeq A \oplus C$. To see so, take $A_0 = C$, we can conclude that there is $i : C \to B$ such that $g \circ i = id_C$, now suppose that we use k to denote $id_B - i \circ g$, then take $A_0 = B$, since $g \circ (id_B - i \circ g) = 0$, there is a $j : B \to A$ such that $f \circ j = id_B - i \circ g$, thus $id_B = f \circ j + i \circ g$, this is shown in the following diagram:

$$\begin{tikzcd}
 & B \arrow[dl, shift left, "j"] \arrow[dr, shift left, "g"] & \\
A \arrow[ur, shift left, "f"] & & C \arrow[ul, shift left, "I"]
\end{tikzcd}$$

we claim that this makes B the direct sum of A and C. We already know that $g \circ f = 0$, $g \circ i = id_C$ and we just showed that $id_B = f \circ j + i \circ g$. If we can show that $j \circ i = 0$, $j \circ f = id$ then by general argument, we can conclude that B is the direct sum. To show this, we first notice that $f \circ j \circ f = f$ since we can precompose f to $f \circ j = id_B - i \circ g$ and use $g \circ f = 0$, now since $Hom(A, A) \to Hom(A, B)$ is injective and the identity compose with f is f, we know that $j \circ f = id_A$. Finally, to show that $j \circ i = 0$, we notice that $j = j - j \circ i \circ g$, thus $j \circ i \circ g = 0$, then precompose with i again we conclude that $j \circ i = 0$. ☐

About direct sums, we can also prove that the direct sum of two distinguished triangles are distinguished:

Proposition 1.35. Suppose that

$$A_1 \to B_1 \to C_1 \to A_1[1]$$

and

$$A_2 \to B_2 \to C_2 \to A_2[1]$$

are distinguished, then

$$A_1 \oplus A_2 \to B_1 \oplus B_2 \to C_1 \oplus C_2 \to (A_1 \oplus A_2)[1]$$

is also distinguished.

Proof. Notice that we can complete $A_1 \oplus A_2 \to B_1 \oplus B_2$ to a distinguished triangle:

$$A_1 \oplus A_2 \to B_1 \oplus B_2 \to D \to (A_1 \oplus A_2)[1]$$

We want to estabilish such a diagram:

$$\begin{tikzcd}
A_1 \oplus A_2 \arrow[r] \arrow[d, "id"] & B_1 \oplus B_2 \arrow[r] \arrow[d, dashed, "id"] & C_1 \oplus C_2 \arrow[r] \arrow[d] & (A_1 \oplus A_2)[1] \arrow[d, "id"'] \\
A_1 \oplus A_2 \arrow[r] & B_1 \oplus B_2 \arrow[r] & D \arrow[r] & (A_1 \oplus A_2)[1]
\end{tikzcd}$$

where $C_1 \oplus C_2 \to D$ is an isomorphism. Notice that for $i = 1, 2$, we have the following diagram:

$$\begin{tikzcd}
A_i \arrow[r] \arrow[d, "id"] & B_i \arrow[r] \arrow[d, dashed, "id"] & C_i \arrow[r] \arrow[d, dashed, "\varphi_i"] & A_i \arrow[d, "id"'] \\
A_1 \oplus A_2 \arrow[r] & B_1 \oplus B_2 \arrow[r] & D \arrow[r] & (A_1 \oplus A_2)[1]
\end{tikzcd}$$

which we can complete to get φ_i, then we define $C_1 \oplus C_2 \to D$ to be $\varphi_1 \oplus \varphi_2$.

Similarly, we can get ψ_i from D to C_i which gives us a morphism from D to $C_1 \oplus C_2$, thus we have the following diagram:

$$\begin{tikzcd}
A_1 \oplus A_2 \arrow[r] \arrow[d, dashed, "id"] & B_1 \oplus B_2 \arrow[r] \arrow[d, dashed, "id"] & C_1 \oplus C_2 \arrow[r] \arrow[d, dashed, "\varphi"] & (A_1 \oplus A_2)[1] \arrow[d, dashed, "id"] \\
A_1 \oplus A_2 \arrow[r] \arrow[d, dashed, "id"] & B_1 \oplus B_2 \arrow[r] \arrow[d, dashed, "id"] & D \arrow[r] \arrow[d, dashed, "\psi"] & (A_1 \oplus A_2)[1] \arrow[d, dashed, "id"] \\
A_1 \oplus A_2 \arrow[r] & B_1 \oplus B_2 \arrow[r] & C_1 \oplus C_2 \arrow[r] & (A_1 \oplus A_2)[1]
\end{tikzcd}$$

if we can show that $\psi \circ \varphi$ is an isomorphism, we are done, since it's easy to show that $\varphi \circ \psi$ is an isomorphism via morphisms of triangles are isomorphisms if two vertical morphisms are isomorphisms (make sure you understand this!).

Now to show that $\psi \circ \varphi$ is indeed an isomorphism, notice that for any $X \in \mathscr{D}$, we can apply $Hom(X, -)$ to this diagram, by adding one row $(B_1 \oplus B_2)[1] \overset{id}{\to} (B_1 \oplus B_2)[1]$ to the diagram and use five lemma and Yoneda to conclude what we want. □

Now let's introduce another important definition in this section, exact functors in the context of derived categories.

Definition 1.36. An **additive functor**:

$$F : \mathscr{D} \to \mathscr{D}'$$

between triangulated categories $\mathscr{D}$ and $\mathscr{D}'$ is called **exact** if the following two conditions are satisfied.

1. F commutes with shift functors in the sense that there is a functor isomorphism:

$$F \circ T_{\mathscr{D}} \simeq T_{\mathscr{D}'} \circ F$$

2. F maps distinguished triangles to distinguished triangles. To be precise, for any distinguished triangle in $\mathscr{D}$:

$$A \to B \to C \to A[1]$$

it is mapped to a distinguished triangle in $\mathscr{D}'$:

$$F(A) \to F(B) \to F(C) \to F(A)[1]$$

here we identify $F(A)[1]$ with $F(A[1])$ via the functor isomorphism $F \circ T_{\mathscr{D}} \simeq T_{\mathscr{D}'} \circ F$ via the diagram:

$$FA \longrightarrow FB \longrightarrow FC \longrightarrow F(A[1]) \xrightarrow{iso} FA[1]$$

Again, this definition can be modified if one is only interested in studying k-linear categories, for example, the shift functor is k-linear and when we say exact functors, we also would like it to be k-linear and not just additive.

We have seen before that in an abelian category, if we have an adjoint relation between two functors $F \dashv G$, then F must be right exact and G must be left exact. However, even we assume F is an exact functor, from the adjoint relation, we still only can deduce G is left exact not necessarily exact. However, if we work with triangulated categories, we have a similar but stronger result:

Proposition 1.37. Let $F : \mathscr{D} \to \mathscr{D}'$ be an **exact functor** between two triangulated categories. If $F \dashv H$ then $H : \mathscr{D} \to \mathscr{D}'$ is exact.

Similarly, if $G \dashv F$, then $G : \mathscr{D} \to \mathscr{D}'$ is exact.

Proof. We still only prove one part of the proposition since the other half is similar. We assume that $F \dashv H$, and we want to show that H is exact. First, we need to show that H commutes with the shift functors.

