Final Project: Intro To Homological Algebra

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Abstract

This article serves as a basic introduction to some of the most basic notions and techniques frequently used in homological algebra. We will focus on several results on the properties of resolutions that are central to our later treatment of derived functors. We will also introduce what is (universal) (co)homological δ-functor and relate it to derived functors. Finally, we will introduce two concrete derived functors called Ext and Tor then talk about the related notion of balanced functors. We will also include some applications of Ext and Tor. Finally, we end our discussion by proving the universal coefficient theorem.

1 Introduction

Since this paper is mainly focused on giving a good introductory treatment of homological algebra with a wide range of notions raised when necessary. In this introduction part, we want to describe what homological algebra is good for (at least according to [Wei94]) and how can it be applied in the context of commutative algebra by giving some examples you will see when you are working with some important notions in commutative algebra.

First, the tools we will develop in this paper mostly provide certain obstructions that can happen when want to know if a certain object has good property. If the obstruction is 0, the desired property happens, as you will see in the application of Ext. Here, Ext^1_R is an obstruction for a exact sequence being split. Another example is if $A \subset B$ are abelian groups we want to know when is nA the intersection of A and nB. Sometimes it happens, but sometimes it doesn't, and $Tor(B/A, \mathbb{Z}/n)$ will be the obstruction. You will find more of these examples as you work with the notions for a longer time.

In commutative algebras, you will also see many applications of homological algebra. One example is the treatment of regular local rings, which is the algebraic foundation of regular schemes in algebraic geometry. A result that's due independently to both Auslander-Buchsbaum and Serre, which relates projective dimension (a homological algebra notion) to the dimensions of a ring and depth of a module. This is one of the first successful applications of homological algebra in commutative algebra.

2 Basics Review

To discuss homological algebra, Abelian Category should be introduced to reach the most generality. Throughout this paper, readers will be assumed to be familiar with modules over a not necessarily commutative ring R and basic definitions of Abelian category(you can find pretty good reference in [Vak] and in [Wei94]). We will try to give definitions as general as possible, as long as the argument is not horribly complicated and still interesting. However, as what we will see below, under suitable conditions, an arbitrary Abelian category is like R-modules. Our main examples and applications will still be done in R-modules. Below, R will always be a fixed ring and $\mathcal{A}$ will always be an Abelian Category stated otherwise.

2.1 (co)Chain Complex and (co)Homology in an Abelian Category $\mathcal{A}$

The category of chain complex of in an Abelian Category $\mathcal{A}$ is similar to what we have for the category of chain complex of R-modules in class, we won't bother here to redefine it. But if we work in an abstract Abelian category, things can be complicated since we don't have a underlying set to deal with. Prop 2.2 shows how the argument can get complicated if we are only allowed to use universal properties.

Definition 2.1. Suppose that $A \longrightarrow B$ is a monomorphism in $\mathcal{A}$, we define the quotient to be the kernel of the cokernel of this arrow, i.e. $Ker(coKer(A \longrightarrow B))$

Proposition 2.2. Given a (co)chain complex in $\mathcal{A}$, there is a unique monomorphim $Imd_{n+1} \longrightarrow Kerd_n$ induced by d_n's.

Proof. A schematically sketch is given below:

$$\begin{tikzcd}
Ker(coKer(d_{n+1})) = Im(d_{n+1}) \arrow[rr, dashed, "\exists!"] \arrow[dr] & & Ker(d_n) \arrow[dl] & \\
C_{n+1} \arrow[u, dashed, "\exists!"] \arrow[r, "d_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[dr] & C_{n-1} \arrow[r] & \cdots \\
& & coKer(d_{n+1}) \arrow[u, dashed, "\exists!"] &
\end{tikzcd}$$

use universal property and follow your nose to define the induced maps. It's is a monomorphism since we know that Ker is momomorphism and if composition of a morphism followed by a monomorphism is monomorphism, then the morphism is a monomorphism. □

Remark 2.3. If the above induced unique morphism $Im(d_{n+1}) \longrightarrow Ker(d_n)$ is an isomorphism, then we will say that the sequence is **exact** at C_n, if it is exact for each C_n, we will say C is **exact** or **acyclic**

Definition 2.4. Given a chain complex or cochain complex, we define the n-th homology and n-th cohomology by the quotient:

$$H_n(C) = Z_n/B_n \qquad H^n(C) = Z_n/B_n$$

where the subscript indicate it's homology (coming from a chain complex) and the superscript indicates the cohomology (coming from a cochain complex). Abelian category is closed under taking kenel and cokernel, thus this definition makes sense. We will call a (co)chain complex acyclic if $H_n(C) = 0$ ($H^n(C) = 0$) for all n.

It's quite difficult to deal with abstract abelian categories just as the proposition above illustrated, a straightforward injection from the image of R-module maps to kernel can be as complicated as the categorical diagram in an abstract abelian category. However, when we are dealing with small Abelian Categories, we will see that it's just a subcategory of R-modules for some fixed not necessarily commutative ring R, as the following theorem we won't prove here shows. You can find details in Theorem 4.4 in [Mit64]

Theorem 2.5. Freyd-Mitchell Embedding Theorem If $\mathcal{A}$ is a small Abelian Category, then there is an exact, fully faithful functor from $\mathcal{A}$ into R-modules which embedds $\mathcal{A}$ as a full subcategory in the sense that $Hom_{\mathcal{A}}(M, N) \cong Hom_R(M, N)$

One implication of this theorem is that, since exact functors preserves epimophisms and monomorphisms, when we are trying to prove some "local" property that only depends on a small full subcategory of $\mathcal{A}$, we can just quote this theorem to embed it into R-modules, where we have concrete sets and maps. For simplicity, we will form now on assume all of the categories we are dealing with are small to avoid unnecessary discussions on whether this property is "local" or not.
The last result we want to say for categorical nonsense is:

Proposition 2.6. The opposite category of an abelian category $\mathcal{A}$ is still abelian.

Proof. First, opposite category of an additive category is additive since $h^{op}(g^{op}+g'^{op})f^{op} = h^{op}(g+g')^{op}f^{op} = (f(g+g')h)^{op} = (fgh+fg'h)^{op} = h^{op}g^{op}f^{op} + h^{op}g'^{op}f^{op}$. Zero object is preserved and kernel and cokernel are switched. □

Proposition 2.7. If $A \longrightarrow B \longrightarrow C$ is exact in $\mathcal{A}$ then the corresponding $C^{op} \longrightarrow B^{op} \longrightarrow A^{op}$ is exact in $\mathcal{A}^{op}$.

Proof. Consider the diagram:

$$\begin{tikzcd}
Kerf^{op} & & Img^{op} \arrow[ll, dashed] \arrow[dl] \\
A^{op} & B^{op} \arrow[l, "f^{op}"] \arrow[dl] & C^{op} \arrow[l, "g^{op}"] \\
coKerg^{op} & &
\end{tikzcd}
\qquad
\begin{tikzcd}
coKerf \arrow[rr, dashed] & & coKer(Kerg \longrightarrow B) \\
A \arrow[r, "f"] & B \arrow[ul] \arrow[ur] \arrow[r, "g"] & C \\
Kerg \arrow[ur] & &
\end{tikzcd}$$

By Thm 2.5, we can do it in R-modules, thus the right hand side $coKerf \longrightarrow coKer(Kerg \longrightarrow B)$ is an isomorphism, since going to the opposite preserves isomorphism, $Img^{op} \longrightarrow Kerf^{op}$ is an isomorphism. □

2.2 Constructions on (co)Chain Complex and the double (co)chain complex

Definition 2.8. Translation of (co)Chain Complexes Given a chain complex C, we define the p-th translate of C to be $C[p]$ where $C[p]_n = C_{p+n}$ with differentials $-1^{|p|}$. For cochain complexes, we have $C[p]^n = C^{n-p}$ with differentials $-1^{|p|}$.

Easy to see that the translation shifts homology and cohomology in the way that $H^nC[p] = H^{n-p}(C)$ and $H_n(C[p]) = H_{n+p}(C)$. If we have $f : C \longrightarrow D$ a chian or cochain map, by shifting the indexes of f we can make the translation a functor.

Definition 2.9. Mapping Cone of a (co)Chain Map Let $f : B \longrightarrow C$ be a map of chain complexes, we define the mapping cone of f, denoted $cone(f)$, to be the chain complex whose degree n part is $B_{n-1} \oplus C_n$ and the differential is defined to be:

$$d(b, c) = (-db, dc - f(b))$$

For maps of cochain complexes, we also define it's mapping cone $cone(f)$ to be the cochain complex whose n-th component is $B_{n+1} \oplus C_n$ with the same differential formula. When $f = id_C$, we will use $cone(C)$ to denote the mapping cone.

The reason we want the notion of mapping cone is that we have the following useful criterion of showing a certain (co)chain map is a quasi-isomorphism.

Proposition 2.10. A (co)chain map of R-modules is an quasi-isomorphism iff the mapping cone $cone(f)$ is exact (or in other words ayclic).

To prove this, we first describe a general way to fit every map $f_* : H_*(B) \longrightarrow H_*(C)$ into a long exact sequence of homology groups.
Consider the following exact sequence of chain complexes with differentials indicated on the arrows:

$$0 \longrightarrow C \xrightarrow{c \mapsto (0,c)} cone(f) \xrightarrow{(b,c) \mapsto -b} B[-1] \longrightarrow 0$$

Then the long exact sequence of chain complexes and the fact $H_{n+1}(B[-1]) = H_n(B)$ give us:

$$\cdots \to H_{n+1}(cone(f)) \to H_n(B) \xrightarrow{\partial} H_n(C) \to H_n(cone(f)) \to H_{n-1}(B) \xrightarrow{\partial} H_{n-1}(C) \to \cdots$$

We will show that ∂ is f_*, thus we see that ∂ is isomorphisms iff the cone is exact(acyclic). To see this, consider the lift in $cone(f)$ of b, a cycle in B_n, which is $(-b, 0)$, then we apply the differential in $cone(f)$ to get $(db, f(b)) = (0, f(b))$, this implies $f(b)$ will be mapped to $(0, f(b))$, thus $\partial[b] = [f(b)] = f_*[b]$, we are done.

Definition 2.11. Truncation of a (co)Chain Complex Given a chain complex C, we define the good truncation of C below n to be:

$$\tau_n C = \begin{cases} 0 & i < n \\ Z_i & i = n \\ C_i & i > n \end{cases}$$

Definition 2.12. Double (co)Chain Complexes A double chain complex in $\mathcal{A}$ is a family of objects $\{C_{p,q}\}$ in $\mathcal{A}$ together with horizontal maps $d^h : C_{p,q} \longrightarrow C_{p-1,q}$ and vertical maps $d^v : C_{p,q} \longrightarrow C_{p,q-1}$ such that $d^h \circ d_h = 0$ and $d^v \circ d_v = 0$, also $d^v d^h + d^h d^v = 0$. It's useful to picture a double complex as a lattice where each row and column is a chain complex, though due to space reason, we won't draw it here.

Dually, we have the definition of a cochain double complex, with all the arrow reversed. Here, we didn't define $d^v d^h = d^h d^v$, but introduced the anticommutative condition, this is for the convenience of:

Definition 2.13. Total (co)Chain Complex Suppose that $\{C_{p,q}\}$ is a double chain complex, define the total complex $Tot(C) = Tot^{\prod}(C)$ or $Tot^{\oplus}(C)$ by:

$$Tot^{\prod}(C)_n = \prod_{p+q=n} C_{p,q} \qquad Tot^{\oplus}(C)_n = \bigoplus_{p+q=n} C_{p,q}$$

together with maps:$d_n : Tot^{\prod}(C)_n \longrightarrow Tot^{\prod}(C)_n$ or $d_n : Tot^{\oplus}(C)_n \longrightarrow Tot^{\oplus}(C)_n$ defined by $d^h + d^v$. To be more specific, it's defined by the following diagram:

$$\begin{tikzcd}
 & Tot^{\oplus}(C)_{n-1} = \bigoplus_{p+q=n-1} C_{p,q} & \\
C_{p,q-1} \arrow[ur] & Tot^{\oplus}(C)_n \arrow[u, dashed] & C_{p-1,q} \arrow[ul] \\
 & C_{p,q} \arrow[ul] \arrow[ur] \arrow[u, hook] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Tot^{\prod}(C)_n \arrow[dl] \arrow[dr] \arrow[d, dashed] & \\
C_{p,q-1} \arrow[dr] & Tot^{\prod}(C)_{n+1} = \prod_{p+q=n+1} C_{p,q} \arrow[d] & C_{p-1,1} \arrow[dl] \\
 & C_{p,q} &
\end{tikzcd}$$

You can check that this is indeed a chain map just because we have the anticommutative condition. Dually, we have similar definition for double cochain complexes.