To see why this is true, we again use Yoneda's lemma, consider the following functorial isomorphisms:

$$\begin{aligned} Hom(A, HT'B) \simeq Hom(FA, T'B) \simeq Hom(T'^{-1}FA, T'^{-1}T'B) \simeq\simeq Hom(T'^{-1}FA, B) \\ Hom(F \circ T^{-1}A, B) \simeq Hom(T^{-1}A, HB) \\ \simeq Hom(A, THB) \end{aligned}$$

Recall the remark we made before where we mentioned that for an equivalence, it's inverse is both left and right adjoint to it, this helps understand some of the isomorphisms above. Beside this, we are only using the fact that $F \dashv H$. Notice that every isomorphism is functorial, thus we have a functorial isomorphism:

$$Hom(A, HT'B) \simeq Hom(A, THB)$$

then Yoneda's lemma tells you that $HT' \simeq TH$, which proves that T commutes with shift functors.

Next we need to show that it maps distinguished triangles to distinguished triangles. Let $A \to B \to C \to A[1]$ be a distinguished triangle, we know that by the axioms that we can complete the induced morphism $HA \to HB$ to a distinguished triangle:

$$HA \to HB \to C_0 \to HA[1]$$

Notice that F is exact, then we apply F to the above distinguished triangle to get:

$$FHA \to FHB \to FC_0 \to FHA[1]$$

we tacitly use $F(HA[1]) \simeq FHA[1]$ since F commutes with the shift functor.

Recall that there is a natural transformation between FH and id, we use it to get:

$$\begin{tikzcd} FHA \arrow[r] \arrow[d] & FHB \arrow[r] \arrow[d] & FC_0 \arrow[r] & FHA[1] \\ A \arrow[r] & B \arrow[r] & C \arrow[r] & A[1] \end{tikzcd}$$

which can be completed to a morphism of triangles via **TR3**:

$$\begin{tikzcd} FHA \arrow[r] \arrow[d] & FHB \arrow[r] \arrow[d] & FC_0 \arrow[r] \arrow[d, dashed, "\xi_0"] & FHA[1] \arrow[d] \\ A \arrow[r] & B \arrow[r] & C \arrow[r] & A[1] \end{tikzcd}$$

next we apply H to the diagram above to obtain:

$$\begin{tikzcd} HFHA \arrow[r] \arrow[d] & HFHB \arrow[r] \arrow[d] & HFC_0 \arrow[r] \arrow[d, dashed, "H\xi_0"] & HFHA[1] \arrow[d] \\ HA \arrow[r] & HB \arrow[r] & HC \arrow[r] & HA[1] \end{tikzcd}$$

recall that we also have a natural transformation between $id \to HF$, thus we can add one more row to the diagram above:

$$\begin{tikzcd} HA \arrow[r] \arrow[d] \arrow[dd, dotted, bend right, "id"'] & HB \arrow[r] \arrow[d] \arrow[dd, dotted, bend right, "id"'] & C_0 \arrow[r] \arrow[d, dashed, "h_{C_0}"] & HA[1] \arrow[d] \\ HFHA \arrow[r] \arrow[d] & HFHB \arrow[r] \arrow[d] & HFC_0 \arrow[r] \arrow[d, dashed, "H\xi_0"] & HFHA[1] \arrow[d] \\ HA \arrow[r] & HB \arrow[r] & HC \arrow[r] & HA[1] \end{tikzcd}$$

we still use the notation we used when we talked about adjounction and h_{C_0} is the same as the h we talked about before.

Now recall that, by a preceding proposition, the composition of the first two vertical column gives you identity, which is indicated by dotted arrows in the above diagram, also similarly, the composition of the lats vertical column is also identity, then if $h_{C_0} \circ H\xi_0$ is an isomorphism, the last row is isomorphic to a distinguished triangle, then it's a distinguished triangle. Notice that we can not directly conclude $h_{C_0} \circ H\xi_0$ is an isomorphism using the above proposition which states that if we have morphism between two distinguished two triangles with two vertical arrows being isomorphisms, then the lats one is also an isomorphism. We need to prove $HA \to HB \to HC \to HA[1]$ is an isomorphism! We can not use it right now!

To show $h_{C_0} \circ H\xi_0$ is indeed an isomorphism, notice that for any A_0, using the fact that $A \to B \to C \to A[1]$ is distinguished, we have an exact sequence:

$$Hom(FA_0, B) \longrightarrow Hom(FA_0, C) \longrightarrow Hom(FA_0, A[1])$$

then using the fact that $F \dashv H$, we get the following diagram:

$$\begin{tikzcd}
Hom(FA_0, B) \arrow[r] \arrow[d] & Hom(FA_0, C) \arrow[r] \arrow[d] & Hom(FA_0, A[1]) \arrow[d] \\
Hom(A_0, HB) \arrow[r] & Hom(A_0, HC) \arrow[r] & Hom(A_0, HA[1])
\end{tikzcd}$$

then we conclude that the second row is also exact(make sure you know what are the arrows in the second row are). Also, we can extend this sequence above to $Hom(A_0, HA)$, thus if we apply $Hom(A_0, -)$ to obtain the following diagram:

$$\begin{tikzcd}
Hom(A_0, HA) \arrow[r] \arrow[d, dashed, "id"] & Hom(A_0, HB) \arrow[r] \arrow[d, dashed, "id"] & Hom(A_0, C_0) \arrow[r] \arrow[d, dashed, "h_{C_0} \circ H\xi_0 \circ"] & Hom(A_0, HA[1]) \arrow[r] \arrow[d, dashed, "id"] & Hom(A_0, HB[1]) \arrow[d, dashed, "id"] \\
Hom(A_0, HA) \arrow[r] & Hom(A_0, HB) \arrow[r] & Hom(A_0, HC) \arrow[r] & Hom(A_0, HA[1]) \arrow[r] & Hom(A_0, HB[1])
\end{tikzcd}$$

with the rows being exact, then the five lemma for abelian groups tells us the middle arrow is an isomorphism for every A_0, then Yoneda's lemma implies that $h_{C_0} \circ H_{\xi_0}$ is an isomorphism. $\square$

We also have the notion of subtriangulated category, for example, if a subcategory $\mathscr{D}' \subset \mathscr{D}$ of a triangulated category is called a triangulated subcategory if $\mathscr{D}'$ admits the structure of a triangulated category such that the inclusion is exact. Here we require the shift functor $T : \mathscr{D} \to \mathscr{D}$ maps $\mathscr{D}'$ to $\mathscr{D}'$, i.e. the triangulated structure of $\mathscr{D}'$ comes from the triangulated structure of $\mathscr{D}$.

If further more, $\mathscr{D}'$ is a full subcategory of $\mathscr{D}$, we have a criterion to decide if it's a triangulated subcategory of $\mathscr{D}$:

Proposition 1.38. In the above set up, $\mathscr{D}'$ is a triangulated subcategory iff $\mathscr{D}'$ is invariant under the shift functor and for any distinguished triangle in $\mathscr{D}$:

$$A \to B \to C \to A[1]$$

with $A, B \in \mathscr{D}'$, there is an object $C' \in \mathscr{D}'$ such that $C \simeq C'$.

Proof. Let's first assume that $\mathscr{D}'$ is a triangulated subcategory, then using the triangulated structure of $\mathscr{D}'$ and the fact that it's a full subcategroy, we know that $A \to B$ can be completed to a distinguished triangle:

$$A \to B \to C' \to A[1]$$

then consider the following diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, dashed, "id"] & B \arrow[r] \arrow[d, dashed, "id"] & C \arrow[r] & A[1] \arrow[d, dashed, "id"] \\
A \arrow[r] & B \arrow[r] & C' \arrow[r] & A[1]
\end{tikzcd}$$

using the triangulated structure of $\mathscr{D}$, we can complete this diagram to add an arrow from C to C' and the arrow is necessarily an isomorphism by a previous proposition.

Conversely, let's say $C \simeq C'$, we want to show that $\mathscr{D}'$ is a triangulated subcategory. We will left some details to the reads and we claim that we only have to show that if $f : A \to B$ is an arrow in $\mathscr{D}'$, it can be completed to a triangle in $\mathscr{D}'$. Using the triangulated structure in $\mathscr{D}$, we know that there is always a C such that:

$$A \to B \to C \to A[1]$$

is distinguished in $\mathscr{D}$, then since we assumed that $C \simeq C'$ for some $C' \in \mathscr{D}'$, we have the following diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "id"] & B \arrow[r] \arrow[d, "id"] & C \arrow[r] \arrow[d, "iso"] & A[1] \arrow[d, "id"] \\
A \arrow[r] & B & C' & A[1]
\end{tikzcd}$$

then you can choose arrow $B \to C'$ and $C' \to A[1]$ such that the diagram commutes, thus the bottom is also distinguished in $\mathscr{D}$, then if we define in the beginning that every distinguished triangle in $\mathscr{D}$ such that it's also in $\mathscr{D}'$ is distinguished in $\mathscr{D}'$, then the above argument is enough for $\mathscr{D}'$ to be a triangulated subcategory. $\square$

Here are a few definitions related to the notion of subtriangulated category:

Proposition 1.39. A full triangulated subcategory $\mathscr{D}' \subset \mathscr{D}$ is called admissible if the inclusion has a right adjoint $\mathscr{D} \to \mathscr{D}'$, i.e. there exist functorial isomorphisms $Hom_{\mathscr{D}}(A, B) \simeq Hom(\mathscr{D}')(A, \pi B)$ for all $A \in \mathscr{D}'$ and $B \in \mathscr{D}$.

The orthogonal complement of a(n admissible) subcategory $\mathscr{D}' \subset \mathscr{D}$ is the full subcategory $\mathscr{D}'^{\perp}$ of all objects $C \in \mathscr{D}$ such that $Hom(B, C) = 0$ for all $B \in \mathscr{D}'$.

Remark 1.40.

1. First, the orthogonal complement defined above should be called the right complement since the objects in $\mathscr{D}'^{\perp}$ appears in the right slot of $Hom(-, -)$. Of course, there is also left orthogonal complement, but since we will not use this notion later, we won't mention it. Through out this notes, orthogonal complement will refer to right orthogonal complement.

2. Notice that, the right adjoint functor $\pi : \mathscr{D}' \to \mathscr{D}$ is right adjoint to the inclusion, which is exact by assumption since an admissible subcategory is in the first place a full triangulated subcategory.

3. Notice that in the definition, we only mentioned that the orthogonal complement is a full subcategory, but we don't know in general if it's a triangulated full subcategory. However, it turns out that if we are looking at the orthogonal complement of an admissible subcategory, it's automatically triangulated.

   To see this, according to the preceding remark, we need to show that $\mathscr{D}'^{\perp}$ is invariant under shift and for any distinguished triangle $A \to B \to C \to A[1]$ with $A, B \in \mathscr{D}'^{\perp}$, C is isomorphic to some $C' \in \mathscr{D}'^{\perp}$. Let's first deal with the first condition: say $B \in \mathscr{D}'$ and $C \in \mathscr{D}'^{\perp}$, then we know that by definition: $Hom(B, C) = 0$, now notice that since $\mathscr{D}'$ is invariant under shift, we know that $Hom(B, C[i]) \simeq Hom(B[-i], C) = 0(T, T^{-1}$ adjoint to each other), thus $\mathscr{D}'^{\perp}$ is indeed invariant under shift.

   To show the second criterion, notice that if we have a distinguished triangle $C_1 \to C_2 \to C_3 \to C_1[1]$ in $\mathscr{D}$ with $C_1, C_2 \in \mathscr{D}'^{\perp}$, notice that if we apply $Hom(A, -)$ to this sequence, we get an exact sequence which shows that $Hom(A, C_3) = 0$ for every $A \in \mathscr{D}'$, thus $C_3 \in \mathscr{D}'^{\perp}$, thus we even don't have to find some object in $\mathscr{D}'^{\perp}$ such that it's isomorphic to C_3 since itself is in $\mathscr{D}'^{\perp}$.

4. There is a more explicit way to determine when a full triangulated subcategory is admissible. Precisely, a full triangulated subcategory $\mathscr{D}' \subset \mathscr{D}$ is admissible iff for all $A \in \mathscr{D}$ there exists a distinguished triangle:

   $$B \to A \to C \to B[1]$$

   with $B \in \mathscr{D}'$ and $C \in \mathscr{D}'$.

First suppose $\mathscr{D}'$ is admissible, then the adjunction property of π allow us to map the identity in $Hom_{\mathscr{D}'}(\pi(A), \pi(A))$ to a morphism in $Hom_{\mathscr{D}}(\pi(A), A)$, then if we define $B = \pi(A)$, the image gives us a morphism $B \to A$ and we can complete it into a distinguished triangle:

$$B \to A \to C \to B[1]$$

we claim that C is in $\mathscr{D}'^{\perp}$. To see this is true, we apply $Hom(B', -)$ to the distinguished triangle to get an exact sequence:

$$Hom(B', B) \to Hom(B', A) \to Hom(B', C)$$

Then notice that $Hom(B', B) = Hom(B', \pi(A)) \simeq Hom(B', A)$, then the above sequence becomes:

$$Hom(B', A) \to Hom(B', A) \to Hom(B', C)$$

$Hom(B', \pi(A)) \to Hom(B', A)$ is $f \mapsto (g_A \circ if) = g_A \circ f$. Notice that g_A is just $B \to A$ we obtained above, thus $Hom(B', A) \to Hom(B', A)$ is the identity, then you extend the exact sequence to conclude that $Hom(B', C) = 0$ for all $B' \in \mathscr{D}'$.

Then conversely, if we assume that for any $A \in \mathscr{D}$, such a distinguished triangle $B \to A \to C \to B[1]$ is given, we define $\pi(A) = B$ **** later, how to show that it doesn't depend on the choic of triangles*

5. Again, let $\mathscr{D}$ be a triangulated category, then we claim that an admissible subcategory occurs whenever there is a fully faithful exact functor $F : \mathscr{D}' \to \mathscr{D}$ that admits a right adjoint. Indeed, in this case, we claim that F defines an equivalence between $\mathscr{D}'$ and an admissible subcategory of $\mathscr{D}$. Consider the full subcategory of $\mathscr{D}$ such that each object of which is isomorphism to some FA for some $A \in \mathscr{D}'$, if we denote this full subcategory by $\mathscr{D}''$ notice that F defines an equivalence between $\mathscr{D}'$ and $\mathscr{D}''$. Let's first show that $\mathscr{D}''$ is indeed a triangulated subcategory. First, it's easy to see why it's invariant under shift. We need to show that if $A \to B \to C \to A[1]$ is distinguished with A, B in $\mathscr{D}''$, then C is isomorphic to something in $\mathscr{D}'$. Consider the following diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "iso"'] & B \arrow[r] \arrow[d, "iso"] & C \arrow[r] \arrow[d, dashed] & A[1] \arrow[d, dashed, "iso"] \\
FF^{-1}A \arrow[r] & FF^{-1}B \arrow[r] & C' \arrow[r] & FF^{-1}A[1]
\end{tikzcd}$$

then we still need to show that it's admissible, i.e., find a right adjoint for the inclusion. Let $G : \mathscr{D} \to \mathscr{D}'$ be the right adjoint of F, we will use this to create a right adjoint for $i : \mathscr{D}'' \to \mathscr{D}$. Consider the following sequence of isomorphisms, for any $A \in \mathscr{D}''$ and $B \in \mathscr{D}$:

$$Hom(A, B) \simeq Hom(FF^{-1}A, B) \simeq Hom(F^{-1}A, GB) \simeq Hom(FF^{-1}A, FGB) \simeq Hom(A, FGB)$$

notice that each isomorphism is functorial in both slots, thus we see that FG is the right adjoint of the inclusion. Here we view F as a functor from $\mathscr{D}'$ to $\mathscr{D}''$.