We also have the notion of maps of double complexes $f : C \longrightarrow D$, which is just a family of maps $\{f_{p,q}\}$ connecting each $C_{p,q} \longrightarrow D_{p,q}$ such that each square you can imagine there commutes. Similar for double cochain complex.

Remark 2.14. Notice that the way we define differentials for $Tot^{\prod}$ and $Tot^{\oplus}$ is dual to each other. If we reverse all the arrows and go to the opposite category, we find that limits and colimits interchange, thus we see that in the dual, differential in $Tot^{\prod}$ is the dual of the differentials in $Tot^{\oplus}$ without the dual. Similar reasoning explains in the dual differentials in $Tot^{\oplus}$ is the dual of the differentials in $Tot^{\prod}$ without the dual.

Lemma 2.15. Suppose that $f : C \longrightarrow D$ is a map of double complex, then it induces a chain map of total complex defined by universal properties of direct sums and direct products.

The following lemma turns out to be useful when we study Ext and Tor we will define later:

Lemma 2.16. Acyclic Assembly Lemma Suppose that C is a double chain complex in R-modules, then: If i): C is a upper half plan complex with exact columns or ii) right half plane with exact rows, then $Tot^{\prod}(C)$ is acyclic.
Similarly, If iii) C is a upper half plan complex with exact rows or iv) right half plane with exact columns, then $Tot^{\oplus}(C)$ is acyclic.

Proof. We claim that it suffices to show i). Once we showed this, notice that $C'_{p,q} = C_{q,p}$ with the same map as in C is still a double complex(it's just C where the role of row and column is reversed), then $Tot^{\oplus}(C')_n = \bigoplus_{p+q=n} C'_{p,q} = \bigoplus_{p+q=n} C_{q,p} = \bigoplus_{p+q=n} C_{p,q}$, again, it's easy to check that the differential is also the same, hence they have the same total complex. A enlightening way to think of this is that reversing the row and column doesn't affect the arrows and the total complex only cares about what are the arrows. Similarly, iv) implies iii), thus we are left with i) implies iv). Suppose that we are in case iv), then we define the good truncation of $C_{p,q}$ denoted $\tau_n C$ to be the good truncation of each column below degree n, after the truncation and a translation, it becomes a first quadrant double complex, thus the product an direct sum total complex coincide, thus i) implies that $\tau_n C$ is acyclic. Notice that each cycle in $Tot^{\oplus}(C)$ is in the cycle of some $\tau_n C$, then we can find an boundary b in the total complex of the truncated double complex such that it's mapped to the cycle, b is also in the total complex of the untruncated one.

In case i), we are allowed to translate left or right, thus we only have to prove that $H_0(Tot)$ is 0. A schematic proof of i) is the following diagram, the key condition that makes the proof work is that all objects below the 0-th row is 0. Our goal here is that given $(c_{-p,p})$ a cycle, we want to produce a sequence of elements such that it's mapped to $(c_{-p,p})$ under the differential. If we choose $0 = b_{1,0}$, we only have to choose $b_{0,1}$, since columns are exact, there is a $b_{0,1}$ that's mapped to $c_{0,0}$. Suppose that $b_{-p+1,p}$ is chosen, then $dv(c_{-p.p} - d^h(b_{-p+1,p})) = d^v(c_{-p,p}) + d^h(c_{-p+1,p-1}) + d^h d^h(\cdots) = 0$, then we again use exactness of columns to choose $b_{-p,p+1}$.

$$\begin{tikzcd}
c_{-2,2} & b_{-1,2} \arrow[l, mapsto] \arrow[d, mapsto] & & \\
& c_{-1,1} & b_{0,1} \arrow[l, mapsto] \arrow[d, mapsto] & \\
& & c_{0,0} \arrow[d, mapsto] & 0 = b_{1,0} \arrow[l, mapsto] \\
& & 0 &
\end{tikzcd}$$

□

Remark 2.17. Notice that given a double cochain complex, we can prove a similar result saying that $Tot^{\prod}(C)$ is acyclic if i) C is a upper half-plane cochain complex with exact rows or ii) C is a right half-plane complex with exact columns. $Tot^{\oplus}$ is acyclic if iii) C is a upper half-plane double cochain complex with exact columns or iv) C is a right half-plane double cochain complex with exact rows. The way we do it is we first go to the dual category, then the cochain complex becomes a chain complex. Notice that going to dual doesn't affect exactess and doesn't affect whether it's upper half-plane or whatever position condition. Thus, the acyclic assembly lemma for double chain complexes works and the total chain complex is acyclic. Recall remark 2.14, we know that this total chain complex is acyclic iff the total cochain complex is acyclic.

3 Projective and Injective Resolutions

Definition 3.1. An object P in $\mathcal{A}$ is said to be a projective object if it satisfies the following universal lifting property:
Given any epimorphism $f : A \longrightarrow B$ and any morphism $\gamma : P \longrightarrow B$, there is at least one morphism $\beta : P \longrightarrow A$ such that the following diagram commutes:

$$\begin{tikzcd}
& P \arrow[dl, dashed, "\beta"'] \arrow[d, "\gamma"] & \\
A \arrow[r, "f"] & B \arrow[r] & 0
\end{tikzcd}$$

where the arrow from B to 0 indicates that f is an epimorphism.

Similarly, we also have the dual definition of injective objects in $\mathcal{A}$.

Definition 3.2. An object I is said to be an injective object if it satisfies the following universal lifting property dual to that of projective objects:
Given any monomorphism $f : B \longrightarrow A$ and any morphism $\gamma : B \longrightarrow I$, there is at least one morphism $\beta : A \longrightarrow I$ such that the following diagram commutes:

$$\begin{tikzcd}
& I & \\
0 \arrow[r] & B \arrow[u, "\gamma"] \arrow[r, "f"'] & A \arrow[ul, dashed, "\beta"']
\end{tikzcd}$$

Here are some useful characterizations of projective(injective) objects which are related the notion of **balanced functor** we will talk about later:

Lemma 3.3. In $\mathcal{A}$, M is a projective object iff $Hom_{\mathcal{A}}(M,-)$ is an exact functor. I is an injective object iff $Hom_{\mathcal{A}}(-,I)$ is an exact functor.

Proof of this is done in R-modules via Thm2.5.
It's good to have projective and injective objects since we can have resolutions:

Definition 3.4. Projective and Injective Resolutions Let M be an object in $\mathcal{A}$, a left resolution of M is a chain complex P such that $P_i = 0$ for $i < 0$ together with a map $\epsilon : P_0 \longrightarrow M$ such that the below augmented complex is exact, it's a projective resolution if each $P-i$ is projective:

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\epsilon} M \longrightarrow 0 \longrightarrow \cdots$$

Dually, a right resolution of M is a cochain complex I, such that $P_i = 0$ for all $i < 0$, together with a map $\epsilon : M \longrightarrow I_0$ such that the following augmented cochain complex is exact, it's an injective resolution if each I_i is injective:

$$0 \longrightarrow M \xrightarrow{\epsilon} I_0 \longrightarrow I_1 \longrightarrow \cdots$$

Remark 3.5. By reversing the arrows, these two notions are dual in the sense that the opposite category, injective and projective are reversed so it is for injective resolution and projective resolution.

Is it always possible to give projective(injective resolutions) for any objects M in $\mathcal{A}$?

Proposition 3.6. If an abelian Category has enough projectives(injectives), then for any object M, there is a projective(injective) resolution for M.

Proof. By Thm 2.5, we still work in R-modules, then this is a result we proved in class. □

Remark 3.7. For every fixed ring R, there are enough projective objects since free R modules are projective, while having enough injectives is a little intricate. We will list the results leading to it but we won't give the proof.

Lemma 3.8. Baer's Criterion A right R module E is injective iff for every right ideal J, any map $J \longrightarrow E$ can be extended $R \longrightarrow E$. In particular, for a PID such as $\mathbb{Z}$, E is injective iff it's divisible i.e. given any nonzero $r \in R$ and $a \in E$ there is some b such that $a = br$.

Lemma 3.9. $\mathbb{Q}/\mathbb{Z}$ is injective in the category of Abelian Groups since it's divisible.

Lemma 3.10. $I(A) = (\mathbb{Q}/\mathbb{Z})^{Hom_{\mathbf{Ab}}(A,\mathbb{Q}/\mathbb{Z})}$ being product of injectives is injective (it's just the universal property of direct product) and the canonical map $\epsilon_A : A \longrightarrow I(A)$ defined by $a \mapsto (\phi(a))$ is injective since for any $a \in A$ nonzero, there is a $f : a\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z}$ with $f(a) \neq 0$ such that it extend to $\mathbb{Q}/\mathbb{Z}$.

Lemma 3.11. Suppose that $\mathcal{A}, \mathcal{B}$ are Abelian Categories, let $R : \mathcal{B} \longrightarrow \mathcal{A}$ is an additive functor that's right adjoint to $L : \mathcal{A} \longrightarrow \mathcal{B}$ which is an exact functor, then suppose that I is an injective in $\mathcal{B}$, $R(I)$ will be injective in A. Dually, suppose that L is left adjoint to R which is exact, then $L(P)$ is projective if P is projective.

A good reference of all the above lemma can be found in [Wei94].
One important fact about these two types of resolutions we will use repeatedly in our development in the theory of derived functors is the following comparison theorem:

Theorem 3.12. Comparison Theorem for Projective Resolutions
Let $P \xrightarrow{\epsilon} M$ be a projective resolution of M and $f' : M \longrightarrow N$ a map in $\mathcal{A}$. Then for every resolution $Q \xrightarrow{\eta} N$ of N there is a chain map $f : P \longrightarrow Q$ lifting f in the sense that $\eta \circ f_0 = f' \circ \epsilon$.The chain map f is unique in the sense that any g satisfies this will be homotopic to f. In particular, this works when $P \xrightarrow{\epsilon} M$ is a projective resolution of M.

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, dashed, "f_2"] & P_1 \arrow[r, "d_1"] \arrow[d, dashed, "f_1"] & P_0 \arrow[r, "\epsilon"] \arrow[d, dashed, "f_0"] & M \arrow[r] \arrow[d, "f'"] & 0 \arrow[d, dashed, "0"] \\
\cdots \arrow[r] & Q_2 \arrow[r, "d_2"] & Q_1 \arrow[r, "d_1"] & Q_0 \arrow[r, "\eta"] & N \arrow[r] & 0
\end{tikzcd}$$

Proof. Recall Thm 2.5, we still work in R-modules. Then the way we proved this in class for free resolutions still works since the lifting property for free objects still works fine for projective objects. $\square$

Dually, we also have a comparison theorem for injective objects.