The following proposition is going to be used later and it's a good place for us to introduce some notation:

Proposition 1.41. Let $A \in \mathscr{D}$ be an object in a triangulated category $\mathscr{D}$, then if we denote $A^{\perp}$ by the full subcategroy that consists of all the following objects:

$$A^{\perp} = \{B \in \mathscr{D} \quad | \quad Hom(A, B[i]) = 0 \quad \forall i \in \mathbb{Z}\}$$

then it's a triangulated full subcategory. Also, if we denote by $\langle A \rangle$ the smallest triangulated full subcategory containing A then $A^{\perp} \simeq \langle A \rangle^{\perp}$.

Proof. From the definition, we know that $A^{\perp}$ is invariant under the shift functor. Therefore, we only need to show that for any distinguished triangle:

$$A_1 \to A_2 \to A_3 \to A_1[1]$$

with A_1, A_2 in $A^{\perp}$, A_3 is isomorphic to some objects in $A^{\perp}$. Notice that if you apply $Hom(A, -)$ to this sequence, then an easy argument shows that for all $i \in \mathbb{Z}^+$, i.e. all non-negative integers, $Hom(A, A_3[i]) = 0$. Also, notice that we can shift backwards, then the same argument shows that we are done.

To show the second assertion, notice that the objects in $\langle A \rangle^{\perp}$ is contained in $A^{\perp}$. The reason is simple, since $\langle A \rangle$ is invariant under shifts then you have:

$$Hom(B, C[i]) \simeq Hom(B[-i], C) = 0$$

then we know that if $C \in \langle A \rangle^{\perp}$, then $C \in A^{\perp}$. Conversely, we want to show that if $C \in A^{\perp}$, then $C \in \langle A \rangle^{\perp}$. The goal is to prove for every $B \in \langle A \rangle$, $Hom(B, C) = 0$. The trick here is to define:

$${}^{\perp}C = \{D \in \mathscr{D} : Hom(D, C[i]) = 0 \quad \forall i \in \mathbb{Z}\}$$

We claim that this is also a full triangulated subcategory. First, it's invariant under shifts since if $D \in {}^{\perp}C$, then $Hom(D, C[i]) = 0$, then by repeatedly applying T or T^{-1} you get $Hom(D[n], C[i+n]) = 0$ for $n, i \in \mathbb{Z}$. This proves that it's closed under shifts. Moreover, if you have a triangle:

$$A_1 \to A_2 \to A_3 \to A_1[1]$$

with $A_1, A_2 \in {}^{\perp}C$, similarly you apply $Hom(-, C[i])$, we are done.

Now, we claim that $A \in {}^{\perp}C$, which means that $Hom(A, C[i]) = 0$. By assumption, $C \in A^{\perp}$, thus by definition $Hom(A, C[i]) = 0$, thus A indeed is in ${}^{\perp}C$, then since $\langle A \rangle$ is the smallest triangulated subcategory containing A, then $\langle A \rangle \subset {}^{\perp}C$, thus every $B \in \langle A \rangle$, $Hom(B, C[i]) = 0$, then take $i = 0$, we are done. □

The notion of equivalence that is important for us is the following:

Definition 1.42. Two triangulated categories $\mathscr{D}$ and $\mathscr{D}'$ are equivalent if there exists an exact equivalence $F : \mathscr{D} \to \mathscr{D}'$.

If $\mathscr{D}$ is a triangulated category, the set $Aut(\mathscr{D})$ of isomorphism classes of exact equivalences $F : \mathscr{D} \to \mathscr{D}$ forms the group of autoequivalences of $\mathscr{D}$.

Before we finish this subsection, we briefly discuss the fact that why Serre functor and triangulated categories are a good match. In geometric settings, the way you construct a Serre functor tells you that it's going to be exact, however, this is more general than just in geometry. It turns out that, this holds for every triangulated category:

Proposition 1.43. Any Serre functor over a triangulated category over a field k is exact.

Proof. **** later* □

1.3 Equivalences of triangulated categories

In this section, we discuss the criteria that allows us to decide when an exact functor is fully faithful or even an equivalence, this continues the discussion we had before, but in the context of triangulated categories. We first begin by the definition of a spanning class:

Definition 1.44. A collection Ω of objects in a triangulated category $\mathscr{D}$ is a spanning class of $\mathscr{D}$ or spans $\mathscr{D}$ if for all $B \in \mathscr{D}$, the following two conditions hold:

1. If $Hom(A, B[i]) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $B = 0$.
2. If $Hom(B[i], A) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $B = 0$.

Proposition 1.45. If $\mathscr{D}$ is endowed with a Serre functor, then the two conditions above are equivalent, thus in the presence of a Serre functor, we only need to require one of the two.

Proof. As before, we still assume that all Hom sets are finite dimensional.

Let's first assume the first condition, now if $Hom(B[i], A) = 0$, notice that $Hom(A, S(B[i]))^* = 0$, we proved that S is exact, thus it commutes with the shift functor, thus we have $Hom(A, SB[i])^* = 0$, which implies that $Hom(A, SB[i]) = 0$ thus by condition 1, we know that $SB = 0$, then since S is an equivalence, $B = 0$. The proof where we assume condition 2 is similar. □

Proposition 1.46. Let $F : \mathscr{D} \to \mathscr{D}'$ be an exact functor between triangulated categories with left and right adjoints $G \dashv F \dashv H$.

Suppose that Ω is a spanning class of $\mathscr{D}$ such that for all objects $A, B \in \Omega$ and all $i \in \mathbb{Z}$ the natural homomorphisms are bijective:

$$F : Hom(A, B[i]) \to Hom(FA, F(B[i]))$$

then F is fully faithful.

Proof. First recall that G, H are both exact due to a previous proposition, which we will use through out the proof.

WE shall use the following commutative diagram for adjoint functors $G \dashv F \dashv H$:

$$\begin{tikzcd} Hom(A,B) \arrow[r, "h_B\circ"] \arrow[d, "\circ g_A"] \arrow[dr, "F"] & Hom(A,HFB) \arrow[d, "iso"] \\ Hom(GFA,B) \arrow[r, "iso"] & Hom(FA,FB) \end{tikzcd}$$

This holds for any $A, B \in \mathscr{D}$. We first show that for any $A \in \Omega$ in the spanning class, $g_A : GFA \to A$ is an isomorphism. In order to see this, we choose any distinguished triangle as below:

$$GFA \to A \to C \to GFA[1]$$

now we apply $Hom(-, B)$ to the sequence for arbitrary $B \in \mathscr{D}$ to get:

$$Hom(C, B[i]) \longrightarrow Hom(A, B[i]) \longrightarrow Hom(GFA, B[i])$$

via the top diagram, we can add to this sequence:

$$\begin{tikzcd} Hom(C,B[i]) \arrow[r] & Hom(A,B[i]) \arrow[r, "\circ g_A"] \arrow[dr, "F"] & Hom(GFA,B[i]) \arrow[d, dashed, "iso"] \\ & & Hom(FA,FB[i]) \end{tikzcd}$$

Notice that if F is in general an isomorphism, we can conclude g_A is an isomorphism by Yoneda's lemma, however, this is what we are trying to prove. To get around this, we know that when $A, B \in \Omega$, F is indeed an isomorphism, thus we know that when $B \in \Omega$, we know that $Hom(C, B[i]) = 0$ for all $i \in \mathbb{Z}$. Then by apply T or T^{-1} multiple times and use $T^{-1}T \simeq id$, $TT^{-1} \simeq id$, we get the conclusion that $Hom(C[i], B]) = 0$ for any $i \in \mathbb{Z}$ and $B \in \Omega$, then since Ω spanns $\mathscr{D}$, we know that $C = 0$, then $Hom(C, -)$ is 0, then for arbitrary $B \in \mathscr{D}$, we know that:

$$Hom(A, B[i]) \overset{\circ g_A}{\to} Hom(GFA, B[i])$$

is an isomorphism, then take $i = 0$, we conclude from Yoneda's lemma g_A is an isomorphism for all $A \in \Omega$. (Or you can use the previous proposition which states that $C = 0$ iff the first arrow is an isomorphism) Please pay special attention that this is only for $A \in \Omega$, not all $A \in \mathscr{D}$.

Now, notice that, g_A being an isomorphism for $A \in \Omega$, plus the top diagram, implies that $Hom(A, B) \overset{h_B\circ}{\to} Hom(A, HFB)$ is also a bijection for any $A \in \Omega$ and $B \in \mathscr{D}$. Then again, using a distinguished triangle of the form:

$$B \to HFB \to C \to B[1]$$

We can apply $Hom(A, -)$ or $Hom(A[i], -)$ to this diagram and, similar to the above argument, get:

$$\begin{tikzcd} Hom(A[i],B) \arrow[r, "h_B\circ"] \arrow[dr, "F"'] & Hom(A[i],HFB) \arrow[r] \arrow[d, "iso"] & Hom(A[i],C) \\ & Hom(FA[i],FB) & \end{tikzcd}$$

again, we can deduce $C = 0$, thus h_B is an isomorphism, then the top diagram shows that F is an isomorphism for all $A, B \in \mathscr{D}$ $\square$

The above proposition helps us to determine when an exact functor is fully faithful, but another question is that when the functor is assumed to be fully faithful, when is it an equivalence? The following lemma gives you the first criterion, even though hard to check, it will help us get another criterion later, which is extremely useful.

Lemma 1.47. Let $F : \mathscr{D} \to \mathscr{D}''$ be a fully faithful exact functor between triangulated categories and suppose that F has a right adjoint $F \dashv H$, then F is an equivalence iff for any $C \in \mathscr{D}'$, the triviality of $H(C)$, i.e. $HC = 0$ implies that $C = 0$.

Proof. First, we need to recall that for a fully faithful functor which admits a right adjoint, how are we going to be able to say that it's an equivalence.

Notice that we already know that F is fully faithful, then we need to show that for any $B \in \mathscr{B}'$, there is something A in $\mathscr{D}$ such that B is isomorphic to FA. Recall that in a previous proposition, we show that if $F \dashv H$, then F is fully faithful is equivalent to $id \simeq H \circ F$ via h, however, this doesn't help here since H of something lands back in $\mathscr{D}$, but we would like an isomorphism in $\mathscr{D}'$. Then what about g, where we have:

$$g_B : FHB \to B$$

if this is an isomorphism for every B, we are done. However, we can not use the the remark since that will require H to be fully faithful, thus we need to find another way to do it, and after we prove g_B is indeed an isomorphism, we can conclude that H is indeed fully faithful and H would be an inverse of F in this case.

Notice that $g_B : FHB \to B$ can be completed to a distinguished triangle:

$$FHB \to B \to C \to FHB[1]$$

also recall that H, being the right adjoint of an exact functor, is itself exact, then we apply H to this triangle:

$$HFHB \to HB \to HC \to HFHB[1]$$

recall that we know $H(g_B) \circ h_{HB} = id_{HB}$ and we already know that h_{HB} is always an isomorphism, thus $H(g_B)$ is an isomorphism, which is the arrow $HFHB \to HB$ in the above diagram, thus $HC = 0$, then by assumption $C = 0$, hence $FHB \to B$ is an isomorphism. This deals with $HC = 0$ implies $C = 0$ implies that F is an equivalence.

Before proving the converse, we need to make a side remark. The above argument only shows that H is a right inverse of F, but this may not imply H is also a left inverse of F. However, since F is fully faithful and H is it's right adjoint, we automatically have $id \simeq H \circ F$.

Now, for the converse, if F is an equivalence, we know that if $HC = 0$ then $FHC = 0$ then using g_C, we know that $C = 0$, we are done. ☐

Using similar methods, we can prove that the case if F has a left adjoint:

Proposition 1.48. Again, let $F : \mathscr{D} \to \mathscr{D}'$ be an exact fully faithful functor between two triangulated categories and suppose that F has a right adjoint $G \dashv F$, then F is an equivalence iff for any $C \in \mathscr{D}'$, the triviality of GC, implies that $C = 0$.

Proof. This time, since F is still fully faithful, from the fact that it has a left adjoint, we know that G is F's left inverse. Again, suppose that $GC = 0$ implies that $C = 0$. For any $B \in \mathscr{D}'$, consider the arrow $h_B : B \to FGB$, we still complete it to a distinguished triangle:

$$B \to FGB \to C \to B[1]$$

since G is exact, we apply G to the above triangle:

$$GB \to GFGB \to GC \to GB[1]$$

notice that $g_{GB} \circ G(h_B) = id$, then since g_{GB} is an isomorphism, $G(h_B)$ is an isomorphism, then $GC = 0$, then $C = 0$, hence h_B is an isomorphism. ☐

Now we introduce another important set of definitions in triangulated category, called decomposition and decomposable:

Definition 1.49. A triangulated category $\mathscr{D}$ is **decomposed** into triangulated subcategories $\mathscr{D}_1$ and $\mathscr{D}_2$ if the following three conditions hold:

1. Both categories $\mathscr{D}_1$ and $\mathscr{D}_2$ contains objects non-isomorphic to 0.

2. For all $A \in \mathscr{D}$, there is a distinguished triangle:

$$B_1 \to A \to B_2 \to B_1[1]$$

   with $B_i \in \mathscr{D}_i$ for $i = 1, 2$.

3. $Hom(B_1, B_2) = Hom(B_2, B_1) = 0$ for all $B_1 \in \mathscr{D}_1$ and $B_2 \in \mathscr{D}_2$.

A triangulated category that can not be decomposed is called **indecomposable**.

From the definition, we see that we don't really care about if $\mathscr{D}_1$ and $\mathscr{D}_2$ are full not not, we only care what kind of objects they have. Later we will see that the derived category (which is triangulated) of a an integral scheme is indecomposable.

The following two propositions helps us understand this notion of decomposition better:

Proposition 1.50. Condition 2 above, in the presence of Condition 3 is equivalent to the fact that A is the direct sum of B_1 and B_2.

Proof. Recall a proposition we proved earlier, we only need to show that the arrow in the triangle $B_2 \to B_1[1]$ is trivial. Notice that being a triangulated subcategroy, it's invariant under shift, then $B_1[1]$ is still in $\mathscr{D}_1$, then Condition 3 implies what we want. ☐

Proposition 1.51. The requirement in Condition 2 above is symmetric in the order of $\mathscr{D}_1$ and $\mathscr{D}_2$, i.e. we can require for any $A \in \mathscr{D}$, there is a distinguished triangle:

$$B_2 \to A \to B_1 \to B_2[1]$$

with $B_i \in \mathscr{D}_i$ for $i = 1, 2$.