Theorem 3.13. Comparison Theorem for Injective Resolutions
Let $M \xrightarrow{\epsilon} I$ be an injective resolution of M and $f' : N \longrightarrow M$ a map in $\mathcal{A}$. Then for every resolution $N \xrightarrow{\eta} Q$ of N there is a chain map $f : Q \longrightarrow I$ lifting f in the sense that $f_0 \circ \eta = \epsilon \circ f'$.The chain map f is unique in the sense that any g satisfies this will be homotopic to f. In particular, this works when $M \xrightarrow{\epsilon} I$ is an injective resolution of M.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & I_0 \arrow[r] & I_1 \arrow[r] & I_2 \\
0 \arrow[r] \arrow[u] & N \arrow[r] \arrow[u, dashed, "f'"] & Q_0 \arrow[r] \arrow[u, dashed, "f_0"] & Q_1 \arrow[r] \arrow[u, dashed, "f_1"] & Q_2 \arrow[u, dashed, "f_2"]
\end{tikzcd}$$

Proof. By working in $\mathcal{A}^{op}$, we get a projective version of this, then go back to $\mathcal{A}$. $\square$

The last result we will use later about these resolutions are the following lemma, which provides us another way to create new projective resolutions from already exsisting ones:

Lemma 3.14. Horseshoe Lemma Suppose that we are given a commutative diagram:

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
\cdots \arrow[r] & P_2' \arrow[r] & P_1' \arrow[r] & P_0' \arrow[r, "\epsilon'"] & A' \arrow[r] \arrow[d] & 0 \\
 & & & & A \arrow[d] & \\
\cdots \arrow[r] & P_2'' \arrow[r] & P_1'' \arrow[r] & P_0'' \arrow[r, "\epsilon''"] & A'' \arrow[r] \arrow[d] & 0 \\
 & & & & 0 &
\end{tikzcd}$$

with the column exact and the rows being a projective resolution of A' and A'', then we have a projective resolution $P \xrightarrow{\epsilon} A$ such that $P_n = P_n' \oplus P_n''$ and a exact sequence of complexes $0 \to P' \hookrightarrow P \to P'' \to 0$ where the left map is inclusion and the right map is projection in each degree. Moreover, the diagram below commutes:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & A' \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & A'' \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & P_\bullet' \arrow[r] \arrow[u] & P_\bullet \arrow[r] \arrow[u] & P_\bullet'' \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Proof. Key ingredient here is snake lemma and lifting property of projective objects. We will inductively construct $P_n = P_n' \oplus P_n''$. First, consider:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Ker\epsilon' \arrow[r] & P_0' \arrow[r, "\epsilon'"] \arrow[d, "\iota"] & A' \arrow[r] \arrow[d, "\iota_A"] & 0 \\
0 \arrow[r] & Ker\epsilon \arrow[r] & P_0' \oplus P_0'' \arrow[r, dashed, "\epsilon"] \arrow[d, "\pi"] & A \arrow[d, "\pi_A"] & \\
0 \arrow[r] & Ker\epsilon'' \arrow[r] & P_0'' \arrow[r, "\epsilon''"] \arrow[d] \arrow[ur, dashed] & A'' \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Here we give maps connecting A', A, A'' for later use. Notice that $P_0'' \to A''$ is surjective thus lifting property gives a map $P_0'' \to A$, P_0' also has a map to A, which is the composition $\iota_A\epsilon'$. Thus, use the fact that $P_0' \oplus P_0''$ is both a direct sum and direct product, we get a induced map $P_0' \oplus P_0'' \to A$ that makes everything commutes. This is a morphism of a SES, then take the kernels, the snake lemma gives us maps connecting kernels and since $P_0' \to A'$ and $P_0'' \to A''$ are surjective, the connecting maps between kernels make it a SES and ϵ is surjective since the cokernel is 0:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Ker\epsilon' \arrow[r] \arrow[d] & P_0' \arrow[r, "\epsilon'"] \arrow[d, "\iota"] & A' \arrow[r] \arrow[d, "\iota_A"] & 0 \\
0 \arrow[r] & Ker\epsilon \arrow[r] \arrow[d] & P_0' \oplus P_0'' \arrow[r, dashed, "\epsilon"] \arrow[d, "\pi"] & A \arrow[d, "\pi_A"] & \\
0 \arrow[r] & Ker\epsilon'' \arrow[r] \arrow[d] & P_0'' \arrow[r, "\epsilon''"] \arrow[d] \arrow[ur, dashed] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Now, notice that concatenate $P_1' \twoheadrightarrow Ker\epsilon'$ and $P_1'' \twoheadrightarrow Ker\epsilon''$ still makes the rows exact, then we do the same procedure to get:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Kerd_1' \arrow[r] \arrow[d] & P_1' \arrow[r, "d_1'"] \arrow[d] & Ker\epsilon' \arrow[r] \arrow[d] & P_0' \arrow[r, "\epsilon'"] \arrow[d, "\iota"] & A' \arrow[r] \arrow[d, "\iota_A"] & 0 \\
0 \arrow[r] & Kerd_1 \arrow[r] \arrow[d] & P_1 \arrow[r, dashed, "d_1"] \arrow[d] & Ker\epsilon \arrow[r] \arrow[d] & P_0' \oplus P_0'' \arrow[r, dashed, "\epsilon"] \arrow[d, "\pi"] & A \arrow[d, "\pi_A"] & \\
0 \arrow[r] & Kerr_1'' \arrow[r] \arrow[d] & P_1'' \arrow[r, "d_1''"] \arrow[d] \arrow[ur, dashed] & Ker\epsilon'' \arrow[r] \arrow[d] & P_0'' \arrow[r, "\epsilon''"] \arrow[d] \arrow[ur, dashed] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

Then keep doing this process to get what we want. $\square$

Remark 3.15. The above way is a quick way to see that there exist such a resolution with desired property, but if we are willing to some work, we can obtain an more explicit construction which carries some ideas

that's also useful in Thm 5.7.
To begin with, our expectation is that we can find differentials between $P'_n \oplus P''_n \to P'_{n-1} \oplus P''_{n-1}$ such that when we put in A and a augmented morphism $P'_0 \oplus P''_0$, such that $P'_\bullet \oplus P''_\bullet \xrightarrow{\epsilon} A$ is a projective resolution and the following commute:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & A' \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & A'' \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & P'_\bullet \arrow[r] \arrow[u] & P_\bullet \arrow[r] \arrow[u] & P''_\bullet \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

However, our work reduces to just construct the connecting morphisms in $P'_\bullet \oplus P''_\bullet$ and the augmented morphism ϵ such that the above diagram commute, since once we have it, since $P'_\bullet$ and $P''_\bullet$ are resolutions, the LES of homology associated to the augmented chain complex shows that $P'_\bullet \oplus P''_\bullet \xrightarrow{\epsilon} A$ is a resolution.
We still inductively construct these maps. First, we need construct maps $P'_0 \oplus P''_0$ to A, notice that this is equivalent of giving $\epsilon'_0 : P'_0 \to A$ and $\epsilon''_0 : P''_0 \to A$. If we want the diagram to commute: we follow the arrows to get:

$$\iota_A \epsilon' = \epsilon'_0 \qquad \pi\epsilon'' = \pi_A \epsilon''_0$$

Notice P''_0 is projective, then it's lifting property gives us ϵ''_0.
Then we keep going and define maps connecting objects in $P'_\bullet \oplus P''_\bullet$. Notice that giving a map $P'_n \oplus P''_n$ to $P'_{n-1} \oplus P''_{n-1}$ is the same as giving how P'_n goes to P'_{n-1} and P''_{n-1} and giving how P''_n goes to P'_{n-1} and P''_{n-1} since finite direct sums and finite direct products are the same and we just use universal property. Thus it's given by a matrix:

$$\begin{bmatrix} f_1 & f_2 \\ g_1 & g_2 \end{bmatrix}$$

We need the diagram to commute, let's call the differentials in $P'_\bullet$ and $P''_\bullet$ d' and d'', thus we need the following to hold:

$$f_1 x' = d'x' \qquad g_1 x' = 0 \qquad g_2 x'' = d''x''$$

for all $x' \in P'_n$ and $x'' \in P''_n$, thus we have the matrix is $\begin{bmatrix} d' & f_2 \\ 0 & d'' \end{bmatrix}$, thus our task is to determine f_2 such that the square of the matrix d is 0(We already made this commute!) Therefore, compute the square, we have it's $\begin{bmatrix} d'd' & d'f_2 + f_2 d'' \\ 0 & d''d'' \end{bmatrix}$, thus we need $d'f_2 + f_2 d'' = 0$ and remember that f_2 goes from P''_n to P'_{n-1} for $n \geqslant 2$. This also applies to $n = 1$ with a little adaption: first compute the composition of $\epsilon \circ d_1$, notice that the image of (x', x'') is $\epsilon(d'x' + f_2 x'', d''x'') = \epsilon'_0 d'x' + \epsilon'_0 f_2 x'' + \epsilon''_0 d''x'' = \epsilon'_0 f_2 x'' + \epsilon''_0 d''x''$, thus we want $\epsilon'_0 f_2 + \epsilon''_0 d'' = 0$. Now we notice that since $\pi\iota\epsilon' = \pi\epsilon'_0 = 0$ and ϵ' is surjective, the image of $\iota\epsilon'$ is $Ker\pi_A$, on the other hand since $\pi_A \epsilon''_0 = 0$, the image of $\epsilon''_0 f_2$ is in $Ker\pi_A$, thus the lifting property of P'_0 gives the desired map. Later on we just need to do induction: if we want $d'f_2 + f_2 d''$ and notice that the second f_2 is already defined, notice that $df_2 d'' = f_2 d'' d'' = 0$, thus similar to the above argument, we get the desired f_2.

Remark 3.16. Using prop 2.7 and prop 2.6, we can prove similar results for injective resolutions.

4 Left and Right Derived Functors

4.1 General definition

Definition 4.1. Left Derived Functor Suppose that $F : \mathcal{A} \longrightarrow \mathcal{B}$ is a covariant right exact functor, suppose that $\mathcal{A}$ has enough projectives, then we define the left derived functor $L_i F$ as:

$$L_i F(A) = H_i(F(P))$$

where we pick any projective resolution $P \xrightarrow{\epsilon} A$. Schematically, we take homology in the following complex:

$$\cdots \longrightarrow F(P_2) \xrightarrow{Fd_2} F(P_1) \xrightarrow{Fd_1} F(P_0) \xrightarrow{0} 0$$

Proposition 4.2. Left Derived Functors are well defined Up to Natural Isomorphism Suppose that we have P and Q two different projective resolutions for A, $H_i(F(P)) \cong H_i(F(Q))$.

Proof. By Comparison Theorem 3.12, we have a chain map $f : P \longrightarrow Q$ lifting id_A and any map f' lifting id_A will be homotopic to f. Thus, after we apply F, they are still homotopic, thus induce the same map $H_i(Ff) = f^*$. We will show this is an isomorphism. To see this, notice we also have a chain map $g : Q \longrightarrow P$ lifting id_A, then the composition $g \circ f$ is a chain map from P to P lifting id_A. Notice that id_P also lifts id_A, thus $f^* \circ g^* = id^*$. Similarly, we have $g^* \circ f^* = id^*$, which proves f^* is a canonical isomorphism. □

Corollary 4.3. $L_0F(A) \cong FA$ and if A is projective $L_iF(A) = 0$ for $i \neq 0$.

Proof. Since F is right exact, consider $F(P_1) \longrightarrow F(P_0) \longrightarrow F(A) \longrightarrow 0$, it's exact at $F(A)$ and $F(P_0)$, thus $F(A) \cong F(P_0)/Im(F(P_1)) = H_0(F(P))$ by definition. If A is projective, choose the resolution $0 \longrightarrow A \longrightarrow A \longrightarrow 0$, easy to see that $L_iF(A) = 0$ for $i \neq 0$. □

Definition 4.4. Right Derived Functors Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a covariant left exact functor. Suppose that $\mathcal{A}$ has enough injectives, then we define the right derived functor R^iF as:

$$R^iF(A) = H^i(F(I))$$

where $I \overset{\epsilon}{\longrightarrow} A$ is a chosen injective resolution of A. Similar to left derived functors, comparison theorem for injective resolutions show that this is well defined and $R^0F(A) \cong F(A)$, if A is injective, then $R^iF(A = 0$ for $i \neq 0$. We also have F-acyclic objects.