Proof. Let's assume Condition 2 to show that we can switch since the other way around is similar.

By assumption, for any $A \in \mathscr{D}$, we have:

$$B_1 \to A \to B_2 \to B_1[1]$$

and A is the direct sum of B_1 and B_2. In this case, we claim that the following diagram:

$$B_1 \to A \to B_2 \to B_1[1]$$

is also distinguished, notice that it's isomorphic to $B_1 \to B_1 \oplus B_2 \to B_2 \to B_1[1]$ thus we only have to prove it is distinguished. Notice that $B_1 \to B_1 \to 0 \to B_1$ and $0 \to B_2 \to B_2 \to 0$ are both distinguished, and you can see that the direct sum of the two is the triangle we claim to be distinguished. Recall the proposition we proved before about direct sum of distinguished triangles, we are done. **** probably there is another way to show this without using the fact that direct sums of distinguished triangles are distinguished* ☐

Now we fulfill our earlier promise to further discuss how to determine if a fully faithful exact functor is an equivalence or not, providing an easier criterion:

Proposition 1.52. Let $F : \mathscr{D} \to \mathscr{D}''$ be a fully faithful exact functor between triangulated categories. Suppose that $\mathscr{D}$ contains objects not isomorphic to 0 and $\mathscr{D}'$ is indecomposable.

Then F is an equivalence iff F has a left adjoint $G \dashv F$ and a right adjoint $F \dashv H$ such that for any object $B \in \mathscr{D}'$, $H(B) = 0$ implies that $GB = 0$.

Proof. In order to prove the proposition, we introduce two full triangulated subcategories $\mathscr{D}_1', \mathscr{D}_2' \subset \mathscr{D}$. We first describe what are the objects of these two subcategories and then show that they are indeed triangulated.

$\mathscr{D}_1'$ is the subcategory where consisting of all objects B that are isomorphic to $F(A)$ for some $A \in \mathscr{D}$. Equivalently, $\mathscr{D}_1'$ consists of all $B \in \mathscr{D}'$ such that $FHB \simeq B$ where the isomorphism is induced by adjunction. This comes from F being fully faithful, then if $B \simeq FA$ then $HB \simeq HFA \simeq A$, here $HFA \simeq A$ uses the assumption that F is fully faithful and $F \dashv H$.

For $\mathscr{D}_2$, we define it to be the the subcategory consisting of all objects C in $\mathscr{D}'$ such that $HC = 0$. Notice that since F is exact then H is exact, then $\mathscr{D}_2$ is invariant under shifts. It's also easy to see why $\mathscr{D}_1$ is invariant under shifts. I will let you to check another condition such that they are indeed triangulated.

Then recall the arguments in the preceding criterion for showing F is an equivalence, notice that for any $B \in \mathscr{D}'$, we can decompose it by a distinguished triangle:

$$B_1 \to B \to B_2 \to B[1]$$

with $B_i \in \mathscr{D}_i'$ Furthermore, for all $B_1 \in \mathscr{D}_1'$ and all $B_2 \in \mathscr{D}_2'$, we have:

$$Hom(B_1, B_2) \simeq Hom(FHB_1, B_2) \simeq Hom(HB_1, HB_2) = 0$$

also:

$$Hom(B_2, B_1) \simeq Hom(B_2, FHB_1) \simeq Hom(GB_2, HB_1) = 0$$

the second sequence of isomorphism is true since B_2 satisfies $HB_2 = 0$, then by assumption $GB_2 = 0$.

However, we assumed in the beginning that $\mathscr{D}'$ is indecomposable, thus either $\mathscr{D}_1'$ or $\mathscr{D}_2'$ os trivial, i.e. one of them only consists of objects isomorphic to 0.

If we assume that $\mathscr{D}_1'$ is trivial, then then for any $A \in \mathscr{D}$, FA is trivial, thus HFA is trivial, then since F is fully faithful, $HFA \simeq A$, which is also trivial, contradicting the fact that $\mathscr{D}$ is not trivial. Therefore, $\mathscr{D}_2$ must be trivial, then since $B_2 \in \mathscr{D}_2'$ is trivial, $B_1 \rightharpoonup B$ is an isomorphism, thus we have for any object $B \in \mathscr{D}$, we have $B \simeq FHB$ via the adjunction, then H is an inverse of F. ☐

Remark 1.53. The criterion above can be best applied when $H = G$, i.e. when the left adjoint and right adjoint are the same functor:

$$H \dashv F \dashv H$$

in this case, if $\mathscr{D}'$ is indecomposable, another condition is automatically satisfied, then we are done.

In the end of this subsection, we combine what we discussed before with Serre functors and see what happens in the presence of a Serre functor:

Corollary 1.54. Let $F : \mathscr{D} \to \mathscr{D}''$ be an exact functor between triangulated categories with left adjoint $G \dashv F$ and right adjoint $F \dashv H$, further assume Ω is a spanning class of $\mathscr{D}$ satisfying the following conditions:

1. For all $A, B \in \Omega$, the natural homomorphisms:

$$Hom(A, B[i]) \to Hom(FA, FB[i])$$

   are bijective for all $i \in \mathbb{Z}$.

2. The categories $\mathscr{D}, \mathscr{D}'$ admit Serre functors $S_{\mathscr{D}}$ and $S_{\mathscr{D}'}$ such that for all $A \in \Omega$, one has $F(S_{\mathscr{D}}(A)) \simeq S_{\mathscr{D}'}(FA)$ or equivalently for objects in Ω, F commutes with Serre functors.

3. $\mathscr{D}'$ is indecomposable and $\mathscr{D}$ is nontrivial.

Then F is an equivalence.

Proof. Notice that condition one tells that F is fully faithful by a previous proposition. We would like to apply the preceding proposition, thus we need to verify that $HB = 0$ implies $GB = 0$, this is done as follows:

Suppose that $HB = 0$, notice that for any $A \in \Omega$, we have:

$$0 = Hom(A, HB) \simeq Hom(FA, B) \simeq Hom(B, SFA)^* \simeq Hom(B, FSA)^* \simeq Hom(GB, SA)^* \simeq Hom(A, GB)$$

therefore $Hom(A, GB) = 0$ for every $A \in \Omega$, then by the definition of the spanning class, $GB = 0$, then we are done. ☐

Remark 1.55. Recall that by a previous remark, when both categories are endowed with Serre functors, given a left adjoint, you are able to construct a right adjoint and vice versa, thus in the above corollary, we only have to assume the existence of one adjoint.

1.4 Exceptional sequences and orthogonal decompositions

The last subsection of this chapter deals with the abstract notion of semi-orthogonal decomposition of a triangulated category. The reason we want it is that in many geometric situations, the derived category (which is also triangulated) is indecomposable. However, there are also situations where we can find a weaker decomposition of derived categories called semi-orthogonal decomposition, which is the topic of this section. To discuss this notion, we can not live without the following definition:

Definition 1.56. An object $E \in \mathscr{D}$, a k-linear triangulated category is called exceptional if:

$$Hom(E, E[l]) = \begin{cases} k & \text{for} \quad l = 0 \\ 0 & \text{for} \quad l \neq 0 \end{cases}$$

An exceptional sequence is a sequence $E_1, \cdots, E_n$ of exceptional objects such that $Hom(E_i, E_j[l]) = 0$ for all $i > j$ and all l, in other words:

$$Hom(E_i, E_j[l]) = \begin{cases} k & \text{if } l = 0, i = j \\ 0 & \text{if } i > j \text{ or if } l \neq 0, i = j \end{cases}$$

An exceptional sequence is full if $\mathscr{D}$ is generated by $\{E_i\}$, i.e. any full triangulated subcategory containing all objects E_i is equivalent to $\mathscr{D}$ via inclusion.