Remark 4.5. Based on the proof of the above corollary 4.5, we find that for any left resolution $P \longrightarrow A$, $F(A) \cong H_0(FP)$, also for any right resolution $A \longrightarrow I$, $F(A) \cong H^0(FI)$.

Remark 4.6. The above definition doesn't immediately show that derived functors are really functors though it's not hard to guess that it's indeed an additive functor since both F and H is. The proof is again another exercise of comparasion Theorem 3.12, we will omit it here.

4.2 Ext and Tor

Finally, we are in good position to introduce Ext and Tor as derived functors.

Definition 4.7. Ext In R-Modules, $Hom_R(A, -)$ is covariant and left exact, thus we can define $Ext^i_R(A, B) = R^iHom(A, -)(B)$ for a R-module B. That is, we find an injective resolution of B, apply $Hom_R(A, -)$ then take cohomology.

Definition 4.8. Tor In R-modules, $- \otimes_R B$ is always right exact, thus we can define $Tor^R_i(A, B) = L_i(- \otimes_R B)(A)$. That is we find a projetive resolution of A, take tensor product $- \otimes_R B$, then take homology.

Given this definition, you might ask what if we use $A \otimes_R -$ and $Hom_R(-, A)$ since they are also right and left exact (though $Hom_R(-, A)$ is contravariant). We first make sense of what is the derived functor of a contravariant functor and we will later see that, using $- \otimes_R A$ and $Hom_R(-, A)$ give us nothing new in the sense that swapping the position of A gives us isomorphic results.

Definition 4.9. Suppose that $F : \mathcal{A} \longrightarrow \mathcal{B}$ is a contravariant left exact functor. Define the right derived functor of F to be $R^iF(A) = H^i(F(P))$ where $P \longrightarrow A$ is a projective resolution of A. Similarly, if F is right exact, define the left derived functor $L_iF(A) = H_i(F(I))$ where $M \longrightarrow I$ is a injective resolution of M.

The reason we define it like this is: a left exact contravariant functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is the same as given a left exact covariant functor $F^{op} : \mathcal{A}^{op} \longrightarrow \mathcal{B}$. We know how to define right derived functor for F^{op}. We choose a injective resolution in A^{op} apply F^{op}, then take cohomology in $\mathcal{B}$. This translates to choose a projective resolution, apply F then take cohomology in $\mathcal{B}$.

Remark 4.10. When we are working over $\mathbb{Z}$ and abelian groups, or when the ring R is clear from context, for our convenience we will always write Tor_n and Ext^n, omitting the ring R in the super or subscript.

4.3 Balancing Ext and Tor: Left and Right Balanced Functors

In this section, we answer the question raised above about how to show that using $-\otimes_R B$, $A\otimes_R -$, $Hom_R(-,A)$ and $Hom_R(B,-)$ give us isomorphic Ext and Tor modules. We will first do the special case just for $-\otimes_R B$ and $A\otimes_R -$. Then will provide a general definition and show that our treatment of the tensor product generalizes as desired.

Definition 4.11. Tensor Product of Chain Complexes Suppose that P and Q are two chain complex in R-module. Recall definition 2.12, we can build a double chain complex out of P, Q by taking tensor product $P_p \otimes_R Q_q$ and set horizontal maps to be $d\otimes 1$ and vertical maps to be $(-1)^p \otimes d$ to match with anticommutativity convention.

It's straight forward to check that if $f : A \longrightarrow B$ and $g : C \longrightarrow D$ are two maps chain complexes, then we have a natural map of double complexes $f\otimes_R g : A\otimes_R C \longrightarrow B\otimes_R D$, which is $f_p \otimes_R g_q$ on $A_p \otimes_R B_q$.

Theorem 4.12. $L_n(A\otimes_R -)(B) \cong L_n(-\otimes_R B)(A) = Tor_n^R(A,B)$

Proof. The first step when we compute $L_n(A\otimes_R -)(B)$ and $L_n(-\otimes_R B)(A)$ is that we pick projective resolutions $P \xrightarrow{\epsilon} B$ and $Q \xrightarrow{\eta} A$. Consider the double chain complex $P\otimes_R Q$, which is a first quadrant double chain complex. Then by viewing A and B as chain complexes which is concentrated at 0-th degree, we can also produce $A\otimes_R Q$ and $P\otimes_R B$. Moreover, by viewing a resolution as quasi isomorphisms from P, Q to M, N(concentrated at 0), we get induced morphisms of double chain complexes, as illustrated in the following diagram:

$$\begin{tikzcd}
A\otimes_R Q_1 \arrow[d] & P_0\otimes_R Q_1 \arrow[d] \arrow[l, phantom, "\Leftarrow \epsilon\otimes_R id_Q ="] & P_1\otimes_R Q_1 \arrow[l] \arrow[d] \\
A\otimes_R Q_0 & P_0\otimes_R Q_0 & P_1\otimes_R Q_0 \arrow[l] \arrow[dl, phantom, "\Downarrow id_P\otimes_R\eta"] \\
 & P_0\otimes_R B & P_1\otimes_R B \arrow[l]
\end{tikzcd}$$

Under this set up, we see that $Tot^{\oplus}(A\otimes_R Q) = A\otimes_R Q$ since each term has only one nonzero summand. Similarly, $Tot^{\oplus}(P\otimes_R B) = P\otimes_R B$. Recall lemma 2.15, we have induced maps on total complexes:

$$A\otimes_R Q = Tot^{\oplus}(A\otimes_R Q) \xleftarrow{Tot^{\oplus}(\epsilon\otimes_R id_Q)} Tot^{\oplus}(P\otimes_R Q) \xrightarrow{Tot^{\oplus}(id_P\otimes_R\eta)} Tot^{\oplus}(P\otimes_R B) = P\otimes_R B$$

We will show that $Tot^{\oplus}(\epsilon\otimes_R id_Q)$ and $Tot^{\oplus}(id_P\otimes_R \eta)$ are quasi-isomorphism, thus induce isomorphism on homology. Recall Prop 2.10, we only have to show that the mapping cone of $Tot^{\oplus}(\epsilon\otimes_R id_Q)$ and $Tot^{\oplus}(id_P\otimes_R\eta)$ are exact. Good news is that, here their cones have simple descriptions: If now we consider a new double chain complex C by adding on $A\otimes_R Q$ to the -1 column of $P\otimes_R Q$ and adding a negative sign to all of the differentials in $A\otimes_R Q$. Also, check that $Tot^{\oplus}(C)[1]$ is the mapping cone of $Tot^{\oplus}(\epsilon\otimes_R id_Q)$. Though it's just straightforward checking, a somewhat more illustrating way of interpreting this is by staring at the diagram below and see that the differential in a mapping cone is very similar to the differential of the total complex of a double complex.

$$\begin{tikzcd}
B_{n-2} & B_{n-1} \arrow[l, "-d"'] \arrow[dl, "-f"] \\
C_{n-1} & C_n \arrow[l, "+d"']
\end{tikzcd}$$

Therefore, we only have to show that $Tot^{\oplus}(C)[1]$ is acyclic. Notice that Q_i is projective, thus $-\otimes_R Q_i$ is exact, thus it has exact rows since we begin with an resolution. Argument for $Tot^{\oplus}(id_P\otimes_R\eta)$ are similar. $\square$

It turns out that the approach we use to show that Ext doesn't care whether it's $Hom_R(A,-)$ or $Hom_R(-,B)$ we use to compute it is very similar to the above proof. These kinds of arguments are special cases of arguments we have for left and right balanced functors:

Definition 4.13. Right and Left Balanced Functors[CE99] Suppose $\mathcal{C}$ is an abelian category that T is a left exact functor between abelian categories $\mathcal{C} \times \cdots \times \mathcal{C} \longrightarrow \mathcal{D}$ that has p slots of variables. When we fix any $p-1$ variables and let the other 1 vary, it becomes covariant or contravariant. We will call this functor right balanced if:

1. If any covariant variable is replaced by an injective object, T becomes exact when we let any other one variable vary.

2. If any contravariant variable is replaced by an projective object, T becomes exact when we let any other one variable vary.

If T is right exact and we interchange "Projective" and "Injective" throughout in the two conditions above, we get the definition of a left balanced functor.

A natural example of a right balanced functor is $Hom(-,-)$, it has two slots, if we fix the first one, it becomes covariant in the second slot while if we fix the second one it becomes contravariant in the first slot. Also, if we fix the first contravariant slot to be a projective module M, recall lemma 3.3, $Hom_R(M,-)$ is exact, if we fix the second covariant to be an injective module I, $Hom_R(-,I)$ is exact. A natural example of left balanced functor is $-\otimes_R -$, as you can check.
The following proof follows the idea in a University of Michigan Homological Algebra Notes:

Theorem 4.14. [613] In the above set up, if T is a right balanced functor. For any i,j, we have natural isomorphisms $R^*T(A_1,\cdots,-_i,\cdots,A_p)(A_i) \cong R^*T(A_1\cdots,-_j,\cdots,A_p)(A_j)$ where the $-$ indicates that we fix all the other variables and treat the T as a functor in one variable.

Proof. First notice that when we fix all the other variables with only A_i and A_j variable, we get a functor $T' : \mathcal{C}\times\mathcal{C} \longrightarrow \mathcal{D}$. Therefore we only have to prove the special case when the functor is of two variables. Recall 4.9, if T is contravariant in the first variable, choose a projective resolution $P \xrightarrow{\epsilon} A_1$, if it's covariant, choose a injective resolution $A_1 \xrightarrow{\epsilon} P$, similarly if T is contravariant in the second variable choose a projective resolution $Q \xleftarrow{\eta} A_2$, if it's covariant in the second variable then choose an injective reslotion $A_2 \xleftarrow{\eta} Q$. Mimic the proof of Thm 4.12, we first get a double complex:

$$\begin{tikzcd}
\cdots & \cdots & \cdots & \\
T(A_1,Q_1) \arrow[u] & T(P_1,Q_0) \arrow[u] \arrow[l, Rightarrow] \arrow[r] & T(P_1,Q_1) \arrow[u] \arrow[r] & \cdots \\
T(A_1,Q_0) \arrow[u] & T(P_0,Q_0) \arrow[u] \arrow[l, Rightarrow, "T(\epsilon,id_Q)"'] \arrow[r] \arrow[d, Rightarrow, "T(id_P,\eta)"'] & T(P_0,Q_1) \arrow[u] \arrow[r] \arrow[d, Rightarrow] & \cdots \\
0 & T(P_0,A_2) \arrow[r] & T(P_1,A_2) \arrow[r] & \cdots
\end{tikzcd}$$

Here we will again prove that the induced maps below are quasi-isomorphisms:

$$T(A_1,Q) = Tot^{\prod}(T(A_1,Q)) \xleftarrow{Tot^{\prod}(T(\epsilon,id_Q))} Tot^{\prod}(T(P,Q)) \xrightarrow{Tot^{\prod}(T(id_P,\eta))} Tot^{\prod}(T(P,A_2)) = T(P,A_2)$$

Consider we add $T(A_1,Q)$ to the -1 column of $T(P,Q)$ with all the differentials being it's negative to get a new double cochain complex C, similar arguments in Thm 4.12 shows $Tot^{\prod}(C)[-1]$ is the mapping cone of $Tot^{\prod}(T(\epsilon,id_Q))$, it's a acyclic by remark 2.17. Similarly we have the argument for the other induced map. □

Corollary 4.15. $R^*Hom_R(-,B)(A) \cong Ext^*_R(A,B) \cong R^*Hom_R(A,-)(B)$

Proof. Recall $Hom_R(-,-)$ is a natural example of right balanced functor. □

Readers can state the version of the above theorem for left balanced functors and prove it just like Thm 4.12, we omit it here. Afterwards, Thm 4.12 will be a corollary of it.
As promised in the beginning, in terms of computation, it doesn't matter to choose $-\otimes_R B$, $A\otimes_R -$, $Hom_R(A,-)$ or $Hom_R(-,B)$, so we won't bother to mention which one we choose when we do computation.

5 Universal δ-Functors

The main result of this section will be: When a derived functor exists, it's a universal δ-(co)homological functor. Before we introduce the notion of (universal) δ-Functors, let's look at a more familiar case to get a feeling about what we will define:

Proposition 5.1. Suppose $0 \to A \to B \to C \to 0$ is a SES of chain complexes, then there exist an associated LES with natural connecting maps $\partial : H_n(C) \to H_{n-1}(A)$. It's natural in the sense that given:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

we have the following diagram commute:

$$\begin{tikzcd} \cdots \arrow[r] & H_{n+1}(C) \arrow[r, "\partial"] \arrow[d, "f_{n+1*}"] & H_n(A) \arrow[r] \arrow[d, "f_{n*}"] & H_n(B) \arrow[r] \arrow[d, "g_{n*}"] & H_n(C) \arrow[r, "\partial"] \arrow[d, "h_{n*}"] & H_{n-1}(A) \arrow[r] \arrow[d, "f_{n-1*}"] & \cdots \\ \cdots \arrow[r] & H_{n+1}(C') \arrow[r, "\partial"] & H_n(A') \arrow[r] & H_n(B') \arrow[r] & H_n(C') \arrow[r, "\partial"] & H_{n-1}(A') \arrow[r] & \cdots \end{tikzcd}$$

Remark 5.2. We will omit the proof since it can be embedded into R-modules and it's just standard arguments, but when we work in R-modules, we would like a formula we can use for the connecting map $\partial : H_{n+1}(C) \to H_n(A)$. The way it works is take $c \in C_{n+1}$ that it a cycle, the pull it back to $b \in B_{n+1}$ such that $f_{n+1}b = c$, apply the differential, you will find db is in Z_n, which will represent the image of $\partial[a]$.

This mechanism can actually be generalized. Notice that here given a SES $0 \to A \to B \to C \to 0$ of chain complexes, for each n, we have a functor H_n and we have a natural connecting map ∂ connecting $H_{n+1}(C) \to H_n$, this leads to our definition:

Definition 5.3. (co)Homological δ-Functors Fix $\mathcal{A}$ and $\mathcal{B}$, two abelian categories. A (covariant) homological δ-functor between $\mathcal{A}$ and $\mathcal{B}$ is a collection of additive functors $T_n : \mathcal{A} \to \mathcal{B}$ for $n \geqslant 0$, together with morphisms $\delta_n : T_n(C) \to T_{n-1}(A)$ defined for each SES $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, (here we make the convention that $T_n = 0$ for $n < 0$)such that for each SES as abovem we have a long exact sequence:

$$\cdots \longrightarrow T_{n+1}(C) \xrightarrow{\delta} T_nA \longrightarrow T_nB \longrightarrow T_nC \xrightarrow{\delta} T_{n-1}A \longrightarrow \cdots$$

(so T_0 is right exact). Moreover, we require these morphism δ_n to be natural in the sense that given a morphism of SES from $0 \to A \to B \to C \to 0$ to $0 \to A' \to B' \to C' \to 0$, we have a commuting diagram:

$$\begin{tikzcd} T_{n+1}(C) \arrow[r, "\delta"] \arrow[d] & T_n(A) \arrow[d] \\ T_{n+1}(C') \arrow[r, "\delta"'] & T_n(A') \end{tikzcd}$$

We have similar definition of δ-cohomological functors. For our convenience, we will sometimes call δ's defined above connecting morphisms.

Example 5.4. The above definition and remark shows H_n and H^n are δ-(co)homological functors.

Definition 5.5. Morphisms of δ functors Suppose T and S are two homological δ functors, a morphism $F : T \to S$ of homological δ-functors is a collection of natural transformations between $T_n \to S_n$ that it commutes with the connecting morphism δ in the sense that we have the following commuting diagram:

$$\begin{tikzcd} T_{n+1}(C) \arrow[r, "\delta"] \arrow[d, "F_{n+1}"'] & T_n(A) \arrow[d, "F_n"'] \\ S_{n+1}(C) \arrow[r, "\delta"'] & S_n(A) \end{tikzcd}$$

The consequence of this above definition is that given any SES $0 \to A \to B \to C \to 0$, we have a commuting ladder diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & T_{n+1}(C) \arrow[r, "\delta"] \arrow[d, "F^C_{n+1}"'] & T_n(A) \arrow[r] \arrow[d, "F^A_n"'] & T_n(B) \arrow[r] \arrow[d, phantom, "F^B_n"] & T_n(C) \arrow[r, "\delta"] \arrow[d, phantom, "F^C_n"] & T_{n-1}(A) \arrow[r] \arrow[d, "F^A_{n-1}"] & \cdots \\
\cdots \arrow[r] & S_{n+1}(C) \arrow[r, "\delta"'] & S_n(A) \arrow[r] & S_n(B) \arrow[r] & S_n(C) \arrow[r, "\delta"'] & S_{n-1}(A) \arrow[r] & \cdots
\end{tikzcd}$$

Beyond this definition, we would like some uniqueness condition on these functor T_n's defined above:

Definition 5.6. Universal (co)Homological δ-Functors A homological δ-functor S is said to be universal if given any other homological δ-functor T and any natural transformation $F_0 : T_0 \to S_0$, there exists a unique morphism of homological δ-functors $\{F_n : T_n \to S_n\}$ extending F_0. Similarly, we can define it for cohomological δ-functors.

Now, let's show the main result:

Theorem 5.7. Left Derived Functors Are Universal δ-Functors The left derived functor of a right exact functor F is a universal homological δ-functor.

Proof. Here, we only show the exsistence of the connecting morphism and the naturality. We will not prove Universality since we won't use it in this paper, but it can be found here in []. The key fact here will be horseshoe lemma 3.14 and the fact that we have a connecting morphism for homology prop 5.1.
Given a SES $0 \to A' \to A \to A'' \to 0$, recall the horseshoe lemma 3.14 gives us:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & A' \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & A'' \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & P'_\bullet \arrow[r] \arrow[u] & P_\bullet \arrow[r] \arrow[u] & P''_\bullet \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Notice that $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$ splits, thus when we apply and additive functor to $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$, it remains exact, hence we have:

$$0 \to FP'_\bullet \to FP_\bullet \to FP''_\bullet \to 0$$

, then apply homology to get a LES associated to homology according to prop 5.1. As for the naturality of this connecting map, assume that we have a commutative diagram of SES and projective resolutions $\epsilon' : P' \to A'$, $\epsilon'' : P'' \to A''$, $\eta' : Q' \to B'$ and $\eta'' : Q'' \to B''$ as illustrated below:

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d, dashed, "f'"] & A \arrow[r] \arrow[d, dashed, "f"] & A'' \arrow[r] \arrow[d, dashed, "f''"] & 0 \\
0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0
\end{tikzcd}$$

Then Comparison Theorem 3.12 and Horseshoe Lemma 3.14 give us the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & P'_\bullet \arrow[r] \arrow[d, dashed, "\epsilon'"] \arrow[ddd, dashed, bend right=60] & P_\bullet \arrow[r] \arrow[d, dashed, "\epsilon"] & P''_\bullet \arrow[r] \arrow[d, dashed, "\epsilon''"] \arrow[ddd, dashed, bend left=60] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d, dashed, "f'"] & A \arrow[r] \arrow[d, dashed, "f"] & A'' \arrow[r] \arrow[d, dashed, "f''"] & 0 \\
0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0 \\
0 \arrow[r] & Q'_\bullet \arrow[r] \arrow[u, dashed, "\eta'"] & Q_\bullet \arrow[r] \arrow[u, dashed, "\eta"] & Q''_\bullet \arrow[r] \arrow[u, dashed, "\eta''"] & 0
\end{tikzcd}$$

As you can tell, $P_\bullet$ and $Q_\bullet$ comes from Horseshoe lemma 3.14 and the map $P'_\bullet \to Q'_\bullet$ $P''_\bullet \to Q''_\bullet$ comes from comparison lemma lifting f' and f'' resp. We claim that we will have the a map $G : P_\bullet \to Q_\bullet$ and a commutative diagram of SES of chain complexes:

$$\begin{tikzcd}
0 \arrow[r] & P'_\bullet \arrow[r] \arrow[d, dashed, "G'"] & P_\bullet \arrow[r] \arrow[d, dashed, "G"] & P''_\bullet \arrow[r] \arrow[d, dashed, "G''"] & 0 \\
0 \arrow[r] & Q'_\bullet \arrow[r] & Q_\bullet \arrow[r] & Q''_\bullet \arrow[r] & 0
\end{tikzcd}$$

where G' and G'' are those maps we mentioned above from comparison lemma 3.12. As we explained in the begining, since the top and bottom row are split exact, thus when we apply F, an additive functor, it's still exact, thus we have:

$$\begin{tikzcd}
0 \arrow[r] & FP'_\bullet \arrow[r] \arrow[d, dashed, "FG'"] & FP_\bullet \arrow[r] \arrow[d, dashed, "FG"] & FP''_\bullet \arrow[r] \arrow[d, dashed, "FG''"] & 0 \\
0 \arrow[r] & FQ'_\bullet \arrow[r] & FQ_\bullet \arrow[r] & FQ''_\bullet \arrow[r] & 0
\end{tikzcd}$$

Then the naturality of connecting map of the derived functor is just the naturality of connecting map of homology. Thus we only have to construct $G : P_\bullet \to Q_\bullet$. The way to construct it is almost the same as what we did in remark 3.15, and you can find the details in [Wei94]. ∎

Remark 5.8. Using dual results of comparison lemma and horseshoe lemma, you can prove that right derived functors are also universal cohomological-δ-functors.

Definition 5.9. F Acyclic Objects F is still a right exact functor. $Q \in \mathcal{A}$ is called F acyclic if $L_iF(Q) = 0$ for all $i \neq 0$, that is all higher degree derived functor vanish on Q. A F-Acyclic resolution of A is a left resolution $Q \longrightarrow A$ such that each Q is F-Acyclic.

An trivial example is that if P is a projective object, it's obviously F acyclic the previous corollary. A less trivial example is that if M is a flat R-module, we will see later that M is $\otimes_R B$- or $A \otimes_R -$-acyclic. The following is a good news that we can compute the derived functor of F using an F-acyclic resolution if we don't have projective resolutions available.

Proposition 5.10. Dimension Shifting and F-acyclic Resolution Lemma Suppose $0 \to M \to P \to A \to 0$ is exact with P being projective or F-acyclic. Then $L_iF(A) \cong L_{i-1}F(M)$ for $i \geqslant 2$ and $L_1F(A)$ is the kernel of $FM \to FP$. More generally, if we have an exact sequence where P_i is projective or F-acyclic:

$$0 \to M_m \to P_m \to P_{m-1} \cdots \to P_0 \to A \to 0$$

Then, $L_iF(A) \cong L_{i-m-1}F(M_m)$ for all $i \geqslant m+2$ and $L_{m+1}FA = Ker(FM_m \to FP_m)$. We can also conclude that if $P \longrightarrow A$ is a F-acyclic resolution, then $L_iF(A) = H_i(F(P))$.