The first result we give is about how excptional objects helps us create admissible triangulated categories:

Proposition 1.57. Let $\mathscr{D}$ be a k-linear triangulated category such that for any $A, B \in \mathscr{D}$, the vector space $\oplus_i Hom(A, B[i])$ is finite dimensional.

If $E \in \mathscr{D}$ is exceptional, then the objects $\oplus_i E[i]^{\oplus j_i}$ form an admissible triangulated subcategory of $\mathscr{D}$.

Proof. We will denote this full subcategory by $\langle E \rangle$, let's first show that it's indeed triangulated.

Notice that the shift functor is k-linear, and in particular additive, thus it commutes with direct sums, then by definition, you see $\langle E \rangle$ is closed under shifts. Again, we still need to check that this subcategory is closed under taking cones. To show this, we need to know how to take cones in the category of k-linear vector spaces, which we will omit here and you can check books on homological algebra for this. This proves that the category consisting of the prescribed objects are indeed a full triangulated subcategory containing E, and you see that any triangulated subcategory which contains E should contains the objects $\oplus_i E[i]^{\oplus j_i}$, which justifies the notation $\langle E \rangle$, the smallest triangulated subcategory that contains E.

Now, to show that it's admissible, we use a previous remark, for a full triangulated subcategory $\mathscr{D}' \subset \mathscr{D}$, it's admissible iff for any $A \in \mathscr{D}$, there is a distinguished triangle:

$$B \to A \to C \to B[1]$$

with $B \in \mathscr{D}'$ and $C \in \mathscr{D}'^{\perp}$. In our question $\mathscr{D}'$ is $\langle E \rangle$, thus by a previous proposition, we know that $\langle E \rangle^{\perp} = E^{\perp}$. Knowing this, consider the following canonical morphism:

$$\bigoplus\nolimits_i Hom(E, A[i]) \otimes E[-i] \longrightarrow A$$

we need to explain a little bit on this morphism, first on the notation and what exactly is the morphism and in the end why it's called canonical. First, what does the tensor notation mean here considering we don't know whether tensor exists or not. This stuff can be characterized by a universal property and usually after we write down the universal property, we construct an explicit object and shows that it satisfies the universal property. In any k-linear category $\mathscr{D}$, for V a finite dimensional k-vector space, we define $V \otimes X$ where $X \in \mathscr{D}$ to be the object that is characterized by the following universal property:

$$Hom(V \otimes X, Y) \simeq Hom(V, Hom(X, Y))$$

In practice, we will define $V \otimes X = \oplus_{i=1}^{dimV} X$ and you can see that in this case, $V \otimes X \to Y$ is characterized by how each copy of X goes to Y, by choosing a basis of V, we can see that each of the copy correspond to a basis element, explaining $Hom(V, Hom(X,Y))$ correspond to $V \otimes X \to Y$.

Now in our case $Hom(E, A[i]) \otimes E[i] \to A[i]$ correspond to $Hom(Hom(E, A[i]), Hom(E, A[i]))$, then the identity $id : Hom(E, A[i]) \to Hom(E, A[i])$ correspond to a morphism $Hom(E, A[i]) \otimes E \to A[i]$, which is the canonical morphism. Then apply $(T^{-1})^i$ to get $Hom(E, A[i]) \otimes E[-i] \to A$.

Now, this morphism can be completed to a distinguished triangle:

$$\bigoplus_i Hom(E, A[i]) \otimes E[-i] \longrightarrow A \longrightarrow B \longrightarrow \cdots$$

we now apply $Hom(E, -)$ to this triangle to get the usual LES associated to a distinguished triangle, we claim this LES tells that $Hom(E, B[t]) = 0$ for every t. For simplicity, let's consider $t = 0$, and other cases are the same. The segment of the long exact sequence that is relevant is:

$$Hom(E, \bigoplus_i Hom(E, A[i]) \otimes E[-i]) \longrightarrow Hom(E, A) \longrightarrow Hom(E, B)$$

since E is exceptional, only when $i = 0$ can we get nontrivial arrows. Then we get:

$$Hom(E, Hom(E, A) \otimes E) \longrightarrow Hom(E, A) \longrightarrow Hom(E, B)$$

We claim that the first arrow is an isomorphism. This is because that by choosing a basis for $Hom(E, A)$, we have an explicit description of everything, I will let you use this to show that the first arrow can be put in such a diagram:

$$\begin{tikzcd}
Hom(E, Hom(E, A) \otimes E) \arrow[r] \arrow[d, "iso"] & Hom(E, A) \\
Hom(E, A) \otimes Hom(E, E) \arrow[r, "iso"] & Hom(E, A) \otimes k \simeq Hom(E, A) \arrow[u, "id"]
\end{tikzcd}$$

then after you shift this distingusihed triangle, you always get the fact that if you apply $Hom(E,)$ to $\bigoplus_i Hom(E, A[i] \otimes E[t-i] \to A[t]$, the connecting morphisms are all isomorphisms, thus we will get LES of the form:

$$A \xrightarrow{iso} B \longrightarrow Hom(E, B[t]) \longrightarrow D \xrightarrow{iso} E$$

which tells you that the stuff in the middle $Hom(E, B[t]) = 0$, which means that $B \in E^{\perp} = \langle E \rangle^{\perp}$. We are done. $\square$

Now, the concept of a (full) exceptional sequence is generalized by the following definition:

Definition 1.58. A sequence of full admissible triangulated subcategories

$$\mathscr{D}_1, \cdots, \mathscr{D}_n \subset \mathscr{D}$$

is semi-orthonormal if for all $i > j$, we have:

$$\mathscr{D}_j \subset \mathscr{D}_i^{\perp}$$

Such a sequence defines a semi-orthogonal decomposition of $\mathscr{D}$ is $\mathscr{D}$ is generated by these $\mathscr{D}_i$'s, i.e. via inclusion, $\mathscr{D}$ is equivalent to the smallest full triangulated subcategory containing all of $\mathscr{D}_i$.

Remark 1.59. Notice that the notion of semi-orthogonal and semi-orthogonal decomposition also make since for non-admissible full triangulated categories. However, on the other hand, if we do have admissible conditions, things will be nicer, as the following examples and propositions will show.

Example 1.60. 1. Let $\mathscr{D}' \subset \mathscr{D}$ be a full admissible triangulated category, then if we define $\mathscr{D}_2 = \mathscr{D}'$ and $\mathscr{D}_1 = \mathscr{D}'^{\perp}$, we know by a previous proposition that $\mathscr{D}_1$ is indeed a triangulated full subcategory. We claim that $\mathscr{D}_1$ and $\mathscr{D}_2$ forms a semi-orthogonal decomposition of $\mathscr{D}$. To see this, let's first notice that these two categories are indeed semi-orthogonal. By definition, we need to check that $\mathscr{D}_1 \subset \mathscr{D}_2^{\perp}$, but this is just literally how we defined them. Then we need to show that the smallest full tirangulated subcategory containing $\mathscr{D}_1, \mathscr{D}_2$ via the inclusion is equivalent to $\mathscr{D}$. To see this, notice that we proved that for any $A \in \mathscr{D}$, we have a distinguished triangle:

$$B \to A \to C \to B[1]$$

with $B \in \mathscr{D}_2$ and $C\mathscr{D}_1$, by rotating this triangle, we have the following new triangle:

$$C[-1] \to B \to A \to C$$

Notice that A now is the cone of $C[-1] \to B$, which should be in the triangulated category containing $\mathscr{D}_1, \mathscr{D}_2$, we are done.