Proof. Consider the LES associated to $0 \to M \to P \to A \to 0$, which is:

$$\cdots \longrightarrow L_2F(A) \longrightarrow L_1F(M) \longrightarrow L_1F(P) \longrightarrow L_1F(A) \longrightarrow F(M) \longrightarrow F(P) \longrightarrow F(A) \longrightarrow 0$$

Notice that P is F-acyclic, thus $L_iF(P) = 0$ for $i \geqslant 1$. Thus, we have $L_iFA \cong L_{i-1}FM$ for $i \geqslant 2$ and the desired kernel in the statement.
Now, suppose that we have $0 \to M_m \to P_m \to P_{m-1} \cdots \to P_0 \to A \to 0$, we take the kernel M_0 of $P_0 \to A$ to be in the first case. Thus, repeat our argument we get $L_iF(A) \cong L_{i-1}FM_0$. Then consider $0 \to M_1 = Ker \to P_1 \to M_0 \to 0$, which is still exac. Then still apply the argument above to get $L_iF(M_0) \cong L_{i-1}M_1$, thus we have $L_iFA \cong L_{i-1}FM_0 \cong L_{i-2}M_1$, also $L_2FA = Ker(FM_1 \to FP_1)$ keep repeating this process until we get $L_iFA = L_{i-m-1}FM_m$ and $L_{m+1}FA = Ker(FM_m \to FP_m)$.
Finally, notice that $L_0FA \cong FA$ and recall remark 4.5, we know that $FA \cong H_0(FP)$ for any resolution, therefore also for F-acyclic resolution. $L_1FA = Ker(FM_0 \longrightarrow FP_0)$, consider the exact sequence $FP_2 \to FP_1 \to FM_0 \to 0$ coming from $P_2 \to P_1 \to M_0 \to 0$ and F is right exact. Thus $FM_0 \cong FP_1/Fd(FP_2)$, hence $L_1FA \cong Ker(FM_0 \longrightarrow FP_0) = Ker(FP_1/Fd(FP_2) \to FP_0) = H_1(FP)$. For any $m \geqslant 1$, now we have $F_{m+1}FA = Ker(FM_m \to FP_m)$, use the exact sequence $FP_{m+2} \to FP_{m+1} \to FM_m \to 0$ and similar argument as above to conclude. ∎

6 Applications of Ext, Tor and Universal Coefficient Theorem

In this section, we first collect some properties of Ext and Tor, then use that to do some computations, finally hopefully show Künneth formula and universal coefficients theorem.

6.1 Basic Properties of Derived Functors

Let's first quote a theorem without proof, though a good guide by which you can prove this theorem yourself can be found in chapter 1 in [Vak].

Theorem 6.1. Let $\mathcal{A}$, $\mathcal{B}$ be arbitrary categories, let $L : \mathcal{A} \longrightarrow \mathcal{B}$ be the left adjoint to $R : \mathcal{B} \longrightarrow \mathcal{A}$. Then:

1. If $A : I \longrightarrow \mathcal{A}$ and $LA : I \longrightarrow \mathcal{B}$ have colimits, then $L(colim_{i\in I} A_i) \cong colim_{i\in I} L(A_i)$
2. If $B : I \longrightarrow \mathcal{B}$ and $RB : I \longrightarrow \mathcal{A}$ have limits, then $R(Lim_{i\in I} B_i) = Lim_{i\in I} R(B_i)$

Notice $-\otimes_R B$ and $A\otimes_R -$ is left adjoint to $Hom_R(B,-)$ or $Hom_R(A,-)$ and homology commutes with direct sums(though homology doesn't commute with arbitrary colimit). The immediate corollary is that:

Corollary 6.2. Suppose that F is a left adjoint and $\{A_i\}$ is a set of objects in an Abelian category $\mathcal{A}$, then we have: $L_*F(\oplus_i A_i) \cong \oplus_i L_*F(A_i)$. In particular, we have $Tor^R_*(A, \oplus_i B_i) \cong \oplus_i Tor^R_*(A, B_i)$ and $Tor^R_*(\oplus A_i, B) \cong \oplus_i Tor^R_*(A_i, B)$.

We need a couple preceding results before we proceed to the next property.

Theorem 6.3. Yoneda Lemma Yoneda embedding reflects exactness: Let $\mathcal{A}$ be an abelian category, then a sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is exact in $\mathcal{A}$ if for every $M \in \mathcal{A}$, we have $Hom_{\mathcal{A}}(M, A) \xrightarrow{\alpha_*} Hom_{\mathcal{A}}(M, B) \xrightarrow{\beta_*} Hom_{\mathcal{A}}(M, C)$ is exact in **Ab**.

There are several stuff that share the name: Yoneda lemma. If you are curious about what exactly it is and why there is a embedding, you can take a look at chapter 1 in [Vak], where you are able to figure it out yourself.

Proof. Let work in R-modules by Thm 2.5. Let $M = A$, we see that $\beta\alpha = \beta_*\alpha_*(id_A) = 0$. Now, let $M = Ker(\beta)$, then we know that the canonical monomorphism $\iota : Ker(\beta) \hookrightarrow B$ satisfies, $\beta\iota = 0 = \beta_*(\iota)$, thus from the exactness we know there is a $\sigma \in Hom_{\mathcal{A}}(M, A)$ such that $\alpha\sigma = \iota$, thus $im(\iota) = Ker(\beta) \subset Im(\alpha)$. ☐

Theorem 6.4. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and suppose that $L : \mathcal{A} \longrightarrow \mathcal{B}$ is left adjoint to $R : \mathcal{B} \longrightarrow \mathcal{A}$. Then L is right exact and R is left adjoint

Since tensor is left adjoint to Hom, the above theorem is another proof of tensor is right exact and Hom is left exact.

Proof. Suppose $0 \to B \to B' \to B'' \to 0$ exact, since L and R are adjoint to each other, we have a commutative diagram with vertical isomorphisms:

$$\begin{array}{ccccccc}
0 \longrightarrow & Hom(LA, B) & \longrightarrow & Hom(LA, B') & \longrightarrow & Hom(LA, B'') \\
& \updownarrow & & \updownarrow & & \updownarrow \\
0 \longrightarrow & Hom(A, R(B)) & \longrightarrow & Hom(A, R(B')) & \longrightarrow & Hom(A, R(B''))
\end{array}$$

The top row is exact since $Hom(LA, -)$ is left exact, thus the bottom row is exact for any A, thus Thm 6.3 tells $0 \to R(B) \to R(B') \to R(B'')$ is exact, thus R is left exact. Notice that when we consider $L^{op} : \mathcal{A}^{op} \longrightarrow \mathcal{B}^{op}$ it becomes a right adjoint, thus L^{op} is left exact, thus L is right exact. ☐

Lemma 6.5. Let I be a fixed category, which is always a filtered set, such as a poset. Suppose that $\mathcal{A}$ is an Abelian Category, then colimit is left adjoint to the diagonal functor $\Delta : \mathcal{A} \longrightarrow \mathcal{A}^I$ a right adjoint functor from $\mathcal{A}^I \longrightarrow \mathcal{A}$

Theorem 6.6. For a fixed filtered set I, the direct limit of R-modules are exact, considered as functors from $(R-mod)^I \longrightarrow R-mod$.

It's worth noting that not every Abelian category satisfies this property, for those categories who have arbitrary direct sums over a set and satisfy the above theorem, we will say this category satisfies the **AB5 axiom**.

Proof. Lemma 6.5 and Thm 6.4 tells us that colimit is right exact, thus we only have to show it's left exact. In our case, we need to show that if $A \to B$ is a monic natural transformation i.e. each $A_i \to B_i$ is a monic morphism, we want to show that $colim_{\to}(A_i) \longrightarrow colim_{\to}(B_i)$ is injective in R-modules. If $a \in colim_{\to} A_i$ which is mapped to 0, then we know that a is the image of $a_i \in A_i$ for some $i \in I$, then we know that the image of a is the image of $t_i(a_i) \in B_i$, thus there will be some $\phi : i \to j$ such that $0 = \phi(t_i(a_i)) = t_i\phi(a_i)$, hence $\phi(a_i) = 0$ since t_i is monic, thus a is 0 in $colim_{\to}(A_i)$. □

We want all the above results to show:

Proposition 6.7. Let $\mathcal{A}$ be a Abelian Category with enough projectives which satisfies AB5 axiom(such as R-modules). Then let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a left adjoint functor between abelian categories. Then, for every $A : I \longrightarrow \mathcal{A}$ with I being a filtered set $L_*F(colim(A_i)) \cong colim(L_*F(A_i))$. In particular, Tor^R_* satisfies this property.

Proof. Let $P_i \longrightarrow A_i$ be a projective resolution, since colimit is exact, it preserves projective objects and preserves exactness, thus $colim P_i \longrightarrow colim A_i$ is still a projective resolution, thus $L_*F(colim A_i) = H_*(F(colim P_i))$ notice that F is left adjoint, thus by Thm 6.5, we have $H_*(F(colim P_i)) = H_*(colim(FP_i))$, again, use the fact colimit is exact, thus commutes with homology(exact functors commute with universal δ-functor though we don't have space to show it here), thus $H_*(colim(FP_i)) = colim H_*(F(P_i)) = colim L_*F(A_i)$ □

We end this subsection with a technical lemma we will use later that's also connected to colimit(direct limit).

Lemma 6.8. Given any R-module M, we have the set of all of it's finitely generated submodules $M_i \in I$, it's a directed set under inclusion. $colim_{\to} M_i \cong M$.

Proof. We only have to check that M satisfies the universal property of $colim_{\to} M_i$. Notice that we have inclusion maps into M of all the finitely generated submodules. Suppose that there is another module N and we are given $f_i : M_i \to N$ that commutes with the inclusions, if there is any map $f : M \to N$ that commutes with f_i, we only have one choice since if we want to know what is $f(a)$, notice that the submodule $R < a >$ generated by a has $f_{R<a>} : R < a > \to N$. It's easy to check that this is a module map since if we want to check $f(ra + tb)$, we only have to consider the submodule generated by a, b. □

6.2 Applications of Tor and Ext

Our first application will be to use Ext to determine projective and injective objects:

Proposition 6.9. TFAE:

1. B is an injective module.
2. $Hom_R(-, B)$ is an exact functor.
3. $Ext^i_R(A, B) = 0$ for all $i \geqslant 1$ and all R-modules A.
4. $Ext^1_R(A, B) = 0$ for all R-modules A.

Proof. We already mentioned 1) is equivalent to 2).
1)$\Longrightarrow$3): since B is injective, $Ext = RHom_R(A, -)(B)$, thus we can choose the injective resolution to be $0 \to B \to B \to 0$, then it easily follows.
3)$\Longrightarrow$4) is trivial.

4)$\Longrightarrow$2): Suppose that we have a SES: $0 \to D \to E \to F \to 0$, recall 5.8, consider the LES associated to it, we get:

$$\cdots \longrightarrow 0 \longrightarrow Hom_R(F,B) \longrightarrow Hom_R(E,B) \longrightarrow Hom_R(D,B) \longrightarrow Ext^1_R(F,B) = 0 \longrightarrow \cdots$$

We are done. □

Remark 6.10. Similarly, TFAE:

1. A is an projective module.
2. $Hom_R(A,-)$ is an exact functor.
3. $Ext^i_R(A,B) = 0$ for all $i \geqslant 1$ and all R-modules B.
4. $Ext^1_R(A,B) = 0$ for all R-modules B.

Now let's explain why Tor is called Tor form the point of view that it can be used to extract the torsion part of a abelian group. For simplicity, now let's focus on abelian groups for a while:

Lemma 6.11. Suppose that B is an abelian group then $Tor_0(\mathbb{Z}/p\mathbb{Z}, B) = B/pB$ and $Tor_1(\mathbb{Z}/p\mathbb{Z}, B)B_p$, $Tor_i(\mathbb{Z}/p\mathbb{Z}, B) = 0$ for all $i \geqslant 2$, where B_p consists of all elements x in B such that $px = 0$.

Proof. Choose a projective resolution to $0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$ and do computation. □

Proposition 6.12. For all abelian groups A and B:

1. $Tor_1(A,B)$ is a torsion abelian group.
2. $Tor_n(A,B) = 0$ for $n \geqslant 2$.

Proof. First notice that all abelian groups are the direct limit of all it's finitely generated subgroups. Recall prop 6.7, Tor commutes with direct limit, thus we have $Tor_1(A,B) = Tor_1(colim_{\to}A_i, B) = colim_{\to}Tor_1(A_i, B)$ where A_i are it's finitely generated subgroup. Easy to see direct limit of torsion groups is a torsion group. Thus we only have to show this for finitely generated subgroups. Using the structure theorem of finitely generated abelian groups, we can assume that $A = \mathbb{Z}^n \oplus \mathbb{Z}/p_1\mathbb{Z} \oplus \cdots \mathbb{Z}/p_r\mathbb{Z}$, recall that Tor commutes with direct sums (prop 6.7)and the previous lemma 6.11, we know that $Tor_1(\mathbb{Z}^n \oplus \mathbb{Z}/p_1\mathbb{Z} \oplus \cdots \mathbb{Z}/p_r\mathbb{Z}, B) = \oplus_i B_{p_i}$. The second assertion follows using similar arguments. □

Proposition 6.13. $Tor_1(\mathbb{Q}/\mathbb{Z}, B)$ is the torsion group of B for every abelian group B.

Proof. Notice that each finitely generated subgroup of $\mathbb{Q}/\mathbb{Z}$ is torsion, thus according to structure theorem of finitely generated abelian groups, it's a finite group, also notice that if we identity $\mathbb{Q}/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z} = S^1 \subset \mathbb{C}$, we know that every multiplicative finite subgroup of a field is cyclic, thus it's of the form $\mathbb{Z}/k\mathbb{Z}$ for some positive integer k. Thus we have $Tor_1(\mathbb{Q}/\mathbb{Z}, B) = colim_{\to}Tor_1(\mathbb{Z}/k\mathbb{Z}, B) = colim_{\to}B_k$, notice that here the direct limit is taken over all positive integers with $a < b$ iff $a|b$, thus $colim_{\to}B_k$ is isomorphic to the torsion subgroup of B as you can easily check the universal property. □

Corollary 6.14. Suppose that A is an abelian group:

$$Tor_1(A,-) = 0 \Leftrightarrow A \text{ is torsion free} \Leftrightarrow Tor_1(-,A) = 0$$

Then let's see how Tor can interact with flatness:

Proposition 6.15. Let P be a projective module, then it's flat.

Proof. The proof is easy since we already have what we need to simplify the question: In last section, we saw that $Tor_n^R(A,P) = 0$ for $n \neq 0$ if P is a projective module. Consider the LES asociated to Tor and $0 \to A \to B \to C \to 0$, we get:

$$Tor_{n+1}^R(C,P) \longrightarrow Tor_n^R(A,P) \longrightarrow Tor_n^R(B,P) \longrightarrow Tor_n^R(C,P) \longrightarrow Tor_{n-1}^R(A,P)$$

Notice that when $n = 0$, we get $Tor_0^R(A,P) \cong A \otimes_R P$ and $n \neq 0$, they are all 0! Therefore, the following SES is exact:

$$0 \to A \otimes_R P \to B \otimes_R P \to C \otimes_R P \to 0$$

□

Following similar arguments, we have a characterization of flatness using Tor:

Proposition 6.16. Let B be a R-module, then the following are equivalent:

1. B is flat.
2. $Tor_n^R(A,B) = 0$ for all $n \neq 0$ and all R-module A.
3. $Tor_1^R(A,B) = 0$ for all R-module A.

Proof. 1)$\Longrightarrow$2):Choose a projective resolution of A, tensor B, it remains exact since B flat, take homology, we get what we want.
2)$\Longrightarrow$3):Trivial.
3)$\Longrightarrow$1):Essentially what we showed in the previous proposition. □

Lemma 6.17. If $0 \to A \to B \to C \to 0$ is exact with B and C flat, then A is flat.

Proof. Take the associated LES associated to this SES and Tor, then use Prop 6.16. □

As a final application of how Tor can interact with flatness, we prove such a result that can be otherwise very complicated without Tor. First as a preparation:

Lemma 6.18. Suppose that I is an ideal of R, then we have a natural isomorphism $R/I = colim_{\to}(R/I_i)$ where I_i ranges over all of the finitely generated subideals of I with them directed by inclusion.

Proof. Notice that by lemma 6.8, we have $R/I = R/(colim_{\to} I_i)$, thus we only have to show that $R/(colim_{\to} I_i) = colim_{\to}(R/I_i)$, in the later term we know that if $I_i \subset I_j$ there is a natural morphism $R/I_i \to R/I_j$, thus it's still directed by inclusion. Consider the following SES for each I_i:

$$0 \longrightarrow I_i \longrightarrow R \longrightarrow R/I_i \longrightarrow 0$$

Notice that when I_i ranges over all finitely generated subideals, these SES assembles into a SES of functors that goes from R-Modules$^{\{I_i\}}$ to R-Modules, thus recall Thm 6.6, we have the following is still a SES:

$$0 \longrightarrow colim_{\to} I_i \longrightarrow R \longrightarrow colim_{\to}(R/I_i) \longrightarrow 0$$

We are done. □

The following criterion for flatness is what we want to prove:

Theorem 6.19. Flat Module Criterions Let R be a ring, and M is a R-module, TFAE:

1. M is a flat R-module.
2. $Tor_1^R(M, R/I) = 0$ for all ideals I of R.
3. The canonical map $I \otimes_R M \to M$ is injective for all ideals I of R.
4. The canonical map $I \otimes_R M \to M$ is injective for all finitely generated ideals I of R.

Proof. 1)$\Longrightarrow$2): Trivial by corollary 4.5.
2)$\Longrightarrow$3): Consider the SES: $0 \to I \to R \to R/I \to 0$ and the LES associated to it.
3)$\Longrightarrow$1): This is standard results, and you can find it in 10.5.Exercise 25 [DF04]. The hint we give here based on what we have developed is that in part b) of that exercise, we only need to consider finitely generated free modules since any free module $R^{\oplus I}$ is the direct limit of $R^{\oplus J}$ where J is a finite subset of I and Tor commutes with colimits, thus if we want to show that $Tor_1^R(M, F/K) = 0$ where $F = R^I$ is a free R module and K is a submodule. Since we know that any element of K lies in some R^J where J is a finite subset of I, then $K = colim_{\to} K_J$ where $K_J = K \cap R^J$ and $F = colim_{\to} R^J$, thus we have $R/K = colim_{\to}(F_J/K_J)$ following similar argument in lemma 6.18. To show this holds for finitely generated free R-modules, use induction on the rank of F, when the rank is 1, it's just what we have. When the rank is $n+1$, consider the following diagram with exact rows, tensor with M, use the induction hypothesis and snake lemma to see what you can conclude:

$$\begin{tikzcd}
0 \arrow[r] & K \cap R^n \arrow[r, "I"] \arrow[d, hook] & K \arrow[r, "\pi"] \arrow[d, hook] & \pi(K) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & R^n \arrow[r, "I"] & R^{n+1} \arrow[r, "\pi"] & R \arrow[r] & 0
\end{tikzcd}$$

3)$\Longrightarrow$4):Trivial.
4)$\Longrightarrow$2): From 4), we can easily see that $Tor_1(M, R/I) = 0$ for finitely generated I, then use lemma 6.18 to conclude that it holds for every ideal I. $\square$

At this point, our application of Ext is less than Tor, let's just consider one of them, which is related to why Ext is called Ext:

Definition 6.20. Extension Suppose that we work in R-modules. An *extension* ξ of A by B is a SES $0 \to B \to X \to A \to 0$. We will say two extensions are *equivalent* if there is a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, dashed, "id"] & X \arrow[r] \arrow[d, dashed, "iso"] & A \arrow[r] \arrow[d, dashed, "id"] & 0 \\
0 \arrow[r] & B \arrow[r] & X' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

An extension is *split* if it's equivalent to $0 \to B \to B \oplus A \to A \to 0$.

Example 6.21. As an example of extension, we will show that there are exactly p equivalent classes of extensions of $\mathbb{Z}/p\mathbb{Z}$ if p is a prime. Suppose that we have a extension $0 \to \mathbb{Z}/p\mathbb{Z} \to X \to \mathbb{Z}/p\mathbb{Z} \to 0$, it's easy to see that X is a finitely generated abelian group of p^2 elements, thus we know that it's either isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ or to $\mathbb{Z}/p^2\mathbb{Z}$ by the structure theorem, since we are allowed to modify this X by isomorphisms we can assume that $\mathbb{Z}/p\mathbb{Z}$ is injects to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Z}/p^2\mathbb{Z}$, the rest is easy as you can easily figure out what are the maps to $\mathbb{Z}/p\mathbb{Z}$.

Lemma 6.22. If $Ext_R^1(A, B) = 0$ then every extension of A by B is split.

Proof. Given an extension $0 \to B \to X \to A \to 0$, consider the LES associated to it when we apply $Ext_R^*(A, -)$, which is consists of a part $Hom_R(A, X) \to Hom_R(A, A) \xrightarrow{\partial} Ext_R^1(A, B) = 0$ thus we know that there will be a element $\sigma : A \to X$ such that it's a section of $X \to A$, thus $X \cong A \oplus B$. $\square$

Remark 6.23. Well-definedness of $\Theta(\xi)$ We will denote the class of ∂id_A by $\Theta(\xi)$. It is called the obstruction of ξ being split. ξ is split iff $\Theta(\xi)$ vanishes. Also, suppose that we have a equivalent extension ξ', then by the naturality of ∂, we have:

$$\begin{tikzcd}
Hom_R(A, X) \arrow[r] \arrow[d] & Hom_R(A, A) \arrow[r, "\partial"] \arrow[d, "id"] & Ext_R^1(A, B) \arrow[d, "id"] \\
Hom_R(A, X') \arrow[r] & Hom_R(A, A) \arrow[r, "\partial"'] & Ext_R^1(A, B)
\end{tikzcd}$$

Thus $\Theta(\xi) = \Theta(\xi')$ is also an obstruction of ξ' and this obstruction only depends on equivalent classes of ξ. Thus it's well defined.

Theorem 6.24. Extension and Ext Given two R-modules A and B, we have an one-to-one correspondence:

$$\{\text{equivalent classes of extension of } A \text{ by } B\} \leftrightarrow Ext^1_R(A,B)$$

where the split extension correspond to the element 0 in $Ext^1_R(A,B)$. Explicitly, the class of ξ is mapped to $\Theta(\xi)$.

Proof. The proof suggested here [Wei94] doesn't fan out as Weibel intended without a prerequisite result which we won't mention here, instead we will use it's dual version to avoid that minor issue. You can still check Weibel's arguments and you will find it's literally dual to the arguments here.
First, choose a injective objects I and an injection $B \to I$, then we obtain a SES $0 \to B \to I \xrightarrow{\pi} N \to 0$ where N is the quotient. Recall prop 6.9, when we consider the associated LES after apply Ext, we get:

$$Hom_R(A,I) \longrightarrow Hom_R(A,N) \xrightarrow{\partial} Ext^1_R(A,B) \longrightarrow Ext^1_R(A,I) = 0$$

Thus, we have ∂ is surjective and suppose that we have $x \in Ext^1_R(A,B)$, we can choose a $\gamma \in Hom_R(A,N)$ such that $\partial\gamma = x$. Now consider the pullback(fiber-product), of $\gamma : A \to M$ and $\pi : I \to N$ and we put it in a diagram:

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, dashed, "id"] & X \arrow[r, "\sigma"'] \arrow[d, "\tau"] & A \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & B \arrow[r] & I \arrow[r, "\pi"] & N \arrow[r] & 0
\end{tikzcd}$$

where $B \to X$ is induced by $B \to I$ and $B \xrightarrow{0} A$, thus this diagram is commutative. Notice that X is a submodule of $I \oplus A$ where $(x,y) \in X$ satisfies $\tau(x) = \sigma(y)$. Then a diagram chase shows that the top row is indeed exact. Thus, apply $Ext^*_R(A)$ and use the naturality of ∂, we have:

$$\begin{tikzcd}
Hom(A,X) \arrow[r] \arrow[d] & Hom(A,A) \arrow[r, "\partial"] \arrow[d, dashed, "\gamma*"] & Ext^1_R(A,B) \arrow[d, dashed, "id"] \\
Hom(A,I) \arrow[r] & Hom(A,N) \arrow[r, "\partial"] & Ext^1_R(A,B)
\end{tikzcd}$$