2. Let $E_1, \cdots, E_n$ be an exceptional sequence in $\mathscr{D}$, then we already saw that each of the following is a full admissible triangulated subcategory:

$$\mathscr{D}_1 = \langle E_1 \rangle, \cdots, \mathscr{D}_n = \langle E_n \rangle$$

We claim that they form a semi-orthogonal sequence.

To show this, we need to show that for all $i > j$:

$$\langle E_j \rangle \subset \langle E_i \rangle^{\perp} = E_i^{\perp}$$

We only have to show that $E_j \in E_i^{\perp}$, which means that we need to show $Hom(E_j, E_i[t]) = 0$, but this is again literally the definition, we are done.

Moreover, if the exceptional sequence if full, we can conclude that $\mathscr{D}_1 = \langle E_1 \rangle, \cdots, \mathscr{D}_n = \langle E_n \rangle$ forms a semi-orthogonal decomposition.

The following lemma that helps us determine if an admissible semi-orthogonal sequence is a semi-orthogonal decomposition is useful later:

Lemma 1.61. Any semi-orthogonal sequence of full admissible triangulated subcategory $\mathscr{D}_1, \cdots, \mathscr{D}_n \subset \mathscr{D}$ generates $\mathscr{D}$, i.e., defines a semi-orthogonal decomposition of $\mathscr{D}$, iff any object $A \in \mathscr{D}$ with $A \in \mathscr{D}_i^{\perp}$ for all $i = 1, \cdots, n$ is trivial.

Proof. First, assume this sequence is a semi-orthogonal decomposition, we want to show that if A_0 is in $\mathscr{D}_i^{\perp}$, then it's trivial.

Recall that we defined ${}^{\perp}A_0$ before, which is the full subcategory consisting of all $A \in \mathscr{D}$ such that $Hom(A, A_0[i]) = 0$ for all $i \in \mathbb{Z}$. We proved that this is a triangulated subcategory. Now by our assumption, $\mathscr{D}_i \subset {}^{\perp}A_0$, then ${}^{\perp}A_0 \subset \mathscr{D}$ is an equivalence, then A_0 is isomorphic to some stuff B in ${}^{\perp}A_0$, then $Hom(A_0, A_0) = Hom(B, A_0) = 0$, thus $A_0 = 0$ by definition of the zero object.

Conversely, let's assume that $\cap \mathscr{D}_i^{\perp} = \{0\}$, and for simplicity, we prove the case when $n = 2$ and you can use similar method for the general case. Now for any $A_0 \in \mathscr{D}$, since $\mathscr{D}_2$ is admissible, we have a triangle of the form:

$$A \to A_0 \to B \to A[1]$$

with $A \in \mathscr{D}_2$ and $B \in \mathscr{D}_2^{\perp}$. For B, we can also find a distinguished triangle:

$$C \to B \to C' \to C[1]$$

with $C \in \mathscr{D}_1$ and $C' \in \mathscr{D}_1^{\perp}$ since $\mathscr{D}_1$ is admissible.

Not notice that as $C \in \mathscr{D}_1 \subset \mathscr{D}_2^{\perp}$ and $B \in \mathscr{B}_2^{\perp}$, one can easily deduce that $C' \in \mathscr{D}_2^{\perp}$, hence $C' \in \mathscr{D}_1^{\perp} \cap \mathscr{D}_2^{\perp}$, which implies that $C' = 0$, thus $B \simeq C \in \mathscr{D}_1$, then rotate the first distinguished triangle to conclude that A_0 is in (or isomorphic to some objects in) the category containing $\mathscr{D}_1$ and $\mathscr{D}_2$. ☐

The two following propositions helps you understand this decomposition and orthogonality better:

Proposition 1.62. Let $\mathscr{D}_1, \mathscr{D}_2$ be a semi-orthogonal decomposition of length 2, then the inclusion $\mathscr{D}_1 \subset \mathscr{D}_2^{\perp}$ is an equivalence. More generally, if $\mathscr{D}_1, \cdots, \mathscr{D}_n$ is a semi-orthogonal decomposition of length n, then $\mathscr{D}_1 \subset \langle \mathscr{D}_2, \cdots, \mathscr{D}_n \rangle$ is an equivalence.

Proof. Let's first deal with the easier case when we only have a length 2 decomposition. Let's assume that these subcategories are admissible, otherwise, the orthogonal complement may not be a triangulated category.

Let's directly prove the converse inclusion, take X in $\mathscr{D}_2^{\perp}$, we want to show that $X \in \mathscr{D}_1$ or at least, there is something in $\mathscr{D}_1$ that is isomorphic to X. Since $\mathscr{D}_1$ is admissible, we have such a triangle:

$$A \to X \to A' \to A[1]$$

with $A \in \mathscr{D}_1$ and $A' \in \mathscr{D}_1^{\perp}$, notice that $A \in \mathscr{D}_1 \subset \mathscr{D}_2^{\perp}$ and X is also in $\mathscr{D}_2^{\perp}$, thus A' is also in $\mathscr{D}_2^{\perp}$, thus A' is 0, thus $A \to X$ is an isomorphism.

For the more general case, let's first observe that $\mathscr{D}_1 \subset \mathscr{D}_i^{\perp}$ for $i \geqslant 2$. Then since $\langle \mathscr{D}_2, \cdots, \mathscr{D}_n \rangle$ is obtained by finite repetition of shifts, taking cones and direct sums, we know that $\mathscr{D}_1$ is indeed in $\langle \mathscr{D}_2, \cdots, \mathscr{D}_n \rangle^{\perp}$, conversely, let's consider if $X \in \langle \mathscr{D}_2, \cdots, \mathscr{D}_n \rangle^{\perp}$, the first thing we can say is that X is in $\mathscr{D}_i^{\perp}$ for all $i \geqslant 2$, then we can repeat the above procedure for the simpler case in the sense that we first have:

$$A_1 \to X \to A_1' \to A_1[1]$$

with $A_1 \in \mathscr{D}_1, A_1' \in \mathscr{D}_1^{\perp}$, then A_1 is also in $\mathscr{D}_i^{\perp}$ for all $i \geqslant 2$, then like in the simpler case, we can still conclude that A_1' is in $\mathscr{D}_i^{\perp}$, thus $A_i' \simeq 0$, thus $A \to X$ is an isomorphism, we are done. ☐

Proposition 1.63. Suppose that $\mathscr{D}_1, \mathscr{D}_2 \subset \mathscr{D}$ is a semi-orthogonal decomposition of $\mathscr{D}$, then for any object $A \in \mathscr{D}$ with $Hom(A, B) = 0$ for all $B \in \mathscr{D}_1$, A is isomorphic to an object in $\mathscr{D}_2$.

Proof. Again, for A we have the following distinguished triangle:

$$D \to A \to C \to D[1]$$

with $D \in \mathscr{D}_2, C \in \mathscr{D}_1$. Notice that $A \to C$ is the zero morphism, then the following diagram:

$$\begin{tikzcd}
D \arrow[r, "f"] & A \arrow[r, "0"] & C \arrow[r] & D[1] \\
D \arrow[r, "f"] \arrow[u, "id"] & A \arrow[r, "0"] \arrow[u, "id"] & 0 \arrow[r] \arrow[u, "0"] & D[1] \arrow[u, "id"]
\end{tikzcd}$$

shows that $C \simeq 0$, then $D \to A$ is an isomorphism, we are done. ☐

References