Thus, we see that γ and id_A are both mapped to x, thus we see that Θ is surjective.
The above construction gives another map Ψ, which is a set theoretic map from $Ext^1_R(A,B)$ to the set of equivalent classes of extensions. To see this, given a $x \in Ext^1_R(A,B)$, we already constructed a ξ above, we need to show that this is indepedent of the lift. Suppose that $\gamma' : A \to N$ is another lift of x. Then we know that their diffenrence is in the kernel of ∂, thus use exactness in that LES we obtained above, we have a $g \in Hom_R(A,I)$ such that $\pi g = \gamma' - \gamma$. Then consider the the following diagram where $X \to X'$ is induced by the universal property of pullback:

$$\begin{tikzcd}
X \arrow[dr, dashed] \arrow[drr, bend left, "\sigma"] \arrow[ddr, bend right, "g\sigma+\tau"'] & & \\
& X' \arrow[r, "\sigma'"] \arrow[d, dashed, "\tau'"] & A \arrow[d, dashed, "\gamma'"] \\
& I \arrow[r, "\pi"] & N
\end{tikzcd}$$

We claim that the induced map is an isomorphism. Indeed, if $(x,y) \in X$, then it's mapped to $(x+gy, y) \in X'$, thus obviously injective. Also surjective since if we want $(a,b) \in X$, then clealy $(a+gb, b)$ is mapped to it. Also, we have the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, dashed, "id"] & X \arrow[r, "\sigma"] \arrow[d, dashed] & A \arrow[r] \arrow[d, dashed, "id"] & 0 \\
0 \arrow[r] & B \arrow[r] & X' \arrow[r, "\sigma'"] & A \arrow[r] & 0
\end{tikzcd}$$

This is because that $b \in B$ is mapped to $(b,0)$ $(b,0)$, then the induced map will map $(b,0)$ to $(b,0)$ On the other hand, if we are given a extension ξ, lifting property of injective objects gives:

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, hook] & I \arrow[r] & N \arrow[r] & 0 \\ 0 \arrow[r] & B \arrow[r, hook] \arrow[u, dashed, "id"] & X \arrow[r] \arrow[u, dashed] & A \arrow[r] \arrow[u, dashed] & 0 \end{tikzcd}$$

Here, X is the pullback of $I \to N$ and $A \to N$. To see this, call $f : X \to I$ $g : X \to A$ $h : I \to N$ and $k : A \to N$ we will construct a isomorphism between X and the submodule of $P = I \oplus A$ where $(x,y) \in P$ satisfies that their image in N equal. $X \to P$ is defined to be $x \mapsto (f(x), g(y))$ it's injective since if $f(x) = 0$ and $g(y) = 0$ the first $x = 0$ and $y \in B$ then use the commutativity we see that y is also 0. Then suppose that $(i,a) \in P$, we first can find a x such that $g(x) = a$ consider $i - f(x) \in B$, thus we can find a $b \in B$ such that $i - f(x) = b$, thus $i = f(x) + b$, since $gb = 0$, then we have $b + x$ is mapped to (i,a), we are done. Thus, $\Psi(\Theta(\xi)) = \xi$ by definition, thus Θ is injective, we are done. □

6.3 Universal Coefficient Theorem

The first result we want to show is Künneth Formula which helps us compute homology of the tensor product using the original homology.

Theorem 6.25. Künneth Formula for Homology Let P be a chain complex of flat R-modules such that each submodule $d(P_n) = P_{n-1}$ is also flat. Then for every n and every flat R-module M, we have a SES:

$$0 \longrightarrow H_n(P) \otimes_R M \longrightarrow H_n(P \otimes_R M) \longrightarrow Tot_1^R(H_{n-1}(P), M) \longrightarrow 0$$

Proof. Consider the SES: $0 \longrightarrow Z_n \longrightarrow P_n \longrightarrow d(P_n) \longrightarrow 0$, recall lemma 6.17, Z_n is also flat. Notice that $Tor_1^R(dP_n, M) = 0$, we know $0 \longrightarrow Z_n \otimes_R M \longrightarrow P_n \otimes_R M \longrightarrow d(P_n) \otimes_R M \longrightarrow 0$ is exact by examining the LES associated to it. Thus, we have a SES of chain complexes, $0 \longrightarrow Z \otimes_R M \longrightarrow P \otimes_R M \longrightarrow d(P) \otimes_R M \longrightarrow 0$. Take the associated LES of homology, we get:

$$\begin{tikzcd} H_{n+1}(dP \otimes_R M) \arrow[r, "\partial"] \arrow[d, leftrightarrow] & H_n(Z \otimes_R M) \arrow[r] \arrow[d, leftrightarrow] & H_n(P \otimes_R M) \arrow[r] & H_n(dP \otimes_R M) \arrow[r, "\partial"] \arrow[d, leftrightarrow] & H_{n-1}(Z \otimes_R M) \arrow[d, leftrightarrow] \\ dP_{n+1} \otimes_R M & Z_n \otimes_R M & & P_n \otimes_R M & Z_{n-1} \otimes_R M \end{tikzcd}$$

Here, since differential in dP and Z are 0, the homology of $dP \otimes_R M$ and $Z \otimes_R M$ are themselves. Recall remark 5.2, suppose that we have $a \otimes b \in dP_{n+1} \otimes_R M$, a cycle, then notice that $a \otimes b \in P_{n+1} \otimes_R M$ is a lift, apply the differential in $P_{n+1} \otimes_R M$, we get $da \otimes b$ which represents the image in $H_n(Z \otimes_R M) = Z_n \otimes_R M$, thus $\partial = \iota \otimes_R M$ where ι is the inclusion of $d(P_{n+1})$ to Z_n.
Finally, notice that we have a flat resolution: $0 \to d(P_{n+1}) \xrightarrow{\iota} Z_n \longrightarrow H_n(P) \longrightarrow 0$, tensor with M we get $0 \to d(P_{n+1}) \otimes_R M \xrightarrow{\partial} Z_n \otimes_R M \longrightarrow H_n(P) \otimes_R M \longrightarrow 0$ is still exact, hence $H_n(P) \otimes_R M \cong (Z_n \otimes_R M)/(d(P_{n+1}) \otimes_R M)$. On the other hand, recall Prop 5.10, we can use flat resolution to compute Tor, thus $Tor_1^R(H_n(P), M)$ is the 1-st degree homology of $0 \to d(P_{n+1}) \otimes M \xrightarrow{\partial} Z_n \otimes M \to 0$, which is $Ker\partial$. We modify the LES we get in the beginning in an obvious way to get the following SES:

$$0 \to H_n(P) \otimes M = \frac{Z_n \otimes M}{Im\partial} \to H_n(P \otimes M) \to Ker(\partial) = Tor_1^R(H_{n-1}(P), M) \to 0$$

as desired. □

Using this formula, we can state one version of the universal coefficient theorem, the result of main interest in this paper which also turns out to be a corollary of Künneth formula. Here we only consider the case of Abelian groups, since we will use the fact that any subgroup of a free Abelian group is also free:

Theorem 6.26. Universal Coefficient Theorem for Homology Let P be a chain complex of free abelian groups, then for every n and every Abelian group M, Künneth formula above splits noncanonically, yielding a direct sum decomposition:

$$H_m(P \otimes M) \cong (H_n(P) \otimes M) \oplus (Tot_1^{\mathbb{Z}}(H_{n-1}(P), M))$$

Proof. Notice $d(P_n)$ is a subgroup of P_{n-1}, thus free, thus flat and projective. This implies that the surjection $P_n \to d(P_n)$ splits, thus we have a noncanonical decomposition $P_n \cong Z_n \oplus d(P_n)$. Tensor with M, we get $P_n \otimes M \cong Z_n \otimes M \oplus d(P_n) \otimes M$, thus $Z_n \otimes M$ is a direct summand of $P_n \otimes M$. Notice that we have: $Z_n \otimes M \subset P_n \otimes M$ and $Ker(P_n \otimes M \longrightarrow P_{n-1} \otimes M) \subset P_n \otimes M$ and obviously $Z_n \otimes M$ will be mapped to 0, thus we have $Z_n \otimes M \hookrightarrow Ker \hookrightarrow P_n \otimes M \cong Z_n \otimes M \oplus dP_n \otimes M$, thus we have $Z_n \otimes M$ is also a direct summand of $Ker(P_n \otimes M \longrightarrow P_{n-1} \otimes M)$ since it's a easily prove fact that if $A \subset B \subset C$ are Abelian groups and A is a direct summand of C, then A is a direct summand of B(hint: think of the definition and take intersection). Thus $Ker = Z_n \otimes M \oplus C$ for some C a subgroup of $P_n \otimes M$. Notice that $Im(d_{n+1} \otimes 1) = Im(d_{n+1}) \otimes M$, which is a subgroup of $Z_n \otimes M$, then we get:

$$H_n(P \otimes M) = \frac{Ker(d_n \otimes 1)}{Im(d_{n+1} \otimes 1)} = \frac{Z_n \otimes M \oplus C}{Im(d_{n+1} \otimes 1)} \cong H_n(P) \otimes M \oplus C$$

since if we have $0 \to A \hookrightarrow B \to B/A \to 0$, then tensor is right exact, leading to $A \otimes M \to B \otimes M \to (A/B) \otimes M \to 0$.
Finally, consider Künneth formula:

$$0 \longrightarrow H_n(P) \otimes M \longrightarrow H_n(P \otimes M) \cong H_n(P) \otimes M \oplus C \longrightarrow Tor_1^{\mathbb{Z}}(H_{n-1}(P), M) \longrightarrow 0$$

Easy to check the the map on the right end is the inclusion of $H_n(P) \otimes M$ into $H_n(P \otimes M)$ by how we defined it in Thm 6.25, thus $Tor_1^{\mathbb{Z}}(H_{n-1}(P), M) \cong C$, as desired. □

The Künneth Formula Thm 6.25, we gave above is a special case of the following result that's really called Künneth Formula.

Theorem 6.27. Let P and Q be R-modules, let $P \otimes_R Q$ be the direct sum total chain complex of tensor product double chain complex defined in definition 4.11. Then if P_n and dP_n are flat for each n, then we have a SES:

$$0 \to \bigoplus_{p+q=n} H_p(P) \otimes_R H_q(Q) \to H_n(P \otimes Q) \to \bigoplus_{p+q=n} Tor_1^R(H_p(P), H_q(Q)) \to 0$$

Moreover, if $R = \mathbb{Z}$ and P is a complex of free abelian groups, this SES splits noncanonically.

Proof. The proof is basically an adaption of the proof of Thm 6.25 once we have make sense of what is Tor of a chain complex and what is the flatness for a chain complex, which you can find in Exercise 10.9 and Exercise 10.10 in [RR09] and go to Theorem 10.82 in [RR09]. □

Finally, we end our paper by briefly discussing the Künneth Formula for cohomology and the universal theorem for cohomology, this will be the analogue of the Künneth Formula for Hom instead of $\otimes$.

Theorem 6.28. Universal Coefficient Theorem for Homology Let P be a chain complex of projective R-modules such that each dP_n is also projective, then for every n and every R-module M, there is a noncanonically split exact sequence:

$$0 \longrightarrow Ext_R^1(H_{n-1}(P), M) \longrightarrow H^n(Hom_R(P, M)) \longrightarrow Hom_R(H_n(P), M) \longrightarrow 0$$

Proof. First, this is indeed an exact sequence: first notice that since P_n and dP_n are projective we have $0 \to Z_n \to P_n \to dP_n \to 0$ splits thus Z_n is a summand of P_n, thus Z_n is also projective. Then consider the following exact sequence:

$$0 \longrightarrow Hom_R(dP_n, M) \longrightarrow Hom_R(P_n, M) \longrightarrow Hom_R(Z_n, M) \longrightarrow 0$$

It's exact since $P_n \cong dP_n \oplus Z_n$ and use Hom commutes with direct sum(since Hom is a right adjoint thus commutes with limits, here it's direct product. However, in R-modules, finite direct product is finite direct sum). Then everything is exactly the argument in Thm 6.25 with homology replaced by cohomology.
To see it's a split SES, we already know that $P_n \cong Z_n \oplus dP_n$, which is the key ingredient in Thm 6.26 □

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